USCEE REPORT #369

Information Rates of Autoregressive Sources

by

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August 1969

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Research Sponsored by the Army Research Office under Grant 31-124-G929.

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ABSTRACT

The rate distortion function, R(D), of a source can be interpreted as being the average amount of information that must be transmitted about a source for the receiver to be able to approximate the source within an average distortion D. It is demonstrated that for the class of time discrete autoregressive sources the rate distortion function for any difference distortion measure is lower bounded by the rate distortion function of the independent letter source that generates the autoregressive source. Autoregressive sources are constructed by passing such an independent letter source through a time discrete linear filter whose z-transform has only poles. This behavior holds even if the autoregressive source is nonstationary. The lower bound is shown to hold with equality for a non-zero range of small average distortion for two important special cases: the class of possibly nonstationary Gaussian autoregressive processes with a mean square error fidelity criterion and the binary symmetric first order Markov source with an average error per bit fidelity criterion. The positive coding theorem is proven for the possibly nonstationary Gaussian autoregressive process with a constraint on its parameters.

Similar results are presented for the case of an independent letter Gaussian source passed through a more general linear filter

and for the time continuous Gaussian autoregressive process. Some original results on the asymptotic behavior of the eigenvalues of Toeplitz and approximately Toeplitz matrices are derived.

CHAPTER I

INTRODUCTION

The rate distortion function of a source can be thought of as the minimum rate at which sufficient information about the source can be transmitted to approximate it within some average distortion. Conceptually such a function is of great interest in source coding, data compression, bandwidth compression, quantization, and other problems where the simplest possible representation of a source given some required average fidelity criterion is desired. Two limitations on the usefulness of rate distortion theory have been the difficulty in finding accurate mathematical models for real random processes and in evaluating the rate distortion function for any but simple sources.

The purpose of the research presented in this dissertation is to deal with the latter of these two difficulties and hopefully give some insight into the former. The rate distortion function for a class of sources with memory known as autoregressive sources is lower bounded and in some cases evaluated exactly. This class includes many stationary and nonstationary sources and is frequently used as a model for real random processes of interest.

A. The Rate Distortion Function

Consider the model of a communications system given in Figure 1.1. The channel is assumed to be noiseless so that the source coding problem may be separated from the channel coding (error correcting) problem. Let the source alphabet be U and the receiver alphabet be V. Define $d(u, v)$ to be a distortion measure or "cost" of receiving $v \varepsilon V$ when $u \varepsilon U$ was sent. The average distortion is then given by $E_{U,V}[d(u, v)]$, where $E_{U,V}$ is the joint expectation over U and V.

Define a set of conditional probability measures $P = \{p(v|u) : E_{U,V}[d(u, v)] \leqq D\}$. Then the rate distortion function is defined as

$$R(D) = \inf_{p \varepsilon P} I(U, V) \tag{1.1}$$

where $I(U, V)$ is the mutual information between U and V. The rate distortion function for a source can be thought of as giving the "equivalent rate" of the source for a given level of average distortion D. More precisely, if we wish to transmit sufficient information about the source to reconstruct it within average distortion D, then we must have available a channel with capacity $C \geqq R(D)$. Conversely, if we have a channel with capacity $C \geqq R(D)$ available, then the existence of a source code enabling us to send sufficient information over the channel to approximate the source within average distortion D is

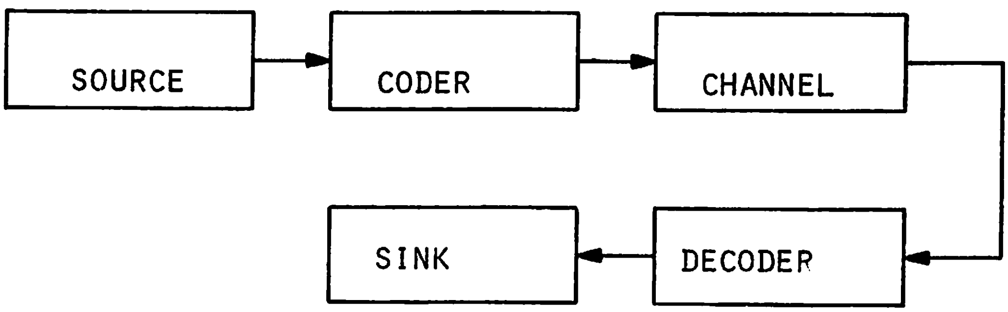


Figure 1.1: Communications System Model

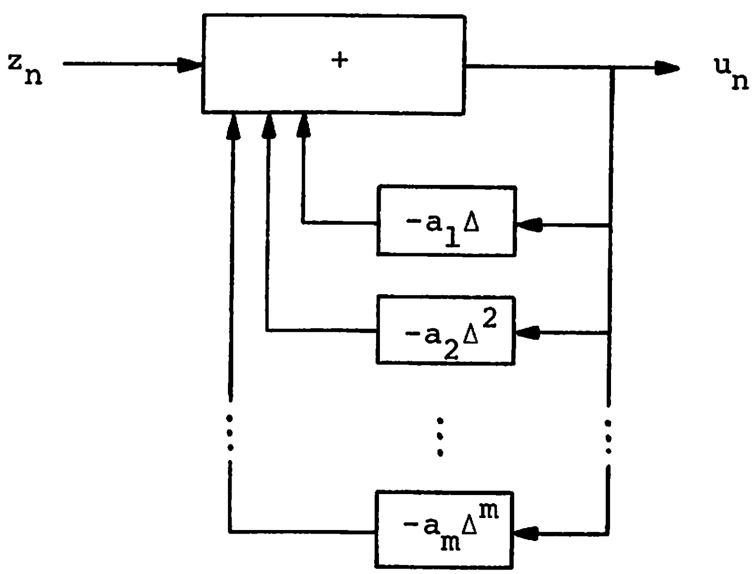


Figure 1.2: Autoregressive Source Model

guaranteed.

The preceding definitions and the precise statements of the coding theorems for source coding subject to a fidelity criterion originated with Shannon [25], [26]. Shannon proved the coding theorems for independent letter sources using random coding arguments and he evaluated the rate distortion function for two independent letter sources: the discrete alphabet, equally probable letter source and the time discrete independent letter Gaussian source.

The coding theorems have been extended to various classes of stationary sources by Pinsker [21], Gallager [9], Goblick [11], Sakrison [22], and Berger [1]. The evaluation of the rate distortion function was extended to stationary Gaussian processes by Kolmogorov [18] and to independent letter discrete alphabets without the equally probable assumption by Gallager [9].

With the exception of the stationary Gaussian process, the rate distortion function has been evaluated only for independent letter, i.e., memoryless, sources. In particular there are no available results for any discrete alphabet sources with memory. Since most data compression techniques involve some sort of prediction the rate distortion function cannot be used to analyze those sources for which data compression is most useful. The only interesting data compression technique for independent letter sources is quantization. The relation of the rate distortion function to quantization and to analogue source digitalization of stationary Gaussian sources has been studied by Goblick and Holsinger [12], McDonald and Schultheiss [20], Kellog [17], and Gish and Pierce [10].

Few results are available for non-stationary processes. Berger [2] independently derived the results of Chapter II concerning the Wiener process and proved a somewhat stronger coding theorem for the Wiener process than that of Chapter III.

In this work it is shown that for the class of autoregressive sources the rate distortion function can be lower bounded and sometimes evaluated exactly in terms of the rate distortion function of a simpler, independent letter source. This class includes examples of sources previously mentioned for which R(D) has not been previously known and some well-known sources.

B. The Source Model

The class of sources dealt with is that of autoregressive sources. A discrete time autoregressive source, $\{u_k\} = u_0, u_1, u_2, \ldots,$ is pictured in Figure 1.2. Such a process satisfies a difference equation of the form

$$u_n = z_n - \sum_{j=1}^{m} a_j u_{n-j} \tag{1.2}$$

where the z_n are independent, identically distributed random variables. If m is allowed to be infinite, then the constraint

$$\sum_{j=0}^{\infty} |a_j|^2 < \infty \tag{1.3}$$

must be satisfied so that (1.2) will converge in mean square.

The conditional density for u_n is simply the density for z_n with the mean shifted by a linear function of the past. Such processes are discussed in Feller [8] and are used extensively by Wiener [27] and Elias [6].

When the alphabet of z_j is discrete, say $K = \{0,1,2,\ldots,k-1\}$, then the constants a_j are assumed to be taken from the same alphabet and the addition is taken to be modulo k.

The most important example of autoregressive sources is the first order autoregressive source, i.e., m=1 in (1.2). This model is frequently used to approximate real, single parameter processes having a smooth autocorrelation function. Chapters II and IV deal with first order autoregressive sources.

In Chapter VI the definition of an autoregressive source is generalized to continuous time and the time continuous Gaussian autoregressive source is studied.

C. The Method

Two main tools are used in this work. The first is theorem 9.4.1 of Gallager [9], which is restated in this section with relevant definitions for reference. This theorem is used in both its discrete and continuous form to bound and to evaluate rate distortion functions. The second technique used involves the theorems on the asymptotic behavior of eigenvalues as considered in the theory of Toeplitz forms of Grenander and Szego [14] and Grenander and Rosenblatt [13]. The important theorems and definitions relating to Toeplitz forms are given in Appendix A. The theorems of Grenander and Szego [14] are given in Appendix A without proof while those generalizations original in this paper are given with proofs. Appendix A is referred to repeatedly in Chapters III, VI and VII.

Define the distortion between n-tuples (column vectors) $\underline{u}$ and $\underline{v}$ to be

$$D(\underline{u},\underline{v}) = n^{-1} d(\underline{u},\underline{v}) = n^{-1} \sum_{j=1}^{n} d(u_j, v_j) \qquad (1.4)$$

where $d(u_k, v_k)$ is the distortion between letters $u_k \varepsilon U$ and $v_k \varepsilon V$, $d: U \times V \rightarrow [0,\infty]$. Define the rate distortion function of the source $\{u_j\}$ by

$$R_u(D) = R(D) = \lim_{n \to \infty} R_n(D) \qquad (1.5a)$$

where

$$R_n(D) = n^{-1} \operatorname{Inf}_{p \varepsilon P} I(\underline{U},\underline{V}) \qquad (1.5b)$$

and

$$P = \{ \text{conditional densities } p(\underline{v}\,|\,\underline{u}) : \bar{D} \leq D \} \qquad (1.5c)$$

where $I(\underline{U},\underline{V})$ is the average mutual information between the ensembles of n-tuples $\underline{U}$ and $\underline{V}$, and $\overline{D} = E_{\underline{U},\underline{V}}[d(\underline{u},\underline{v})]$. The rate distortion function exists only if the limit of (1.5a) exists.

The discrete alphabet version of Theorem 9.4.1 of Gallager [9] is given below. When the continuous alphabet version is required, it is assumed that the obvious changes from sums to integrals and from probability mass functions to probability density functions have been made.

Theorem 1.1

For a given source with alphabet K and letter probabilities Q_k, $k \varepsilon K$, and a given distortion measure $d(k,j) : K \times J \to [0,\infty]$, let

$$R_o(\rho, p) = \sum_k \sum_j Q_k p(j|k) \left\{ \ln \left[\frac{p(j|k)}{\sum_i Q_i p(j|i)} \right] + \rho d(k,j) \right\} \tag{1.6}$$

Then for any $\rho \geqq 0$,

$$\min_{p \varepsilon P} R_o(\rho, p) \geqq \sum_{k \varepsilon K} Q_k \ln(f_k / Q_k) \tag{1.7}$$

where $\underline{f} = (f_1, f_2, \ldots, f_n)^T$ is any vector with non-negative components satisfying the constraints

$$\sum_{k \varepsilon K} f_k e^{-\rho d(k,j)} \leqq 1 \qquad \text{all } j \varepsilon J \tag{1.8}$$

Equality holds in (1.7) iff there exists a set of non-negative numbers $\omega(j)$, $j \varepsilon J$, satisfying the relations

$$1 = (f_k/Q_k) \sum_{j \varepsilon J} \omega(j)\, e^{-\rho d(k,j)}\,, \quad \text{all } k \tag{1.9}$$

and if (1.8) is satisfied with equality for each j for which $\omega(j) > 0$.

The above theorem is essentially just a Kuhn-Tucker minimization of (1.5b) and ρ serves as the Lagrange multiplier. When using the above theorem the letters k and j will be considered to be n-tuples.

D. Outline of Dissertation

In Chapter II the rate distortion function for a time discrete first order autoregressive Gaussian source with a mean square error fidelity criterion is evaluated. For certain values of the single parameter the rate distortion function can be found exactly or lower bounded by evaluating the eigenvalues of the inverse autocorrelation matrix, R^{-1}, of the process. For other values of the parameter the eigenvalues cannot be evaluated exactly and their asymptotic behavior as considered in the theory of Toeplitz forms must be studied. Chapter II presents the motivation for the generalizations of Chapter III while dealing with a simple source to avoid unnecessary complexity. The results are of interest in that they show that $R_u(D)$ of the autoregressive source $\{u_k\}$ can be lower bounded and sometimes given exactly by $R_z(D)$, the rate distortion function for the simpler independent letter source $\{z_k\}$ that generates the autoregressive source. The results are valid even when $\{u_k\}$ is nonstationary, e.g., the

Wiener process.

Chapter III generalizes in a straightforward manner the results of Chapter II first to finite order and then to infinite order time discrete Gaussian autoregressive sources. The behavior of $R_u(D)$ with respect to $R_z(D)$ is shown to be the same as that of the first order source. Finally, the source coding theorem is stated with an added constraint on the parameters of the process. The proof is given in Appendix B.

In Chapter IV the rate distortion function of a binary symmetric first order Markov source with an average probability of error per bit fidelity criterion is evaluated for small distortion. This source is equivalent to a first order binary autoregressive source generated by an independent letter Bernoulli sequence. It is shown that despite the radical differences between this source and that of Chapter III the same relative behavior of $R_u(D)$ and $R_z(D)$ results in both cases.

Chapter V consists of a theorem demonstrating that for time discrete autoregressive processes with a difference distortion measure $R_u(D)$ can always be lower bounded by $R_z(D)$.

In Chapter VI some of the results of Chapter III are extended to continuous time Gaussian autoregressive processes.

In Chapter VII the results of Chapter III are extended to some non-autoregressive and to more general discrete time Gaussian sources.

Chapter VIII concludes the dissertation with a discussion of the major results and suggestions for future research.

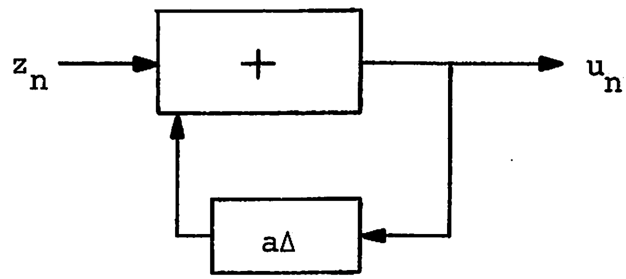
CHAPTER II

THE RATE DISTORTION FUNCTION OF A TIME DISCRETE, GAUSSIAN FIRST ORDER AUTOREGRESSIVE SOURCE WITH A MEAN SQUARE ERROR FIDELITY CRITERION

A. The Source Model

One of the simplest possible time discrete autoregressive sources is the Gaussian first order autoregressive source depicted in Figure 2.1. This source, while simple, is interesting because it can represent either a stationary or nonstationary first order Markov Gaussian source depending on the value of the single parameter a. In this chapter we shall take a somewhat naive approach to evaluating the rate distortion function of this source with a mean square error distortion measure in order to demonstrate the use of Theorem 1.1 and to give the motivation for the techniques used in the more elegant derivation of Chapter III.

The important results of this chapter are twofold: the rate distortion function of the nonstationary Wiener process is found and the behavior of $R_u(D)$ with respect to $R_z(D)$ is first demonstrated. Both of these results were found independently by Berger [2] who analyzed the Wiener process using a similar, but less powerful, approach. The methods of the latter part of this chapter generalize



Δ = Unit Delay

$$p_{z_n}(x) = p_z(x) = (2\pi\sigma^2)^{-\frac{1}{2}}\exp\{-\tfrac{1}{2}x^2/\sigma^2\} \quad \text{all } n$$

Figure 2.1: First Order Gaussian Autoregressive Process

immediately to the problem of Chapter III, while Berger's method does not.

From Figure 2.1 a difference equation for the Gaussian independent letter sequence $\{z_k\}$ and the Gaussian autoregressive sequence $\{u_k\}$ can be written:

$$u_n = au_{n-1} + z_n \tag{2.1}$$

where it is assumed that $u_k = z_k = 0$ for $k \leqq 0$. Assume $a \geqq 0$ for simplicity. The case $a < 0$ can be similarly handled. The sequence is started at $u_0 = 0$ rather than with a stationary distribution since when $a \geqq 1$ the variance of u_n is unbounded as $n \to \infty$ and no stationary distribution exists.

The sequence $\{u_k\}$ is a first order Markov source and the conditional density is simply

$$p(u_n | u_{n-1}, u_{n-2}, \dots) = p(u_n | u_{n-1}) = p_z(u_n - au_{n-1}) = (2\pi\sigma^2)^{-\frac{1}{2}} e^{-(u_n - au_{n-1})^2/2\sigma^2} \tag{2.2}$$

and therefore the density for the n-tuple $\underline{u}$ is

$$q(\underline{u}) = \prod_{k=1}^{n} p(u_k | u_{k-1}) = (2\pi\sigma^2)^{-n/2} \exp\left\{ -\sum_{k=1}^{n} (u_k - au_{k-1})^2/2\sigma^2 \right\} \tag{2.3}$$

Equation (2.3) can be put into the matrix representation of a Gaussian density by defining the n×n matrix

$$A = \begin{bmatrix} 1 & & & & 0 \\ -a & 1 & & & \\ & -a & \cdot & & \\ & & & \cdot & \\ 0 & & & -a & 1 \end{bmatrix} \tag{2.4}$$

so that $A\underline{u} = \underline{z}$. Then

$$R^{-1} = (1/\sigma^2)A^TA = (1/\sigma^2)\begin{bmatrix} 1+a^2 & -a & & & & 0 \\ -a & 1+a^2 & -a & & & \\ & -a & \cdot & & & \\ & & & \cdot & & \\ 0 & & & & \cdot & \\ & & & & 1+a^2 & -a \\ & & & & -a & 1 \end{bmatrix} \tag{2.5}$$

and

$$q(\underline{u}) = (2\pi\sigma^2)^{-n/2}\exp\{-\underline{u}^TA^TA\underline{u}/2\sigma^2\} \tag{2.6}$$

Note that $\det[R^{-1}] = (1/\sigma^2)^n\{\det[A]\}^2 = (1/\sigma^2)^n$. The autocorrelation matrix of $\underline{u}$ is

$$R = [A^TA/\sigma^2] = \sigma^2A^{-1}(A^T)^{-1} = \sigma^2A^{-1}(A^{-1})^T$$

$$= \sigma^2\begin{bmatrix} 1 & & & & 0 \\ a & 1 & & & \\ a^2 & a & & & \\ a^3 & a^2 & a & \cdot & \\ & & & \cdot & \\ a^n & & a^2 & a & 1 \end{bmatrix}\begin{bmatrix} 1 & a & a^2 & \dots & a^n \\ & 1 & a & & \\ & & \cdot & & \\ & & & \cdot & a^2 \\ 0 & & & \cdot & a \\ & & & & 1 \end{bmatrix}$$

$$= \sigma^2\left\{a^{|j-k|}\sum_{m=0}^{\min(k-1,j-1)} a^{2m}\right\} \tag{2.7}$$

When a=1, (2.1) defines a Wiener process and (2.7) becomes the well-known $R = \{\sigma^2[\min(k,j)]\}$. When $0 \leq a < 1$, the statistics of the process approach a stationary distribution. Specifically, for large k, j $R(j,k) \cong \frac{\sigma^2 a^{|j-k|}}{1-a^2}$, which is the j,k^{th} entry of the auto-correlation matrix for a first order Gaussian autoregressive source initiated with the stationary distribution rather than with $u_o = 0$.

B. A Lower Bound for $R_u(D)$

Theorem 1.1 can now be applied to the above source for distortion measure $d(\underline{u},\underline{v}) = \sum_{i=1}^{n} d(u_i, v_i) = \sum_{i=1}^{n} (u_i - v_i)^2$. Define the vector $\underline{s}$ by $s_k = u_k - v_k$ so that (1.8) becomes

$$\int f(s_1+u_1, \ldots, s_n+u_n) \exp\left[-\rho \sum_{k=1}^{n} s_k^2\right] d\underline{s} \leqq 1 \tag{2.8}$$

which suggests a guess of $f(\underline{u}) = (\rho/\pi)^{n/2}$. This $f(\underline{u})$ satisfies (2.8) with equality and will therefore at least yield a lower bound for $R_n(D)$. Substitution of this value into (1.7) results in

$$\min_{p \varepsilon P} nR_{o_n}(\rho, p) \geqq \int d\underline{u}\, q(\underline{u}) \ln\left[(\rho/\pi)^{n/2}/q(\underline{u})\right]$$

$$= \ln(2\rho\sigma^2)^{n/2} + (1/2\sigma^2) \sum_{k=1}^{n} \int dz_k\, z_k^2 \frac{\exp[-\Sigma z_k^2/2\sigma^2]}{(2\pi\sigma^2)^{n/2}}$$

$$= (n/2) \ln 2\rho\sigma^2 e \tag{2.9}$$

where $R_{o_n}(\rho, p)$ is the per letter $R_o(\rho, p)$ of (1.6) for $j = \underline{u}$ and $k = \underline{v}$

and the variable substitution $\underline{z} = A\underline{u}$ has been made. From the definition of $R_o(\rho, p)$

$$R_n(d) \geqq \min_{p \epsilon P} R_{o_n}(\rho, p) - \rho d/n \tag{2.10}$$

so that (2.9) becomes

$$R_n(d) \geqq 1/2 \ln 2\rho\sigma^2 e - \rho d/n \tag{2.11}$$

Maximizing this bound over ρ gives $\rho = n/2d$ yielding the bound

$$R_n(D) \geqq 1/2 \ln \sigma^2/D \tag{2.12}$$

where $D = d/n$, the normalized distortion. Since the right hand side of (2.12) is independent of n the limit of (1.5a) is simply

$$R_u(D) = \lim_{n \to \infty} R_n(D) \geqq \begin{cases} 1/2 \ln[\sigma^2/D] & D \leqq \sigma^2 \\ 0 & D > \sigma^2 \end{cases} \tag{2.13}$$

The right hand side of (2.13) is immediately recognizable as the rate distortion function $R_z(D)$ for the independent letter Gaussian sequence $\{z_k\}$ as found by Shannon [26]. This behavior will be later shown to be common to all discrete time autoregressive sources with difference distortion measures.

