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Optimization of Digital Communication Systems

by

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CONTENTS

1. INTRODUCTION

The complex relationships between the design parameters of communication systems and their physical burdens in terms of their fabrication cost, weight, volume, power requirement, etc., create the need for a unified approach to the optimum design of communication systems. This report outlines an optimization methodology which provides the systems designer with the optimum values of the major parameters of a communication system. These major system parameters are defined to be:

transmitter antenna diameter or gain

receiver antenna diameter or gain

transmitter power

receiver field of view

The optimization technique is extensible to include other design parameters such as receiver temperature or transmission wavelength. But, for consiseness, and for practical reasons to be discussed later, only the major system parameters listed above will be considered in this report.

The optimization methodology is applicable for optical as well as radio systems. In principle any type of modulation or demodulation can be handled if some suitable performance criterion is available. This report discusses the optimization procedure for the most common and practical combinations of digital modulation and detection techniques. For these systems the performance criterion is the probability of detection error. The methodology may include one way links, two way links, and a net of several transmitters and receivers.

Basically, the optimization procedure is to develop system cost relationships as a function of the values of the system parameters. These cost relationships include the fabrication cost of the system components, the cost of placing the components aboard a spacecraft, if necessary, on a cost per unit weight basis, and any other pertinent system costs. This phase of the optimization procedure is, in many respects, the most difficult since in many cases it requires technological predictions. However, a great amount of parametric cost burden data has been gathered for many system components [1-4]. With the cost relationships developed the total system cost is minimized as a function of the values of the major system parameters under the constraint that the performance criterion is achieved.

The communication component burden relationships employed in the optimization procedure may be modeled by power series or be specified numerically. The only requirements are that the burden relationships be monotonic, single-valued, piece-wise differentiable functions of the system parameters. These conditions are usually fulfilled for the four major system parameters listed previously. The conditions are generally not met when attempts are made to express burdens as a function of transmission wavelength. Search procedures are often necessary for transmission wavelength optimization.

The following sections summarize typical burden relationships for communication system components. The burdens are expressed in terms of antenna diameter only. Conversion to gain burdens can be made through the use of formulas given in Appendix A.

Transmitter Antenna. The weight and fabrication cost of a transmitter antenna system are dependent upon the transmitter antenna diameter. A transmitter antenna is usually designed to operate as close to the diffraction limit as possible to achieve the greatest spatial power density at the receiver for a given transmitter aperture diameter. For small transmitter apertures, the weight is proportional to the antenna area, and hence to the square of the aperture diameter. For larger size apertures, as structural supports are added to maintain the rigidity required for diffraction limited operation, the weight dependence becomes volumetric.

Receiver Antenna. The weight and fabrication cost of a receiver antenna system are dependent upon the receiver antenna aperture diameter. At optical frequencies receiver antennas are not normally designed to be diffraction limited, and hence construction and mechanical support tolerances need not be as stringent as for a transmitter antenna.

Transmitter Antenna Pointing System. The transmitter antenna pointing system consists of a gimballed support unit, which points the transmitter antenna toward the receiver. The weight of the transmitter antenna pointing system is relatively insensitive to the transmitter pointing accuracy. Its weight is proportional to the weight of the transmitter antenna, whose weight is dependent upon the transmitter antenna diameter. The fabrication cost of the transmitter pointing equipment is inversely proportional to the transmitter pointing accuracy. The pointing accuracy is usually specified as a fixed percentage of the transmitter bandwidth. Since the transmitter antenna is diffraction limited, the fabrication cost is proportional to the transmitter aperture diameter. The electrical power requirement for the transmitter antenna pointing system is primarily dependent upon the weight of the transmitter antenna.

Receiver Pointing System. The weight of the receiver pointing system is relatively insensitive to the receiver pointing accuracy. Its weight is proportional to the weight of the receiver antenna, which is itself dependent upon the receiver aperture diameter. The fabrication cost of the receiver pointing equipment is inversely proportional to the receiver pointing accuracy, which is a fixed percentage of the receiver field of view. The power supply requirement for the receiver pointing system is primarily dependent upon the weight of the receiver antenna.

Transmitter. For a given transmission wavelength, within limits, the weight and fabrication cost of a transmitter are dependent upon the transmitter power. The electrical input power requirement is directly proportional to the transmitted power.

Transmitter System Power Supply. The fabrication cost and weight of the electrical power supply and conversion equipment at the transmitter are dependent upon the electrical power requirements of the transmitter antenna pointing system, transmitter, and modulator.

Receiver System Power Supply. The fabrication cost and weight of the electrical power supply and conversion equipment at the receiver are dependent upon the power requirements of the receiver pointing system and communications receiver equipment.

2. CLASSIFICATION OF SYSTEMS

In this report communication systems are classified according to the following categories:

Transmission wavelength

radio

optical

Modulation method

PCM amplitude modulation

PCM polarization modulation

PCM frequency modulation

PCM phase modulation

Demodulation method

direct

heterodyne

homodyne

Type of noise

thermal

background radiation

shot

The transmission wavelength division is commonly taken at about 100 microns. For wavelengths shorter than 100 microns the transmitter is usually a laser, the antennas are made of polished reflectors or transparent lenses, and the carrier demodulator is a photodetector. At the radio wavelengths a variety of transmitter oscillators are available, the antennas are generally metal reflectors, horns, or wire assemblies, and the detector is a nonlinear electrical element.

Not all combinations of modulation and demodulation methods are feasible at all transmission wavelengths. Polarization modulation is limited to the optical region because of difficulties in constructing radio frequency polarization modulators. Phase modulation systems must employ a homodyne receiver to perform demodulation.

At radio frequencies noise is principally caused by two physical sources, thermal noise at the antenna load and background radiation from external sources. Both types of noise may be modeled by Gaussian statistics. Optical receiver noise is caused by thermal noise of the photodetector load resistor and resistive elements within the detector, and by detector shot noise which is caused by the randomness of electron emissions induced by laser carrier radiation, background radiation, and detector dark current. Shot noise is modeled by Poisson statistics. In an optical direct detection receiver if the photodetector has an internal current gain mechanism, detector shot noise is usually dominant, otherwise thermal noise predominates. In a heterodyne or homodyne optical receiver the local oscillator power can be made large to achieve shot noise limited operation even without photodetector gain.

For the optimization analysis, communication systems have been divided into four types which are described below.

ROPS - radio communication systems, thermal and background radiation noise, Gaussian statistics.

TOPS - optical communication systems, direct detection, thermal noise, Gaussian statistics.

SOPS - optical communication systems, direct detection, shot noise, Poisson statistics.

HOPS - optical communication systems, heterodyne or homodyne detection, shot noise, Poisson statistics.

The appendicies describe the operation of the four types of communication systems in greater detail.

3. OPTIMIZATION OF ONE WAY LINKS

The first optimization case to be considered is the one way transmission from a transmitter station to a receiver station. Let x, y, z, w represent a set of four physical parameters of the communication system to be optimized, e.g., transmitter antenna gain or diameter, receiver antenna gain or diameter, transmitter power, and receiver field of view. The probability of detection error, P, [5, 6] may then be expressed in terms of the system parameters as*

$$P = f_1(x, y, z, w) \tag{1}$$

Likewise, the total system cost, C, is another function of the system parameters.

$$C = f_2(x, y, z, w)$$

Let P^R be the required probability of detection error. Then, by the method of Lagrange multipliers, to minimize the total system cost and achieve P^R, the dummy function C' is formed [7]

$$C' \equiv C + \Lambda (P^R - P)$$

where Λ is the Lagrange multiplier. Now, setting the partial derivatives of C', with respect to the system parameters, equal to zero yields

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* $f_m(\cdot)$ represents a general function.

$$\frac{\partial C'}{\partial x} = \frac{\partial C}{\partial x} - \Lambda \frac{\partial P}{\partial x} = 0$$

$$\frac{\partial C'}{\partial y} = \frac{\partial C}{\partial y} - \Lambda \frac{\partial P}{\partial y} = 0$$

$$\frac{\partial C'}{\partial z} = \frac{\partial C}{\partial z} - \Lambda \frac{\partial P}{\partial z} = 0$$

$$\frac{\partial C'}{\partial w} = \frac{\partial C}{\partial w} - \Lambda \frac{\partial P}{\partial w} = 0$$

Equating the Λ's gives a set of six characteristic equations.

$$\frac{\partial C}{\partial x}\frac{\partial P}{\partial y} - \frac{\partial C}{\partial y}\frac{\partial P}{\partial x} = 0$$

$$\frac{\partial C}{\partial x}\frac{\partial P}{\partial z} - \frac{\partial C}{\partial z}\frac{\partial P}{\partial x} = 0$$

$$\frac{\partial C}{\partial x}\frac{\partial P}{\partial w} - \frac{\partial C}{\partial w}\frac{\partial P}{\partial x} = 0$$

$$\frac{\partial C}{\partial y}\frac{\partial P}{\partial z} - \frac{\partial C}{\partial z}\frac{\partial P}{\partial y} = 0$$

$$\frac{\partial C}{\partial y}\frac{\partial P}{\partial w} - \frac{\partial C}{\partial w}\frac{\partial P}{\partial y} = 0$$

$$\frac{\partial C}{\partial z}\frac{\partial P}{\partial w} - \frac{\partial C}{\partial w}\frac{\partial P}{\partial z} = 0$$

Any subset of three of these equations solved simultaneously with equation (1) for the required probability of detection error gives the optimum solution of the system parameters. In a particular optimization problem one or more of the system parameters may be held fixed either by desire or because of technological limitations. In this situation the characteristic equations containing the fixed parameters are merely deleted from the simultaneous solution. For some optimization problems it is possible to solve the characteristic equations analytically, but usually recursive digital techniques are required.

