USC-SIPI REPORT #162

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August 1990

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This work was supported by the Garland Division of E-Systems, Inc. under Contract 7307-EP-76673.

A DOMAIN DECOMPOSITION PRECONDITIONER BASED ON A CHANGE TO A MULTILEVEL NODAL BASIS *

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Abstract. We present a domain decomposition method based on a simple change of basis on the interfaces and vertices and we show that this leads to an effective preconditioner compared to the ones previously considered such as the preconditioner by Bramble, Pasciak and Schatz (BPS) [2], and the hierarchical basis domain decomposition (HBDD) preconditioner by Smith and Widlund [8]. Our domain-decomposed preconditioner is based on Bramble, Pasciak and Xu's multilevel nodal basis preconditioner [3]. We show that analytically our method and the HBDD method give the same order of condition number, namely, $O(\log^2 \frac{H}{h})$ for problems with smooth coefficients. Numerically our method is much more effective and it appears to be $O(1)$ for the model problem.

Key words. domain decomposition, hierarchical basis, multilevel nodal basis, preconditioned conjugate gradient methods, Schur Complement

AMS(MOS) subject classifications. 65F10, 65N30

1. Introduction. We consider second-order, self-adjoint, uniformly elliptic partial differential equations on a two dimensional polygonal domain Ω. The problems are solved numerically by using piecewise linear finite elements. The domain is first divided into nonoverlapping subregions Ω_i's which are further divided into triangular finite elements. We denote H the diameter of a typical subregion and h the diameter of its elements.

We begin with the linear system arising from a discretization of the problem and we first eliminate the variables interior to the subregions Ω_i. The resulting reduced system, the Schur complement, involves only the variables associated with Γ, the set of edges and vertices of the subregions. This system is then solved by a preconditioned conjugate gradient method, where the preconditioner is constructed from certain problems associated with the interfaces $\Gamma_{ij} = \partial\Omega_i \cap \partial\Omega_j$ and vertices and a global coarse problem associated with the vertices.

Many preconditioners have been proposed for the subproblems associated with the edges Γ_{ij}. For example, the method by Bramble, Pasciak, and Schatz [2] uses an operator similar to the square root of the Laplacian operator as the subproblem. Recently Smith and Widlund [8] propose a computationally more efficient hybrid preconditioning method which involves only a simple change of basis (between nodal and hierarchical basis) with the unknowns on the edges Γ_{ij}. They show that the new method has a condition number which grows no faster than $C(1 + \log(\frac{H}{h}))^2$, which is comparable to that of the BPS method.

The domain-decomposed preconditioner we consider in this paper is inspired by the work of Smith and Widlund [8]. In the same way that [8] uses the hierarchical basis on the edges to obtain a domain decomposition method, we use the multilevel nodal basis of Bramble, Pasciak, and Xu [3] applied to the reduced system on the interfaces (i.e. the edges *and the vertices*) to obtain our domain decomposition

* This version is dated August 1990. This work was supported in part by the Department of Energy under contract DE-FG-03-87-ER-25037, the Army Research Office under contract DAAL03-88-K-0085, and Office of Naval Research under contract N00014-90-J-1695.

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number is bounded by $O(\log^2 \frac{H}{h})$ for smooth coefficient problems. Numerical experiments, however, show that the condition number appears to be $O(1)$ for the model problem. The computational cost of our method is about the same as that of Smith and Widlund's method.