C. Evaluation of $R_u(D)$

Theorem 1.1 states that for equality to hold in (2.10), and therefore in (2.13), there must exist a valid probability density $w(\underline{v})$ which is a solution to

$$(\rho/\pi)^{n/2} \int w(\underline{v}) e^{-\rho[\underline{u}-\underline{v}]^T[\underline{u}-\underline{v}]} d\underline{v} = (2\pi\sigma^2)^{-n/2} e^{-1/2\, \underline{u}^T R^{-1} \underline{u}} \quad (2.14)$$

Since (2.14) is an n-dimensional convolution integral of $w(\underline{v})$ with a Gaussian density to yield a Gaussian density, $w(\underline{v})$ must be Gaussian itself. Assuming the covariance of $\underline{v}$ to be K, (2.14) will be satisfied iff

$$K = R - (1/2\rho)I = \sigma^2 A^{-1}(A^T)^{-1} - (1/2\rho)I \quad (2.15)$$

where I is the identity matrix.

When K is non-negative definite, $w(\underline{v})$ will be a valid density function and (2.13) will hold with equality. To find the range of the parameter ρ for which K is non-negative definite it is necessary to investigate the eigenvalues of K.

Let $\{\beta_{k,n}\}$ be the eigenvalues of the nxn matrix $A^T A$, i.e., $\beta_{k,n}$ are the solutions to

$$A^T A\underline{x} = \beta\underline{x} \quad (2.16)$$

The eigenvalues of K, $\{\alpha_{k,n}\}$, are then

$$\alpha_{k,n} = \sigma^2/\beta_{k,n} - 1/(2\rho) \quad (2.17)$$

The range of ρ for which K is non-negative definite is simply the range for which $\min_k \alpha_{k,n} \geqq 0$, or $\max_k \beta_{k,n} \leqq 2\rho\sigma^2$.

Since $A^T A$ has terms only on the principal diagonal and the two adjacent diagonals, (2.16) can be written as a set of difference equations with boundary conditions:

$$-ax_{j-1} + (1+a^2)x_j - ax_{j+1} = \beta x_j \qquad 1 \leqq j \leqq n-1 \tag{2.18}$$

$$x_0 = 0, \quad x_n - ax_{n-1} = \beta x_n \tag{2.19}$$

Guessing the solution $x_j = r^j$, (2.18) becomes

$$r^2 - [(1+a^2)/a - \beta/a] r + 1 = (r-r_1)(r-r_2) = 0 \tag{2.20}$$

so that the roots of (2.20) are reciprocals.

Thus

$$x_j = C_1 r_1^j + C_2 r_2^j = C_1 r_1^j + C_2 r_1^{-j} \tag{2.21}$$

where C_1 and C_2 are determined from the boundary conditions. Since $x_0 = C_1 + C_2 = 0$, (2.21) becomes $x_j = C_1(r_1^j - r_1^{-j})$. Setting $r_1 = e^{i\theta}$ gives

$$x_j = C_3 \sin(j\theta) \tag{2.22a}$$

$$\beta = 1+a^2 - 2a\cos(\theta) \tag{2.22b}$$

where θ is given by the second boundary condition, i.e., it is a solution to

$$\sin[(n+1)\theta] = a\sin(\theta n) \tag{2.23}$$

When a=1, i.e., when $\{u_k\}$ is a Wiener process, (2.23) can be solved to obtain

$$\theta_k = \pi(2k-1)/(2n+1) \qquad k=1,\ldots,n \tag{2.24}$$

yielding

$$\beta_k = 2-2\cos[\pi(2k-1)/(2n+1)] = 4\sin^2[\pi(2k-1)/2(2n+1)] < 4 \tag{2.25}$$

For a Wiener process the requirement that K be non-negative definite for large n is then equivalent to the requirement that

$$\min_k \alpha_{k,n} \geqq \sigma^2/4 - 1/2\rho \geqq 0 \tag{2.26}$$

or

$$\rho \geqq 2/\sigma^2 \tag{2.27}$$

Thus the equality will hold in (2.13) iff

$$D \leqq \sigma^2/4 \tag{2.28}$$

This result is surprising in that it says for small enough average distortion a Wiener process and the much simpler independent letter process can be source coded with the same rate.

When a=0, $u_n = z_n$ and (2.23) becomes $\sin[n\theta] = 0$ so that $\theta = \pi/2$ and therefore $R_u(D) = R_z(D)$ when $D \leqq \sigma^2$, which is the entire nonzero region of $R_z(D)$, as one would expect.

When a≠1 or 0 (2.23) is a transcendental equation and cannot be solved exactly. If the solutions can be shown to be real, then (2.22) will still give a bound of the form of (2.25) and therefore of (2.28).

When $0 < a < 1$ the solutions to (2.23) can be shown to be real. Setting $z = e^{i\theta}$ (2.23) becomes

$$z^{2n+1} = f(z) = \frac{1-az}{z-a} \tag{2.29}$$

which is a linear fractional transformation. The transformation maps the interior of the unit circle in the z plane into the exterior of the unit circle in the f(z) plane. Thus if $|z| > 1$, then $z^{2n+1} > 1$ and $(1-az)/(z-a) < 1$, which is a contradiction. Likewise $|z| < 1$ cannot hold. Thus $|z| = 1$ so that θ must be pure real. Since θ is real, $\cos\theta \geqq -1$ so that

$$\beta \leqq (1+a)^2 \tag{2.30}$$

Therefore K will be positive definite iff

$$\rho \geqq (1+a)^2/2\sigma^2 \tag{2.31}$$

or, equivalently, if

$$D \leqq \sigma^2/(1+a)^2 \triangleq D_c \tag{2.32}$$

When $a > 1$, it can be demonstrated that there are complex solutions to (2.23) so that the bounding method of the previous case cannot be used. To find the desired region for all values of a it is clear that the behavior of the solutions of (2.23) must be known at least asymptotically. It will be seen that in the limit $n \to \infty$ solutions to (2.23) become dense in $[0, \pi]$.

Before evaluating $R_u(D)$ in the range where $R_u(D)$ is not equal to

$R_z(D)$ the following facts are pointed out: The eigenvalues of R^{-1} could only be evaluated when dealing with the Wiener process or the original independent letter process. A bound on the eigenvalues was obtained in the case $0 < a < 1$ and this bound was sufficient for evaluating $R_u(D)$ for $D \leqq D_c$. Knowledge of the actual behavior of the eigenvalues and not just a bound will be necessary to continue and so some method of approximating the eigenvalues when they cannot be evaluated exactly must be found. The theory of Toeplitz forms discussed in Appendix A provides such a method.

To evaluate $R_u(D)$ for all ρ, first consider the problem of extending the results for $0 \leqq a \leqq 1$ to the case $\rho \leqq (1+a)^2/2\sigma^2$. Begin by diagonalizing the inverse autocorrelation matrix:

$$R^{-1} = \sigma^2 QBQ^* \tag{2.33}$$

where Q is the unitary matrix whose columns are the eigenvectors of R^{-1} and $B = \{\beta_{j,n}\delta_{ij}\}$, or equivalently, $B(i,j) = \beta_{j,n}\delta_{ij}$ where $B(i,j)$ denotes the i,j^{th} entry of B. Note that $\prod_{j=1}^{n} \beta_{j,n} = \det B = \det A^T A = 1$. We shall make the variable changes $\underline{y} = Q^*\underline{u}$ and $\underline{x} = Q^*\underline{v}$ so the n-tuple $\underline{y}$ consists of independent Gaussian random variables with variance $\sigma^2/\beta_{k,n}$. Assume that the $\beta_{k,n}$ have been ordered to be non-decreasing with k. Theorem 1.1 can be restated in terms of the new variables.

Theorem 2.1

Define $R_{o_n}(\rho, p)$ as the per letter $R_o(\rho, p)$ of (1.6) with $k = \underline{x} = Q^* \underline{v}$ and $j = \underline{y} = Q^* \underline{u}$ then

$$\min_{p \epsilon P} R_{o_n}(\rho, p) \geqq \frac{1}{n} \int d\underline{y}\, q(\underline{y}) \ln[f(\underline{y})/q(\underline{y})] \tag{2.34}$$

where $q(\underline{y}) = (2\pi\sigma^2)^{-n/2} \exp\{-1/2\, \underline{y}\, B\underline{y}\}$ and $f(\underline{y})$ is any function satisfying the constraints

$$\int d\underline{y} f(\underline{y}) e^{-\rho[\underline{y}-\underline{x}]^T[\underline{y}-\underline{x}]} \leqq 1 \text{ all } \underline{x} \tag{2.35}$$

Equality will hold in (2.34) iff there exists a density $w(\underline{x})$ satisfying the equation

$$\int e^{-\rho[\underline{y}-\underline{x}]^T[\underline{y}-\underline{x}]} w(\underline{x}) d\underline{x} = \frac{e^{-(1/2\sigma^2)\underline{y}^T B\underline{y}}}{(2\pi\sigma^2)^{n/2} f(\underline{y})}$$

$$= \frac{e^{-(1/2\sigma^2) \sum_{k=1}^{n} y_k^2 \beta_{k,n}}}{(2\pi\sigma^2)^{n/2} f(\underline{y})} \tag{2.36}$$

and (2.35) is satisfied with equality for all $\underline{x}$ such that $w(\underline{x}) > 0$. This orthogonalization enables us to deal only with the independent zero mean Gaussian random variables y_k.

The K matrix is positive definite iff the lowest eigenvalue of K, $\sigma^2/\beta_{n,n} - 1/2\rho$, is strictly greater than zero. When $\rho = \beta_{n,n}/2\sigma^2$ the K matrix and the B matrix are singular and the variance of x_n is zero, i.e., x_n is no longer random. This suggests that the previous results may be modified for the case $\beta_{n-1,n}/2\sigma^2 \leqq \rho \leqq \beta_{n,n}/2\sigma^2$ by setting $x_n = 0$ by putting a unit impulse $\delta(x_n)$ in the $w(\underline{x})$ density and reducing the B matrix to a non-negative definite $(n-1)\times(n-1)$ matrix, i.e., set

$$w(\underline{x}) = \frac{\delta(x_n)\exp\left[-1/2 \sum_{k=1}^{n-1} x_k^2 2\rho\beta_{k,n}/(2\rho\sigma^2 - \beta_{k,n})\right]}{\prod_{k=1}^{n-1}\left\{2\pi\left(\frac{\sigma^2}{\beta_{k,n}} - \frac{1}{2\rho}\right)\right\}^{1/2}} \tag{2.37a}$$

Inserting (2.37a) into (2.36) yields a modified $f(\underline{y})$ for the new region:

$$f(\underline{y}) = \frac{(\rho/\pi)^{(n-1)/2}\exp\left[-\rho(\beta_{n,n}/2\rho\sigma^2 - 1)y_n^2\right]}{\sqrt{2\pi\sigma^2/\beta_{n,n}}} \tag{2.37b}$$

for $\beta_{n-1,n}/2\sigma^2 \leqq \rho < \beta_{n,n}/2\sigma^2$. For (2.37b) to be a valid $f(\underline{y})$ and give the lower bound of (2.34) it must satisfy the constraint (2.35):

$$\int e^{-\rho[\underline{y}-\underline{x}]^T[\underline{y}-\underline{x}]} f(\underline{y})d\underline{y} = e^{-\rho x_n^2[1 - 2\sigma^2\rho/\beta_{n,n}]} \leqq 1 \tag{2.38}$$

When $w(\underline{x}) > 0$, i.e., when $x_n = 0$, the left hand side of (2.38) is equal to one. When $w(\underline{x}) = 0$, i.e., when $x_n \neq 0$, the left hand side of (2.38) is less than one. Since $f(\underline{y})$ satisfies (2.35), Eq. (2.34) holds true. Since $w(\underline{x})$ and $f(\underline{y})$ satisfy (2.36) and since (2.35) holds with equality when $w(\underline{x}) > 0$, the lower bound of (2.34) must hold with equality. Thus by Theorem 2.1 the assignments of (2.37) yield equality in (2.34) when $\beta_{n-1,n}/2\sigma^2 \leqq \rho < \beta_{n,n}/2\sigma^2$. Before evaluating (2.34) for the assignments of (2.37), the assignments of (2.37) are generalized to the entire range of ρ. It is possible to extend (2.37b) to the region $\beta_{k-1,n}/2\sigma^2 \leqq \rho < \beta_{k,n}/2\sigma^2$ by using

$$f(\underline{y}) = \frac{(\rho/\pi)^{(k-1)/2} \exp\left[-\rho \sum_{j=k}^{n} (\beta_{j,n}/2\rho\sigma^2 - 1) y_j^2\right]}{\prod_{j=k}^{n} \sqrt{2\pi\sigma^2/\beta_{j,n}}} \tag{2.39a}$$

$$\beta_{k-1,n}/2\sigma^2 \leqq \rho < \beta_{k,n}/2\sigma^2 \tag{2.39b}$$

and to define $\beta_0 = 0$, $\beta_{n+1} = \infty$ so that (2.39) is valid for $k = 1, 2, \ldots, n+1$. Equation (2.39) satisfies (2.35) so that inserting (2.39) into (2.34) yields the lower bound

$$\min_{p \in P} R_{o_n}(\rho, p) \geqq \frac{(k-1)}{2n} \ln (2\rho\sigma^2 e) + n^{-1}\rho \sum_{j=k}^{n} \sigma^2/\beta_{j,n} + 1/2n^{-1} \sum_{j=k}^{n} \ln \beta_{j,n} \tag{2.40}$$

which with (2.10) becomes

$$R_n(D) \geqq \frac{k-1}{2n} \ln(2\rho\sigma^2 e) - \rho\left[d/n - n^{-1} \sum_{j=k}^{n} \sigma^2/\beta_{j,n}\right] + 1/2n^{-1} \sum_{j=k}^{n} \ln \beta_{j,n} \tag{2.41}$$

$$\text{for } \beta_{k-1,n}/2\sigma^2 \leqq \rho < \beta_{k,n}/2\sigma^2$$

Maxmizing this bound over ρ yields

$$\rho = \left[(k-1)/2n\right] / \left[(d/n) - n^{-1} \sum_{j=k}^{n} \sigma^2/\beta_{j,n}\right] \tag{2.42}$$

of equivalently

$$D = d/n = (k-1)/2n\rho + n^{-1} \sum_{j=k}^{n} \sigma^2/\beta_{j,n} \tag{2.43}$$

Substituting the optimum ρ of (2.42) into (2.41) the lower bound for $R_n(D)$ becomes

$$R_n(D) \geqq \frac{(k-1)}{2n} \ln \frac{\sigma^2(k-1)/n}{\frac{d}{n} - \frac{1}{n} \sum_{j=k}^{n} \sigma^2/\beta_{j,n}} + \frac{1}{2n} \sum_{j=k}^{n} \ln \beta_{j,n} \tag{2.44a}$$

$$\text{for} \quad \beta_{k-1,n}/2\sigma^2 \leqq \rho < \beta_{k,n}/2\sigma^2 \tag{2.44b}$$

Using (2.42) the range of (2.44) can be written in terms of $d/n = D$ instead of ρ as

$$(k-1)\sigma^2/n\beta_{k,n} + n^{-1} \sum_{j=k}^{n} \sigma^2/\beta_{j,n} \leqq d/n = D_n < (k-1)\sigma^2/n\beta_{k-1,n} + n^{-1} \sum_{j=k}^{n} \sigma^2/\beta_{j,n} \tag{2.45}$$

Note that

$$\sum_{j=1}^{n} \ln \beta_{j,n} = \ln \prod_{j=1}^{n} \beta_{j,n} = \ln \mathrm{Det}\,[A^T A] = \ln(\mathrm{Det}[A])^2 = \ln 1 = 0 \tag{2.46}$$

The assignment

$$w(x) = \left(\prod_{j=k}^{n} \delta(x_j)\right) \prod_{j=1}^{k-1} \left\{\left[2\pi\left(\frac{\sigma^2}{\beta_{j,n}} - \frac{1}{2\rho}\right)\right]^{-1/2} \exp\left[-x_i^2/\left[2\left(\frac{\sigma^2}{\beta_{j,n}} - \frac{1}{2\rho}\right)\right]\right]\right\}$$

$$k=1,2,\ldots,n+1 \qquad (2.47)$$

which is simply the extension of (2.37a) to the region of (2.44b) or (2.45) along with (2.39) can be straightforwardly shown to satisfy the conditions of Theorem 2.1 to yield equality in (2.34) and therefore in (2.40) and (2.44) in the same manner that (2.37) was shown to satisfy (2.36) and (2.38).

Since both R and its inverse can be constructed, the eigenvalues of R^{-1} must be strictly positive. Equations (2.44), (2.45), and (2.46) imply that for $k > 1$

$$\begin{aligned} R_n(D) &> \frac{(k-1)}{2n} \ln \beta_{k-1,n} + (1/2n) \ln \prod_{j=k}^{n} \beta_{j,n} \\ &= \frac{(k-1)}{2n} \ln \beta_{k-1,n} - (1/2n) \ln \prod_{j=1}^{k-1} \beta_{j,n} \\ &= \frac{(k-1)}{2n} \ln \beta_{k-1,n} - (1/2n) \sum_{j=1}^{k-1} \ln \beta_{j,n} \\ &\geq \frac{(k-1)}{2n} \ln \beta_{k-1,n} - \frac{(k-1)}{2n} \ln \beta_{k-1,n} = 0 \end{aligned} \qquad (2.48)$$

When k=1, Equation (2.46) causes (2.44) and (2.45) to reduce to

$$R_n(D) = (1/2n) \sum_{j=1}^{n} \ln \beta_{j,n} = 0 \qquad (2.49a)$$

when

$$D \geqq D_M = n^{-1} \sum_{j=1}^{n} \sigma^2/\beta_{j,n} \tag{2.49b}$$

Thus $R_n(D)$ is bounded away from zero unless k=1, i.e., unless $D \geqq D_M$. This can be interpreted as meaning that the only way to obtain a zero information rate is essentially to set $\underline{x} = \underline{0}$, the all zero vector, and to suffer an average distortion of $D_M = n^{-1} \sum_{j=1}^{n} \sigma^2/\beta_{j,k} = n^{-1} \sum_{j=1}^{n} \overline{y_j^2} = n^{-1} \sum_{j=1}^{n} \overline{u_j^2}$. When $\{u_k\}$ is a Wiener process so that a=1 D_M is unbounded as $n \to \infty$ since $n^{-1} \sum_{j=1}^{n} \overline{u_j^2} = n^{-1} \sum_{j=1}^{n} j\sigma^2 = (n+1)\sigma^2/2$. The possibly unbounded D_M will make it necessary to modify the standard proof of the coding theorem which assumes a finite maximum average distortion.

To take the limit in (2.44) and (2.45) it is necessary to know the asymptotic behavior of the eigenvalues. When dealing with the Wiener process the eigenvalues could be evaluated exactly. The limits can then be found immediately since

$$\lim_{n \to \infty} n^{-1} \sum_{j=[\frac{\theta}{\pi} n]}^{n} F(\beta_{j,n}) = (1/\pi) \int_{\theta}^{\pi} F(\beta(x))\, dx \tag{2.50}$$

where $F(\cdot)$ is any function continuous on

$$[\beta_{k,n}, \beta_{n,n}], \text{ and } \beta(x) = \lim_{n \to \infty} \beta_{k,n} = 2 - 2\cos x$$

where $k = \left[\frac{\theta}{\pi} n\right]$. Expression (2.50) is interesting in that it shows that the limiting forms of the sums of interest can be found in terms of $\beta(x)$, which is simply the Fourier transform of any row or column of the $A^T A$ matrix except those near the upper left or lower right corners. The function $\beta(x)$ can be interpreted as the "spectrum" of the $A^T A$ matrix. The notion of the "spectrum" of a matrix will shortly be made precise and it will be seen that a relation of the form (2.50) holds even when the eigenvalues cannot be evaluated exactly.

With the exception of the lower right hand corner entry, the entries of the R^{-1} matrix depend only on the difference between the row index and the column index, i.e., $R^{-1}(j,k) = R^{-1}(j-k)$ for $\max(j,k) \leq n^{-1}$. Matrices of the form $T(j,k) = T(j-k)$ are known as Toeplitz matrices and a great deal is known about their asymptotic eigenvalue distribution [14],[13]. Furthermore, if a Toeplitz T_n matrix is approximated by a bounded Hermetian matrix H_n in the sense that

$$\lim_{n \to \infty} |T_n - H_n| = 0 \tag{2.51}$$

where the metric is defined by

$$|T_n - H_n|^2 = n^{-1} \sum_{j=1}^{n} \sum_{k=1}^{n} |T_n(k,j) - H_n(k,j)|^2 \tag{2.52}$$

then the asymptotic eigenvalue distribution of H_n is known to be equal to that of T_n (Theorem A4).