For many communication systems the probability of error is related monotonically and uniquely to the signal-to-noise ratio, $\frac{S}{N}$, measured at some point in the communication receiver.

$$P = f_3 \left(\frac{S}{N} \right)$$

The signal-to-noise ratio can then be written as a function of the system parameters

$$\frac{S}{N} = f_4(x, y, z, w) \tag{2}$$

The characteristic equations, for such systems, then reduce to

$$\frac{\partial C}{\partial x} \frac{\partial(\frac{S}{N})}{\partial y} - \frac{\partial C}{\partial y} \frac{\partial(\frac{S}{N})}{\partial x} = 0$$

$$\frac{\partial C}{\partial x} \frac{\partial(\frac{S}{N})}{\partial z} - \frac{\partial C}{\partial z} \frac{\partial(\frac{S}{N})}{\partial x} = 0$$

$$\frac{\partial C}{\partial x} \frac{\partial(\frac{S}{N})}{\partial w} - \frac{\partial C}{\partial w} \frac{\partial(\frac{S}{N})}{\partial x} = 0$$

$$\frac{\partial C}{\partial y} \frac{\partial(\frac{S}{N})}{\partial z} - \frac{\partial C}{\partial z} \frac{\partial(\frac{S}{N})}{\partial y} = 0$$

$$\frac{\partial C}{\partial y} \frac{\partial(\frac{S}{N})}{\partial w} - \frac{\partial C}{\partial y} \frac{\partial(\frac{S}{N})}{\partial w} = 0$$

$$\frac{\partial C}{\partial z} \frac{\partial(\frac{S}{N})}{\partial w} - \frac{\partial C}{\partial w} \frac{\partial(\frac{S}{N})}{\partial z} = 0$$

A simultaneous solution of these equations with equation (2) for the required value of $\frac{S}{N}$ to achieve the desired probability of detection error gives the optimum system parameters.

The following sections describe the characteristic equations for the four major types of communication systems.

3.1 ROPS One Way Links

For all of the major types of radio frequency communication systems listed in Table D-1, the probability of detection error is a unique, monotonic function of the signal-to-noise ratio at the input to the first detector of the receiver. Hence, the set of characteristic equations involving the SNR may be employed.

The receiver antenna of a radio frequency communication system is generally designed to approach the diffraction limit as closely as possible to maximize the receiver gain. For such systems the receiver field of view is inversely proportional to the receiver antenna diameter or gain. Hence, only three system parameters need be optimized; transmitter antenna diameter, d_T, or gain, G_T; receiver antenna diameter, d_R, or gain, G_R; and transmitter power, P_T. From appendix C.2 the SNR expressions for all combinations of d_T, G_T, d_R, and G_R are given by

$$\frac{S}{N} = K_{R1} d_T^2 d_R^2 P_T = K_{42} G_T d_R^2 P_T = K_{R3} d_T^2 G_R P_T = K_{R4} G_T G_R P_T \tag{3}$$

where the constants K_{Ri} are defined in Table C-1. The partial derivatives of the SNR expressions with respect to each of the system parameters are

$$\frac{\partial\left(\frac{S}{N}\right)}{\partial d_T} = \frac{2}{d_T}\left(\frac{S}{N}\right) \qquad \frac{\partial\left(\frac{S}{N}\right)}{\partial G_T} = \frac{1}{G_T}\left(\frac{S}{N}\right)$$

$$\frac{\partial\left(\frac{S}{N}\right)}{\partial d_R} = \frac{2}{d_R}\left(\frac{S}{N}\right) \qquad \frac{\partial\left(\frac{S}{N}\right)}{\partial G_R} = \frac{1}{G_R}\left(\frac{S}{N}\right)$$

$$\frac{\partial\left(\frac{S}{N}\right)}{\partial P_T} = \frac{1}{P_T}\left(\frac{S}{N}\right)$$

Then the sets of characteristic equations for optimization of all combinations of the system parameters are

d_T, d_R, P_T optimization

$$d_T \frac{\partial C}{\partial d_T} - 2P_T \frac{\partial C}{\partial P_T} = 0 \qquad (4)$$

$$d_T \frac{\partial C}{\partial d_T} - d_R \frac{\partial C}{\partial d_R} = 0 \qquad (5)$$

$$2P_T \frac{\partial C}{\partial P_T} - d_R \frac{\partial C}{\partial d_R} = 0$$

G_T, d_R, P_T optimization

$$G_T \frac{\partial C}{\partial G_T} - P_T \frac{\partial C}{\partial G_T} = 0$$

$$2G_T \frac{\partial C}{\partial G_T} - d_R \frac{\partial C}{\partial d_R} = 0$$

$$2P_T \frac{\partial C}{\partial P_T} - d_R \frac{\partial C}{\partial d_R} = 0$$

d_T, G_R, P_T optimization

$$d_T \frac{\partial C}{\partial d_T} - 2P_T \frac{\partial C}{\partial P_T} = 0$$

$$d_T \frac{\partial C}{\partial P_T} - G_R \frac{\partial C}{\partial G_R} = 0$$

$$P_T \frac{\partial C}{\partial P_T} - G_R \frac{\partial C}{\partial G_R} = 0$$

G_T, G_R, P_T optimization

$$G_T \frac{\partial C}{\partial G_T} - P_T \frac{\partial C}{\partial G_T} = 0$$

$$G_T \frac{\partial C}{\partial G_T} - G_R \frac{\partial C}{\partial G_R} = 0$$

$$P_T \frac{\partial C}{\partial P_T} - G_R \frac{\partial C}{\partial G_R} = 0$$

If the partial derivatives of the cost functions are relatively simple power series with integer exponents the characteristic equations can usually be inverted to obtain the optimum values of the system parameters. If inversion is not possible, a recursive solution can be performed. As an example of the recursive solution, consider the case of the optimization of d_T, d_R, and P_T. The procedure is listed below.

A. Select a trial value of d_T.

B. Find a value of P_T which satisfies (4)

C. Find a value of d_R which satisfies (5)

D. If (3) is satisfied for the triplet d_T, d_R, P_T terminate the procedure, and if not select a new trial value of d_T based upon the difference between the actual and desired signal-to-noise ratio.

3.2 TOPS One Way Links

The probability of detection error for thermal noise limited, direct detection, optical communication systems is dependent upon the signal-to-noise ratio at the receiver output. From Appendix C.3 the SNR is

$$\frac{S}{N} = K_T d_T^4 \, d_R^4 \, P_T^2$$

The partial derivatives with respect to system parameters are

$$\frac{\partial\left(\frac{S}{N}\right)}{\partial d_T} = \frac{4}{d_T}\left(\frac{S}{N}\right) \qquad \frac{\partial\left(\frac{S}{N}\right)}{\partial d_R} = \frac{4}{d_R}\left(\frac{S}{N}\right)$$

$$\frac{\partial\left(\frac{S}{N}\right)}{\partial P_T} = \frac{2}{P_T}\left(\frac{S}{N}\right)$$

Then the set of characteristic equations becomes

$$d_T \frac{\partial C}{\partial d_T} - 2P_T \frac{\partial C}{\partial P_T} = 0$$

$$d_T \frac{\partial C}{\partial d_T} - d_R \frac{\partial C}{\partial d_R} = 0$$

$$2P_T \frac{\partial C}{\partial P_T} - d_R \frac{\partial C}{\partial d_R} = 0$$

3.3 SOPS One Way Links

From Table C-2, the only shot noise limited, direct detection, optical, digital communication systems of practical interest are the systems employing amplitude and polarization modulation. The probability of error expression for the PCM/PL system can be written in closed form as a function of the carrier and shot noise power, but not as the carrier to shot noise ratio. Hence, optimization must be performed by the minimization of the probability of error expression. For the PCM/AM system, the probability of error expression is complicated by the fact that the series summation index is a function of the carrier and shot noise power. For this reason numerical partial differentiation is necessary to optimize the PCM/AM system.

