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A Minimum-Phase LU Factorization Preconditioner for Toeplitz Matrices

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A MINIMUM-PHASE LU FACTORIZATION PRECONDITIONER FOR TOEPLITZ MATRICES *

TA-KANG KU† AND C.-C. JAY KUO†

Abstract. A new preconditioner is proposed for the solution of an $N \times N$ Toeplitz system $T_N \mathbf{x} = \mathbf{b}$, where T_N can be symmetric indefinite or nonsymmetric, by preconditioned iterative methods. The preconditioner F_N is obtained based on factorizing the generating function $T(z)$ into the product of two terms corresponding, respectively, to minimum-phase causal and anticausal systems and therefore called the minimum-phase LU (MPLU) factorization preconditioner. Due to the minimum-phase property, $\|F_N^{-1}\|$ is bounded. For rational Toeplitz T_N with generating function $T(z) = A(z^{-1})/B(z^{-1}) + C(z)/D(z)$, where $A(z)$, $B(z)$, $C(z)$ and $D(z)$ are polynomials of orders p_1, q_1, p_2 and q_2, we show that the eigenvalues of $F_N^{-1}T_N$ are repeated exactly at 1 except at most α_F outliers, where α_F depends on p_1, q_1, p_2, q_2 and the number w of the roots of $\tilde{T}(z) = A(z^{-1})D(z) + B(z^{-1})C(z)$ outside the unit circle. A preconditioner K_N in circulant form generalized from the symmetric case is also presented for comparison. The MPLU preconditioner F_N performs better than circulant preconditioner K_N in our numerical experiments.

Key words. LU factorization, minimum phase, preconditioned iterative methods, preconditioner, Toeplitz, Padé approximation.

AMS(MOS) subject classifications. 65F10, 65F15

1. Introduction. Toeplitz matrices arise in many signal processing applications. To solve a general $N \times N$ Toeplitz system of equations $T_N \mathbf{x} = \mathbf{b}$, direct inverse algorithms based on Levinson recurrence [23] with $O(N^2)$ operations have been studied intensively in the past [11], [18], [30], [33]. Superfast algorithms with $(N \log^2 N)$ complexity have also been proposed [1], [3], [4], [16]. Although the computational complexity of these algorithms is lower than that of Gaussian elimination with pivoting, i.e. $O(N^3)$, their stability is still an issue when applied to indefinite or nonsymmetric T_N. It has been shown that these algorithms may become unstable if T_N is not symmetric positive definite (SPD) and well-conditioned [5], [10]. A stable extension of Levinson's algorithm to general Toeplitz matrices has recently been studied by Chan and Hansen [9].

In this research, we consider the use of preconditioned iterative methods for solving a general Toeplitz system $T_N \mathbf{x} = \mathbf{b}$ to reduce the computational complexity as well as to avoid the numerical instability. Various preconditioners in circulant form have been used in the the Preconditioned Conjugate Gradient (PCG) algorithm [6], [8], [17], [19], [29] to solve the SPD Toeplitz system. All the preconditioners can be inverted via fast transform algorithms with $O(N \log N)$ operations. Besides, the spectra of the preconditioned Toeplitz matrices have such a nice clustering property that the PCG method converges in a finite number of iterations for T_N generated by a positive function in the Wiener class [7], [19]. Although it is possible to generalize this preconditioning technique to general Toeplitz matrices in a straightforward way (see §4), the focus of this paper is to develop a novel approach to construct a general Toeplitz preconditioner based on an approximate LU factorization. The resulting preconditioned systems are then solved by various iterative methods such as the Generalized Minimal Residual (GMRES) [27] and the Conjugate Gradient Squared (CGS) [28].

The idea of constructing the LU factorization preconditioner can be simply stated as follows. Consider a banded Toeplitz matrix T_N with a finite-order generating function $T(z) = \sum_{n=-s}^{r} t_n z^{-n}$. The

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$T(z)$ can be factorized into the product $T(z) = z^j L(z^{-1})U(z)$, where $L(z^{-1})$ and $U(z)$ have all roots inside and outside the unit circle, respectively. We associate z^j, $L(z^{-1})$ and $U(z)$ with a shift matrix S_N, a lower and an upper triangular Toeplitz matrices L_N and U_N, correspondingly, and the product $F_N = S_N L_N U_N$ is the desired preconditioner for T_N. The above factorization procedure has been used frequently in the context of digital signal processing [25] to design the minimum-phase causal (or maximum-phase anticausal) linear filter. The F_N is therefore called the minimum-phase LU (MPLU) factorization preconditioner. To generalize the MPLU preconditioning technique to full Toeplitz matrices, we first obtain an approximating rational generating function for the original one with Padé approximation. Since a rational Toeplitz matrix can be transformed to a banded matrix which is nearly Toeplitz, the appropriate MPLU preconditioner can also be constructed.