To make these ideas more precise define the Toeplitz matrix C_n by

$$C_n = \begin{bmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-1} \\ c_{-1} & c_0 & c_1 & & \\ & c_{-1} & \ddots & \ddots & \\ & & & \ddots & c_1 \\ c_{-n+1} & & & c_{-1} & c_0 \end{bmatrix} = \{c_{k-j}\} \tag{2.53}$$

where $\sum_{j=-\infty}^{\infty} |c_j|^2 < \infty$. Define the "spectrum" of C_n by

$$c(x) = \sum_{j=-\infty}^{\infty} c_j e^{ijx} \tag{2.54}$$

so that

$$C_n(j,k) = (1/2\pi) \int_{-\pi}^{\pi} e^{-(j-k)ix} c(x)\,dx \tag{2.55}$$

The eigenvalues of C_n, $\psi_{k,n}$, are asymptotically equally distributed as $c(x)$ with x uniform on $[-\pi, \pi]$, i.e.,

$$\lim_{n\to\infty} n^{-1} \sum_{k=1}^{n} F(\psi_{k,n}) = (1/2\pi) \int_{-\pi}^{\pi} F(c(x))\,dx \tag{2.56}$$

for any F continuous on the interval $[\min_x (c(x)), \max_x (c(x))]$ (Theorem A2). The precise definitions and theorems are given in Appendix A, but for the moment only the simple results needed will be used.

Approximate the matrix $A^T A$ of (2.5) by the Toeplitz matrix, T, which is simply $A^T A$ with its lower right hand corner entry changed to $1+a^2$. Then (2.51) is satisfied by the matrices A^TA and T since $|T-A^TA|^2 = \frac{a^4}{n} \xrightarrow[n\to\infty]{} 0$ so that the eigenvalues of A^TA are asymptotically distributed as $g(x) = 1+a^2-2a\cos x$, i.e.,

$$\lim_{n\to\infty} n^{-1} \sum_{k=1}^{n} F(\beta_{k,n}) = \pi^{-1}\int_0^{\pi} F(1+a^2-2a\cos x)\,dx \qquad (2.57)$$

Only the region $(0,\pi)$ is considered since $g(x)$ is even. This can be interpreted as simply meaning that asymptotically the eigenvalues behave as (2.22b) where the solutions to (2.23) have become dense in $(0,\pi)$. Define the characteristic function

$$\chi_m(x) = \begin{cases} 1 & x > m \\ 0 & x \leqq m \end{cases} \qquad (2.58)$$

Then the functions $\chi_{g(\theta)}(x)/x$ and $\chi_{g(\theta)}(x)\ln x$ can be approximated arbitrarily closely by continuous functions $F_1(x)$ and $F_2(x)$ so long as $g(\theta) > 0$. Applying (2.57) to $F_1(x)$ and $F_2(x)$ and defining $x_j = j\pi/n$, $\theta = x_k$, and $\Delta x = x_j - x_{j-1} = \pi/n$ produces

$$\lim_{n\to\infty} n^{-1} \sum_{j=k}^{n} 1/\beta_{j,n} = \lim_{n\to\infty} \pi^{-1} \sum_{j=k}^{n} \Delta x_j/g(x_j) = \pi^{-1}\int_{\theta}^{\pi} dx/g(x) \qquad (2.59a)$$

$$\lim_{n\to\infty} n^{-1} \sum_{j=k}^{n} \ln\beta_{j,n} = \pi^{-1}\int_{\theta}^{\pi} dx\,\ln g(x) \qquad (2.59b)$$

where $g(x) = 1+a^2-2a\cos x$. Equations (2.59) are the promised relations of the form of (2.50) and now the limit of (2.44), which has been shown to hold with equality, and the limit of (2.45) can be evaluated to obtain

$$R_u(D) = (\theta/2\pi) \ln g(\theta) + (1/2\pi) \int_\theta^\pi dx \ln g(x) \tag{2.60a}$$

$$D = \theta\sigma^2/\pi g(\theta) + (\sigma^2/\pi) \int_\theta^\pi dx/g(x) \tag{2.60b}$$

The above parametric expressions for the rate distortion function hold for any value of a and thus include the earlier results of this chapter. In particular, when $\theta = \pi$, (2.60) becomes

$$\begin{aligned} R_u(D) &= 1/2 \ln \sigma^2/D = R_z(D) \\ 0 \leqq D \leqq D_c &= \sigma^2/\max_{x \epsilon [0,\pi]} g(x) = \\ &\sigma^2/(1+a)^2 \end{aligned} \tag{2.61}$$

which is of the same form as both (2.28) and (2.32). It is interesting that $R_u(D)$ is independent of σ^2, the variance of the independent letter process $\{z_k\}$, except through the parameter θ. As $\theta \to 0$, $R_u(D) \to 0$ and $D \to (\sigma^2/\pi) \int_0^\pi dx/g(x)$, where the integral is taken to be ∞ if it does not exist.

Equation (2.60) can be rewritten in a somewhat more compact form as

$$R_u(D) = (1/2\pi) \int_0^\pi \ln [\max (g(x), g(\theta))] dx \tag{2.62}$$

$$D = \pi^{-1} \int_0^{\pi} dx\, \sigma^2 / \max[g(x), g(\theta)]$$

When $\theta \neq \pi$, Jensen's inequality [15] applied to (2.62) produces

$$\begin{aligned} R_u(D) &= -(1/2\pi) \int_0^{\pi} \ln \{1/\max[g(x), g(\theta)]\} dx \\ &\geqq -1/2 \ln \pi^{-1} \int_0^{\pi} dx/\max[g(x), g(\theta)] \\ &= 1/2 \ln \sigma^2/D \end{aligned} \quad (2.63)$$

where the equality can hold only if $\max[g(x), g(\theta)]$ is independent of x, i.e., if $\theta = \pi$. Thus when $\theta \neq \pi$, (2.63) becomes

$$R_u(D) > 1/2 \ln \sigma^2/D = R_z(D), \quad D > D_c \quad (2.64)$$

D. Discussion

The results of (2.62) can be generalized to any time discrete Gaussian source whose inverse autocorrelation matrix can be approximated by a Toeplitz matrix. This generalization will be formalized in the next chapter.

The most important results of this chapter are (2.61) and (2.64) rather than (2.62). It will be shown in later chapters that this relative behavior of $R_u(D)$ to $R_z(D)$ holds for some non-Gaussian autoregressive sources as well as for all time discrete Gaussian autoregressive sources.

Expressions equivalent to (2.62) were derived independently by Berger [2] who evaluated the eigenvalues of the $R = \{\max(i,j)\}$ matrix using difference techniques. Berger [2] also noted the behavior of the results (2.61) and (2.64) although he attributes this behavior to the fact that the source is Gaussian and not to the fact that it is autoregressive. The method used in this chapter of dealing with the inverse autocorrelation rather than the autocorrelation is more powerful in that it extends readily to all Gaussian autoregressive processes.

CHAPTER III

THE RATE DISTORTION FUNCTION OF A TIME DISCRETE, GAUSSIAN AUTOREGRESSIVE SOURCE WITH A MEAN SQUARE ERROR FIDELITY CRITERION

A. Introduction

In this chapter the results of the previous chapter are generalized first to discrete time finite order Gaussian autoregressive processes and then to those of infinite order. Frequent use of the theorems of Appendix A on Toeplitz forms will be made. Finally, the coding theorem for time discrete autoregressive Gaussian processes with a constraint on their "spectrum" will be stated. The proof of the coding theorem is so similar to that of Theorem 9.7.1 of Gallager [9] that the proof is relegated to Appendix B.

B. The Rate Distortion Function for Finite Order

A time discrete, m^{th} order Gaussian autoregressive source, $\{u_k\}$, is defined by the difference equation

$$u_n = -\sum_{k=1}^{m} a_k u_{n-k} + z_n , \quad n \geq 1 \tag{3.1}$$

where the z_n are zero mean independent Gaussian random variables with variance σ^2. If the roots of the characteristic equation, $x^m +$

$a_1x^{m-1} + \ldots + a_m = 0$, are all of modulus strictly less than one, then the statistics of the process approach a stationary distribution as $n \to \infty$. This condition will not be assumed to hold so that asymptotically nonstationary processes may be included in the results.

Define the $n \times n$ matrix A by

$$A = \begin{bmatrix} 1 & & & & & & \\ a_1 & 1 & & & 0 & & \\ a_2 & a_1 & 1 & & & & \\ \vdots & & & \cdot & & & \\ & & & & \cdot & & \\ a_m & & & & & & \\ 0 & & & & & \cdot & \\ & a_m & & & & a_1 & 1 \end{bmatrix} \tag{3.2}$$

of equivalently,

$$A(k,j) = \begin{cases} a_{k-j} & k-j \geq 0 \\ 0 & k-j < 0 \end{cases} \tag{3.3}$$

When n is not clear from context the A matrix will be subscripted with an n. Using (3.2) the difference equation can be rewritten as

$$\underline{z} = A\underline{u} \tag{3.4}$$

where $\underline{u} = (u_1, u_2, \ldots u_n)^T$.

The autocorrelation matrix of the autoregressive process is

$$R = E[\underline{uu}^T] = \sigma^2 A^{-1} A^{T^{-1}} \tag{3.5}$$

The existence of R is assured since det A = 1 ≠ 0 and therefore A is non-singular. Inversion of (3.5) gives

$$R^{-1} = (1/\sigma^2)A^T A \tag{3.6}$$

Note that since R and R^{-1} are invertable they must both be positive definite and hence have strictly positive eigenvalues. Rewrite (3.6) as

$$R^{-1}(i,j) = \sigma^{-2} \sum_{k=0}^{n-\max(i,j)} a_k a_{k+|i-j|} \tag{3.7}$$

Equation (3.7) has the interesting property that for all entries $\min(i,j) \leqq n-m$

$$R^{-1}(i,j) = R^{-1}(|i-j|) = \sigma^2 \sum_{k=0}^{m} a_k a_{k+|i-j|} \qquad \min(i,j) \leqq n-m \tag{3.8}$$

For large n the entries of the matrix R^{-1} depend only on $|i-j|$ except for those entries in an $m \times m$ square in the lower right hand corner. The matrix of (2.5) is an example of such an "almost Toeplitz" matrix.

As in Chapter II R^{-1} is approximated by a Toeplitz matrix R_T^{-1} where

$$R_T^{-1}(i,j) = \sigma^{-2} E(i-j) \tag{3.9}$$

where

$$E(i,j) = E(i-j) = \sum_{k=0}^{m} a_k a_{k+|i-j|} \tag{3.10}$$

The Toeplitz matrix R_T^{-1} approximates R^{-1} in the metric for matrices defined in (2.52) and Appendix A since

$$\lim_{n\to\infty} | R^{-1} - R_T^{-1} |^2 = \lim_{n\to\infty} n^{-1} \sum_{i=1}^{n} \sum_{j=1}^{n} | \sigma^{-2} \sum_{k=0}^{m} a_k a_{k+|i-j|}$$

$$- \sigma^{-2} \sum_{k=0}^{n-\max(i,j)} a_k a_{k+|i-j|} |^2$$

$$= \lim_{n\to\infty} n^{-1} \sigma^4 \sum_{i=n-m+1}^{n} \sum_{j=n-m+1}^{n} | \sum_{k=n-\max(i,j)+1}^{m} a_k a_{k+|i-j|} |^2$$

$$= \lim_{n\to\infty} n^{-1} \sigma^4 \sum_{j=0}^{m-1} \sum_{i=0}^{m-1} | \sum_{k=\min(i,j)+1}^{m} a_k a_{k+|i-j|} |^2$$

$$= 0 \qquad (3.11)$$

since the a_k's are all finite. Theorem A4 and (3.11) together show that the asymptotic eigenvalue distributions of R^{-1} and R_T^{-1} are the same.

From (3.1) the density function for u_n conditioned on the past is

$$p(u_n | u_{n-1}, u_{n-2}, \ldots) = p(u_n | u_{n-1}, u_{n-2}, \ldots, u_{n-m}) = p_z\left(u_n + \sum_{j=1}^{m} a_j u_{n-j}\right)$$

$$= (2\pi\sigma^2)^{-1/2} \exp\left\{ -\left[u_n + \sum_{j=1}^{m} a_j u_{n-j} \right]^2 / 2\sigma^2 \right\} \qquad (3.12)$$

The probability density function for the n-tuple $\underline{u}$ is then

$$p(\underline{u}) = \prod_{k=1}^{n} p(u_k | u_{k-1}, \ldots, u_{k-m})$$

$$= (2\pi\sigma^2)^{-n/2} e^{-1/2\, \underline{u}^T R^{-1} \underline{u}} \qquad (3.13)$$

where R^{-1} is given by (3.6). Diagonalization of the R^{-1} matrix yields

$$R^{-1} = \sigma^{-2} QBQ^* \tag{3.14}$$

where $B(i,j) = \beta_{i,n}\delta_{ij}$, where the $\beta_{i,n}$ are the eigenvalues of the matrix A^TA ordered so as to be non-decreasing with i, δ_{ij} is the Kronecker delta, and Q is the unitary matrix whose columns are the eigenvectors of R^{-1}.

Defining $\underline{y} = Q^*\underline{u}$, the density for the n-tuple $\underline{y}$ is

$$\begin{aligned} p(\underline{y}) &= (2\pi\sigma^2)^{-n/2} e^{-1/2\, \underline{y}^T B \underline{y}} \\ &= (2\pi\sigma^2)^{-n/2} \exp\left\{-1/2 \sum_{k=1}^{n} y_k^2 \beta_{k,n}\right\} \end{aligned} \tag{3.15}$$

so that we are now dealing with zero mean, independent Gaussian random variables with variance $\sigma^2/\beta_{k,n}$.

The distortion measure between n-tuples $\underline{u}$ and $\underline{v}$ is defined as previously by

$$d(\underline{u},\underline{v}) = \sum_{i=1}^{n} d(u_i - v_i) = \sum_{i=1}^{n} (u_i - v_i)^2 \tag{3.16}$$

so that defining $\underline{x} = Q^*\underline{v}$ yields

$$\begin{aligned} d(\underline{u},\underline{v}) &= d(Q\underline{y}, Q\underline{x}) = (\underline{y}-\underline{x})^T Q^* Q(\underline{y}-\underline{x}) = (\underline{y}-\underline{x})^T(\underline{y}-\underline{x}) \\ &= \sum_{i=1}^{n} (y_i - x_i)^2 = d(\underline{y},\underline{x}) \end{aligned} \tag{3.17}$$

The assignments

$$f(\underline{y}) = \frac{(\rho/\pi)^{[k-1]/2} \exp\left\{-\rho \sum_{j=k}^{n} [\beta_{j,n}/2\rho\sigma^2 - 1] y_j^2\right\}}{\prod_{j=k}^{n} \sqrt{2\pi\sigma^2/\beta_{j,n}}} \tag{3.18a}$$

and

$$w(\underline{x}) = \prod_{j=k}^{n} \delta(x_j) \prod_{i=1}^{k-1} \left\{2\pi\left[\frac{\sigma^2}{\beta_{i,n}} - \frac{1}{2\rho}\right]\right\}^{-1/2} \exp\left\{-\frac{1}{2}x_i^2\left[\frac{\sigma^2}{\beta_{i,n}} - \frac{1}{2\rho}\right]^{-1}\right\} \tag{3.18b}$$

$$\beta_{k-1,n}/2\sigma^2 \leqq \rho < \beta_{k,n}/2\sigma^2 \tag{3.18c}$$

$$k=1,\ldots,n+1$$

satisfy the conditions of Theorem 2.1 to yield

$$R_n(D) = 1/2(k-1)n^{-1}\ln\left\{\frac{\sigma^2(k-1)/n}{\frac{d}{n} - \frac{1}{n}\sum_{j=k}^{n} \sigma^2/\beta_{j,n}}\right\} + 1/2n^{-1}\sum_{j=k}^{n} \ln\beta_{j,n} \tag{3.19}$$

for

$$\frac{(k-1)}{n}\frac{\sigma^2}{\beta_{k,n}} + \frac{1}{n}\sum_{j=k}^{n} \sigma^2/\beta_{j,n} \leqq \frac{d}{n} = D$$

$$< \frac{k-1}{n}\frac{\sigma^2}{\beta_{k-1,n}} + \frac{1}{n}\sum_{j=k}^{n} \sigma^2/\beta_{j,n} \tag{3.20}$$

$$k=1,2,\ldots,n+1$$

where $\beta_{k,n} \leqq \beta_{k+1,n}$, $\beta_{n+1,n} \triangleq \infty$, $\beta_{0,n} \triangleq 0$

i.e., (3.18a) satisfies (2.35) so that $R_n(D)$ is lower bounded by the right hand side of (2.41). Optimizing this lower bound over ρ yields the right hand side of (3.19) as a lower bound for $R_n(D)$ in the range (3.20). Since (3.18a) and (3.18b) satisfy (2.36), the lower bound holds with equality.

When k = 1 (3.19) reduces to

$$R_n(D) = 0 \quad ; \qquad D = D_M \triangleq \frac{1}{n} \sum_{j=1}^{n} \sigma^2/\beta_{j,n} \tag{3.21}$$

When k = n+1, (3.19) becomes

$$\begin{aligned} R_n(D) &= 1/2 \ln \sigma^2/D = R_z(D) \\ 0 \leqq \frac{d}{n} = D &\leqq D_c \leqq \sigma^2/\max_j \beta_{j,n} = \sigma^2/\beta_{n,n} \end{aligned} \tag{3.22}$$

To evaluate $R_u(D) = \lim_{n\to\infty} R_n(D)$ recall that the eigenvalues of R^{-1} are asymptotically distributed as the eigenvalues of R_T^{-1}. From Theorem A2 the eigenvalues of the Toeplitz matrix R_T^{-1}, $\sigma^{-2}\alpha_{j,n}$, are distributed as $\sigma^{-2} g(x) = \sigma^{-2} \sum_{k=-\infty}^{\infty} E(k) e^{ikx}$, the "spectrum" of the matrix R_T^{-1}, with x uniform on $[0, \pi]$. Consider g(x) to be defined on $[0, \pi]$ rather than $[-\pi, \pi]$ since R_T^{-1} is symmmetric and therefore g(x) is an even function. Thus

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} F[\alpha_{k,n}] = \frac{1}{\pi} \int_0^{\pi} F[g(x)]\,dx \tag{3.23}$$

holds for any arbitrary function $F[\cdot]$ continuous on $[0, \max_x g(x)]$ where

$$0 \leq g(x) = \sum_{k=-\infty}^{\infty} \left[\sum_{j=0}^{m} a_j a_{j+k} \right] e^{ikx} = \left| \sum_{k=0}^{m} a_k e^{ikx} \right|^2 < \infty \qquad (3.24)$$

Define the characteristic function $\chi_m(x)$ as in (2.58) and approximate $\chi_{g(\theta)}(x)/x$, $\chi_{g(\theta)}(x) \ln x$, and $\chi_{g(\theta)}(x)$ arbitrarily closely by continuous functions $F_1(x)$, $F_2(x)$ and $F_3(x)$, respectively. Reorder the sequence $g(2\pi k/n)$ into the sequence $g_{j,n}$ which is non-decreasing with j, as are the $\beta_{j,n}$. Define $x_j = 2\pi j/n$, $\Delta x = x_j - x_{j-1} = \pi/n$, and $\theta = x_k$. Then applying (3.23) to $F_1(x)$, $F_2(x)$ and $F_3(x)$ yields

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=k}^{n} 1/\beta_{j,n} = \lim_{n\to\infty} \frac{1}{n} \sum_{j=k}^{n} 1/g_{j,n} = \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \chi_{g(\theta)}[g_{j,n}]/g_{j,n}$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \chi_{g(\theta)}\left[g\left(\frac{2\pi}{n} j\right)\right] / g\left(\frac{2\pi j}{n}\right)$$

$$= \frac{1}{\pi} \int_0^{\pi} dx\, \chi_{g(\theta)}[g(x)]/g(x) = \frac{1}{\pi} \int_{g(x)>g(\theta)} dx/g(x) \qquad (3.25a)$$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=k}^{n} \ln \beta_{j,n} = \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \chi_{g(\theta)}[g_{j,n}] \ln g_{j,n}$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \chi_{g(\theta)}\left[g\left(\frac{2\pi j}{n}\right)\right] \ln \left[g\left(\frac{2\pi j}{n}\right)\right] = \frac{1}{\pi} \int_0^{\pi} \chi_{g(\theta)}[g(x)] \ln g(x)$$

$$= \frac{1}{\pi} \int_{g(x)>g(\theta)} dx \ln g(x) \tag{3.25b}$$

$$\lim_{n\to\infty} \frac{k-1}{n} = \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \{1-\chi_{g(\theta)}[\beta_{j,n}]\} = \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \{1-\chi_{g(\theta)}(g_{j,n})\}$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \left\{1-\chi_{g(\theta)}\left(g\left(\frac{2\pi j}{n}\right)\right)\right\} = \frac{1}{\pi} \int_0^{\pi} \{1-\chi_{g(\theta)}(g(x))\} dx$$

$$= \int_{g(x) \leqq g(\theta)} dx/\pi \tag{3.25c}$$

Evaluating the limits of (3.19) and (3.20) using (3.25) then results in

$$R_u(D) = \left[\int_{x:\, g(x) \leqq g(\theta)} dx/2\pi\right] \ln[g(\theta)] + (2\pi)^{-1} \int_{x:\, g(x)>g(\theta)} dx \ln[g(x)] \tag{3.26a}$$

$$D = \left[\pi^{-1} \int_{x:\, g(x) \leqq g(\theta)} dx\right] \sigma^2/g(\theta) + \pi^{-1} \int_{x:\, g(x)>g(\theta)} dx\, \sigma^2/g(x) \tag{3.26b}$$

Equation (3.26) can be rewritten more conveniently as

$$R_u(D) = (2\pi)^{-1} \int_0^{\pi} dx \ln \{\max [g(x), g(\theta)]\} \tag{3.27a}$$

$$D = \pi^{-1} \int_0^{\pi} dx\, \sigma^2 / \max [g(x), g(\theta)] \tag{3.27b}$$

$$\text{for} \quad 0 \leqq D < D_M = (\sigma^2/\pi) \int_0^{\pi} dx/g(x) \tag{3.27c}$$

Equations (3.27) are identical to (2.62). Thus the parametric relations for $R_u(D)$ and D in terms of the "spectrum" of the inverse autocorrelation matrix are the same for any finite order time discrete Gaussian autoregressive source. It will shortly be shown that (3.27) holds also when the order, m, is allowed to be infinite.