The PCM/PL system is inherently more efficient in terms of transmitter power than the PCM/AM system, without appreciably greater equipment complexity. Optimization of this system is considered below.

PCM/PL Optimization. The probability of detection error for the shot noise limited, direct detection, PCM/PL, optical communication system is of the form

$$P = \frac{1}{2} [1 + Q(a,b) - Q(b,a)]$$

where $Q[\cdot]$ is the Q function defined as

$$Q(a,b) = \int_b^\infty \exp\left\{-\frac{a^2+x^2}{2}\right\} I_0(ax)\, x\, dx$$

The constants a and b are equal to

$$a = \left[K_B d_R^2 \theta_R^2 \right]^{\frac{1}{2}}$$

$$b = \left[K_C d_T^2 d_R^2 P_T + K_B d_R^2 \theta_R^2 \right]^{\frac{1}{2}}$$

where K_B and K_C are defined in Appendix D. The partial derivatives of P with respect to the system parameters may be expressed as

$$\frac{\partial P}{\partial d_T} = \frac{1}{2} \left\{ \frac{\partial Q(a,b)}{\partial b} \frac{\partial b}{\partial d_T} - \frac{\partial Q(b,a)}{\partial b} \frac{\partial b}{\partial d_T} \right\}$$

$$\frac{\partial P}{\partial d_R} = \frac{1}{2}\left\{\frac{\partial Q(a,b)}{\partial a}\frac{\partial a}{\partial d_R} + \frac{\partial Q(a,b)}{\partial b}\frac{\partial b}{\partial d_R} - \frac{\partial Q(b,a)}{\partial a}\frac{\partial a}{\partial d_R} - \frac{\partial Q(b,a)}{\partial b}\frac{\partial b}{\partial d_R}\right\}$$

$$\frac{\partial P}{\partial P_T} = \frac{1}{2}\left\{\frac{\partial Q(a,b)}{\partial b}\frac{\partial b}{\partial P_T} - \frac{\partial Q(b,a)}{\partial b}\frac{\partial b}{\partial P_T}\right\}$$

$$\frac{\partial P}{\partial \theta_R} = \frac{1}{2}\left\{\frac{\partial Q(a,b)}{\partial a}\frac{\partial a}{\partial \theta_R} + \frac{\partial Q(a,b)}{\partial b}\frac{\partial b}{\partial \theta_R} - \frac{\partial Q(b,a)}{\partial a}\frac{\partial a}{\partial \theta_R} - \frac{\partial Q(b,a)}{\partial b}\frac{\partial b}{\partial \theta_R}\right\}$$

The partial derivatives of the Q function with respect to a and b can be written in terms of modified Bessel functions as [8]

$$\frac{\partial Q(a,b)}{\partial a} = bI_1(ab)\exp\left\{-\frac{a^2+b^2}{2}\right\}$$

$$\frac{\partial Q(a,b)}{\partial b} = -bI_0(ab)\exp\left\{-\frac{a^2+b^2}{2}\right\}$$

$$\frac{\partial Q(b,a)}{\partial a} = -aI_0(ab)\exp\left\{-\frac{a^2+b^2}{2}\right\}$$

$$\frac{\partial Q(b,a)}{\partial b} = aI_1(ab)\exp\left\{-\frac{a^2+b^2}{2}\right\}$$

Then, the partial derivatives of a and b with respect to the system parameters are

$$\frac{\partial b}{\partial d_T} = \frac{b^2-a^2}{b\,d_T}$$

$$\frac{\partial a}{\partial d_R} = \frac{a}{d_R} \qquad \frac{\partial b}{\partial d_R} = \frac{b}{d_R}$$

$$\frac{\partial b}{\partial P_T} = \frac{b^2-a^2}{2bP_T}$$

$$\frac{\partial a}{\partial \theta_R} = \frac{a}{\theta_R} \qquad \frac{\partial b}{\partial \theta_R} = \frac{a^2}{b\theta_R}$$

Combining the above equations, after some algebraic manipulation, the characteristic equations reduce to

$$d_T \frac{\partial C}{\partial d_T} - 2P_T \frac{\partial C}{\partial P_T} = 0$$

$$d_T \frac{\partial C}{\partial d_T} - d_R \frac{\partial C}{\partial d_R}\left[1 + \frac{a}{b}\frac{I_1(ab)}{I_0(ab)}\right] = 0$$

$$d_T \frac{\partial C}{\partial d_T} + \theta_R \frac{\partial C}{\partial \theta_R}\left[1 + \frac{b}{a}\frac{I_0(ab)}{I_1(ab)}\right] = 0$$

$$2P_T \frac{\partial C}{\partial P_T} - d_R \frac{\partial C}{\partial d_R}\left[1 + \frac{a}{b}\frac{I_1(ab)}{I_0(ab)}\right] = 0$$

$$2P_T \frac{\partial C}{\partial P_T} + \theta_R \frac{\partial C}{\partial \theta_R}\left[1 + \frac{b}{a}\frac{I_0(ab)}{I_1(ab)}\right] = 0$$

$$d_R \frac{\partial C}{\partial d_R} + \theta_R \frac{\partial C}{\partial \theta_R}\left[\frac{b}{a}\frac{I_0(ab)}{I_1(ab)}\right] = 0$$

3.4 HOPS One Way Links

The probability of detection error expressions for the PCM/AM and PCM/FM heterodyne detection and the PCM/PM homodyne detection optical communication systems are dependent upon the receiver output signal to noise ratio. The SNR expression is

$$\frac{S}{N} = K_H d_T^2 d_R^2 P_T$$

where K_H represents the constant K_{HET} for a heterodyne receiver or K_{HOM} for a homodyne receiver as defined in Appendix C. 5.

The partial derivatives with respect to the system parameters are given by

$$\frac{\partial\left(\frac{S}{N}\right)}{\partial d_T} = \frac{2}{d_T}\left(\frac{S}{N}\right)$$

$$\frac{\partial\left(\frac{S}{N}\right)}{\partial d_R} = \frac{2}{d_R}\left(\frac{S}{N}\right)$$

$$\frac{\partial\left(\frac{S}{N}\right)}{\partial P_T} = \frac{1}{P_T}\left(\frac{S}{N}\right)$$

The characteristic equations are then

$$d_T \frac{\partial C}{\partial d_T} - 2P_T \frac{\partial C}{\partial P_T} = 0$$

$$d_T \frac{\partial C}{\partial d_T} - d_R \frac{\partial C}{\partial d_R} = 0$$

$$2P_T \frac{\partial C}{\partial P_T} - d_R \frac{\partial C}{\partial d_R} = 0$$

4. OPTIMIZATION OF TWO WAY LINKS

In many communication links a single antenna is used both as a transmitting and receiving antenna. Two way communication systems of this type are considered in this section.

For two way links the probability of detection error for transmitting data from station A to station B and from B to A can be expressed in terms of the system parameters.

$$P_A = f_5(d_A, d_B, P_{TA}, \theta_{RB}) \quad (6)$$

$$P_B = f_6(d_A, d_B, P_{TB}, \theta_{RA})$$

where P_A = probability of detection error for transmitting from station A to station B

P_B = probability of detection error for transmitting from station B to station A

d_A = antenna diameter of station A

d_B = antenna diameter of station B

P_{TA} = transmitter power of station A

P_{TB} = transmitter power of station B

θ_{RA} = receiver field of view of station A

θ_{RB} = receiver field of view of station B

The total system cost, C, is now the cost of station A and the cost of station B. Again, by the method of Lagrange multipliers, to minimize the total system cost and to achieve the required probability of detection error in each direction, P_A^R and P_B^R, the dummy function C' is formed.

$$C' \equiv C + \Lambda_A(P_A^R - P_A) + \Lambda_B(P_B^R - P_B)$$

where Λ_A and Λ_B are the Lagrange multipliers. Now, setting the partial derivatives of C', with respect to the system parameters, equal to zero yields

$$\frac{\partial C'}{\partial d_A} = \frac{\partial C}{\partial d_A} - \Lambda_A \frac{\partial P_A}{\partial d_A} - \Lambda_B \frac{\partial P_B}{\partial d_A} = 0$$

$$\frac{\partial C'}{\partial d_B} = \frac{\partial C}{\partial d_B} - \Lambda_A \frac{\partial P_A}{\partial d_B} - \Lambda_B \frac{\partial P_B}{\partial d_B} = 0$$

$$\frac{\partial C'}{\partial P_{TA}} = \frac{\partial C}{\partial P_{TA}} - \Lambda_A \frac{\partial P_A}{\partial P_{TA}} = 0$$

$$\frac{\partial C'}{\partial P_{TB}} = \frac{\partial C}{\partial P_{TB}} - \Lambda_B \frac{\partial P_B}{\partial P_{TB}} = 0$$

$$\frac{\partial C'}{\partial \theta_{RA}} = \frac{\partial C}{\partial \theta_{RA}} - \Lambda_B \frac{\partial P_B}{\partial \theta_{RA}} = 0$$

$$\frac{\partial C'}{\partial \theta_{RB}} = \frac{\partial C}{\partial \theta_{RB}} - \Lambda_A \frac{\partial P_A}{\partial \theta_{RB}} = 0$$