For well-conditioned Toeplitz T_N, we show that the preconditioner F_N is well-conditioned due to the minimum-phase factorization property. Then, the preconditioned matrix $A_N = F_N^{-1}T_N$ is also well-conditioned so that the system $A_N\mathbf{x} = F_N^{-1}\mathbf{b}$ can be stably solved by iterative algorithms. One obvious choice is to form the well-conditioned SPD normal system $A_N^T A_N \mathbf{x} = A_N^T F_N^{-1}\mathbf{b}$ and solve the resulting system by the CG method (known as the CGNR method [15]). Thus, for well-conditioned nonsymmetric Toeplitz systems, numerical stability is easy to obtain by using preconditioned iterative methods.

The spectral clustering properties of the MPLU-preconditioned Toeplitz $F_N^{-1}T_N$ are studied for both banded and rational T_N. We prove that, for rational T_N with generating function $T(z) = A(z^{-1})/B(z^{-1}) + C(z)/D(z)$, where $A(z)$, $B(z)$, $C(z)$ and $D(z)$ are polynomials of orders p_1, q_1, p_2 and q_2, the eigenvalues of $F_N^{-1}T_N$ are repeated exactly at 1 except α_F outliers, where α_F depends on p_1, q_1, p_2, q_2 and the number w of the roots of $\tilde{T}(z) = A(z^{-1})D(z) + B(z^{-1})C(z)$ outside the unit circle. A direct consequence of these spectral properties is that the appropriate preconditioned iterative methods converge in at most $\alpha_F + 1$ iterations. This result should be compared to that of the circulant-preconditioned rational Toeplitz $K_N^{-1}T_N$. In [22], we prove that the eigenvalues of $K_N^{-1}T_N$, except α_K outliers, are clustered in the interval $(1-\epsilon_K, 1+\epsilon_K)$, where the clustering radius ϵ_K is proportional to the magnitude of the last elements used to construct the circulant preconditioner. It can be shown that $\alpha_K \geq \alpha_F$ and $\epsilon_K \geq \epsilon_F = 0$. Thus, the MPLU preconditioner provides better spectral clustering properties for a faster convergence rate.

Since the MPLU preconditioner F_N is a product of the shift matrix S_N and triangular banded Toeplitz matrices L_N and U_N, the preconditioning step $\mathbf{z} = F_N^{-1}\mathbf{r}$ can be achieved with a computational complexity proportional to $O(N)$ only. The total computational complexity for solving a rational Toeplitz system by MPLU-preconditioned iterative methods is $O(N)$, which is lower than the $O(N \log N)$ operations required by the circulant-preconditioned iterative methods and is in the same order as that required by several direct methods [12], [13], [31], [32]. However, there is a drawback of the MPLU preconditioner in the context of parallel processing. That is, the MPLU preconditioning has to be performed sequentially whereas the circulant preconditioning can be easily parallelized.

The outline of this paper is as follows. In §2, the procedure to construct the MPLU preconditioner for banded Toeplitz matrices is described, and the spectral properties of the preconditioned banded Toeplitz are examined. In §3, the MPLU preconditioning technique is generalized to full Toeplitz matrices, including both rational and nonrational cases, and the spectral properties of the MPLU-preconditioned rational Toeplitz are studied. In §4, we compare the MPLU preconditioner with the circulant preconditioner. Finally, numerical results are given in §5 to access the efficiency of the MPLU preconditioner.

2. MPLU preconditioner for banded Toeplitz. Consider a sequence of $m \times m$ Toeplitz matrices T_m, $m = 1, 2, \cdots$, with a generating sequence t_n, $-\infty < n < \infty$, such that

$$T_N = \begin{bmatrix} t_0 & t_{-1} & \cdot & t_{-(N-2)} & t_{-(N-1)} \\ t_1 & t_0 & t_{-1} & \cdot & t_{-(N-2)} \\ \cdot & t_1 & t_0 & \cdot & \cdot \\ t_{N-2} & \cdot & \cdot & \cdot & t_{-1} \\ t_{N-1} & t_{N-2} & \cdot & t_1 & t_0 \end{bmatrix}.$$

The Laurent series

$$T(z) = \sum_{n=-\infty}^{\infty} t_n z^{-n}$$

is known as the generating function of the matrix sequence T_m. We assume that the generating sequence t_n satisfies the following two conditions:

$$(2.1) \qquad |T(e^{i\theta})| = \left|\sum_{-\infty}^{\infty} t_n e^{-in\theta}\right| \geq \delta > 0, \qquad \forall\theta,$$

$$(2.2) \qquad \sum_{-\infty}^{\infty} |t_n| \leq B < \infty.$$

Since $T(e^{i\theta}) = \sum_{-\infty}^{\infty} t_n e^{-in\theta}$ describes the asymptotic eigenvalue distribution of T_m, conditions (2.1) and (2.2) imply that $||T_N||$ and $||T_N^{-1}||$ are bounded and, consequently, that T_N is well-conditioned.