When $D \leqq D_c = \sigma^2 / \max_x g(x)$, (3.27) reduces to

$$R_u(D) = 1/2 \ln \sigma^2/D = R_z(D) \tag{3.28}$$

Note that since g(x) must be bounded, the relation (3.28) is guaranteed to hold over a non-zero interval. When $D_c \leqq D < D_M$, (2.63) and therefore (2.64) hold. In summary

$$\begin{aligned} R_u(D) &= R_z(D) \qquad && 0 \leqq D \leqq D_c \\ R_u(D) &> R_z(D) \qquad && D_c < D \leqq D_M \end{aligned} \tag{3.29}$$

An alternate method for deriving (3.27) is to begin with the equations of Kolmogorov [18] for the rate distortion function of a

Gaussian vector,

$$D = n^{-1} \sum_{k=1}^{n} \min [\psi, \lambda_{k,n}] \tag{3.30a}$$

$$R_n(D) = n^{-1} \sum_{k=1}^{n} \max [0, 1/2 \ln [\lambda_{k,n}/\psi]] \tag{3.30b}$$

where the $\lambda_{k,n}$ are the eigenvalues of the autocorrelation matrix R so that $\lambda_{k,n} = \sigma^2/\beta_{k,n}$. Applying the Topelitz form theorems to (3.30) and using (2.46) produces

$$D = \pi^{-1} \int_0^{\pi} \min [\psi, \sigma^2/g(x)] dx \tag{3.31a}$$

$$R_u(D) = \pi^{-1} \int_0^{\pi} \max [0, 1/2 \ln[\sigma^2/\psi g(x)]] dx \tag{3.31b}$$

which are equivalent to (3.27). The method of this paper, which is essentially to derive the expressions for finite n from the basic Theorem 1.1 is more powerful than the shortcut above in that it can be applied to non-Gaussian sources as shall be seen in Chapter IV. In Chapter V the Kolmogorov expressions for continuous time Gaussian processes will be used as a shortcut rather than deriving them from the appropriate form of Theorem 1.1.

C. The Rate Distortion Function for Infinite Order

The restriction of finite m is now removed. The infinite order Gaussian autoregressive process is defined by the difference equation

$$z_n = \sum_{k=0}^{\infty} a_k u_{n-k} \tag{3.32}$$

$$z_j = u_j = 0 \qquad j \leq 0$$
$$a_0 = 1$$
$$a_j = 0 \qquad j < 0$$

The constraint

$$\sum_{k=0}^{\infty} |a_k|^2 < \infty \tag{3.33}$$

is sufficient to insure convergence in (3.32), but a stronger constraint will be assumed shortly to insure the matrices being dealt with are bounded. The expressions (3.19) and (3.20) remain valid and only the limit as $n \to \infty$ need be evaluated. As before the eigenvalues of R^{-1}, the inverse autocorrelation matrix, will be evaluated by approximating R^{-1} by a Toeplitz matrix.

Define formally

$$g(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx}, \qquad x \epsilon [-\pi, \pi] \tag{3.34a}$$

where

$$c_k = c_{-k} = \sum_{j=0}^{\infty} a_j a_{j+k} \tag{3.34b}$$

and define the n×n Toeplitz matrix T_n by

$$T_n(j,k) = T_n(j-k) = \int_{-\pi}^{\pi} g(x)\, e^{-i(j-k)x}\, dx/2\pi$$

$$= c_{j-k} \quad . \qquad (3.34c)$$

In order to guarantee that T_n is bounded (or equivalently, that g(x) is in L_2), we require that the sequence $\{a_k\}$ converge absolutely, i.e.,

$$\sum_{k=0}^{\infty} |a_k| = S < \infty \qquad (3.35)$$

Then T_n is bounded since

$$|T_n|^2 = \frac{1}{n} \sum_{k=1}^{n} \sum_{j=1}^{n} |c_{k-j}|^2$$

$$\leq \frac{2}{n} \sum_{k=0}^{n-1} (n-k)\, |c_k|^2 \leq 2 \sum_{k=0}^{n-1} |c_k|^2$$

$$\leq 2 \sum_{k=0}^{\infty} |c_k|^2 = 2 \int_{-\pi}^{\pi} |g(x)|^2\, dx/2\pi$$

$$= 2 \sum_{k=0}^{\infty} \left| \sum_{j=0}^{\infty} a_j a_{j+k} \right|^2 \leq 2 \sum_{k=0}^{\infty} \left\{ \sum_{j=0}^{\infty} |a_j a_{j+k}| \right\}^2$$

$$\leq 2 \left\{ \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} |a_j a_{j+k}| \right\}^2 \leq 2 S^4 \qquad (3.36)$$

Furthermore (3.35) implies (3.33) and it insures that g(x) is bounded since

$$|g(x)| = |\sum_{k=-\infty}^{\infty} c_k e^{ikx}| \leqq 2 \sum_{k=0}^{\infty} |c_k|$$

$$\leqq 2 S^2 < \infty \tag{3.37}$$

From Theorem A2 the eigenvalues of T_n are asymptotically distributed as g(x) with x uniform on $[0, \pi]$. If $A^T A$ can be shown to approximate T_n in the sense of the metric of (2.52) and Appendix A, then the asymptotic distribution of the eigenvalues of $A^T A$ will be known from Theorem A4. This result is proven in Theorem 3.1.

Theorem 3.1:

Let $A_n(j,k) = \begin{cases} a_{j-k} & j \geqq k \\ 0 & j < k \end{cases}$, $\sum_{k=0}^{\infty} |a_k| < \infty$ and define c_k, g(x), and T_n as in (3.34). Then

$$\lim_{n \to \infty} |A_n^T A_n - T_n| = 0 \tag{3.38a}$$

Proof: From (3.34) and (3.7)

$$|T_n - A_n^T A_n|^2 = n^{-1} \sum_{k=1}^{n} \sum_{j=1}^{n} | \sum_{k=0}^{\infty} a_k a_{k+|k-j|}$$

$$- \sum_{n=0}^{n-\max(k,j)} a_k a_{k+|k-j|} |^2$$

$$\leq 2 n^{-1} \sum_{k=1}^{n} \sum_{j=1}^{k} | \sum_{k=n-k+1}^{\infty} a_k a_{k+|k-j|} |^2 \tag{3.38b}$$

Making the variable substitutions m = k-j and i = n-k+1 results in

$$|T_n - A_n^T A_n|^2 \leq 2n^{-1} \sum_{i=0}^{\infty} \sum_{m=0}^{\infty} \left\{ \sum_{k=i}^{\infty} |a_k a_{k+m}| \right\}^2 \tag{3.39}$$

Given ϵ, from (3.36) there exists a p such that

$$\sum_{k=p}^{\infty} \left\{ \sum_{j=0}^{\infty} |a_j a_{j+k}| \right\}^2 < \epsilon/8 \tag{3.40}$$

Since

$$|c_k| = |\sum_{j=0}^{\infty} a_j a_{k+j}| \leq \sum_{j=0}^{\infty} |a_j a_{k+j}|$$

$$\leq \sum_{j=0}^{\infty} |a_j|^2 \leq S^2 \tag{3.41}$$

we can pick an $r >> p$ such that

$$\sum_{j=r}^{\infty} |a_j a_{j+k}| \leq \sqrt{\epsilon/(8p)} \tag{3.42}$$

Breaking up the sum in (3.39) and using (3.40) - (3.42) yields

$$|T_n - A_n^T A_n|^2 \leq 2n^{-1} \sum_{i=0}^{r-1} \sum_{m=0}^{p-1} \left\{ \sum_{k=i}^{\infty} |a_k a_{k+m}| \right\}^2$$

$$+ 2n^{-1} \sum_{i=r}^{n} \sum_{m=0}^{p-1} \left\{ \sum_{k=r}^{\infty} |a_k a_{k+m}| \right\}^2$$

$$+ 2n^{-1} \sum_{i=0}^{n} \sum_{m=p}^{\infty} \left\{ \sum_{k=0}^{\infty} |a_k a_{k+m}| \right\}^2$$

$$\leq 2rS^4/n + 2(n-r+1)\,p\epsilon 1/(8\,pn) + 2n\epsilon/(8n)$$

$$\leq \epsilon/2 + 2rS^2/n \tag{3.43}$$

Pick N >>p so that $2rS^2/n < \epsilon/2$ for $n \geq N$. Equation (3.43) then becomes

$$|T_n - A_n^T A_n|^2 \leq \epsilon\,, \quad n \geq N \tag{3.44}$$

Thus (3.38) holds and the theorem is proven.

Applying Theorem 3.1 to (3.19) and (3.20) yields (3.27) with $g(x) = |\sum_{k=0}^{\infty} a_k e^{ikx}|^2$ and thus (3.29) holds and $D_c > 0$ as in the finite order case. The function 1/g(x) will be referred to as the "spectrum" of the process even when the spectrum does not strictly exist, i.e., when 1/g(x) is non-integrable and the process is nonstationary.

D. The Coding Theorem

The rate distortion function has been evaluated for all time discrete Gaussian autoregressive processes in terms of the parameters of the process. The positive coding theorem as proven by Gallager [9] for Gaussian processes only holds for stationary processes. To show that the rate distortion function evaluated in previous sections can still be interpreted as the "equivalent rate" of a source for a given level of average distortion D the coding theorem must be proven for nonstationary Gaussian autoregressive processes. Unfortunately this has not yet been done for the most general case since in order to obtain a bound on the extreme eigenvalue behavior an added constraint on the "spectrum" of the process, $1/g(x)$, is imposed. By assuming that $g(x)$ can have only a single zero instead of the earlier assumption of $g(x) > 0$ almost everywhere a theorem of Grenander and Szego [14], can be used to find the asymptotic behavior of a single extreme eigenvalue. It is essential to study this behavior in order to handle the possibly infinite D_M of the nonstationary case. It is conjectured that the theorem holds true in the more general case mentioned, but the author has been unable to prove it.

Theorem 3.2:

Let R(D) be the mean-square-error rate-distortion function of a time-discrete Gaussian autoregressive process $\{u_k\}$ having an inverse "spectrum" $g(x) \geq 0$, i.e., $g(x)$ is the "spectrum" of the inverse autocorrelation matrix R^{-1}, with the equality holding at most one value of x. Then for any $D > 0$, any $\delta > 0$, and any sufficiently large n there exists a coding mapping source symbol sequences of length n, $M \leq e^{n[R(D) + \delta]}$ code words such that the average distortion per letter between $\underline{u}$ and $\underline{v}(\underline{u})$, the code word into which $\underline{u}$ is mapped, is less than or equal to D.

The proof, which is a straightforward modification of Theorem 9.7.1 of Gallager [9], is given in Appendix B.

Berger [3] has pointed out that one would like a somewhat stronger theorem when the process is nonstationary to allow one to encode long blocks of "super letters" instead of the entire sequence at once. The theorem above applies strictly to a "one shot" transmission of the entire source sequence and does not imply that the source can be broken up into smaller blocks and the same code used to encode each block in the sequence. Berger [2] has proven the stronger form of the coding theorem for the Wiener process using a specific (delta-modulation) coding scheme that tracks the starting point of each block. While the coding theorem of this chapter does not imply a method of implementing an actual source encoder, it does demonstrate the exponential dependence of the source code alphabet size on the rate distortion function.

E. Discussion

The major result of this chapter is (3.29), which has been shown to hold for all time discrete Gaussian autoregressive sources, even when the source is nonstationary. Since the inverse "spectrum" of the process, $\sigma^{-2} g(x)$, must always be bounded, we are guaranteed that $D_c > 0$ so that $R_u(D) = R_z(D)$ over some non-zero interval of sufficiently small D. This also holds true regardless of the stationarity or nonstationarity of the source. This surprising behavior will be seen to hold true for a non-Gaussian autoregressive source in Chapter IV.

In Chapter VI the methods of this chapter are extended to continuous time autoregressive Gaussian sources.

In Chapter VII the Toeplitz forms theorems are applied to time discrete Gaussian moving averages (to obtain the discrete time equivalent of a well-known result) and to a general linearly filtered independent letter Gaussian sequence. The results are of the same general form as those of this chapter, but is is no longer guaranteed that $D_c > 0$.

CHAPTER IV

THE RATE DISTORTION FUNCTION OF A BINARY FIRST ORDER AUTOREGRESSIVE SOURCE WITH AN AVERAGE PROBABILITY OF ERROR PER BIT FIDELITY CRITERION

A. The Model

The rate distortion function of the binary first order autoregressive source of Figure 4.1 is now evaluated for average distortion. The autoregressive sequence $\{u_k\}$ satisfies the difference equation

$$u_n = z_n \oplus u_{n-1} \tag{4.1}$$

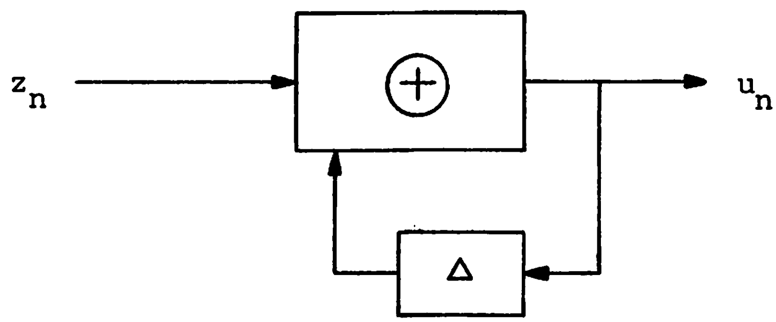
where $\{z_k\}$ is a sequence of independent Bernoulli random variables with probability mass function

$$P_z(k) = \alpha^k (1-\alpha)^{1 \oplus k} \qquad k=0,1 \tag{4.2}$$

and " $\oplus$ " denotes the "exclusive or" operation, i.e., addition in GF(2). As previously assume that $u_i = z_i = 0$ for all non-positive i. This source is simply an autoregressive representation of the well-known binary symmetric first order Markov source of Figure 4.2.

Define the distortion between letters j and k by

$$d(j,k) = 1 - \delta_{jk} \tag{4.3}$$



$$p_{z_n}(k) = \alpha^k (1-\alpha)^{1 \oplus k} \quad , \ k=0,1, \quad \text{all } n$$

Figure 4.1: Binary First Order Autoregressive Source

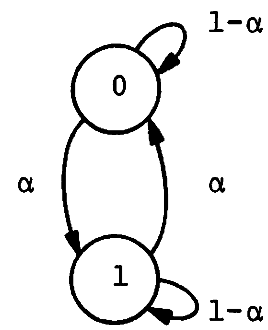


Figure 4.2: State Diagram of Binary Symmetric First Order Markov Source

where δ_{jk} is the Kronecker delta. The distortion between n-tuples $\underline{u}$ and $\underline{v}$ is

$$d(\underline{u},\underline{v}) = \sum_{k=1}^{n} d(u_k, v_k) = \sum_{k=1}^{n} [1-\delta_{u_k, v_k}]$$

$$= W(\underline{u} \oplus \underline{v}) \qquad (4.4)$$

where $W(\underline{k})$ denotes the Hamming weight of the n-tuple $\underline{k}$. The average distortion for a given test channel characterized by the conditional density $p(\underline{v}|\underline{u})$ is then

$$n^{-1} E[d(\underline{u},\underline{v})] = \frac{1}{n} \sum_{\underline{u}} \sum_{\underline{v}} p(\underline{v}|\underline{u}) Q(\underline{u}) \cdot W(\underline{u} \oplus \underline{v}) = P_e \qquad (4.5)$$

where P_e is the average probability of error per bit over the given test channel and

$$Q(\underline{u}) = \prod_{j=1}^{n} \alpha^{u_j \oplus u_{j-1}} (1-\alpha)^{1 \oplus u_j \oplus u_{j-1}} \qquad (4.6)$$

is the probability mass function for the n-tuples $\underline{u}$.

Despite the radical difference between this source and those of the previous chapters, the rate distortion functions of $\{u_k\}$ and $\{z_k\}$ will be seen to exhibit the same behavior, i.e., (3.29) will hold.

It should be pointed out that the model used here is slightly different from that usually used for the binary symmetric Markov source both in that it shows the autoregressive nature of the source and in that it starts with $u_0 = 0$ rather than with the stationary distribution of u_n. Starting with $u_0 = 0$ causes a slightly different behavior of $R_n(D)$ for finite n, but the asymptotic behavior, $R_u(D) = \lim_{n\to\infty} R_n(D)$, is of course the same.

B. The Rate Distortion Function

Gallager [9] uses the discrete version of Theorem 1.1 to find the rate distortion function for discrete alphabet, independent letter sources. For the case of the independent letter Bernoulli sequence $\{z_k\}$ his results reduce to

$$R_z(D) = \begin{cases} H(\alpha) - H(D) & 0 \leqq D \leqq \min[\alpha, 1-\alpha] \\ 0 & \text{otherwise} \end{cases} \tag{4.7}$$

where

$$H(x) = -x \ln x - (1-x) \ln (1-x) \tag{4.8}$$

As in previous chapters $R_u(D)$ will be shown to be lower bounded by $R_z(D)$ and a non-zero region of equality will be demonstrated. Outside of this region $R_u(D)$ is only lower bounded.

Guessing that the f_k of Theorem 1.1 are constant, the constraint (1.8) requires that

$$f_k = f = 1/\sum_{\underline{k}} e^{-\rho d(\underline{k},\underline{j})} = 1/\sum_{\underline{k}} e^{-\rho d(\underline{k},\underline{0})}$$

$$= 1/\sum_{j=1}^{n} \binom{n}{j} e^{-\rho j} = 1/(1+e^{-\rho})^n \tag{4.9}$$

where $\underline{0}$ is the all zero vector. Inserting (4.9) into (1.7) yields

$$\min_{\rho \varepsilon P} R_{o_n}(\rho,p) \geqq n^{-1}[H(\underline{U}) + \ln\{1/(1+e^{-\rho})^n\}] \tag{4.10}$$

where $H(\underline{U})$ is the entropy of the source $-\sum_{\underline{u}} Q(\underline{u}) \ln Q(\underline{u})$.

Since the source considered as producing n-tuples $\underline{u}$ is simply the n^{th} order extension of a first order Markov source with transition probability α, $H(\underline{U}) = nH(U) = nH(\alpha)$. Since $R_n(D) \geqq \min_{p \varepsilon P} R_{o_n}(\rho,p) - \rho D$, (4.10) becomes

$$R_n(D) \geqq H(\alpha) - \ln(1+e^{-\rho}) - \rho D \tag{4.11}$$

Maximizing this lower bound over ρ to get $D = e^{-\rho}/(1+e^{-\rho})$ and noting that (4.11) is independent of n results in

$$R_u(D) \geqq H(\alpha) - H(D) = R_z(D) \tag{4.12}$$

For (4.12) to hold with equality there must be a range of ρ for which there exists a vector $\underline{w}$ with non-negative entries which satisfies (1.9) of Theorem 1.1, i.e., for which

$$\{e^{-\rho d(\underline{k},j)}\}\underline{w} \triangleq E_n \underline{w} = (1+e^{-\rho})^n \underline{Q} \tag{4.13}$$

where the vector $\underline{Q}$ has as its $\underline{k}^{th}$ entry the probability that $\underline{u} = \underline{k}$ as given by (4.6). Thus it is necessary to consider solutions to the equation

$$\underline{w} = (1+e^{-\rho})^n E_n^{-1} \underline{Q} \tag{4.14}$$

Since E_n can be defined recursively by

$$E_n = \begin{bmatrix} E_{n-1} & e^{-\rho}E_{n-1} \\ e^{-\rho}E_{n-1} & E_{n-1} \end{bmatrix} \tag{4.15}$$

$$E_1 = \begin{bmatrix} 1 & e^{-\rho} \\ e^{-\rho} & 1 \end{bmatrix} \tag{4.16}$$

and since

$$E_1^{-1} = (1-e^{-2\rho})^{-1} \begin{bmatrix} 1 & -e^{-\rho} \\ -e^{-\rho} & 1 \end{bmatrix} \tag{4.17}$$

the recursive relation

$$E_n^{-1} = (1-e^{-2\rho})^{-1} \begin{bmatrix} E_{n-1}^{-1} & -e^{-\rho}E_{n-1}^{-1} \\ -e^{-\rho}E_{n-1}^{-1} & E_{n-1}^{-1} \end{bmatrix} \tag{4.18}$$

can be immediately verified. Combining (4.14), (4.18), and (4.17) yields

$$\underline{w} = (1-e^{-\rho})^{-1} \{ (-e^{-\rho})^{W(\underline{k} \oplus \underline{j})} \} \underline{Q} \tag{4.19}$$

The problem of finding the range of ρ for which

$$P_{\underline{j}}^{(n)} = (1-e^{-\rho})^n w_{\underline{j}}^{(n)} = \sum_{\underline{k}} (-e^{-\rho})^{W(\underline{k} \oplus \underline{j})} Q_{\underline{k}}^{(n)} \tag{4.20}$$

is non-negative for all $\underline{j}$ is attacked by restating the problem as a matrix recursion relation. As in (4.20) the superscript n will be added when it is not clear from context.