2. The Multilevel Nodal Basis Algorithm and Domain Decomposition Methods. In this section we provide the necessary background to define the MNBDD algorithm. We consider a second-order, self-adjoint, uniformly elliptic, bilinear form $a(u,v)$ on Ω with Dirichlet condition on $\partial\Omega$:

$$a(u,v) = (f,v) \quad \forall v \in H_0^1(\Omega), \quad u \in H_0^1(\Omega).$$

Let $V^H(\Omega)$ and $V^h(\Omega)$ be the spaces of continuous, piecewise linear functions, on the two triangulations, which vanish on $\partial\Omega$. We use elements which obey certain regularity assumptions, and obtain the discrete variational problem:

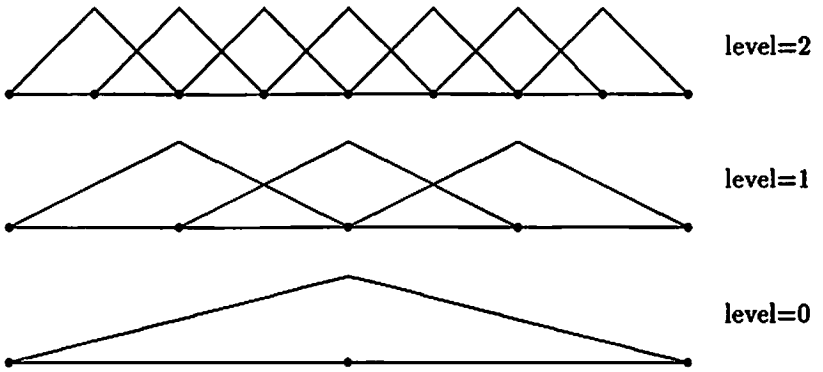
$$a(u_h, v_h) = (f, v_h) \quad \forall v_h \in V^h(\Omega), \quad u_h \in V^h(\Omega).$$

By introducing the standard nodal basis $\{\phi_i\}$ for the space V^h, the above finite dimensional variational problem is reduced to a linear system:

$$Kx = b.$$

Here x is the vector of unknowns x_i, b is the vector of components (f, ϕ_i), and K is the stiffness matrix where $K_{ij} = a(\phi_i, \phi_j)$.

FIG. 1. **Mulitlevel Nodal Basis Functions**



2.1. The Multilevel Nodal Basis Algorithm. The multilevel nodal basis method [3] is given in terms of a set of nested sequence of finite element spaces,

$$V_0^h \subset V_1^h \subset \cdots \subset V_J^h \equiv V^h, \quad J = \log_2 \frac{H}{h} \geq 1,$$

which are successive refinements by a factor of two of $V_0^h = V^H$. Here V_i^h is the set of piecewise linear finite element functions after i levels of refinement from the original coarse triangulation. In other words, V_i^h is the set of piecewise linear functions $\{\phi_i^j\}_{j=1}^{n_i}$ ($n_i = dim V_i^h$) in $V^{2^{J-i}h}$ that satisfies

$$\phi_i^j(y_i^l) = \delta_{jl}, \quad \forall j,l = 1, \cdots, n_i,$$

where $\{y_i^l : l = 1, \cdots, n_i\}$ is the set of all interior nodal points of the triangulation on which V_i^h is defined. In short, $\{\phi_i^j\}_{j=1}^{n_i}$ is the standard nodal basis for the space V_i^h. Figure 2.1 shows the multilevel nodal basis functions in one dimension. The multilevel nodal basis preconditioner M of Bramble, Pasciak, and Xu [3, 6] applied to $v \in V^h$ takes the following form:

$$M^{-1}v = \alpha A_0^{-1} Q_0 v + \sum_{l=1}^{J} \sum_{i=1}^{n_l} (v, \phi_l^i)\phi_l^i \tag{2.1}$$

where the operator A_0 is a discretization of the elliptic operator $-\Delta$ on V_0^h and Q_0 is the standard orthogonal L^2 projection from V_J^h to V_0^h. The parameter α is not present in the original formulation of this preconditioner. Its inclusion here is inspired by the work of Smith and Widlund [8]. It can be shown that this modified version has the same order of condition number bound as the unmodified one (i.e. $\alpha = 1$)) provided that α is independent of h and H.