Equating the Λ_A's and Λ_B's, yields a set of eight equations with six unknowns

$$\frac{\partial P_A}{\partial d_A}\left\{\frac{\partial C}{\partial d_B}\frac{\partial P_B}{\partial P_{TB}} - \frac{\partial C}{\partial P_{TB}}\frac{\partial P_B}{\partial d_B}\right\} - \frac{\partial P_A}{\partial d_B}\left\{\frac{\partial C}{\partial d_A}\frac{\partial P_B}{\partial P_{TB}} - \frac{\partial C}{\partial P_{TB}}\frac{\partial P_B}{\partial d_A}\right\} = 0$$

$$\frac{\partial P_A}{\partial d_A}\left\{\frac{\partial C}{\partial d_B}\frac{\partial P_A}{\partial \theta_{RA}} - \frac{\partial C}{\partial \theta_{RA}}\frac{\partial P_B}{\partial d_B}\right\} - \frac{\partial P_A}{\partial d_B}\left\{\frac{\partial C}{\partial d_A}\frac{\partial P_A}{\partial \theta_{RA}} - \frac{\partial C}{\partial \theta_{RA}}\frac{\partial P_B}{\partial d_A}\right\} = 0$$

$$\frac{\partial P_B}{\partial d_A}\left\{\frac{\partial C}{\partial d_B}\frac{\partial P_A}{\partial P_{TA}} - \frac{\partial C}{\partial P_{TA}}\frac{\partial P_A}{\partial d_B}\right\} - \frac{\partial P_B}{\partial d_B}\left\{\frac{\partial C}{\partial d_A}\frac{\partial P_A}{\partial P_{TA}} - \frac{\partial C}{\partial P_{TA}}\frac{\partial P_A}{\partial d_A}\right\} = 0$$

$$\frac{\partial P_B}{\partial d_A}\left\{\frac{\partial C}{\partial d_B}\frac{\partial P_A}{\partial \theta_{RB}} - \frac{\partial C}{\partial \theta_{RA}}\frac{\partial P_A}{\partial d_B}\right\} - \frac{\partial P_B}{\partial d_B}\left\{\frac{\partial C}{\partial d_A}\frac{\partial P_A}{\partial \theta_{RB}} - \frac{\partial C}{\partial \theta_{RB}}\frac{\partial P_A}{\partial d_A}\right\} = 0$$

$$\frac{\partial P_A}{\partial P_{TA}}\frac{\partial C}{\partial \theta_{RB}} - \frac{\partial P_A}{\partial \theta_{RB}}\frac{\partial C}{\partial P_{TA}} = 0$$

$$\frac{\partial P_B}{\partial P_{TB}}\frac{\partial C}{\partial \theta_{RA}} - \frac{\partial P_A}{\partial \theta_{RA}}\frac{\partial C}{\partial P_{TB}} = 0$$

$$\frac{\partial P_A}{\partial P_{TA}}\left\{\frac{\partial C}{\partial d_B}\frac{\partial P_B}{\partial P_{TB}} - \frac{\partial C}{\partial P_{TB}}\frac{\partial P_B}{\partial d_B}\right\} - \frac{\partial P_A}{\partial d_B}\frac{\partial C}{\partial P_{TA}}\frac{\partial P_B}{\partial P_{TB}} = 0$$

$$\frac{\partial P_A}{\partial P_{TA}}\left\{\frac{\partial C}{\partial d_A}\frac{\partial P_B}{\partial P_{TB}} - \frac{\partial C}{\partial P_{TB}}\frac{\partial P_B}{\partial d_A}\right\} - \frac{\partial P_A}{\partial d_A}\frac{\partial C}{\partial P_{TA}}\frac{\partial P_B}{\partial P_{TB}} = 0$$

Solution of a subset of four of these equations along with equations (6) and (7) giving the required probability of detection error yields the optimum values of the six system parameters.

5. OPTIMIZATION OF MULTIPLE STATION LINKS

The general communications network may be modeled as a net of M transmitter stations communicating along all possible paths with N receiver stations. The probability of detection error for transmission from the m th transmitter to the n th receiver is

$$P_{m,n} = f_7(d_{Tm}, d_{Rn}, P_{Tm}, \theta_{Rn}) \tag{8}$$

where d_{Tm} = antenna diameter of transmitter station m

P_{Tm} = transmitter power of transmitter station m

d_{Rm} = antenna diameter of receiver station n

θ_{Rn} = receiver field of view of receiver station n

The total system cost, C, is the sum of the cost of the M transmitters and N receivers. By the method of Lagrange multipliers

$$C' = C + \sum_{m=1}^{M} \sum_{n=1}^{N} \left[P^R_{m,n} - P_{m,n} \right] \Lambda_{m,n}$$

where $P^R_{m,n}$ is the required probability of detection error for transmission from station m to station n and $\Lambda_{m,n}$ is one of M·N Lagrange multipliers. Setting the partial derivatives of C', with respect to the system parameters, equal to zero gives

$$\frac{\partial C'}{\partial d_{Tm}} = \frac{\partial C}{\partial d_{Tm}} - \sum_{n=1}^{N} \Lambda_{m,n} \frac{\partial P_{m,n}}{\partial d_{Tm}} = 0$$

$$\frac{\partial C'}{\partial P_{Tm}} = \frac{\partial C}{\partial P_{Tm}} - \sum_{n=1}^{N} \Lambda_{m,n} \frac{\partial P_{m,n}}{\partial P_{Tm}} = 0$$

$$\frac{\partial C'}{\partial d_{Tm}} = \frac{\partial C}{\partial d_{Tm}} - \sum_{n=1}^{N} \Lambda_{m,n} \frac{\partial P_{m,n}}{\partial d_{Tm}} = 0$$

$$\frac{\partial C'}{\partial \theta_{Rn}} = \frac{\partial C}{\partial \theta_{Rn}} - \sum_{m=1}^{M} \Lambda_{m,n} \frac{\partial P_{m,n}}{\partial \theta_{Rn}} = 0$$

This set of $2(M+N)$ equations in conjunction with the $M \cdot N$ probability of error equations of (8) must be solved simultaneously to determine the $2(M+N)$ system parameters and the $M \cdot N$ Lagrange multipliers. The following example illustrates the mutliple station optimization procedure.

ROPS Multiple Station Optimization Example

Consider a radio frequency communication link consisting of a single transmitter station and two diffraction limited receiver stations. The probability of detection error expressions are

$$P_{11} = f_8(d_{T1}^2, d_{R1}^2, P_{T1}) \tag{9}$$

$$P_{12} = f_9(d_{T1}^2, d_{R2}, P_{T1}) \tag{10}$$

The cost equation can be written as

$$C' = C + \Lambda_1(P_{11}^R - P_{11}) + \Lambda_2(P_{12}^R - P_{12})$$

Partial differentiation with respect to the system parameters gives the following equations

$$\frac{\partial C}{\partial d_{T1}} = \Lambda_1 \frac{\partial P_{11}}{\partial d_{T1}} + \Lambda_2 \frac{\partial P_{12}}{\partial d_{T1}}$$

$$\frac{\partial C}{\partial d_{R1}} = \Lambda_1 \frac{\partial P_{11}}{\partial d_{R1}}$$

$$\frac{\partial C}{\partial P_{T1}} = \Lambda_1 \frac{\partial P_{11}}{\partial P_{T1}} + \Lambda_2 \frac{\partial P_{12}}{\partial P_{T1}}$$

$$\frac{\partial C}{\partial d_{R2}} = \Lambda_2 \frac{\partial P_{12}}{\partial d_{R2}}$$

The Lagrange multipliers can be eliminated from the above equations giving two characteristic equations

$$\frac{\partial C}{\partial d_{T1}} \frac{\partial P_{11}}{\partial d_{R1}} \frac{\partial P_{12}}{\partial d_{R2}} - \frac{\partial C}{\partial d_{R2}} \frac{\partial P_{11}}{\partial d_{R1}} \frac{\partial P_{12}}{\partial d_{R1}} - \frac{\partial C}{\partial d_{R}} \frac{\partial P_{11}}{\partial d_{T1}} \frac{\partial P_{12}}{\partial d_{R2}} = 0$$

$$\frac{\partial C}{\partial P_{T1}} \frac{\partial P_{11}}{\partial d_{R1}} \frac{\partial P_{12}}{\partial d_{R2}} - \frac{\partial C}{\partial d_{R1}} \frac{\partial P_{11}}{\partial P_{T1}} \frac{\partial P_{12}}{\partial d_{R2}} - \frac{\partial C}{\partial d_{R2}} \frac{\partial P_{11}}{\partial d_{R1}} \frac{\partial P_{12}}{\partial P_{T1}} = 0$$

Simultaneous solution of these equations with the two probability of error expressions of equations (9) and (10) gives the optimum values of the four system parameters.