Define

$$F_{\underline{j}}^{(n)}(k_n) = \sum_{\underline{k}_{n-1}} Q_{\underline{k}}^{(n)} (-e^{-\rho})^{W(\underline{k} \oplus \underline{j})} \tag{4.20}$$

where the summation is over all possible (n-1)-tuples, $\underline{k}_{n-1}$, with k_n held fixed. Then

$$P_{\underline{j}}^{(n)} = F_{\underline{j}}^{(n)}(0) + F_{\underline{j}}^{(n)}(1) \tag{4.21}$$

so that $P_{\underline{j}}^{(n)}$ can be considered as the sum of the coordinates of the vector $\underline{F}_{\underline{j}}^{(n)} = (F_{\underline{j}}^{(n)}(0),\ F_{\underline{j}}^{(n)}(1))^T$. Breaking up the sum of (4.20) shows that the vector $\underline{F}_{\underline{j}}^{(n)}$ must satisfy the recursion relation

$$\underline{F}_{\underline{j}}^{(n+1)} = \begin{bmatrix} (1-\alpha)(-e^{-\rho})^{j_{n+1}} & \alpha(-e^{-\rho})^{j_{n+1}} \\ \alpha(-e^{-\rho})^{1 \oplus j_{n+1}} & (1-\alpha)(-e^{-\rho})^{1 \oplus j_{n+1}} \end{bmatrix} \underline{F}_{\underline{j}}^{(n)} \tag{4.22}$$

with the initial condition

$$\underline{F}_0^{(0)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{4.23}$$

which corresponds to the fact that $u_0 = 0$.

Form a set of vectors by beginning with vector $\underline{F}_0^{(0)}$ and multiplying it by one of the matrices

$$M_0 = \begin{bmatrix} 1-\alpha & \alpha \\ -\alpha e^{-\rho} & -(1-\alpha)e^{-\rho} \end{bmatrix} \tag{4.24a}$$

or

$$M_1 = \begin{bmatrix} -(1-\alpha)e^{-\rho} & -\alpha e^{-\rho} \\ \alpha & 1-\alpha \end{bmatrix} \tag{4.24b}$$

depending on the value of j_1. Continuing in this way construct a set of 2^n vectors $\{\underline{v}\}$. From (4.21) there will exist a vector $\underline{w}$ with non-negative entries satisfying (4.19) iff the entire set of vectors $\{\underline{v}\}$ lies above the -45° line, the unshaded region of Figure 4.3

Instead of working with the matrices M_0 and M_1, note that $M_1 = GM_0G$, where $G = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, and consider the matrices M_0 and G. The set of vectors generated by M_0 and G is larger than that generated by M_0 and M_1, e.g., $G\underline{F}_0^{(0)}$, but either set lies above the -45° line iff the other one does so that only G and M_0 need be considered.

The problem can now be restated as follows: Find the range of ρ for which there exists a cone with an axis along the 45° line in the unshaded region of Figure 4.3 such that repeated applications of M_0 or G in any order on any vector within the cone yields another vector

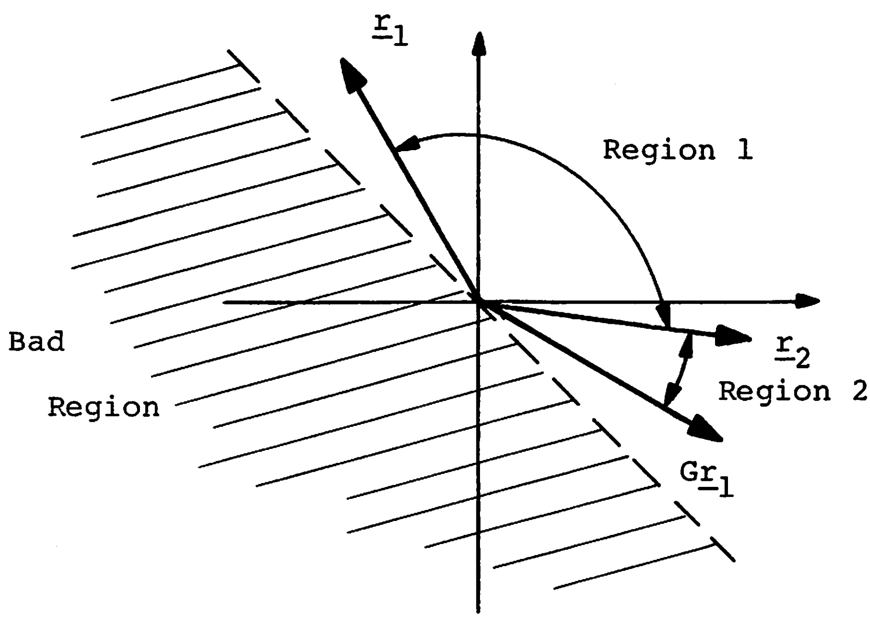


Figure 4.3: Two Dimensional Vector Space

within the cone. No vector is allowed to lie in the bad (shaded) region after k operations since that would lead to one of the final 2^n vectors being in the bad region.

To find when such a cone exists the two cases $0 \leqq \alpha \leqq 1/2$ and $1/2 < \alpha \leqq 1$ are considered separately.

$0 \leqq \alpha \leqq 1/2$

When $\alpha \leqq 1/2$ we investigate the eigenvalues and eigenvectors of the matrix GM_0. The eigenvalues are the solutions of the equation

$$\delta^2 - \alpha\delta(1-e^{-\rho}) + e^{-\rho}(1-2\alpha) = 0 \qquad (4.25)$$

and are given by

$$\delta_i = \frac{\alpha(1-e^{-\rho}) \pm \sqrt{\alpha^2(1-e^{-\rho})^2 - 4(1-2\alpha)\,e^{-\rho}}}{2} \qquad (4.26)$$

where δ_1 is assumed to be the smaller eigenvalue. The eigenvectors are given by

$$\underline{r}_i = \begin{bmatrix} r_{i_1} \\ r_{i_2} \end{bmatrix} = C_i \begin{bmatrix} (1-\alpha)\,e^{-\rho} \\ -(\alpha\,e^{-\rho} + \delta_i) \end{bmatrix} = C_i' \begin{bmatrix} \delta_i - \alpha \\ 1-\alpha \end{bmatrix} \qquad (4.27)$$

where C_i and C_i' are appropriate normalization constants.

Case 1: $\alpha^2(1-e^{-\rho})^2 - 4(1-2\alpha)\,e^{-\rho} \geqq 0$ (4.28) or, equivalently,

$$e^{-\rho} \leqq 1+2(1-2\alpha)/\alpha^2 - 2\sqrt{(1-2\alpha)/\alpha^2 + [(1-2\alpha)/\alpha^2]^2} \qquad (4.29)$$

In this case the eigenvalues of GM_0 are real, non-negative, and satisfy the inequality

$$\delta_1/\delta_2 < 1 \qquad (4.30)$$

Choose the C_i so that $\underline{r}_1$ lies in the second quadrant and $\underline{r}_2$ lies in the fourth quadrant. It will be shown that the desired cone, C, exists and is the cone between $\underline{r}_1$ and $G\underline{r}_1$. Since $\delta_1 < \delta_2$ $\underline{r}_2$ is in the cone and $\underline{F}_0^{(0)}$ is obviously in the cone. If $\underline{v} \in C$, then $G\underline{v} \in C$. Thus to prove that C is the desired cone it need only be shown that $M_0\underline{v} \in C$ when $\underline{v} \in C$. If $\underline{v}$ is in region 1 of Figure 4.3 then $\underline{v} = a\underline{r}_1 + b\underline{r}_2$, where a and b are non-negative. Then $GM_0\underline{v} = \delta_1 a\underline{r}_1 + \delta_2 b\underline{r}_2$ is also in region 1 since δ_1 and δ_2 are both non-negative. Thus $GGM_0\underline{v} = M_0\underline{v} \in C$.

If $\underline{v}$ is in region 2, then $\underline{v} = -a\underline{r}_1 + b\underline{r}_2$ for some non-negative a and b. Thus $GM_0\underline{v} = \underline{v}' = \delta_1 a\underline{r}_1 + \delta_2 b\underline{r}_2$ also lies in Region 2. From the law of sines for a plane triangle

$$\frac{a}{b} = \frac{\sin(A)}{\sin(\pi - C - A)} \qquad (4.31)$$

where $A = \angle(\underline{r}_2, \underline{v})$, the angle between $\underline{r}_2$ and $\underline{v}$, and $C = \pi - \angle(\underline{r}_2, -\underline{r}_1)$. Denoting the new angle opposite side $\delta_1 a$ by A', i.e., $A' = \angle(\underline{r}_2, \underline{v}')$, we have again by the law of sines

$$\left(\frac{\delta_1}{\delta_2}\right)\frac{a}{b} = \frac{\sin(A')}{\sin(\pi - C - A')} < \frac{a}{b} \tag{4.32}$$

so that $A' < A$ and Region 2 is mapped into itself by GM_0. Thus $M_0\underline{v} \in C$ for any $M_0\underline{v} \in C$.

The above argument for Region 2 can also be applied to Region 1 since the inequality (4.30) implies that any vectors above the $\underline{r}_1, -\underline{r}_1$ line will be rotated by GM_0 toward $\underline{r}_2$. Likewise any vector below the $\underline{r}_1, -\underline{r}_1$ line will be rotated toward the $-\underline{r}_2$ vector. Hence C is the largest cone above the -45^o line that is closed under GM_0.

Case 2: $\alpha^2(1-e^{-\rho})^2 - 4(1-2\alpha)e^{-\rho} < 0$ (4.33)

It will be shown that when (4.33) hold no cone of the desired type exists by showing that any $\underline{v}$ in the good region can be carried into the bad region by some sequence of operations.

When (4.33) holds, the eigenvalues and eigenvectors are complex conjugates. Any $\underline{v}$ in the good region can be written as $\underline{v} = a\underline{r}_1 + b\underline{r}_2$ where a and b must be complex conjugates since $\underline{v}$ is a real vector. Consider the vector

$$\underline{v}_{(k)} = (GM_0)^k\underline{v} = a\underline{r}_1\delta_1^k + b\underline{r}_2\delta_2^k \tag{4.34}$$

By conjugate symmetries this becomes

$$\underline{v}_{(k)} = 2\,\mathrm{Re}\{a\underline{r}_1\delta_1^k\} \tag{4.35}$$

The sum of the coordinates of $\underline{v}_{(k)}$ is

$$S_k = 2 \operatorname{Re} \{ a (r_{1_1} + r_{1_2}) \delta_1^k \} \tag{4.36}$$

Defining the polar forms

$$a(r_{1_1} + r_{1_2}) = r_a e^{j\theta_a} , \qquad r_a > 0 \tag{4.37}$$

$$\delta_1 = r_{\delta'} e^{j\theta_\delta} , \qquad r_\delta > 0 \tag{4.38}$$

Equation (4.36) can be rewritten as

$$S_k = 2 r_a r_\delta \cos (\theta_a + k\theta_\delta) \tag{4.39}$$

Since θ_δ is non-zero, there exists a value of k = K such that $\cos (\theta_a + K\theta_\delta)$ and hence S_K are negative and no cone of the desired type exists, the solution of $\underline{w}$ of (4.19) will not have all of its entries non-negative, and therefore the equalities in (4.10) - (4.12) will not hold. Thus, when $\alpha \leqq 1/2$, (4.29) is a necessary and sufficient condition for equality in (4.12).

$\underline{1/2 < \alpha \leqq 1}$

When $\alpha > 1/2$ consider the matrix M_0 instead of GM_0. The eigenvalues are the solutions of the equation

$$\nu^2 - \nu(1-\alpha)(1-e^{-\rho}) + e^{-\rho}(2\alpha-1) = 0 \tag{4.40}$$

which are

$$\nu_i = \frac{(1-\alpha)(1-e^{-\rho}) \pm \sqrt{(1-\alpha)^2(1-e^{-\rho})^2 - 4e^{-\rho}(2\alpha-1)}}{2} \tag{4.41}$$

where it is assumed that ν_1 is the smaller eigenvalue. The eigenvectors are

$$\underline{e}_i = \begin{bmatrix} e_{i_1} \\ e_{i_2} \end{bmatrix} = c_i \begin{bmatrix} -\alpha \\ 1-\alpha-\nu_i \end{bmatrix} = c_i' \begin{bmatrix} (1-\alpha)e^{-\rho}+\nu_i \\ -\alpha e^{-\rho} \end{bmatrix} \tag{4.42}$$

As when a $\leqq 1/2$ two cases are considered.

Case 1: $$(1-\alpha)^2(1-e^{-\rho})^2 - 4e^{-\rho}(2\alpha -1) \geqq 0 \tag{4.43a}$$

or equivalently

$$e^{-\rho} \leqq 1+2(2\alpha-1)/(1-\alpha)^2-2\sqrt{(2\alpha-1)/(1-\alpha)^2+\left[\frac{2\alpha-1}{(1-\alpha)^2}\right]^2} \tag{4.43b}$$

In this case the ν_i are both real, non-negative, and

$$\nu_1/\nu_2 < 1 \tag{4.44}$$

Choose the c_i so that both $\underline{e}_1$ and $\underline{e}_2$ lie in the fourth quadrant as in Figure 4.4.

A cone of the desired type, C, will be shown to exist and to be the cone between $\underline{e}_1$ and $G\underline{e}_1$. Let $\underline{v} \varepsilon C$, then $G\underline{v} \varepsilon C$. It need only be shown that for any $\underline{v} \varepsilon C$ it is also true that $M_0 \underline{v} \varepsilon C$. If $\underline{v}$ is in region 1 of Figure 4.4, then $\underline{v} = a\underline{e}_1 + b\underline{e}_2$, where a and b are non-negative. Then $M_0 \underline{v}$ will also lie in region 1 since the eigenvalues of M_0 are non-negative.

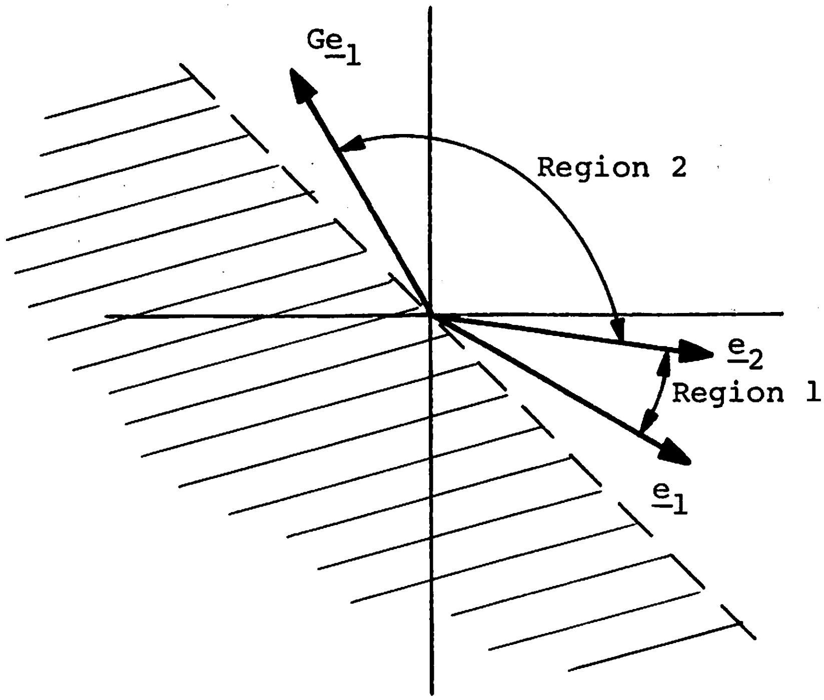


Figure 4.4: Two Dimensional Vector Space

When $\underline{v}$ lies in region 2 we can write $\underline{v} = -a\underline{e}_1 + b\underline{e}_2$ for some non-negative a and b. Thus $M_0\underline{v} = \underline{v}' = -a\nu_1\underline{e}_1 + b\nu_2\underline{e}_2$ must also lie in region 2. From the sine law

$$\frac{a}{b} = \frac{\sin A}{\sin(\pi - C - A)} \tag{4.45}$$

where $A = \angle(\underline{e}_2, \underline{v})$ and $C = \pi - \angle(e_2 - e_1)$. Denoting the new angle opposite side $\nu_1 a$ by $A' = \angle(\underline{e}_2, \underline{v}')$, the sine law can be applied again to obtain

$$\frac{a}{b}\left(\frac{\nu_1}{\nu_2}\right) = \frac{\sin A'}{\sin(\pi - C - A)} < \frac{a}{b} \tag{4.46}$$

where the inequality follows from (4.44). Thus $A' < A$ and therefore $\underline{v}' = M_0\underline{v} \in C$.

Case 2: $$(1-\alpha)^2(1-e^{-\rho})^2 - 4e^{-\rho}(2\alpha - 1) \geqq 0 \tag{4.47}$$

When (4.49) holds, the proof that no cone C of the desired type exists is identical to that of Case 2 when $\alpha \leqq 1/2$, i.e., some sequence of operations taking any $\underline{v}$ in the good region into the bad region can be constructed.

C. Summary of Results and Discussion

The results of the previous section can be summarized as follows:

$$R_u(D) = R_z(D) \qquad 0 \leqq D \leqq D_c \tag{4.48}$$

$$R_u(D) > R_z(D) \qquad D_c < D \tag{4.49}$$

$$\text{where } D_c = e^{-\rho_c}/(1+e^{-\rho_c}) > 0 \tag{4.50}$$

$$\text{and } e^{-\rho_c} = 1+2|1-2m|/m^2-2\sqrt{|1-2m|/m^2+[(1-2m)/m^2]^2} \tag{4.51}$$

and

$$m = \min[\alpha, 1-\alpha] \tag{4.52}$$

Equation (4.49) is only a good lower bound for $D \leqq m = \min[\alpha, 1-\alpha]$, the D_M of the $\{z_k\}$ sequence, since for greater D, $R_z(D) = 0$. Also $R_u(D) = 0$ for $D \geqq 1/2$ since a zero information rate can be obtained with $P_e = 1/2$ by simply sending the all zero sequence. Alternately, since $\lim_{\rho \to 0} e^{-\rho}/(1+e^{-\rho}) = 1/2$ the D_M of the $\{u_k\}$ sequence is 1/2.

When $\alpha = 1/2$, $\{u_k\}$ is an independent letter source and $R_u(D) = R_z(D)$ for $0 \leqq D \leqq D_c = 1/2 = D_M$, i.e., over the entire range of non-zero $R_z(D)$. As an example of the numbers involved in (4.48) - (4.52), when $\alpha = 3/8$, then $D_c = 1/10$.

There are two new and important results of this chapter. The combination of (4.7) and (4.48) - (4.52) is the first evaluation known to the author of the rate distortion function for a discrete alphabet source with memory. Thus it can be used to compare data compression schemes such as generalized run length encoding and backwards sequential decoding with a bound on performance. Such results are of interest since this source is simple but both commonly used and important. The second and most important result is the fact that (4.48) and (4.49) are identical to (3.29), the corresponding relations for time discrete Gaussian autoregressive sources so that it appears that this behavior is due to the autoregressive character of the source rather than the Gaussian nature of the source of Chap. 3. In the next chapter it will be shown that the lower bound part of this behavior generalizes to all time discrete autoregressive sources with difference distortion measures.

CHAPTER V

A LOWER BOUND FOR THE RATE DISTORTION FUNCTION OF A TIME DISCRETE AUTOREGRESSIVE SOURCE WITH A DIFFERENCE DISTORTION MEASURE

It is shown in this chapter that some of the behavior of the examples of the previous chapters generalizes to all time discrete autoregressive sources. Specifically, it is shown that the rate distortion function of the general autoregressive source defined in Chapter I is lower bounded by the rate distortion function of the generating independent letter process when the distortion depends only on the difference between letters, i. e., $d(u,v) = d(u-v)$.

<u>Theorem 5.1</u>

Let $\{u_n\}$ be a sequence of random variables satisfying the difference equation

$$z_n = \sum_{k=0}^{\infty} a_k u_{n-k} \quad ; \quad u_k = 0 \quad k \leq 0 \tag{5.1}$$

where $a_0 = 1$, the a_k satisfy the constraint $\sum_{k=0}^{\infty} |a_k|^2 < \infty$, and the sequence $\{z_n\}$ is a sequence of independent, zero mean random variables having a probability measure $q_{z_n}(\cdot) = q_z(\cdot)$ for all n. Assume that $f_z(u)$ and $w(v)$ satisfying Theorem 1.1 for the $\{z_n\}$ source exist so that

$$\min_{p \varepsilon P} R_{z_0}(\rho, p) = \int q_z(u) \ln [f_z(u)/q_z(u)] du \quad (5.2)$$

$$\int f_z(u)\, e^{-\rho d(u-v)} du \leqq 1 \quad \text{all } v \quad (5.3)$$

and therefore that $R_z(D)$ is known. Then

$$R_u(D) \geqq R_z(D) \quad (5.4)$$

and if $R_z(0)$ is finite, then $R_u(0) = R_z(0)$.