The operator M^{-1} involves transformations between standard nodal basis and multilevel nodal basis (the second term in the above equation) as well as the solution of the problem corresponding to the original coarse triangulation (the first term in the equation). In matrix form, this preconditioner $\hat{M}$ can be written as:

$$\hat{M}^{-1} = GD^{-1}G^T$$

where G and G^T represent the transformation matrices from multilevel nodal basis to nodal basis and vice versa, respectively; and $D^{-1} = blockdiag[I_{n_J}, I_{n_{J-1}}, \cdots, I_{n_1}, \alpha A_0^{-1}]$ solves the elliptic problem on the coarse level while leaving the other levels unchanged. It was proved in [3] that for smooth coefficient problems, the condition number of the preconditioned system $\kappa(\hat{M}^{-1}K) \leq CJ^2$ where C is a constant independent of h and H. In addition, this algorithm requires only $O(n_J)$ operations where n_J is the dimension of the finite element space V^h.

The matrix G transforms the input vector from multilevel nodal basis to nodal basis. The dimension of the multilevel nodal basis is $m \equiv n_J + n_{J-1} + \cdots + n_1 + n_0$ while that of the standard nodal basis is $n = n_J$. Thus, G is a rectangular matrix of size $n \times m$ (which is unlike the square transformation matrices for hierarchical basis). Let $v = \{v_l^i, i = 1, \cdots, n_l\}_{l=0}^J \in R^M$ where v_l^i is the value at the nodal point y_l^i corresponding to level l, then

$$Gv = \sum_{l=0}^{J} \sum_{i=1}^{n_l} v_l^i \phi_l^i$$

where $\phi_l^i, i = 1, 2, \cdots n_l$ is the set of basis functions in V_l^h.

The algorithm for G as applied to a vector v (of dimension m) is as follow:

Algorithm G : $v_i \rightarrow u, i = 0, \cdots J$
  for $l = 0, \cdots, J-1$
    $v_{l+1} := v_{l+1} + I_l^{l+1} v_l$
  end for
  $u = v_J$
end G

Here the I_l^{l+1} matrix is obtained from the choice of ϕ_i's. On a two-dimensional uniform domain using triangular elements, it corresponds to a seven-point interpolation operator.

2.2. Domain Decomposition Methods. Domain decomposition methods generally split the space V^h into $N+1$ subspaces

$$V^h = V_{har}^h \oplus V_0^h(\Omega_1) \oplus \cdots \oplus V_0^h(\Omega_N).$$

For each subregion Ω_i, we thus have a subspace $V_0^h(\Omega_i) = V^h \cap H_0^1(\Omega_i)$. The elements of V_{har}^h are piecewise, discrete harmonic functions, i.e. they are orthogonal, in the sense of the bilinear form $a(\cdot, \cdot)$, to all the other subspaces.

First we partition the stiffness matrix K and vector x into those corresponding to the interior of the subregions and the edges and vertices. We then have

$$Kx = \begin{bmatrix} K_I & K_{IB} \\ K_{IB}^T & K_B \end{bmatrix} \begin{bmatrix} x_I \\ x_B \end{bmatrix} = \begin{bmatrix} b_I \\ b_B \end{bmatrix}.$$

If we apply block Gaussian elimination to eliminate the interior points, we obtain the following reduced system or Schur complement for the edges and interfaces

$$S_B x_B = (K_B - K_{IB}^T K_I^{-1} K_{IB}) x_B = b_B - K_{IB}^T K_I^{-1} b_I = \hat{b}_B.$$

The Schur complement matrix S_B is in general dense. However, it is not necessary to generate this matrix since, in the conjugate gradient iteration, this matrix is needed only in terms of matrix vector products which can be computed by solving each subregion once and collecting the solution on the interfaces and vertices.