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APPENDIX A

Carrier Power Equation

In a communication system the relationship between transmitted and received signal power, described by the carrier power equation, provides a characterization of propagation in the communication channel, free space propagation loss, and attenuation losses in the system components.

If a transmitter system with output power, P_A, feeds an ideal lossless isotropic antenna (i.e. one that radiates uniformly in all directions) the spatial power density at the receiver located at a range R from the transmitter is

$$\frac{P_A}{4\pi R^2}$$

The power density from a directive antenna, that concentrates energy at a point in space, is

$$\frac{P_A G_T}{4\pi R^2}$$

where G_T is the transmitter antenna directive gain defined at its maximum value. The power received by the antenna is

$$(P_R)_{MAX} = \frac{G_T P_A \tau_a A_{eR}}{4\pi R^2}$$

where A_{eR} is the effective area of the receiver antenna and τ_a is the atmospheric transmissivity. The effective antenna area is related to the actual receiver antenna area, A_R, by the receiver antenna aperture efficiency ρ_R,

$$A_{eR} = \rho_R A_R$$

The aperture efficiency is unity for an antenna that is uniformly excited by its feed. Antenna theory gives the relationship between the receiver antenna power gain and effective area as

$$G_R = \frac{4\pi A_{eR}}{\lambda^2}$$

An equivalent relationship exists for transmitting antennas

$$G_T = \frac{4\pi A_{eT}}{\lambda^2}$$

where the effective area of the transmitter antenna, A_{eT}, is related to the actual area, A_T, by the transmitter antenna aperture efficiency, ρ_T

$$A_{eT} = \rho_T A_T$$

The received power may then be written as

$$(P_R)_{MAX} = \frac{G_T G_R \tau_a P_A \lambda^2}{16\pi^2 R^2}$$

The transmitter power, P_T, is related to the antenna output power, P_A, by the transmitter system transmissivity which describes ohmic losses, mismatching of the antenna and transmitter, and radiation spillover from the feed.

$$\tau_t \equiv \frac{P_A}{P_T}$$

There are similar losses in the receiver system described by the receiver system transmissivity, τ_r. As a conservative estimate, the received power is usually taken as $\frac{1}{2}(P_R)_{MAX}$ to account for antenna beam pointing errors. Then, the received carrier power is

$$P_C = \frac{\tau_t \tau_a \tau_r \lambda^2 G_T G_R P_T}{32 \pi^2 R^2}$$

For circular transmitter and receiver antennas of diameters, d_T, and, d_R, respectively, the received carrier power may be expressed as

$$P_C = \frac{\tau_t \tau_a \tau_r \rho_T d_T^2 G_R P_T}{32 R^2}$$

or

$$P_C = \frac{\tau_t \tau_a \tau_r \rho_R G_T d_R^2 P_T}{32 R^2}$$

or

$$P_C = \frac{\pi^2 \tau_t \tau_a \tau_r \rho_T \rho_R d_T^2 d_R^2 P_T}{32 \lambda^2 R^2}$$

APPENDIX B

Background Radiation Power Equation

Background radiation is due to sources which are at an elevated temperature producing self-emissions, and sources which reflect radiation from other hot bodies. Several common units of background radiation measurements are listed below.

$W(\lambda)$ - spectral radiant emittance - radiant power into a hemisphere, per unit area of source in hemisphere, per unit wavelength interval.

$N(\lambda)$ - spectral radiance - radiant power into a unit solid angle, per unit projected area of source in hemisphere, per unit wavelength interval.

$H(\lambda)$ - spectral irradiance - radiant power incident upon a surface, per unit surface area, per unit wavelength interval.

The spectral radiant emittance and spectral radiance are related by

$$W(\lambda) = \pi N(\lambda)$$

Table B-1 lists equations for the background radiation power at the input to a communication receiver in terms of $W(\lambda)$, $N(\lambda)$, and $H(\lambda)$. If background radiation measurements are made in terms of frequency units $W(f)$, $N(f)$, and $H(f)$, conversion of the fomulas in Table B-1 can be made

Source Relationship	Expression	Background Radiation Measurement
any source	$P_B = \tau_a \tau_r \rho_R \lambda_i \frac{\pi}{4} d_R^2 H(\lambda)$	spectral irradiance
	$P_B = \tau_a \tau_r \rho_R B_i \frac{\pi}{4} d_R^2 H(f)$	
spherical source of diameter, d_S, not filling receiver field of view.	$P_B = \tau_a \tau_r \rho_R \lambda_i \frac{\pi}{16} \frac{d_S^2 d_R^2}{R^2} W(\lambda)$	spectral radiant emittance
	$P_B = \tau_a \tau_r \rho_R B_i \frac{\pi}{16} \frac{d_S^2 d_R^2}{R^2} W(f)$	
	$P_B = \tau_a \tau_r \rho_R \lambda_i \frac{\pi^2}{16} \frac{d_S^2 d_R^2}{R^2} N(\lambda)$	spectral radiance
	$P_B = \tau_a \tau_r \rho_R B_i \frac{\pi^2}{16} \frac{d_S^2 d_R^2}{R^2} N(f)$	
extended source filling receiver field of view, θ_R.	$P_B = \tau_a \tau_r \rho_R \lambda_i \frac{\pi}{16} \theta_R^2 d_R^2 W(\lambda)$	spectral radial emittance
	$P_B = \tau_a \tau_r \rho_R B_i \frac{\pi}{16} \theta_R^2 d_R^2 W(f)$	
non-diffraction limited receiver	$P_B = \tau_a \tau_r \rho_R \lambda_i \frac{\pi^2}{16} \theta_R^2 d_R^2 N(\lambda)$	spectral radiance
	$P_B = \tau_a \tau_r \rho_R \lambda_i \frac{\pi^2}{16} \theta_R^2 d_R^2 N(f)$	
extended source filling receiver field of view, θ_R.	$P_B = \tau_a \tau_r \rho_R \lambda_i \frac{\pi}{16} \lambda^2 W(\lambda)$	spectral radiant emittance
	$P_B = \tau_a \tau_r \rho_R B_i \frac{\pi}{16} \frac{c^2}{f^2} W(f)$	
diffraction limited receiver	$P_B = \tau_a \tau_r \rho_R B_i \frac{\pi^2}{16} \lambda^2 N(\lambda)$	spectral radiance
	$P_B = \tau_a \tau_r \rho_R B_i \frac{\pi^2}{16} \frac{c^2}{f^2} N(f)$	

Table B-1. Expressions for Background Radiation Power at Receiver Input for Circular Antennas.

through the relations.

$$df = \frac{-c}{\lambda^2} d\lambda$$

$$W(\lambda) d\lambda = -W(f) df$$

$$N(\lambda) d\lambda = -N(f) df$$

$$H(\lambda) d\lambda = -H(f) df$$

Background radiation sources are often modeled by Planck's law of blackbody radiation. In wavelength units, for unpolarized radiation, Planck's law can be written in terms of the spectral radiant emittance as

$$W(\lambda) d\lambda = \frac{2\pi c^2 h}{\lambda^5} \frac{d\lambda}{\exp\left\{\frac{hc}{\lambda k T_s}\right\} - 1}$$

where c = velocity of light

λ = wavelength

T_s = absolute temperature of source

h = Planck's constant

k = Boltzmann's constant

At radio frequencies $hc \ll \lambda k T_s$ and Planck's law can be expressed in frequency units as

$$W(f) df \approx \frac{2\pi f^2 k T_s df}{c^2}$$

Background radiation measurements at radio frequencies are often expressed in terms of an equivalent blackbody temperature at a given frequency even though the physical source may not obey Planck's law. Table B-2 lists equations for P_B for a blackbody source at radio frequencies.