Proof:

If $R_z(0)$ is finite, then $R_u(0) = R_z(0)$ since $\underline{u}$ is simply a measure preserving linear transformation of $\underline{z}$. The n-tuple $\underline{u}$ will have a density

$$q_{\underline{u}}(\underline{x}) = \prod_{k=1}^{n} q_z(x_k + \sum_{j=1}^{\infty} a_j x_{k-j}) \quad (5.5)$$

where $x_k = 0$ for $k \leqq 0$. Theorem 1.1 states that

$$\min_{p \varepsilon P} [nR_{n_0}(\rho, p)] \geqq \int q_{\underline{u}}(x) \ln [f_{\underline{u}}(\underline{x})/g_{\underline{u}}(\underline{x})] d\underline{x} \quad (5.6)$$

where $f_{\underline{u}}(\underline{x})$ must satisfy the constraint

$$\int f_{\underline{u}}(\underline{x})\, e^{-\rho d(\underline{x}, \underline{v})} d\underline{x} \leqq 1 \quad \text{all } v \quad (5.7)$$

Choosing

$$f_{\underline{u}}(\underline{x}) = \prod_{k=1}^{n} f_z(x_k + \sum_{k=1}^{\infty} a_j x_{k-j}) \quad (5.8)$$

the constraint (5.7) becomes

$$\int\int\cdots\int dx_1\ldots dx_n \prod_{k=1}^{n} f_z(x_k + \sum_{j=1}^{\infty} a_j x_{k-j})\, e^{-\rho d(x_k - v_k)} =$$

$$\int\cdots\int dx_1\ldots dx_{n-1} \prod_{k=1}^{n-1} f_z(x_k + \sum_{j=1}^{\infty} a_j x_{k-j})\, e^{-\rho d(x_k - v_k)} \cdot$$

$$\int dx_n\, f_z(x_n + \sum_{j=1}^{\infty} a_j x_{n-j})\, e^{-\rho d(x_n - v_n)} \leqq 1 \quad \text{all } \underline{v} \qquad (5.9)$$

Since the last integration is only over x_n make the variable changes $x'_n = x_n + \sum_{j=1}^{\infty} a_j x_{n-j}$ and $v'_n = v_n + \sum_{j=1}^{\infty} a_j x_{n-j}$ to obtain

$$\int dx_n\, f_z(x_n + \sum_{j=1}^{\infty} a_j x_{n-j})\, e^{-\rho d(x_n - v_n)}$$

$$= \int dx'_n\, f_z(x'_n)\, e^{-\rho d(x'_n - v'_n)} \leqq 1 \qquad (5.10)$$

where the last inequality follows from the fact that (5.3) must hold for v'_n. Proceeding iteratively in this way verifies that (5.7) is indeed satisfied. Substituting (5.8) into (5.6) produces

$$\min_{p \varepsilon P} [nR_{n_0}(\rho, p)] \geqq$$

$$\int d\underline{x} \prod_{k=1}^{n} q_z(x_k + \sum_{j=1}^{\infty} a_j x_{k-j}) \ln \left\{ \frac{\prod_{k=1}^{n} f_z(x_k + \sum_{j=1}^{\infty} a_j x_{k-j})}{\prod_{k=1}^{n} q_z(x_k + \sum_{j=1}^{\infty} a_j x_{k-j})} \right\} \qquad (5.11)$$

which after some manipulation yields

$$\min_{p} [R_0(\rho, p)] \geqq \int dx\, q_z(x) \ln [f_z(x)/q_z(x)] \tag{5.12}$$

This lower bound for R_0 of the $\{u_n\}$ process is identical to the right hand side of (5.2) so that maximizing over ρ yields (5.4) and the Theorem is proven.

The above theorem can easily be proven for discrete alphabets $K = \{0, 1, \ldots, k-1\}$ by making the obvious changes of integrals to sums, probability density functions to probability mass functions, and using the discrete version of Theorem 1.1. In the discrete case the a_k's are also drawn from K and the arithmetic is modulo k.

One would like to have an intuitive explanation to reinforce the lower bound of (5.4). Since the independent letter source is already the "best" way to encode a source (each letter carries mutual information equal to the average mutual information of the source) one would suspect that any kind of processing of the z_k sequence to obtain the redundant u_k sequence could only result in requiring at least as great and possibly an even higher information rate to obtain any given average distortion. This argument is inadequate, however, since the relationship between the average distortion of the z_k sequence and that of the u_k sequence is not clear except when D=0 or when $u_k = z_k$. As an example, when D=0 in the binary source of Chapter IV it is known that an optimum source coding makes successive source letters, independent, i.e., encode the transitions (the z_k sequence) and then reconstruct the u_k sequence at the receiver. When $D > 0$ this method no longer works as a single error in the z_k sequence can cause a long run of errors, and therefore a high D, in the u_k sequence. Likewise just encoding the transitions of a Wiener process with $E[D] \leqq D^*$ results in an error build up in the reconstructed u_k sequence which eventually causes the average distortion of the u_k sequence to be infinite rather than less than D^*.

In Chapter VIII an alternate explanation is discussed for the $R_u(D) - R_z(D)$ behavior in Gaussian autoregressive sources, but as yet no satisfactory explanation for this behavior can be given for the general time discrete autoregressive source.

Theorem 5 is limited in that it does not give necessary and sufficient conditions on $q_z(\cdot)$ and $d(x-v)$ for the existence of a non-zero interval of equality in the bound of (5.4). Since the methods of Chapters III and IV are quite different, they do not suggest a unified approach for showing the existence of a non-zero interval of equality in the more general case. The results of the previous chapters do suggest that for many interesting autoregressive sources the lower bound of (5.4) will hold with equality for sufficiently small D. Even when one cannot demonstrate a region of equality the bound of (5.4) may yield a useful lower bound for R(D) of a complicated source with memory in terms of a much simpler memoryless source. When $R_z(0)$ is finite, the continuity of the rate distortion function assures that the bound becomes tighter as D becomes smaller.

CHAPTER VI

THE RATE DISTORTION FUNCTION OF A TIME CONTINUOUS GAUSSIAN AUTOREGRESSIVE SOURCE WITH A MEAN SQUARED ERROR FIDELITY CRITERION

A. The Time Continuous Autoregressive Source

The definition of time discrete autoregressive processes of Chapter I is generalized by defining the class of time continuous autoregressive functions $w(t)$, $t \in [0,T]$,

$$\int_0^T a(t,s)w(s)ds = n(t) \qquad , \quad 0 \leqq t,s \leqq T \tag{6.1}$$

Where $n(t)$ is a white noise process, the continuous time analogue of an independent letter source, and $a(t,s)$ satisfies the relation

$$a(t,s) = \begin{cases} a(t-s) & T_m \geqq t-s \geqq 0 \\ 0 & \text{elsewhere} \end{cases} \tag{6.2}$$

Furthermore it is assumed that $a(t,s)$ has an inverse, i.e., that there exists a function $a^{-1}(t,s)$ satisfying the integral equation

$$\int_0^T a^{-1}(t,x)\, a(x,s)\, dx = \delta(t-s) \tag{6.3}$$

The kernel $a(t,s)$ is merely the integral analogue of the matrix A of

Chapter III. In Chapter III m was allowed to be infinite if the a_k's satisfied a certain constraint. In the same manner T_m can be allowed to be infinite by constraining the a(t, s) in an appropriate manner, but in this paper only finite T_m will be considered for simplicity.

The generating source, n(t), is now assumed to be a white Gaussian process with spectral density σ^2. From (6.1)

$$E[n(t)n(z)] = \sigma^2 \delta(t-z) = \iint\limits_{0 \leqq s,x \leqq T} ds\, dx\, a(t,s)\, a(z,x)\, R(s,x) \qquad t,z \geqq 0 \tag{6.4}$$

where R(s, x) is the autocorrelation kernel of w(t). Multiplying both sides of (6.4) by $a^{-1}(x,t)a^{-1}(y,z)$ and integrating over t and z yields

$$\sigma^2 \int_0^T dz\, a^{-1}(x,z)\, a^{-1}(y,z) = R(x,y) \qquad T \geqq x,y \geqq 0 \tag{6.5}$$

Inverting both sides of (6.5) and using (6.2) gives

$$R^{-1}(x,y) = \sigma^{-2} \int_0^{T-\max[x,y]} dz\, a(z)\, a(z+|x-y|) \tag{6.6}$$

which is the integral analogue of (3.6). As in Chapter III the inverse autocorrelation kernel can be approximated by a Toeplitz kernel $R_e^{-1}(x,y)$ where

$$R_e^{-1}(x,y) = R_e^{-1}(|x-y|) = \sigma^{-2} \int_0^{T_m} dz\, a(z)a(z+|x-y|) \qquad (6.7)$$

which has a Fourier Transform

$$g(x) \triangleq \sigma^2 \int_{-\infty}^{\infty} R_e^{-1}(s)\, e^{-isx}\, ds \qquad (6.8)$$

B. The Rate Distortion Function

The rate distortion function can be derived in a manner exactly analogous to the method of Chapter III. In fact, the derivation of Section 9.7 of Gallager [9] of the rate distortion function for stationary Gaussian processes still holds true if the eigenvalues of the autocorrelation kernel are considered to be roots of the integral equation

$$\int_0^T R(t,s)f(s)\,ds = \lambda f(t) \qquad , \quad 0 \leqq t \leqq T \qquad (6.9)$$

where it is not assumed that $R(t,s) = R(t-s)$. With this modification the parametric equations for $R(D)$ and D for finite T of Gallager [9] (Equations (9.7.40) and (9.7.41)) hold for all time continuous Gaussian autoregressive processes as previously defined.

If $\lambda_k(0,T)$ are the solutions to (6.9), then $\beta_k(0,T) = 1/\lambda_k(0,T)$ are the solutions to the integral equation

$$\int_0^T R^{-1}(t,s)f(s)\,ds = \beta f(t) \qquad , \quad 0 \leqq t \leqq T \qquad (6.10)$$

Define $\beta_k'(0,T)$ as the solutions to

$$\int_0^T R_e^{-1}(t,s)f(s)\,ds = \beta' f(t) \qquad ,0 \leq t \leq T \tag{6.11}$$

Then from the theorems on Toeplitz forms, $\beta_k(0,T)$ and $\beta_k'(0,T)$ are known to be asymptotically equally distributed as $\sigma^{-2}g(x)$ with x distributed with density $1/\pi$ on the non-negative real axis. As in Chapter III only non-negative x need be considered since g(x) is even. The Toeplitz theorems can then be applied to evaluate the limit of the above mentioned parametric relations for R(D) and D as $T \rightarrow \infty$.

As an alternative to actually carrying out the above derivation define

$$\lim_{\substack{T\rightarrow\infty \\ k/T\rightarrow x/\pi}} \lambda_k(0,T) = \lambda(x) \tag{6.12}$$

and use a result of Berger [4] showing that the Kolmogorov equations

$$D = \pi^{-1} \int_0^\infty \min[\theta, \lambda(x)]\,dx \tag{6.13}$$

$$R_w(D) = \pi^{-1} \int_0^\infty \max[0, 1/2 \ln(\lambda(x)/\theta)]\,dx \tag{6.14}$$

hold if $\lambda(x)$ is defined as in (6.12) even when the process is non-stationary. Putting (6.13) and (6.12) into the more convenient form of (3.27) and using Theorem A10 yields finally

$$D = (\sigma^2/\pi) \int_0^\infty dx/\max\,[g(\theta), g(x)] \qquad (6.15)$$

$$R_w(D) = (2\pi)^{-1} \int_0^\infty dx \ln \{\max\,[g(\theta), g(x)]\} \qquad (6.16)$$

$$g(x) = \int_\infty^\infty ds\, e^{-isx} \int_0^{T_m} a(z)a(z+|s|)\,dz \qquad (6.17)$$

$$= \left| \int_0^{T_m} a(s)e^{-isx} ds \right|^2$$

which are analogous to (3.27) and (3.24).

As in Chapters II and III $R_w(D)$ may be lower bounded by applying Jensen's inequality to (6.15) and (6.16) to obtain

$$R_w(D) \geqq 1/2 \ln (\sigma^2/D) \qquad (6.18)$$

where the equality will hold iff $\max\,[g(\theta), g(x)]$ is independent of x, i.e., if $g(\theta) = \max_x\,[g(x)]$. Unlike the time discrete case g(x) need not be bounded so that there is no guarantee that the equality will ever hold in (6.18). Furthermore, the inequality of (6.18) does not have the interpretation of (2.63) since the right hand side of (6.18) is the rate distortion function of a white Gaussian process which has been uniformly sampled at unit time intervals and not the rate distortion function of the white generating process, $R_n(D)$. The rate distortion function of white Gaussian noise, $R_n(D)$, is infinite for all finite average distortion.

Thus despite the fact that the bound of (6.18) looks almost identical to the lower bounds for $R_u(D)$ in Chapters II and III, it is not true that $R_w(D) \geq R_n(D)$ for time continuous autoregressive processes. In fact the opposite inequality holds. In addition, the bound of (6.18) may be an exceedingly poor one, even for small D.

C. Examples and Discussion

As an example, consider the case

$$a(t) = \sum_{k=0}^{m} u_k(t)\, a_k \tag{6.19}$$

where $u_k(t)$ are the k^{th} singularity functions or formal derivatives of the unit impulse (Dirac delta) function [19]. In this case (6.1) becomes

$$\sum_{k=0}^{m} (d^k/dt^k)\, w(t)\, a_k = n(t) \qquad , \ t \geq 0 \tag{6.20}$$

Equation (6.19) of interest since it is an m^{th} order differential equation analogous to the m^{th} order difference equation in (3.1). If m=2, $a_0 = 0$, and $a_1 = 1$ then $a(t) = u_1(t)$, the unit doublet, and (6.20) becomes

$$dw/dt = n(t) \qquad , \ t \geq 0 \tag{6.21}$$

the equation for the time continuous Wiener process starting at the origin. In this case (6.17) yields $g(x) = x^2$, which is unbounded, and from (6.15) and (6.16)

$$R(D) = 2\sigma^2/\pi^2 D \tag{6.22}$$

as was found by Berger [2]. Since g(x) is unbounded the lower bound of (6.18) never holds with equality. Furthermore $R_w(D)$ does not approach the lower bound as $D \to 0$.

The results of this chapter are not meant to be complete. They are intended merely to show how the methods and results of Chapters II and III can be applied to continuous time Gaussian autoregressive sources.

A lower bound was obtained for the rate distortion function of a time continuous Gaussian autoregressive source using Jensen's inequality as in previous chapters. This lower bound is not equal to the rate distortion function of the white generating process, $R_n(D)$, as in the time discrete case. In addition, the lower bound may hold nowhere with equality and may be a poor bound even for small average distortion, as is the case with the time continuous Wiener process.

The coding theorem for continuous time Gaussian autoregressive processes is identical in statement to that of Theorem 9.7.1 of Gallager [9] with the exception that the source need not be stationary, it need only be autoregressive as defined in this chapter. The proof requires the same restriction as that of Chapter III, i.e., that g(x) can have at most a single zero. The proof itself is identical to that of Gallager with a manipulation like that used in Appendix B using Theorem A3 to handle possibly infinite maximum average distortions.

CHAPTER VII

FURTHER RESULTS ON THE RATE DISTORTION FUNCTIONS OF TIME DISCRETE GAUSSIAN SOURCES

A. Introduction

In this chapter the rate distortion function of time discrete linearly filtered independent letter Gaussian sources with a mean squared error fidelity criterion is considered. The autoregressive source previously considered was an example of such a source. In this chapter another example, the Gaussian moving average source, will be considered first. Since any stationary non-deterministic process can be represented as a moving average [14],[18] this model is simply an alternate description of time discrete stationary Gaussian processes. The continuous time equivalent result is well known [18],[19]. The derivation of the parametric equations for the rate distortion function is simply sketched by showing how the methods of Chapter III are modified to handle the moving average case.

In Section C the results of Chapter III and Section B of this chapter are extended to the case of an autoregressive filter and a moving average filter in cascade, that is, a general linear filter. Unfortunately the most general linear filter presents difficulties that are yet unsolved and so a restriction is placed on the "spectrum" of the possibly nonstationary process in order to obtain the desired results. As in Section B, the derivation will consist of merely modifying the method of Chapter III.

B. The Moving Average Source

A moving average source is defined by the difference equation

$$u_n = \sum_{k=0}^{\infty} b_{n-k} z_k \tag{7.1}$$

where $b_0 = 1$, $b_j = 0$ for negative j, the z_k are independent zero mean Gaussian random variables with variance σ^2, z_j and u_j are zero for non-positive j, and the b_j's satisfy the constraint

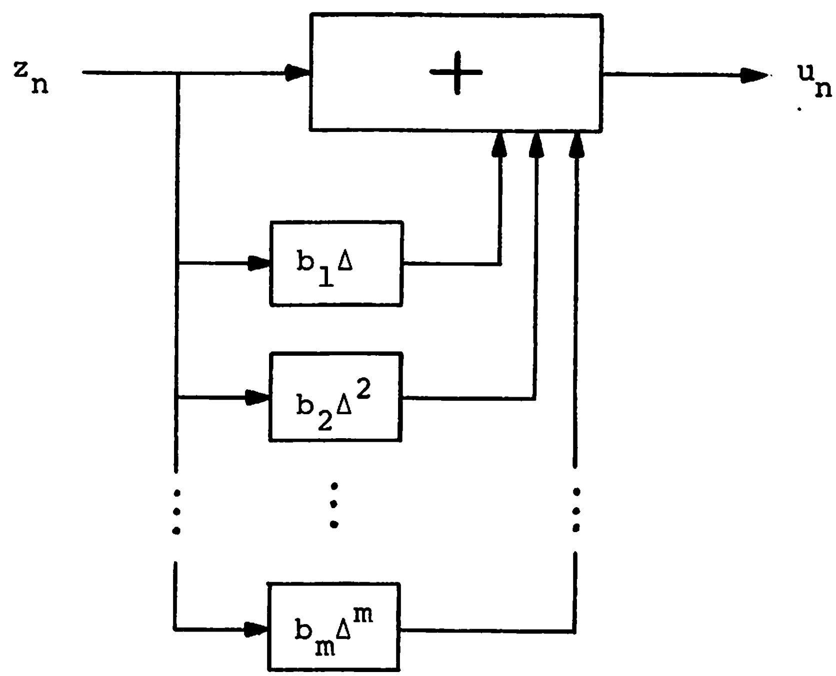
$$\sum_{k=0}^{\infty} | b_k | < \infty \tag{7.2}$$

so that (7.1) converges in the mean. This source is pictured in Figure 7.1. Define the matrix

$$B = \begin{bmatrix} 1 & & & & & \\ b_1 & 1 & & & 0 & \\ b_2 & b_1 & 1 & & & \\ \cdot & & & \cdot & & \\ \cdot & & & & \cdot & \\ \cdot & & & & & \cdot \\ b_n & b_{n-1} & & & b_1 & 1 \end{bmatrix} \tag{7.3}$$

Then (7.1) can be rewritten as

$$\underline{u} = B\underline{z} \tag{7.4}$$



Δ = Unit Delay

Figure 7.1: The Moving Average Source

The autocorrelation is simply

$$R = E[\underline{u}\underline{u}^T] = BE[\underline{z}\underline{z}^T]B^T = \sigma^2 BB^T \tag{7.5}$$

The matrix $\sigma^2 BB^T$ can be shown by the method of Section C of Chapter III to approximate the Toeplitz matrix

$$R_e = \{(\sigma^2/2\pi) \int_{-\pi}^{\pi} f(x)\, e^{-ix(k-j)}\, dx\} \tag{7.6}$$

where

$$f(x) = |\sum_{k=0}^{\infty} b_k e^{ikx}|^2 \tag{7.7}$$

is the "spectrum" of the process. If the process $\{u_k\}$ is started with u_0 being a random variable with the stationary marginal density $\lim_{n\to\infty} f_{u_n}(\cdot)$ rather than with $u_0 = 0$, then the process $\{u_k\}$ is actually stationary rather than being asymptotically stationary, i.e., having its statistics approach those of a stationary process as $n \to \infty$. In this case $R=R_e$ and is itself a Toeplitz matrix rather than an approximation of a Toeplitz matrix and $\sigma^2 f(x)$ is in fact the power spectral density of the process. Using the Wald decomposition a moving average source can be used to represent any stationary source which concurs with the above arguments since for our case a Toeplitz autocorrelation corresponds to a stationary process.

The derivation of the rate distortion function for finite n is identical to that of Chapter III if $\sigma^2 \lambda_{k,n}$, the eigenvalues of the autocorrelation matrix R, are substituted throughout for $\sigma^2/\beta_{k,n}$, where $\beta_{k,n}/\sigma^2$ are the eigenvalues of the inverse autocorrelation matrix, R^{-1}. To evaluate the limit of the expression for $R_n(D)$ as $n \to \infty$ the Toeplitz form theorems are again invoked, i.e., $\lambda_{k,n}$ is asymptotically distributed as f(x) with x uniform on $[0,\pi]$. Thus the parametric equations

$$R_u(D) = (2\pi)^{-1} \int_0^{\pi} dx \ln \{1/\min[f(x), f(\theta)]\} \tag{7.8}$$

$$D = (\sigma^2/\pi) \int_0^{\pi} dx \min [f(x), f(\theta)] \tag{7.9}$$

hold by arguments essentially identical to those of Chapter III. The only difference is that (7.6) and (7.7) involve $\sigma^2 f(x)$, the "spectrum" of the autocorrelation matrix and the actual power spectral density of the process if the process is begun with its stationary density, rather than with $\sigma^{-2} g(x)$, the "spectrum" of the inverse autocorrelation matrix. Expressions (7.8) and (7.9) are the discrete time equivalents of the well-known result of Kolmogorov [18] and Gallager [9].

Jensen's inequality can be applied to (7.8) to obtain

$$R_u(D) = -(2\pi)^{-1} \int_0^{\pi} dx \ln \{ \min [f(x), f(\theta)] \}$$

$$\geq -1/2 \ln \{ \pi^{-1} \int_0^{\pi} dx \min [f(x), f(\theta)] \}$$

$$= 1/2 \ln (\sigma^2/D) \qquad (7.10)$$

where the equality can hold iff $\min [f(x), f(\theta)]$ is independent of x, i.e., if $f(\theta) \leq \min_x f(x)$. Thus for a moving average source

$$R_u(D) = R_z(D) \qquad 0 \leq D \leq D_c = \sigma^2 \min_x f(x) \qquad (7.11)$$

$$R_u(D) > R_z(D) \qquad D_c \leq D < D_M = (\sigma^2/\pi) \int_0^{\pi} f(x)\,dx \qquad (7.12)$$

There are two important differences between (7.11) - (7.12) and the corresponding expressions for the autoregressive source. Since f(x) may be zero, D_c may be zero so that the equality of (7.11) may not hold over a non-zero interval. From (7.2) f(x) must be bounded so that D_M is always finite unlike the autoregressive case. Since it is the possibly infinite D_M that causes the difficulties in proving the coding theorem for autoregressive sources, the proof for moving average Gaussian sources is simply the obvious discrete time equivalent of the proof of Theorem 9.7.1 of Gallager [9] without the modifications of Appendix B.