2.3. Multilevel Nodal Basis Domain Decomposition Preconditioner. In this section, we combine the ideas from previous sections to derive a new domain decomposition algorithm. The symmetric form of the preconditioned system using multilevel nodal basis preconditioner can be written as:

$$D^{-\frac{1}{2}}G^T K G D^{-\frac{1}{2}}\tilde{x} = \tilde{b}$$

where $GD^{-\frac{1}{2}}\tilde{x} = x$ and $\tilde{b} = D^{-\frac{1}{2}}G^T b$. Let us partition the unknowns corresponding to the multilevel nodal basis $\tilde{x}$ into those on the subregion interior $\tilde{x}_I$ and those on the interface $\tilde{x}_B$ and we eliminate the subregion interior variables $\tilde{x}_I$, we obtain the reduced system:

$$\tilde{S}_B \tilde{x}_B = \tilde{b}_B,$$

where $\tilde{S}_B$ is the Schur complement of $D^{-\frac{1}{2}}G^T K G D^{-\frac{1}{2}}$ after eliminating x_I. Here we can also decompose G according to the interior and interface unknowns so that

$$G = \begin{bmatrix} G_I & G_{IB} \\ G_{BI} & G_B \end{bmatrix}$$

where G_I and G_B are the transformation (rectangular) matrices involving subregion interior points and interface points respectively.

The above formulation requires many arithmetic operations when the Schur complement is applied to a vector during the conjugate gradient iterations. By using the following lemmas, the above system can be reduced to a simpler form.

LEMMA 2.1. *G represents a change of basis which leaves the space of variables on Γ invariant (i.e. $G_{BI} = 0$).*

Proof. G_{BI} represents the contribution of the multilevel nodal bases in the subdomain interior to the nodal basis on the interfaces during the transformation. Recall from previous sections that

$$u \equiv Gv = \sum_{l=0}^{J}\sum_{i=1}^{n_J} v_l^i \phi_l^i \tag{2.2}$$

where $\{\phi_l^i, i = 1, 2, \cdots n_l\}$ is the set of basis functions in V_l^h, and $v_l^i, i = 1, \cdots, n_l$ is the set of values defined on the nodal points x_l^i. Let $u = (u_I, u_B)^T$ and $v = (v_I, v_B)$ be the partitionings according to subdomain interior and interfaces where $u \in R^n$ and $v \in R^m$. If we evaluate the above expression at node y_j^i on the interface Γ, we obtain

$$u_B^i = (G_B v_B)^i + (G_{BI} v_I)^i = \sum_{l=0}^{J}\left\{\sum_{j\in\Gamma}(v_B)_l^j(\phi_l^j)_i + \sum_{j\notin\Gamma}(v_I)_l^j(\phi_l^j)_i\right\}$$

where $(\phi_l^j)_i$ is the value of the basis function at node j evaluated at node i on level l. It can be verified that

$$(\phi_l^j)_i = 0 \quad \forall j \in \Omega_k \text{ and } i \in \Gamma$$

since all multilevel nodal basis functions at the nodes interior to the subregions vanish on the interface. In other words, the second term in the summation of equation 2.2 is identically equal to 0. Thus $G_{BI} = 0$ and $u^i = (G_B v_B)^i$. □

The following lemma is based on a similar one from Smith and Widlund's paper [8]. The proof can be obtained by a straightforward matrix manipulation.

LEMMA 2.2. *Let $G_{BI} = 0$ and $D = blockdiag(D_I, D_B)$, then*

$$\tilde{S}_B = D_B^{-\frac{1}{2}} G_B^T S_B G_B D_B^{-\frac{1}{2}}.$$

The above two lemmas imply that if we first eliminate the interior variables and then transform to the multilevel nodal basis, we will be solving the same linear system as before (transformation to multilevel nodal basis and then do the elimination). As a result, we are solving the following simpler and smaller system:

$$D_B^{-\frac{1}{2}} G_B^T S_B G_B D_B^{-\frac{1}{2}} \tilde{x}_B = \tilde{b}_B,$$

which is equivalent to solving the Schur complement system $S_B x_B = b_B$ with the preconditioner $M^{-1} = G_B D^{-1} G_B^T$. We call the preconditioner using this new formulation the multilevel nodal basis domain decomposition (MNBDD) preconditioner. This method offers several possible advantages over the standard multilevel nodal basis method by Bramble, Pasciak and Xu [3]. The conjugate gradient iteration is carried out over a much smaller set of unknowns and we will show that the condition number is smaller. The solution of the subproblems on the interfaces is easily parallelizable since they are independent. One possible drawback, however, is that now it is necessary to solve each subregion exactly in each iteration which adds more computational overhead.