Source Relationship	Expression
spherical source of diameter, d_S, not filling receiver field of view, θ_R.	$P_B = \frac{\pi^2 \tau_a \tau_r \rho_R d_S^2 d_R^2 k T_S B_i}{8 R^2 \lambda^2}$ $P_B = \frac{\tau_a \tau_r d_S^2 G_R k T_S B_i}{8R^2}$
extended source filling receiver field of view, θ_R non-diffraction limited receiver	$P_B = \frac{\pi^2 \tau_a \tau_r \rho_R \theta_R^2 d_R^2 k T_S B_i}{8\lambda^2}$ $P_B = \frac{\tau_a \tau_r \theta_R^2 G_R k T_S B_i}{8}$
extended source filling receiver field of view, θ_R diffraction limited receiver	$P_B = \frac{\pi^2 \tau_a \tau_r \rho_R k T_S B_i}{8}$

Table B-2. Expressions for Background Radiation Power at the Receiver Input for Circular Antennas and Blackbody Sources at Radio Frequencies.

APPENDIX C

Signal-to-Noise Ratio Expressions

C.1 Noise Sources

The two sided noise power spectral density of a resistor, R_L, in curren units is

$$G_{i_T}(f) = \frac{2kT}{R_L}$$

where k is Boltzmann's constant and T is the resistor temperature. The shot noise produced by a photodetector has a two sided noise power spectral density

$$G_{i_H}(f) = qI_P$$

where q is the electronic charge and I_P is the average photodetector current. This equation applies for a photoemissive or photovoltaic detector. The noise spectral density of a photoconductive detector is twice as large. Other noise sources are usually dominated by thermal and shot noise.

C.2 ROPS Systems

Figure C-1 illustrates the equivalent circuit of a radio frequency receiver. In this circuit

v_R = equivalent rms antenna signal voltage

v_L = equivalent rms input signal voltage

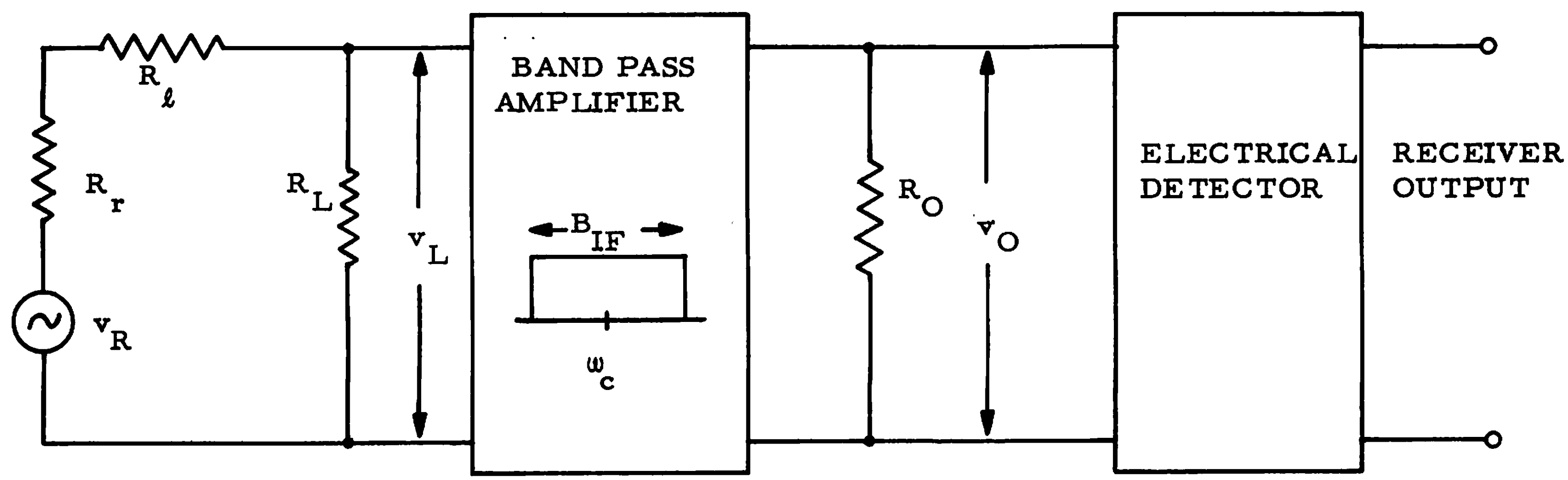

FIGURE C-1. EQUIVALENT CIRCUIT OF RADIO RECEIVER

R_r = antenna radiation resistance

R_ℓ = antenna loss resistance

R_L = antenna load resistance

R_O = bandpass amplifier output load resistance

The bandpass amplifier is assumed to be a zonal filter of bandwidth, B_i, with a voltage gain, G. The carrier power at the receiver, P_C, and the input signal voltage, v_L, are related by

$$P_C = \frac{v_L^2}{R_L}$$

The maximum power transfer between the antenna and load occurs when $R_L = R_r + R_\ell$. The output signal voltage, v_O, is

$$v_O \equiv GV_L = G\sqrt{P_C R_L}$$

The signal output power, S, is then

$$S = \frac{V_O^2}{R_L} = G^2 P_C \frac{R_L}{R_O}$$

Following the same procedure, the noise power at the filter output due to background radiation is equal to

$$N_B = G^2 P_B \frac{R_L}{R_O}$$

where P_B is defined in Table B-1.

The radiation resistance, R_r, is a fictitious resistor employed in the equivalent circuit to form the relationship between P_C and v_R, and hence is not a source of thermal noise. Neglecting the antenna loss resistance, R_ℓ, the thermal noise input to the bandpass amplifier is due only to the antenna load resistor R_L. The output thermal noise power due to R_L is

$$N_T = G^2 kTB_i \frac{R_L}{R_O}$$

If the amplifier gain is high the thermal noise contribution to the amplifier output load resistor, R_O, may be neglected.

Then, the signal-to-noise ratio at the amplifier output is

$$\frac{S}{N} = \frac{P_C}{P_B + kTB_i}$$

Substituting for the expressions for background radiation power from Table B-2, and for the equations for carrier power from Appendix A yields:

Source not filling field of view

$$\frac{S}{N} = \frac{K_{R1} d_T^2 d_R^2 P_T}{K_{D1} + d_R^2} = K_{R2} \frac{G_T d_R^2 P_T}{K_{D2} + d_R^2} \pm \frac{K_{R3} d_T^2 G_R P_T}{K_{D3} + G_R} = \frac{K_{R4} G_T G_R P_T}{K_{D4} + G_R}$$

Extended source, non-diffraction limited receiver

$$\frac{S}{N} = \frac{K_{R1} d_T^2 d_R^2 P_T}{K_{D1} + d_R^2 \theta d_R^2} = \frac{K_{R2} G_T d_R^2 P_T}{K_{D2} + d_R^2 \theta d_R^2} = \frac{K_{R3} d_T^2 G_R P_T}{K_{D3} + G_R \theta_R^2} = \frac{K_{R4} G_T G_R P_T}{K_{D4} + G_R \theta_R^2}$$

Extended source, diffraction limited receiver

$$\frac{S}{N} = K_{R1} d_T^2 d_R^2 P_T = K_{R2} G_T d_R^2 P_T = K_{R3} d_T^2 G_R P_T = K_{R4} G_T G_R P_T$$

The constants K_{Ri} and K_{Di} are listed in Table C-1. In the case of an extended background radiation source and a diffraction limited receiver an equivalent antenna temperature is often defined as

$$T_e \equiv \frac{\pi^2 \tau_a \tau_r \rho_R T_S}{8} + T$$

C.3 TOPS Systems

Figure C-2 contains the equivalent circuit of a direct detection optical receiver. In this circuit

i_P = instantaneous detector current

R_L = detector load resistance

Under thermal noise limited operating conditions the detector current contributions due to background radiation and dark current are negligible and the average detector current is

$$I_P = \frac{\eta q}{hf_c} P_C$$

where η is the quantum efficiency. This equation must be modified by a frequency dependent attenuation factor if a photoconductive detector is employed. The receiver output signal power for peak optical carrier