Since results have been presented for the rate distortion function of an independent letter Gaussian sequence $\{z_k\}$ passed through an autoregressive filter and for $\{z_k\}$ passed through a moving average filter, it is now natural to study the behavior of an independent letter source passed through an autoregressive filter and a moving average filter in cascade, i.e., through a general linear filter. This is the subject of the next section.

C. The Rate Distortion Function of a Linearly Filtered Independent Letter Gaussian Source

Consider now the sequence $\{u_k\}$ satisfying the difference equation

$$\sum_{k=0}^{\infty} b_{n-k} z_k = \sum_{k=0}^{\infty} a_{n-k} u_k \qquad (7.13)$$

where the z_k's are independent, zero mean Gaussian random variables, $z_j = u_j = 0$ for non-positive j, $b_j = a_j = 0$ for negative j, $b_0 = a_0 = 1$, and the b_j's and a_j's satisfy the constraints

$$\sum_{k=0}^{\infty} |a_k| < \infty \qquad (7.14)$$

$$\sum_{k=0}^{\infty} |b_k| < \infty \qquad (7.15)$$

so that both sides of (7.13) converge in the mean. Both the autoregressive source of Chapter III and the moving average source of Section B of this chapter are examples of (7.13). Define

$$f(x) = \left| \sum_{k=0}^{\infty} b_k e^{ikx} \right|^2 \tag{7.16}$$

$$g(x) = \left| \sum_{k=0}^{\infty} a_k e^{ikx} \right|^2 \tag{7.17}$$

Define the matrices A and B as in (3.2) and (7.3), respectively. Then (7.13) can be rewritten as

$$B\underline{z} = A\underline{u} \tag{7.18}$$

The autocorrelation of the process is

$$R = E[\underline{uu}^T] = \sigma^2 A^{-1} B B^T (A^{-1})^T \tag{7.19}$$

Once again the rate distortion function for finite n can be found exactly as in Chapter III or Section B of this chapter in terms of $\sigma^2 \lambda_{k,n}$, the eigenvalues of R. The problem now arises in taking the limit as $n \to \infty$. In previous cases either R or R^{-1} could be approximated by a Toeplitz matrix. It is now possible that neither R nor R^{-1} approximates a Toeplitz matrix.

The eigenvalues of the product of matrices are unaffected by a cyclic rotation of the matrices so that the eigenvalues of R are the same as those of $R' = \sigma^2 (A^{-1})^T A^{-1} B B^T = \sigma^2 (AA^T)^{-1} BB^T$ and hence it is sufficient to investigate R'. The matrix R' is of the form $\sigma^2 R_1^{-1} R_2$, where R_1 and R_2 can both be approximated by Toeplitz matrices T[f(x)] and T[g(x)] where

$$T[f(x)] = \left\{ (2\pi)^{-1} \int_{-\pi}^{\pi} f(x)\, e^{-i(k-j)x}\, dx \right\} \tag{7.20}$$

It is possible, however, that $R_1^{-1} R_2$ does not approximate a Toeplitz matrix since if $g(x) = 0$ anywhere then by Theorem A8 $T[g(x)]^{-1}$ need not be approximately Toeplitz and hence R_1^{-1} may not approximate a Toeplitz matrix. Alternatively, the condition $g(x) > 0$ everywhere implies that $T[g(x)]^{-1}$ has the same asymptotic eigenvalue distribution as $T[1/g(x)]$, which is a Toeplitz matrix. In this case Theorem A8 shows that the eigenvalues of $T[f(x)]\,T[1/g(x)]$ asymptotically equal to those of $T[f(x)/g(x)]$ so that Corollary A3 implies that $R_1^{-1} R_2$ has the same asymptotic eigenvalue distribution as the Toeplitz matrix $T[f(x)/g(x)]$. Thus the asymptotic eigenvalue distribution of $R_1^{-1} R_2$ is known and is equal to $f(x)/g(x)$ with x uniform on $[0, \pi]$ (since $f(x)$ and $g(x)$ are even functions). A similar argument shows that if $f(x) > 0$ everywhere, then the eigenvalues of R^{-1} are asymptotically distributed as $g(x)/f(x)$ with x uniform on $[0, \pi]$. Thus if either $f(x)$ or $g(x)$ is required to be strictly greater than zero everywhere rather than almost everywhere then Corollary A3 can be applied to obtain the asymptotic behavior of the eigenvalues of R or R^{-1} and the rate distortion function is found as in Chapter III to be

$$R_u(D) = (2\pi)^{-1} \int_0^{\pi} dx \ln m(x, \theta) \tag{7.21}$$

$$D = \pi^{-1} \int_0^{\pi} dx\, \sigma^2 / m(x, \theta) \tag{7.22}$$

$$m(x, \theta) = \max\{[g(x)/f(x)], [g(\theta)/f(\theta)]\}$$

$$= 1/\min\{[f(x)/g(x)], [f(\theta)/g(\theta)]\} \tag{7.23}$$

Jensen's inequality may be applied to (7.21) - (7.22) to obtain

$$R_u(D) = R_z(D) \qquad 0 \leqq D \leqq D_c = \sigma^2 \min_x [f(x)/g(x)] \tag{7.24}$$

$$R_u(D) > R_z(D) \qquad D_c \leqq D < D_M = (\sigma^2/\pi) \int_0^{\pi} dx\, f(x)/g(x) \tag{7.25}$$

This general source shares the disadvantages of both the autoregressive and moving average sources since D_c may be zero and D_M may be infinite.

When f(x) and g(x) are both allowed to equal zero on a set of measure zero the previous method breaks down since neither R nor R^{-1} necessarily approximates a Toeplitz matrix. The author has been unable to prove the conjecture that so long as $\ln[f(x)]$ and $\ln[g(x)]$ are integrable, then the eigenvalues of $R_1^{-1} R_2$ are equally distributed as f(x)/g(x) up to some cutoff, i.e., if the eigenvalues of $R_1^{-1} R_2$ are denoted as $\rho_{k,n}$, then the sequence $\min[\rho_{k,n}, M]$ is asymptotically distributed as $\min[f(x)/g(x), M]$ with x uniform on $[0, \pi]$. If this conjecture is true, then the relations (7.21) - (7.23) hold true so long as f(x) and g(x) are strictly greater than zero a.e., i.e., in all cases of possible interest (non-deterministic processes).

D. Discussion

The results of this chapter are of interest in that they demonstrate that the methods of Chapter III can be applied both to different and more general source models. Numerous new problems are suggested by these results. The first is to remove the restriction that either f(x) or g(x) be strictly greater than zero everywhere. The second and likely the most interesting problem would be to consider the linear filter as an autoregressive filter and a moving average filter in cascade and to find the relations between the rate distortion functions at the input, output, and between the two filters. Studying this behavior for the special case where the autoregressive and moving average filters are the inverses of each other may yield insight into how the average distortion of a source affects that of the same source passed through a filter. Thirdly, results of the form of Chapter V might be extendable to the general linear filter case. Fourthly, the analogous results for continuous time linearly filtered white noise might be of interest.

CHAPTER VIII

CONCLUSIONS AND SUGGESTIONS FOR FUTURE RESEARCH

The main result of this work is twofold. The rate distortion functions of a general time discrete autoregressive source, $\{u_k\}$, and of the independent letter source which generates it, $\{z_k\}$, satisfy the inequality

$$R_u(D) \geqq R_z(D) \tag{8.1}$$

For two interesting cases the equality was shown to hold over a non-zero interval, i.e.,

$$R_u(D) = R_z(D) \qquad 0 \leqq D \leqq D_c \tag{8.2}$$

$$R_u(D) > R_z(D) \qquad 0 < D_c \leqq D < D_M \tag{8.3}$$

While most of the work involved the Gaussian source, the binary source of Chapter IV is probably the most interesting result in that the behavior of the Gaussian source had been shown for the Wiener process independently by Berger [2]. The result for the binary symmetric first order Markov source is the only evaluation of a rate distortion function of a discrete alphabet source with memory known to the author. It would have been logical to guess the correct behavior of the Gaussian autoregressive source from that of the Wiener process and a closer look at the expressions for stationary time discrete Gaussian process; but infering and demonstrating that the same relative behavior of $R_u(D)$ and $R_z(D)$ holds for a discrete alphabet source is original here.

Similar results were discussed for time continuous and non-autoregressive sources, but the chief results remain those of Chapters III, IV, and V.

The behavior of (8.1) - (8.3) is quite surprising. The lower bound was discussed in Chapter V, but no satisfactory intuitive explanation can yet be given. The region of equality is even more surprising then the inequality in that it says that for a given average distortion a complicated, redundant source can be coded at as low an information rate as can an independent letter, i.e., memoryless, source carrying the same information. Why this equality should hold for any non-zero average distortion is unclear. For the special case of Gaussian processes the behavior can be given an alternate argument. Berger [2] has given an explanation for this behavior in the Wiener process by showing that a lower bound involving the minimum mean squared error (MMSE) one step prediction error is tight for sufficiently low distortion. His lower bound is equivalent to that of Chapter III when the MMSE one step prediction error is recognized as being simply the variance of the independent letter process.

Bunin [5], who independently noted the $R_u(D)$-$R_z(D)$ behavior for stationary finite order autoregressive Gaussian processes having a monotonic spectral density, gives an alternate derivation of this behavior for such processes also involving the MMSE one step prediction error. Both of these arguments involve the Gaussian rather than the autoregressive nature of the source and neither of these arguments appears to apply to the binary source of Chapter IV. As has been pointed out, the $R_u(D)$-$R_z(D)$ is not unique to Gaussian sources and any worthwhile explanation for this behavior must depend on the autoregressive, not the Gaussian, character of the source. The major shortcoming of this paper is that no such explanation is given.

The shortcoming just mentioned leads naturally to a list of problems and areas for future research:

1. An intuitive explanation for the behavior of $R_u(D)$ and $R_z(D)$ as demonstrated in Chapters III, IV, and V would be of great interest in understanding autoregressive sources, in suggesting actual coding techniques for such sources, and in approaching the next problem.

2. Theorem 5 is weak and needs to be strenghthened before it can be used to actually evaluate rather than simply bound $R_u(D)$. Specifically, necessary and sufficient conditions on $f_z(\cdot)$ and $d(u,v)$ for the bound to hold with equality over a non-zero region are required. A more unified approach to the demonstration of the

region of equality for the different sources of Chapters III and IV is likely a prelude to such a theorem.

3. Schemes for achieving the rate distortion bound for Gaussian processes using noiseless feedback have been studied by Elias [7], Schalkwijk and Bluestein [24], and Bunin [5]. Similar schemes for the binary source would be of interest. Such feedback arguments may give insight into $R_u(D)$-$R_z(D)$ behavior. In addition, actual coding schemes such as generalized run length encoding or using a sequential decoder as a source encoder might be analyzed for low average distortion and compared to the $R_u(D)$ of Chapter IV. Berger [2] has evaluated the information rate of a delta modulation scheme for a Wiener process and has compared it with the rate distortion function. He is currently working on extensions of this coding scheme to autoregressive Gaussian sources. Analogous schemes may be applicable to the binary source.

4. The various problems discussed in Chapter VII, i.e., the general linear filter, should be investigated further in the manner discussed there.

5. Evaluating the rate distortion function for a source with memory is a parallel problem to finding the capacity of a channel with memory. It is likely that many of the results and observations of this paper can be applied to the capacity problem.

6. The results of Chapter IV may be extendable to larger

alphabets and higher order Markov sources. Preliminary investigations show that the matrix recursion method of Chapter IV leads to some exceedingly messy mathematics so that it is likely a difficult problem. Probably a better and simpler method for demonstrating the region of equality is required before generalizations are possible.

7. Preliminary studies show that a modified mean square error can be defined analogous to that of Sakrison [23] and that many of the results of Chapter III would generalize to such a fidelity criterion.

8. Since the autoregressive source is frequently used as a model for a real random process, it would be of interest to study the relation between the rate distortion function of a process and that of an optimally chosen autoregressive model that approximates the process. The relation of optimal and nearly optimal coding techniques of the two sources would also be of interest. As an approach to this problem the rate distortion function of a high order autoregressive process and the rate distortion function of an optimally chosen low order autoregressive process that approximates the higher order process could be compared.

APPENDIX A

THE ASYMPTOTIC EIGENVALUE DISTRIBUTION OF TOEPLITZ MATRICES

This appendix presents a brief introduction to and summary of the theory of the asymptotic eigenvalue distribution of Toeplitz and approximately Toeplitz matrices. The important theorems of Grenander and Szego [14] (hereafter referred to as G and S) relevant to the dissertation are given in this appendix without proof. In addition, several generalizations and related theorems that do not appear to have been previously published are given with proofs.

Let f(x) be a continuous, integrable, bounded function on $[-\pi, \pi]$ such that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} dx \ln f(x) > -\infty \tag{1}$$

so that $f(x) > 0$ a.e. The collection of all such functions will be referred to as the class C. Define the n by n Toeplitz matrix $T = T_n = T[f(x)]$ by

$$T[f(x)] = \{c_{k,j}\} = \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, e^{-i(k-j)x} dx \right\} = \{ c_{k-j} \} \tag{2}$$

The subscript n will be added to T when it is not clear from context.

The matrix T must be positive definite since

$$\underline{x}^T T \underline{x} = \sum_{k=1}^{n} \sum_{j=1}^{n} c_{k-j} x_k x_j$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \sum_{k=1}^{n} x_k e^{i(k-1)x} \right|^2 f(x)\,dx > 0 \tag{3}$$

since $f(x) > 0$ a.e. Thus $\det T > 0$, T is invertable for all n, and the eigenvalues of T, $\lambda_{k,n}$, are strictly greater than zero for all n.

The following definitions are useful for studying the asymptotic behavior of $\lambda_{k,n}$:

Definition A1: Consider the non-negative sequences $\{a_{k,n}\}$ and $\{b_{k,n}\}$, $k=1,2,\ldots,n$. Assume both sequences are upper bounded by K independent of n. The sets $\{a_{k,n}\}$ and $\{b_{k,n}\}$ are said to be asymptotically equally distributed as $n \to \infty$ if

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \left[F(a_{k,n}) - F(b_{k,n}) \right] = 0 \tag{4}$$

where $F(t)$ is any arbitrary continuous function on $[0, K]$.

Definition A2: Two metrics are defined to clarify the idea of one matrix approximating another. Let A_n be a Hermetian, $n \times n$ matrix with eigenvalues $\alpha_{j,n}$, then

$$\| A_n \| = \max_j |\alpha_{j,n}| \tag{5}$$

$$|A| = \left\{ \frac{1}{n} \sum_{k=1}^{n} \sum_{j=1}^{n} |A(k,j)|^2 \right\} = \left\{ \frac{1}{n} \sum_{k=1}^{n} \alpha_{k,n}^2 \right\}^{\frac{1}{2}} \tag{6}$$

where $A(k,j)$ denotes the k, j^{th} entry of A.

<u>Lemma A1:</u> $$|A_n B_n| \leqq \|A_n\| \cdot |B_n| \tag{7}$$

Lemma A1 is proven in 7.3.a of G and S and is used extensively in this Appendix.

When two matrices satisfy the relations

$$\|A_n\|,\ \|B_n\|,\ |A_n|,\ |B_n| < \infty \tag{8}$$

$$\lim_{n\to\infty} |A_n - B_n| = 0 \tag{9}$$

They will be said to approximate each other. Approximation will be denoted by

$$A_n \sim B_n \tag{10}$$

which is simply shorthand for (8) - (9).

<u>Theorem A1:</u> Let $f(x) \varepsilon C$. Denote the minimum of the quadratic form $\underline{x}^T T_n(f) \underline{x}$ under the side condition $x_1 = 1$ by m_n. Then

$$m_n = (\det T_n)/(\det T_{n-1}) \tag{11}$$

and

$$\lim_{n\to\infty} m_n = \exp\left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \ln f(x)\,dx \right\} \tag{12}$$

Theorem A1 is a combination of theorems 2.2 and 3.1 of G and S and is used in the proof of Theorem A5.

Theorem A2: Denote by m and M the essential lower and upper bounds of an f(x) in class C. If $F[\lambda]$ is any continuous function defined in the interval $m \leqq \lambda \leqq M$, then

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} F(\lambda_{k,n}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} F[f(x)]dx \tag{13}$$

i.e., the sequence $\{\lambda_{k,n}\}$ and $\left\{f\left(-\pi + \frac{2k\pi}{n+1}\right)\right\}$ are equally distributed. Furthermore, if the eigenvalues are ordered in a non-decreasing manner

$$\lambda_{1,n} \leqq \lambda_{2,n} \leqq \cdots \leqq \lambda_{n,n} \tag{14}$$

then

$$\lambda_{1,n} \geqq m \quad , \quad \lambda_{n,n} \leqq M \tag{15}$$

and

$$\lim_{n \to \infty} \lambda_{1,n} = m \, , \quad \lim_{n \to \infty} \lambda_{n,n} = M \tag{16}$$

This powerful theorem is Theorem 5.2 of G and S and it is also discussed in Section 3.5 of Grenander and Rosenblatt [13]. Theorem A2 is the basic theorem on the asymptotic eigenvalue distribution of Toeplitz matrices. Theorem A10 is the equivalent theorem for Toeplitz Kernels.

Theorem A3: Let $f(x) \varepsilon C$. Let $\min_x f(x) = f(x_o) = m$ and let

$x = x_o$ be the only value of x (mod 2π) for which this minimum is achieved. Moreover, assume that f(x) has a continuous second derivative in a certain neighborhood of x_o and that $f''(x_o) \neq 0$. Then for fixed k and $n \to \infty$

$$\lambda_{k,n} - m \cong f(x_o + 2\pi/n) - f(x_o) \cong ck^2\pi^2/n^2 \qquad (17)$$

where $\qquad c = \frac{1}{2} f''(x_o)\ , \qquad k=1,2,3,\ldots$

Theorem A3 is Theorem 5.4 of G and S and is used only in the proof in Appendix B.

Theorem A4: Given a Toeplitz matrix K_n and an Hermetian matrix L_n such that $K_n \sim L_n$. If all moments of the eigenvalues of either matrix converge, then L_n and K_n will have the same asymptotic eigenvalue distribution.

Theorem A4 is a slight modification of Theorem 7.4 of G and S. The proof is unaltered. This is the basic theorem for finding the asymptotic eigenvalue distribution of approximately Toeplitz matrices.

Theorem A5: Given $f(x) \varepsilon C$, then

$$\lim_{n \to \infty} [\det T_n[f(x)]]^{1/n} = \exp\left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} dx \ln f(x) \right\} \qquad (18)$$

Proof: From Theorem A1

$$\lim_{n\to\infty}\left[\det T_n/\det T_{n-1}\right] = \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} dx \ln f(x)\right\} \tag{19}$$

which implies (18).

Theorem A5 is a generalization of Theorem 5.2cii of G and S which is restricted to the case $\lambda_{k,n} \geqq m > 0$. Theorem A5 is valid for $\lambda_{k,n} > m \geqq 0$. The above theorem is used to evaluate $\frac{1}{2\pi}\int_{-\pi}^{\pi} dx \ln f(x)$ and can be used to construct a proof of Theorem A2.

Theorem A6: Let $f(x) \varepsilon C$ with the added restriction that $f(x) \geq m > 0$. Then the eigenvalues of $T^{-1}[f(x)]$ are asymptotically distributed as $1/f(x)$ with x uniform on $[-\pi, \pi]$. Furthermore

$$T^{-1}[f(x)] \sim T[1/f(x)] \tag{20}$$

i.e., $T^{-1}[f(x)]$ approximates a Toeplitz matrix.

Proof: Apply Theorem A2 to $F(\lambda) = G(1/\lambda)$ where $G(1/\lambda)$ is continuous on $[1/M, 1/m]$. Then

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} G(1/\lambda_{k,n}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} G[1/f(x)]\,dx \tag{21}$$

since $1/\lambda_{k,n}$ are the eigenvalues of $T^{-1}[f(x)]$, the first part of the theorem is proven. Note that if m=0 the above proof fails, e.g., if $G(t) = t$, then $F(\lambda) = G(1/\lambda) = 1/\lambda$ is not continuous at the point $\lambda = m = 0$ and therefore not continuous on $[m,M]$. To prove the second part of the theorem $T_n[f] = \{c_{k-j}\}$ is approximated using circulant matrices as in section 7.6 of G and S. It is there shown that if the following definitions are made

$$V_n = \left\{ n^{-\frac{1}{2}} e^{2\pi i \frac{jk}{n}} \right\} \tag{22}$$

$$f_p(x) = \sum_{k=-p}^{p} (1 - |k|/p)\, c_k\, e^{ikx} \tag{23}$$

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ikx} f(x)\, dx \tag{24}$$

$$D_n = \{ f_p(2\pi k/n)\, \delta_{jk} \} \tag{25a}$$

$$L_n(f) = L_n = V_n^* D_n V_n \tag{25b}$$

then

$$L_n(f) \sim T_n(f) \tag{26}$$

for sufficiently large p. Note that V_n is a unitary matrix, $f_p(x)$ is the $p^{\underline{th}}$ Cesàro sum, and $f_p(2\pi k/n)$ are the eigenvalues of L_n.