Here the algorithm for G_B algorithm is similar to the one for G shown before except now the basis functions used in the evaluation are restricted only to those on the interfaces. The operation of G_B on a vector v_B is defined by:

$$G_B v_B = \sum_{j=0}^{J} \sum_{i=1}^{m_j} (v_B)_j^i \phi_j^i$$

where $\phi_j^i, i = 1, 2, \cdots m_j$ is the set of basis functions on Γ.

The MNBDD algorithm can be summarized as follow:

Algorithm MNBDD : input = r, output = $z = M_B^{-1} r$
  Perform partial change of basis to the MN basis $v = G_B^T r$
  solve the coarse grid problem $y = D_B^{-1} v$
  Perform change of basis back to the nodal basis $z = G_B y$
end MNBDD

We need one more lemma from [8].

LEMMA 2.3. *Let K be symmetric and positive definite. Then the condition numbers of K and its Schur complement satisfy*

$$\kappa(Schur(K)) \leq \kappa(K).$$

Using Lemma 2.3 as well as the condition number bounds from [3], we arrive at the following main theorem:

THEOREM 2.4. $\kappa(M_B^{-1} S_B) \leq O(\log^2(\frac{H}{h}))$ *for smooth coefficient problems.*

Proof. By using Lemma 2.1 and 2.2 we obtain

$$\tilde{S}_B = D_B^{-\frac{1}{2}} G_B^T S_B G_B D_B^{-\frac{1}{2}}.$$

By using Lemma 2.3, we obtain

$$\begin{aligned} \kappa(D_B^{-\frac{1}{2}} G_B^T S_B G_B D^{-\frac{1}{2}}) &= \kappa(\tilde{S}_B) \\ &\leq \kappa(D^{-\frac{1}{2}} G^T K G D^{-\frac{1}{2}}), \end{aligned}$$

which is bounded by $O(\log^2 \frac{H}{h})$, see [3]. □

It is also proved in [3] that the condition number bound in Theorem 2.4 depends also on the elliptic regularity of the problem. For example, for smooth coefficient problems on convex polygonal domains, the condition number is bounded by $O(\log \frac{H}{h})$; and for certain discontinuous problems, the condition number is bounded by $O(\log^3 \frac{H}{h})$.

The hierarchical basis domain-decomposed (HBDD) [8] algorithm is similar to the BPS algorithm by Bramble, Pasciak and Schatz [2]. The only difference is that the HBDD uses hierarchical basis preconditioner for the edges while the BPS uses variants of Dryja's preconditioner [5]. The MNBDD algorithm, however, has one important difference; namely, in addition to the use of multilevel nodal basis as preconditioners on the edges, the MNBDD algorithm also implements multilevel nodal basis on the vertices. This introduces some redundancy on the vertices. This redundancy may be the reason for its improved performance (see next section).

3. Numerical Results.

3.1. Two-subdomain Example. For two-subdomain case, since there is one edge and no vertex, it is not necessary to solve a coarse problem.. We use the two-subdomain example of [8] with different right hand side and we compare our results with those reported in [8]. We also include results from using the Dryja's ($l_0^{1/2}$) [5] preconditioner. We use the domain $\bar{\Omega} = \bar{\Omega}_1 \cup \bar{\Omega}_2$ where $\bar{\Omega}_1$ and $\bar{\Omega}_2$ are unit squares aligned along an edge $\Gamma = \bar{\Omega}_1 \cap \bar{\Omega}_2$. We use the standard uniform mesh and the usual five point discretization for the Laplacian and the iteration counts are listed in Table 3.1 (where n is the number of unknowns on the interface). The right hand side is such that the solution is $x(x-1)y(y-1)$ and the stopping criterion is when the relative 2-norm of the residual falls below 10^{-6}. The initial guess used is $u^{(0)} = 1.0$. We observe that while the iteration count for the HBDD continues to grow with larger n, the other preconditioners seem to be bounded independent of n. While the iteration counts using the HBDD preconditioners grow with n, we see that the iteration counts using MNBDD seem to gradually levels off. Overall, we see the MNBDD preconditioner performs relatively well compared to the others.