Case		K_{R1}	K_{D1}
Source not filling field of view	d_T, d_R	$\frac{\tau_t \rho_T}{4 d_S^2 k T_S B_i}$	$\frac{8 \lambda^2 R^2 T}{\tau_a \tau_r \rho_R \pi^2 d_S^2 T_S}$
	G_T, d_R	$\frac{\tau_t \lambda^2}{4\pi^2 d_S^2 k T_S B_i}$	"
	d_T, G_R	$\frac{\tau_t \rho_T}{4 d_S^2 k T_S B_i}$	$\frac{8 R^2 T}{\tau_a \tau_r d_S^2 T_S}$
	G_T, G_R	$\frac{\tau_t \lambda^2}{4\pi^4 d_S^2 k T_S B_i}$	"
Extended source, non-diffraction limited receiver	d_T, d_R	$\frac{\tau_t \rho_T}{4 R^2 k T_S B_i}$	$\frac{8 \lambda^2 T}{\pi^2 \tau_a \tau_r \rho_R T_S}$
	G_T, d_R	$\frac{\tau_t \lambda^2}{4\pi^2 R^2 k T_S B_i}$	"
	d_T, G_R	$\frac{\tau_t \rho_T}{4 R^2 k T_S B_i}$	$\frac{8 T}{\tau_a \tau_r k T_S}$
	G_T, G_R	$\frac{\tau_t \lambda^2}{4\pi^2 R^2 k T_S B_i}$	"
Extended source, diffraction limited receiver	d_T, d_R	$\frac{\pi^2 \tau_t \tau_a \tau_r \rho_T \rho_R}{32 \lambda^2 R^2 k T_e B_i}$	——
	G_T, d_R	$\frac{\tau_t \tau_a \tau_r \rho_R}{32 R^2 k T_e B_i}$	——
	d_T, G_R	$\frac{\tau_t \tau_a \tau_r \rho_T}{32 R^2 k T_e B_i}$	——
	G_T, G_R	$\frac{\tau_t \tau_a \tau_r \lambda^2}{32 \pi^2 R^2 k T_e B_i}$	——

Table C-1. Signal-to-Noise Ratio Constants for ROPS Systems

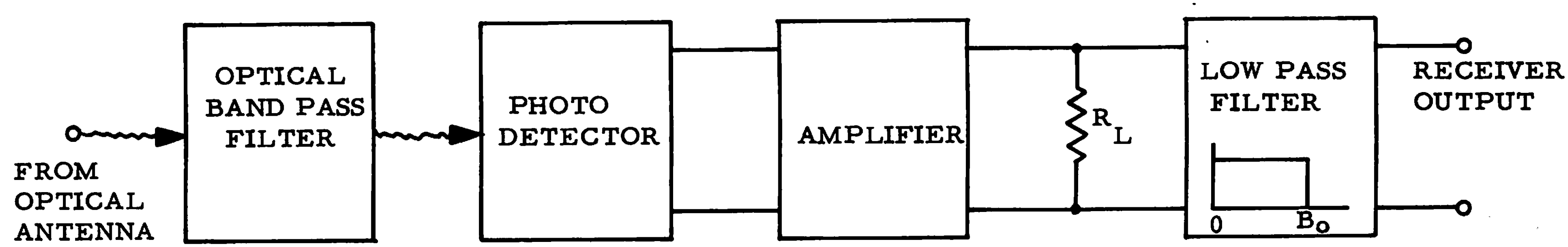

FIGURE C-2. EQUIVALENT CIRCUIT OF A DIRECT DETECTION OPTICAL RECEIVER.

intensity is then

$$S = I_P^2 R_L = \left(\frac{\eta q}{hf_c}\right)^2 R_L P_C$$

The thermal noise power at the receiver output due to the load resistor, R_L is

$$N_T = 4kTB_O$$

Then, the SNR becomes

$$\frac{S}{N} = \frac{\left(\frac{\eta q}{hf_c}\right)^2 R_L}{4kTB_O} P_C$$

Substitution for the carrier power from Appendix A gives

$$\frac{S}{N} = k_T d_T^2 d_R^2 P_T$$

where $$k_T \equiv \frac{\pi^2 \tau_t \tau_a \tau_r \rho_T \rho_R R_L}{128\, R^2 kTB_O} \left(\frac{\eta q}{hc}\right)^2$$

C.4 SOPS Systems

In a shot noise limited, direct detection optical receiver the photodetector current is composed of a signal component proportional to the carrier power, a background radiation component proportional to the background radiation power incident upon the detector surface, and a component due to the detector dark current. The average detector current is

$$I_P = \left(\frac{\eta q}{hf_c}\right) P_C + \left(\frac{\eta q}{hf_c}\right) P_B + I_D$$

The peak signal power at the receiver output is equal to

$$S = \left(\frac{\eta q}{hf_c}\right)^2 R_L P_C^2$$

while the shot noise power at the receiver output is

$$N_H = 2qB_O R_L \left\{ \left(\frac{\eta q}{hf_c}\right) P_C + \left(\frac{\eta q}{hf_c}\right) P_B + I_D \right\}$$

The signal-to-noise ratio is not a reliable performance criterion for shot noise limited direct detection optical receiver when P_B or I_D are large. Probability of detection error expressions are usually written in terms of the average number of signal and shot noise photoelectrons counts per bit period.

C. 5 HOPS Systems

Heterodyne Receiver. Figure C-3 contains an equivalent circuit of a heterodyne optical receiver. In the heterodyne receiver the carrier mixes with the local oscillator to produce a difference frequency containing the carrier modulation. The signal voltage at the bandpass filter output assuming perfect spatial alignment of the carrier and local oscillator wave is

$$v_{IF} = \frac{2\eta q}{hf_c} \sqrt{P_C P_O}\, R_L \cos\left[(\omega_o - \omega_c)t + (\Phi_o - \Phi_c)\right]$$

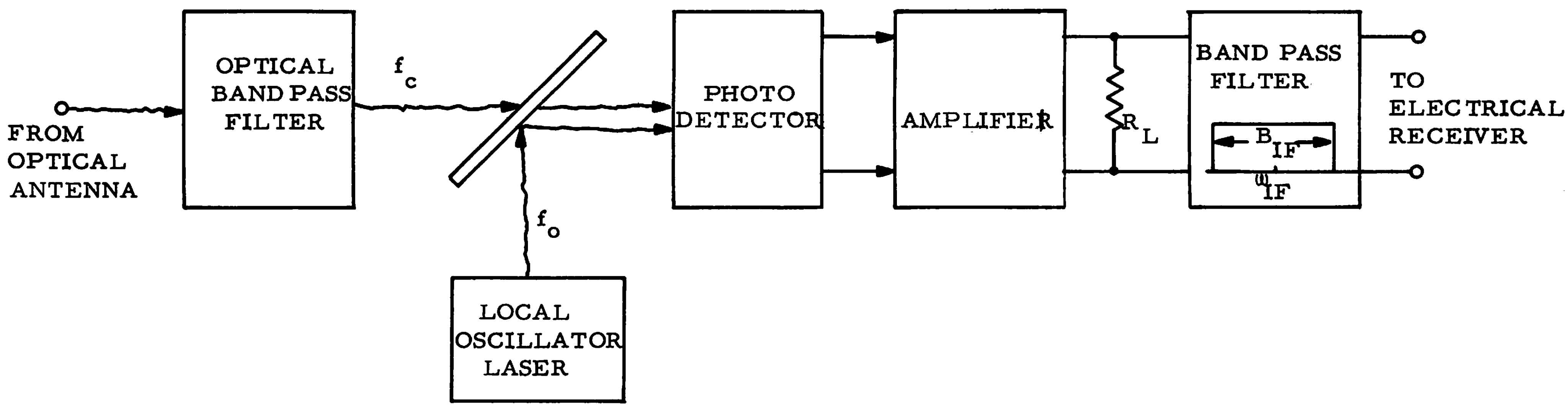

FIGURE C-3. EQUIVALENT CIRCUIT OF A HETERODYNE DETECTION OPTICAL RECEIVER.

where P_O, ω_o, Φ_c and P_C, ω_c, Φ_c are the average power, angular frequency, and phase angle of the local oscillator and carrier, respectively. The average signal power at the filter output for peak carrier intensity is then

$$S = 2\left(\frac{\eta q}{hf_c}\right)^2 P_C P_O R_L$$

Background radiation does not mix with the local oscillator due to its lack of coherence. If the local oscillator power is high the detector shot noise contributions due to the carrier, background radiation, and dark current become negligible. Under this condition the shot noise power is

$$N_H = 2q^2 B_{IF} R_L \left(\frac{\eta}{hf_c}\right) P_O$$

The receiver output signal-to-noise ratio then reduces to

$$\frac{S}{N} = \frac{\eta P_C}{hf_c B_{IF}}$$

Upon substitution for the carrier power

$$\frac{S}{N} = K_{HET} d_T^2 d_R^2 P_T$$

where

$$K_{HET} \equiv \frac{\pi^2 \tau_t \tau_a \tau_r \rho_T \rho_R \eta}{32\ hc\ \lambda R^2 B_{IF}}$$

Homodyne Receiver. In the homodyne receiver shown in Figure C-4 the local oscillator is set at the carrier frequency and the signal