Thus

$$|L_n^{-1}(f) - T_n^{-1}(f)| = |T_n^{-1} T_n L_n^{-1} - T_n^{-1} L_n L_n^{-1}|$$

$$\leqq \| T_n^{-1} \| \cdot \| L_n^{-1} \| \cdot |T_n - L_n|$$

$$\leqq \left(\frac{1}{m}\right)^2 |T_n - L_n| \tag{27}$$

so that (26) implies that

$$T_n^{-1}(f) \sim L_n^{-1}(f) \tag{28}$$

Since the eigenvalues of $L_n^{-1}(f)$ are $1/f_p(2\pi k/n)$ then

$$L_n^{-1} = V_n^* D_n^{-1} V_n \tag{29}$$

Define $L_n[1/f]$ by

$$(1/f)_p(x) = \sum_{k=-p}^{p} (1-|k|/p)\, b_k e^{ikx} \tag{30}$$

$$b_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} dx\, e^{-ikx}/f(x) \tag{31}$$

$$B_n = \{ (1/f)_p(2\pi k/n)\, \delta_{jk} \} \tag{32}$$

$$L_n(1/f) = V_n^* B_n V_n \tag{33}$$

Then

$$| L_n^{-1}(f) - L_n(1/f)|^2$$

$$= \frac{1}{n}\sum_{j=1}^{n} | 1/f_p(2\pi j/n) - (1/f)_p(2\pi j/n)|^2 \tag{34}$$

But as in (26)

$$L_n(1/f) \sim T_n(1/f) \tag{35}$$

where $T_n(1/f)$ is the Toeplitz matrix generated by $1/f(x)$. Therefore

(35), (21), and (26) yield

$$\lim_{p,n\to\infty} \frac{1}{n} \sum_{k=1}^{n} (1/f)_p(2\pi k/n) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} 1/\lambda_{k,n}$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} 1/f_p(2\pi k/n) \tag{36}$$

so that

$$\lim_{p,n\to\infty} \frac{1}{n} \sum_{k=1}^{n} [1/f_p(2\pi k/n) - (1/f)_p(2\pi k/n)]$$

$$= \lim_{p\to\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} dx\, [1/f_p(x) - (1/f)_p(x)] = 0 \tag{37}$$

Taking the limit in (34) using (37) yields

$$L_n^{-1}(f) \sim L_n(1/f) \tag{38}$$

Finally, combining (28), (35), and (37) gives

$$|T_n^{-1}(f) - T_n(1/f)| \leqq |T_n^{-1}(f) - L_n^{-1}(f)| +$$

$$|L_n^{-1}(f) - L_n(1/f)| + |L_n(1/f) - T_n(1/f)| \xrightarrow[n\to\infty]{} 0 \tag{39}$$

and (20) is proven.

Theorem A6 strongly required the strict positivity of f(x) and gives a necessary and sufficient condition for the inverse of a Toeplitz matrix to itself approximate the Toeplitz matrix $T_n[1/f(x)]$.

The condition is necessary since if $f(x) = 0$ anywhere, $1/f(x)$ will not be integrable and $T_n[1/f]$ and $L_n[1/f]$ cannot be constructed.

Corollary A1: Let $f(x) > 0$ everywhere and $A_n \sim T_n[f]$ where A_n is bounded and Heremetian. Then

$$A_n^{-1} \sim T_n(1/f) \tag{40}$$

and therefore from Theorem A4 the eigenvalues of A_n^{-1} are asymptotically distributed as $1/f(x)$ with x uniform on $[-\pi, \pi]$.

Proof:

$$|A_n^{-1} - T_n^{-1}(f)| = |T_n^{-1}(f)\, T_n(f)\, A_n^{-1} - T_n^{-1}(f)\, A_n A_n^{-1}|$$

$$\leqq \|T_n^{-1}\| \cdot \|A_n^{-1}\| \cdot |T_n(f) - A_n| \xrightarrow[n\to\infty]{} 0 \tag{41}$$

Note that $f(x) > 0$ everywhere is required for $\|T_n^{-1}\|$ to be bounded. Applying (39) and (41) gives

$$|A_n^{-1} - T_n(1/f)| \leqq |A_n^{-1} - T_n^{-1}(f)| + |T_n^{-1}(f) - T_n(1/f)|$$

$$\xrightarrow[n\to\infty]{} 0 \tag{42}$$

and (40) is proven.

Corollary A1 has the satisfying interpretation that if a matrix approximates a Toeplitz matrix generated by $f(x)$, then the inverse approximates another Toeplitz matrix generated by $1/f(x)$. The condition $f(x) > 0$ everywhere was again vital to the proof.

<u>Theorem A7</u>: Let f(x) and g(x) be both of the class C having M as an upper bound. Then

$$\lim_{n \to \infty} | T_n[f] T_n[g] - T_n[fg] | = 0 \tag{43}$$

and the eigenvalues of the matrix $T_n[f] T_n[g]$ are asymptotically distributed as f(x) g(x) with x uniform on $[-\pi, \pi]$.

<u>Proof</u>: Define $L_n[f]$ and $L_n[g]$ as previously. Then

$$\lim_{n \to \infty} | L_n[f] - T_n[f] | = 0 \tag{44}$$

$$\lim_{n \to \infty} | L_n[g] - T_n[g] | = 0 \tag{45}$$

and

$$| L_n[f] - L_n[g] - L_n[fg] |^2 =$$

$$\frac{1}{n} \sum_{k=1}^{n} | f_p(\frac{2\pi k}{n}) g_p(\frac{2\pi k}{n}) - (fg)_p(\frac{2\pi k}{n}) |^2 \tag{46}$$

From (43) and a procedure similar to that of (34) - (38) shows that the sum of (46) goes to zero as $n \to \infty$. Then

$$| T_n[f] T_n[g] - T_n[fg] | \leq | T_n[f] T_n[g] - T_n[f] L_n[g] |$$

$$+ | T_n[f] L_n[g] - L_n[f] L_n[g] | + | L_n[f] L_n[g] - L_n[fg] |$$

$$\leqq M \cdot | T_n(g) - L_n(g) | + M | T_n[f] - L_n[f] |$$

$$| L_n[f] L_n[g] - L_n[fg] | \xrightarrow[n \to \infty]{} 0 \tag{47}$$

Corollary A2: Let the bounded, Hermitian matrices A_n and B_n approximate $T_n[f]$ and $T_n[g]$, respectively. Then

$$\lim_{n \to \infty} |A_n B_n - T_n[fg]| = 0 \tag{48}$$

and the eigenvalues of $A_n B_n$ are asymptotically distributed as $f(x) g(x)$ with x uniform on $[-\pi, \pi]$.

Proof: Use (47) to obtain

$$|A_n B_n - T_n[fg]| \le |A_n B_n - A_n T_n[g]|$$

$$+ |A_n T_n[g] - T_n[f] T_n[g]| + |T_n[f] T_n[g] - T_n[fg]|$$

$$\xrightarrow[n \to \infty]{} 0 \tag{49}$$

Theorem A8: Let f(x), g(x) be in the class C with $g(x) > 0$ everywhere. Then

$$\lim_{n \to \infty} | T[f(x)] T^{-1}[g(x)] - T[f(x)/g(x)] | = 0 \tag{50}$$

and the eigenvalues of $T[f(x)] T^{-1}[g(x)]$ are asymptotically distributed as $f(x)/g(x)$ with x uniform on $[-\pi, \pi]$.

Proof: Simply combine Theorems A6 and A7.

Corollary A3: Define A_n and B_n as in Corollary A2. Then

$$\lim_{n \to \infty} | A_n B_n^{-1} - T_n[f/g] | = 0 \tag{51}$$

and the eigenvalues of $A_n B_n^{-1}$ are asymptotically distributed as $f(x)/g(x)$ with x uniform on $[-\pi, \pi]$.

Proof: Exactly as in Corollary A2.

Note that once again $g(x) > 0$ everywhere was required to insure $1/g(x)$ was bounded so that B_n^{-1} could be handled.

Theorem A9: Let f(x) be of the class C and let $\lambda_{k,n}$ be the eigenvalues of $T_n[f(x)]$. Define $\chi_p(x)$ to be the characteristic function of p, a subset of $[-\pi, \pi]$, i.e.,

$$\chi_p(x) = \begin{cases} 1 & x \in p \\ 0 & x \notin p \end{cases} \tag{52}$$

Assume also that $F[\lambda]$ is an arbitrary continuous function on $[0, \max_x f(x)]$. Then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} F[\lambda_{k,n}]\, \chi_p\left(\frac{2\pi k}{n}\right) = \frac{1}{2\pi} \int_{-\pi}^{\pi} F[f(x)]\, \chi_p(x)\, dx \tag{53}$$

Proof: Define

$$c(x) = \begin{cases} 1 & x \in p \\ \epsilon & x \notin p \end{cases} \tag{54}$$

and $\xi_{k,n}$ as the eigenvalues of the Toeplitz matrix $T_n[c]$. Then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} F(\xi_{k,n}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} F[c(x)]\,dx = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} F\left[c\left(\frac{2\pi k}{n}\right)\right] \tag{55}$$

From Theorem A7

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} F\left[\lambda_{k,n} c\left(\frac{2\pi k}{n}\right)\right] = \frac{1}{2\pi} \int_{-\pi}^{\pi} F[f(x)c(x)]\,dx \tag{56}$$

Combining (56) and (54) yields

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} F[\lambda_{k,n}]\chi_p\left(\frac{2\pi k}{n}\right) + \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} F[\epsilon\lambda_{k,n}]\chi_{\bar{p}}\left(\frac{2\pi k}{n}\right)$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} F[f(x)]\chi_p(x)\,dx + \frac{1}{2\pi} \int_{-\pi}^{\pi} F[\epsilon f(x)]\chi_{\bar{p}}(x)\,dx \tag{57}$$

Taking the limit of (57) as $\epsilon \to 0$ and using the fact that $F[\lambda]$ is continuous at the origin yields (53).

Theorem A9 shows that not only are the eigenvalues of $T_n[f]$ dense in $[-\pi, \pi]$, but on any subset of $[-\pi, \pi]$.

Theorem A10: If f(x) is a real-valued, bounded and integrable function, the eigenvalues of the integral equation

$$\int_0^T K(s-t)\,\varphi(t)\,dt = \lambda\varphi(s), \qquad 0 < s, T \tag{58}$$

have asymptotically the distribution of the values of f(x) where x is distributed with density $(2\pi)^{-1}$ along the real axis.

Theorem A10 is the equivalent theorem of A2 for Kernels and is Theorem 8.6 of G and S. An equivalent theorem is given by Kac, Murdock, and Szego [16].

APPENDIX B

PROOF OF THEOREM 3.2

For $D^* \geqq D_{MAX}$ the theorem is trivial. For $D^* < D_{MAX}$ choose θ to satisfy (3.27b). For any $\delta > 0$ let n be large enough to satisfy

$$R_n(D_n^*) \leqq R(D) + \delta/4 \tag{1}$$

$$D_n^* \leqq D^* + \delta/4 \tag{2}$$

where the k in (3.19) - (3.20) is chosen so that $\beta_{k-1,n} \leqq g(\theta) < \beta_{k,n}$. Consider an ensemble of codes with

$$M' = \exp\{n[R(D^*) + (7/8)\delta]\} \tag{3}$$

code words chosen independently according to the output probabilities of the test channel yielding $R_n(D_n^*)$. The code words are of the form $\underline{x}_m = (x_{m_1}, x_{m_2}, \ldots, x_{m_n})^T$ where the x_{m_j} are independent, zero mean Gaussian random variables with variance

$$E[x_{m_j}^2] = \begin{cases} \sigma^2/\beta_{j,n} - 1/(2\rho) & j \leqq k-1 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

since for every code word $x_{m_j} = 0$ for $j \geqq k$. Then

$$d(\underline{y}, \underline{x}_m) = \sum_{j=1}^{k-1} (y_j - x_{m_j})^2 + \sum_{j=i}^{n} y_j^2 \tag{5}$$

and we need only consider

$$d_1(\underline{y},\underline{x}_m) = \sum_{j=1}^{k-1} (y_j - x_{m_j})^2 \tag{6}$$

The average distortion per letter for any code in the ensemble is then

$$E[D_n] = n^{-1}\left\{ E[d_1(\underline{y},\underline{x}(\underline{y})] + \sum_{j=k}^{n} \sigma^2/\beta_{j,n} \right\} \tag{7}$$

Define

$$\hat{R} = R_n(D_n^*) + \delta/2 \tag{8}$$

$$\hat{D}_1 = n^{-1} \sum_{j=1}^{k-1} (1/2\rho) + \delta/2 \tag{9}$$

Let $P_c[d_1 > n\hat{D}_1]$ be the probability, over the ensemble of codes and source waveforms, that $d_1[\underline{y},\underline{x}(\underline{y})] > n\,\hat{D}_1$. From Gallager's Lemma 9.3.1 [9]

$$P_c[d_1 > n\,\hat{D}_1] \leqq P_c(A) + \exp[-M' e^{-n\hat{R}}] \tag{10}$$

where

$$A = \{\underline{y},\underline{x} : I(\underline{y},\underline{x}) > \hat{R} \quad \text{or} \quad d_1(\underline{y},\underline{x}) > n\,\hat{D}_1\} \tag{11}$$

and $P_t(A)$ is the probability of event A in the test channel.

On the test channel $I[\underline{Y},\underline{X}] = n\,R_n(D_n^*)$ and $E[d_1(\underline{x},\underline{y})] = \sum_{j=1}^{k-1} (1/2\rho)$ so that $P_t(A)$ can be upper bounded using Chebyshev's inequality by

$$P_t(A) \leqq \left(\frac{k-1}{n}\right) \frac{(1 + 1/(2\rho^2))}{n(\delta/2)^2} \tag{12}$$

in the same manner as Gallager's (9.7.54) - (9.7.61). Combining (3), (3.19), and (8) produces

$$M' \geqq e^{n[\hat{R} + \delta/8]} - 2 \tag{13}$$

which with (12) yields

$$P_c[d_1 > n\hat{D}_1] \leqq \left(\frac{k-1}{n}\right) \frac{(1 + 1/(2\rho^2))}{n(\delta/2)^2} + \exp\left\{-e^{(\delta/8)n}\right\} \tag{14}$$

Thus far the proof is an exact parallel of Gallager's. We now diverge from his proof to handle the possibly infinite D_{MAX} of the autoregressive case.

Consider now five codes for which (14) holds: V_i, $i=1,\ldots,5$. Define the distortion between the first k-1 letters of a source word and the code word in V_i into which it is mapped as $D[\underline{y}, \underline{x}_i(\underline{y})]$. Define $B_i = \{\underline{y} : d_1(\underline{y}, \underline{x}_i(\underline{y})) > \hat{D}_1\}$ and $B = \bigcap_{i=1}^{5} B_i$. Form a super code as follows: If $\underline{y} \in \overline{B}$, map $\underline{y}$ into the closest code word in any of the V_i's. If $\underline{y} \in B$, map it into the all zero code word $\underline{x}_o$.

Note that the super code has $5M' + 1 < 6M'$ words. For the super code

$$E[d_1(\underline{y}; \underline{x}(\underline{y}))] \leqq n\hat{D}_1 + P(B) \sum_{j=1}^{k-1} \overline{d_j} \tag{15}$$

where $\bar{d}_j$ is the average distortion or value of y_j^2 given that $\underline{y} \in B$. Assuming that large values of y_j^2 belong to sequences $\underline{y} \in B$ we can apply Gallager's method of (9.7.66) - (9.7.68) [9] to obtain

$$P(B)\,\bar{d}_j \leqq \sqrt{3}\,(\sigma^2/\beta_{j,n})\sqrt{P(B)} \qquad (16)$$

Since the codes are independently chosen $P(B) = P^5(B_i)$ and (15) becomes

$$E[d_1(\underline{y}, \underline{x}(\underline{y}))] \leqq n\left\{ \hat{D}_1 + [3D^2_{MAX}(n)\,P^5(B_i)]^{\frac{1}{2}} \right\} \qquad (17)$$

where $D_{MAX}(n) = n^{-1}\sigma^2 \sum_{k=1}^{n} \beta^{-1}_{k,n}$. The artifice of five independent codes instead of one was used merely to get the fifth power of $P(B_i)$.

Combining (17), (19), (7), (3.20) and (2.42) gives

$$E[D_n] \leqq D^* + (3/4)\delta + \sqrt{3P^5(B_i)}\; D_{MAX}(n) \qquad (18)$$

where $P(B_i)$ is upper bounded by the right hand side of (14). Since

$$D_{MAX}(n) \leqq \sigma^2/\beta_{1,n} \qquad (19)$$

and since $g(x)$ is constrained to have only a single zero, Theorem A3 can be applied to obtain

$$D_{MAX}(n) \leqq K\,n^2 \qquad (20)$$

for some sufficiently large K. Taking the limit in (18) now leaves

$$\lim_{n\to\infty} E[D_n] \leqq D^* + (3/4)\delta + \lim_{n\to\infty} \{3P^5(B_i)[n^2/K]^2\}^{\frac{1}{2}} \tag{21}$$

The last term is bounded by

$$\lim_{n\to\infty} \left\{ 3\left[((k-1)/n)(1+1/(2\rho^2))/(n\frac{\delta^2}{4}) + e^{-e^{n\delta/8}} \right]^5 (n^2/K^2)^2 \right\}^{\frac{1}{2}} \tag{22}$$

which goes to zero since $\lim_{n\to\infty} n^2 \exp\{-je^{n\delta/8}\} = 0$ for any j and

since $\lim_{n\to\infty} \left\{\left[\left(\pi^{-1} \int_{x:g(x)\leqq g(\theta)} dx\right)(1+1/(2\rho^2))/(n\delta^2/4)\right]^5 n^4\right\}^{\frac{1}{2}} = 0.$ Thus

by taking n sufficiently large the right hand side of (21) can be upper bounded by $D^* + \delta$ so that $E[D_n] \leqq D^* + \delta$. Note that if $P(B_i)$ was not raised to the fifth power the last term would diverge. Finally, we have

$$M \leqq 6M' = \exp\{n[R(D^*) + (7/8)\delta + (\ln 6/n)]\} \tag{23}$$

Picking n sufficiently large

$$M \leqq e^{n[R(D^*) + \delta]} \tag{24}$$

BIBLIOGRAPHY

1. T. Berger "Rate Distortion Theory for Sources with Abstract Alphabets and Memory," Information and Control, Vol. 13, pp. 254-273, September 1968.

2. T. Berger, "Information Rates of Wiener Processes," presented at the International Symposium on Information Theory, Ellenville, N.Y., January 1969.

3. T. Berger, Private Communication.

4. T. Berger, "Rate Distortion Functions of Nonstationary Gaussian Processes with Applications to Diffusion," Tech. Memo TP-63, Raytheon Company, Wayland, Mass., November 1967.

5. B. Bunin, "The Rate Distortion Function of Correlated Gaussian Processes," Ph.D. Thesis, Brooklyn Polytechnic Institute, June 1970.

6. P. Elias, "Predictive Coding," Transactions of I.R.E., IT-1, 62, 1955.

7. P. Elias, "Channel Capacity Without Coding," in Lectures on Communication System Theory, E. J. Baghdady (Ed.), McGraw-Hill, New York.

8. W. Feller, An Introduction to Probability Theory and Its Applications, Volume II, John Wiley & Sons, Inc., New York, 1966.

9. R. G. Gallager, Information Theory and Reliable Communication, John Wiley and Sons, Inc., New York, 1968.

10. H. Gish and J. N. Pierce, "Asymptotically Efficient Quantizing," IEEE Transactions on Information Theory, Vol. IT 14, Number 6, September 1968, pp. 676-683.

11. T. J. Goblick, Jr., "A Coding Theorem for Time-Discrete Analog Data Sources," IEEE Transactions on Information Theory, Vol. IT-15, No. 3, May 1969, pp. 401-407.

12. T. J. Goblick and J. L. Holsinger, "Analog Source Digitization: A Comparison of Theory and Practice," IEEE Transactions on Information Theory (correspondence), Vol. IT 13, No. 2, April 1969, pp. 323-326.

13. U. Grenander and M. Rosenblatt, Statistical Analysis of Stationary Time Series, John Wiley and Sons, New York, 1957.

14. U. Grenander and G. Szego, Toeplitz Forms and Their Applications, Univ. of California Press, Berkeley and Los Angeles, 1958.

15. G. H. Hardy, J. E. Littlewood, G. Polya, Inequalities, Cambridge University Press, Cambridge, 1967.

16. M. Kac, W. L. Murdock, and G. Szego, "On the Eigenvalues of Certain Hermetian Forms," Journal of Rational Mechanics and Analysis, Vol. 2, pp. 767-800.

17. W. C. Kellog, "Information Rates in Sampling and Quantization," IEEE Transactions on Information Theory, Vol. IT-13, No. 3, July 1967, pp. 506-511.

18. A. N. Kolmogorov, "On the Shannon Theory of Information Transmission in the Case of Continuous Signals," IRE Transactions of Information Theory, Vol. IT-2, pp. 102-108, December 1956.

19. M. D. Lighthill, Introduction to Fourier Analysis and Generalized Functions, Cambridge University Press, Cambridge, 1964.

20. R. A. McDonald and P. M. Schultheiss, "Information Rates of Gaussian Signals Under Criteria Constraining the Error Spectrum," Proceedings of the IEEE (Correspondence), Vol. 52, April 1964, pp. 415-416.

21. M. S. Pinsker, Information and Information Stability of Random Variables and Processes, Holden-Day, San Francisco, 1964.

22. D. J. Sakrison, "The Rate Distortion Function for a Class of Sources," Space Sciences Laboratory Technical Memo, Series No. 9, Issue N. 35, University of California, Berkeley, 1968.

23. D. J. Sakrison, "The Rate Distortion Function of a Gaussian Process with a Weighted-Square Error Criterion," IEEE Trans. on Information Theory, IT-14, May 1968, pp. 506-508.

24. P. Schalkwijk and L. Bluestein, "Transmission of Analog Waveforms Through Channels with Feedback," IEEE Transactions on Information Theory, Vol. IT-13, 1967.

25. C. E. Shannon and W. Weaver, A Mathematical Theory of Communication, Univ. of Illinois Press, Urbana, 1949.

26. C. E. Shannon, "Coding Theorems for a Discrete Source with a Fidelity Criterion," IRE National Convention Record, Part 4, pp. 142-163, 1959.

27. N. Wiener, Extrapolation, Interpolation, and Smoothing of Stationary Time Series, M.I.T. Press, Cambridge, Mass., 1949.