TABLE 1
iteration count versus n

n	No precond	Dryja	MNBDD	HBDD
8	4	4	4	4
16	8	6	7	7
32	16	6	9	8
64	27	6	9	10
128	39	6	9	12

3.2. Many-subdomain Example. Next we consider the case of many subdomains. The unit square Ω is subdivided uniformly into $k \times k$ square subdomains and the same model problem is solved using uniform meshes. We compare our results with a set of experiments reported in [8]. We also compare the condition numbers as well as iteration counts between our method, Smith and Widlund's method, and Bramble, Pasciak and Schatz's method (BPS). For HBDD method, it is reported in [8] that the contribution of the coarse problem should be scaled by a factor $\alpha \approx 3.6$ to achieve fastest convergence. We also experiment with different α in our preconditioner and the α that gives fastest convergence in our experiment is about 0.5. The numerical results using this α are included under the method MNBDD($\alpha = 0.5$). The right hand side f is constructed such that the solution is $u(x,y) = x(x-1)y(y-1)$ and the stopping criterion is 10^{-5}. Again, the initial guess is $u^{(0)} = 1.0$. The results are shown in Table 3.2.

Our first observation is that our condition number results for the HBDD method agree very well with those reported in [8]. We also observe that the condition numbers using the MNBDD method are much lower than the BPS and HBDD methods. The condition numbers grow very slowly with n while this is not the case with the BPS and HBDD methods. With the use of the scaling factor $\alpha = 0.5$, we observe an even better performance and in fact, the condition numbers appear to be $O(1)$.

Through the numerical experiment on the model problem, we have showed that the MNBDD preconditioner offers good convergence rates (better than BPS and HBDD for the model problem used in our experiment) as well as low computational cost ($O(n)$ for MNBDD and HBDD and $O(n \log n)$ for BPS). The BPS, HBDD, and MNBDD algorithms have a similar property that the edge preconditioners do not take into account the variation of coefficients in the original equation. It is not clear whether the good condition number behavior will hold for problems with variable coefficients and it is our intention to pursue this issue in the near future.

TABLE 2
Condition Numbers and iteration counts for the Many-subdomain Case

grid	number of subdomains	HBDD($\alpha = 3.6$)		BPS		MNBDD($\alpha = 1$)		MNBDD($\alpha = 0.5$)	
		κ	iter	κ	iter	κ	iter	κ	iter
32x32	2x2	9.62	11	11.85	11	2.27	8	2.24	7
32x32	4x4	7.96	11	8.75	14	2.88	9	2.19	8
32x32	8x8	5.30	10	6.08	12	3.09	9	2.10	7
64x64	2x2	12.68	13	16.47	12	2.34	8	2.32	8
64x64	4x4	11.84	13	13.03	15	2.96	9	2.28	8
64x64	8x8	8.52	12	9.79	15	3.21	10	2.21	8
64x64	16x16	5.41	10	6.32	13	3.21	10	2.11	7
128x128	4x4	16.49	15	17.92	18	3.01	9	2.35	8
128x128	8x8	12.54	15	14.18	16	3.30	10	2.35	8
128x128	16x16	8.69	13	10.21	15	3.31	10	2.24	8
128x128	32x32	5.42	10	6.36	13	3.22	10	2.11	7
256x256	4x4	21.90	17	23.45	19	3.03	9	2.39	8
256x256	8x8	17.30	17	19.33	18	3.34	10	2.43	8
256x256	16x16	12.72	15	14.79	17	3.37	10	2.36	8
256x256	32x32	8.69	13	10.27	15	3.30	10	2.24	8
256x256	64x64	5.37	10	6.37	12	3.21	10	2.09	7

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