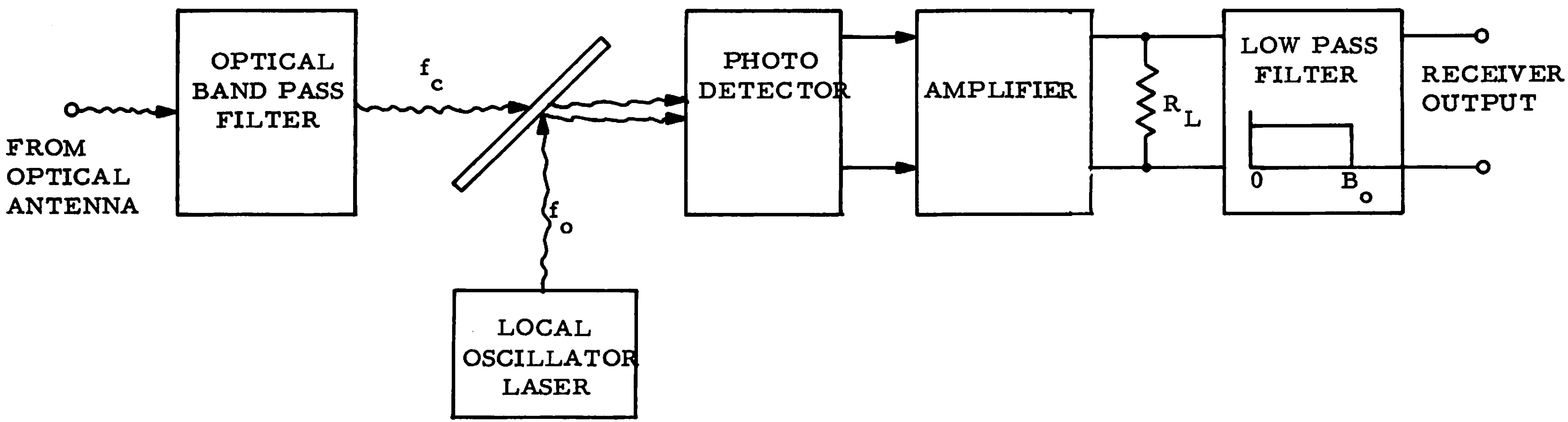

FIGURE C-4. EQUIVALENT CIRCUIT OF A HOMODYNE DETECTION OPTICAL RECEIVER.

voltage at the receiver output is

$$V_O = 2\frac{\eta q}{hf_c}\sqrt{P_C P_O}\, R_L \cos(\Phi_o - \Phi_c)$$

The peak signal power is then

$$S = 4\left[\frac{\eta q}{hf_c}\right]^2 P_O P_C R_L$$

and the shot noise power for a strong local oscillator is

$$N_H = 2q^2 B_O R_L \left(\frac{\eta}{hf_c}\right) P_O$$

The signal-to-noise ratio may then be written as

$$\frac{S}{N} = \frac{2\eta P_C}{hf_c B_o}$$

Again, upon substitution for P_C

$$\frac{S}{N} = K_{HOM} d_T^2 d_R^2 P_T$$

where

$$K_{HOM} \equiv \frac{\pi^2 \tau_t \tau_a \tau_r \rho_T \rho_R \eta}{16\, hc\, \lambda R^2 B_O}$$

APPENDIX D

Probability of Detection Error Expressions

D.1 Radio Systems

Expressions for the probability of detection error for digital radio frequency communication systems are listed in Table D-1. [5] These equations are derived under the conservative assumption that the radio receiver output bandwidth is set equal to the inverse of the bit period. In all cases the probability of error is monotonically related to the signal-to-noise ratio at the input to the radio receiver detector.

D.2 Optical Systems

Table D-2 contains expressions for the probability of detection error for digital, optical communication systems [6]. All of the expressions can be written in closed form except for the PCM/AM direct detection system for which the probability of error is

$$P_e = \frac{1}{2}\left\{1 - \sum_{k=k_D}^{\infty} \frac{\exp\{-\mu_{H,B}\}}{k!}\left[(\mu_{S,B} + \mu_{H,B})^k \exp(-\mu_{S,B}) - (\mu_{H,B})^k\right]\right\}$$

where k_D is the largest integer value of

$$\frac{\mu_{S,B}}{\ln\left[1 + \dfrac{\mu_{S,B}}{\mu_{H,B}}\right]}$$

and where

Modulation Method	Demodulation Method	
	Direct	Homodyne
PCM/AM	$\frac{1}{2}\left\{1 + Q\left[0, \sqrt{2 + \frac{1}{2}\frac{S}{N}}\right] - Q\left[\sqrt{2\frac{S}{N}}, \sqrt{2 + \frac{1}{2}\frac{S}{N}}\right]\right\}$	$\frac{1}{2}\left[1 - \text{erf}\sqrt{\frac{1}{4}\frac{S}{N}}\right]$
PCM/FM	$\frac{1}{2}\exp\left\{-\frac{1}{2}\frac{S}{N}\right\}$	$\frac{1}{2}\left[1 - \text{erf}\sqrt{\frac{1}{2}\frac{S}{N}}\right]$
PCM/PM	phase demodulation not possible by direct detection	$\frac{1}{2}\left[1 - \text{erf}\sqrt{\frac{S}{N}}\right]$

$\frac{S}{N}$ = signal-to-noise ratio at input to radio receiver detector.

$$\text{erf}\{x\} \equiv \frac{2}{\pi}\int_0^x \exp\{-y^2\}\,dy$$

$$Q(ab) \equiv \int_0^\infty \exp\left\{-\frac{a^2 + x^2}{2}\right\} I_0(ax)\,x\,dx$$

Table D-1. Probability of Detection Error Expressions for Digital Radio Communication Systems

Modulation Method	Type of Noise	Demodulation Method		
		Direct	Heterodyne	Homodyne
PCM/AM	thermal	$\frac{1}{2}\left\{1 - \mathrm{erf}\left[\frac{q\mu_{S,B}}{4\sqrt{2}}\sqrt{\frac{R_L}{kT\tau_B}}\right]\right\}$ $\frac{1}{2}\left\{1 - \mathrm{erf}\left[\frac{1}{2\sqrt{2}}\sqrt{\frac{S}{N}}\right]\right\}$	A	A
	shot	C	$\frac{1}{2}\{1 + Q(0,b) - Q(a,b)$ $a = \sqrt{\mu_{S,B}}$ $b = \sqrt{2 + \frac{\mu_{S,B}}{4}}$	A
PCM/PL	thermal	$\frac{1}{2}\left\{1-\mathrm{erf}\left[\frac{q\mu_{S,B}}{4}\sqrt{\frac{R_L}{kT\tau_B}}\right]\right\}$ $\frac{1}{2}\left\{1-\mathrm{erf}\left[\frac{1}{2}\sqrt{\frac{S}{N}}\right]\right\}$	A	A
	shot	$\frac{1}{2}\{1 + Q(a,b) - Q(b,a)\}$ $a = \sqrt{2\mu_{H,B}}$ $b = \sqrt{2(\mu_{S,B} + \mu_{H,B})}$	A	A
PCM/FM	thermal	B	A	A
	shot	B	$\frac{1}{2}\exp\left\{-\frac{\mu_{S,B}}{4}\right\}$ $\frac{1}{2}\exp\left\{-\frac{1}{2}\left(\frac{S}{N}\right)\right\}$	A
PCM/PM	thermal	B	B	A
	shot	B	B	$\frac{1}{2}\left\{1-\mathrm{erf}\sqrt{\mu_{S,B}}\right\}$ $\frac{1}{2}\left\{1-\mathrm{erf}\left[\frac{1}{\sqrt{2}}\sqrt{\frac{S}{N}}\right]\right\}$

Notes: (A) Not practical system; (B) Modulation and demodulation methods not compatible; (C) See text.

Table D-2. Probability of Detection Error Expressions for Digital Optical Communication Systems.

$\mu_{S,B}$ = average number of signal photoelectrons emitted by the detector per bit period

$\mu_{H,B}$ = average number of photoelectrons emitted by the detector per bit period due to background radiation and dark current

The signal and shot noise photoelectron counts are related to the carrier power, background radiation power, and detector dark current by

$$\mu_{S,B} \equiv \left(\frac{\eta \tau_B}{hf_c}\right) P_C$$

$$\mu_{H,B} \equiv \left(\frac{\eta \tau_B}{hf_c}\right) P_B + \left(\frac{\tau_B}{q}\right) I_D$$

where τ_B is the bit period.

For PCM polarization modulation when the detector dark current is negligible and when the background radiation source completely fills the receiver field of view, the constants a and b of Table D-2 can be written as

$$a = \left[K_B d_R^2 \theta_R^2 \right]^{1/2}$$

$$b = \left[K_C d_T^2 d_R^2 P_T + K_B d_R^2 \theta_R^2 \right]^{1/2}$$

where $K_B \equiv \dfrac{\pi^2 \eta \tau_a \tau_r \lambda_i \lambda N(\lambda)}{4hc}$

$$K_C \equiv \frac{\pi^2 \tau_t \tau_a \tau_r}{16 hc \lambda R^2}$$