The system of equations

$$(2.3) \qquad T_N \mathbf{x} = \mathbf{b}$$

can be solved by various iterative methods. To accelerate the convergence rate, a preconditioner P_N is introduced to solve the preconditioned system of equations

$$(2.4) \qquad P_N^{-1} T_N \mathbf{x} = P_N^{-1} \mathbf{b},$$

where P_N is the preconditioner used to approximate T_N. In this section, we focus on the case where T_N is banded with lower bandwidth r and upper bandwidth s, i.e. $t_n = 0$ if $n < -s$ or $n > r$, $t_{-s} t_r \neq 0$, and $r + s = d < N$.

2.1. Construction of the preconditioner. We can use a direct method to factorize T_N,

$$(2.5) \qquad T_N = \mathcal{L}_N \mathcal{U}_N,$$

where $\mathcal{L}_N$ and $\mathcal{U}_N$ are lower and upper triangular matrices, respectively. The exact factorization (2.5) with the Levinson-type algorithms requires $O(dN)$ and $O(N^2)$ operations for rational and full Toeplitz, respectively [12], [31]. If T_N is not symmetric positive definite, the numerical stability of these algorithms cannot be guaranteed. Instead of performing the exact factorization, we propose to factorize T_N approximately as

$$(2.6) \qquad T_N \approx S_N L_N U_N = F_N,$$

where S_N is a shift matrix and L_N and U_N are, respectively, lower and upper triangular banded Toeplitz matrices. Our objectives include that the approximate factorization (2.6) can be achieved by a stable algorithm with operations independent of N, that F_N approximates T_N well, and that $||F_N^{-1}||$ is bounded. Then, the F_N can be used as a preconditioner in preconditioned iterative methods.

To derive the approximate factorization, it is convenient to consider the problem in the Z-transform domain and ignore the boundary effect arising in a Toeplitz system. When T_N is banded with lower bandwidth r and upper bandwidth s, its generating function can be expressed as

$$(2.7) \qquad T(z) = \sum_{n=-s}^{r} t_n z^{-n} = t_{-s} z^s \prod_{i=1}^{d} (1 - z_i z^{-1}),$$

where $d = r + s$ and z_i is a root of $T(z)$. From (2.1), we know that $|z_i| \neq 1$. If $T(z)$ has w roots outside the unit circle, we can factorize $T(z)$ as

$$(2.8) \qquad T(z) = z^{s-w} L(z^{-1}) U(z),$$

where

$$L(z^{-1}) = \prod_{|z_i|<1} (1 - z_i z^{-1}), \qquad U(z) = t_{-s} \prod_{|z_i|>1} (z - z_i).$$

Note that the above factorization has a special feature, namely, all zeros of $L(z^{-1})$ (or $U(z)$) are inside (or outside) the unit circle. The following example is used to illustrate the factorization procedure (2.8).

Example 1: Let A_N be an $N \times N$ tridiagonal Toeplitz matrix with $t_1 = 1.5$, $t_0 = -6.5$ and $t_{-1} = 2$. Then, we have

$$T(z) = 1.5z^{-1} - 6.5 + 2z = 2z(1 - 0.25z^{-1})(1 - 3z^{-1}) = L(z^{-1})U(z),$$

where

$$L(z^{-1}) = 1 - 0.25z^{-1}, \qquad U(z) = 2z - 6.$$

Since $r = s = w = 1$ in this example, the term z^{s-w} in (2.8) is equal to 1. □

Let us associate the right-hand-side of the factorization (2.8) with the following matrices

$$L(z^{-1}) \longleftrightarrow L_N, \quad U(z) \longleftrightarrow U_N, \quad z^{s-w} \longleftrightarrow S_N \equiv E_N^{s-w}, \tag{2.9}$$

where L_N and U_N are $N \times N$ lower and upper triangular Toeplitz matrices with generating functions $L(z^{-1})$ and $U(z)$, respectively, and E_N is the $N \times N$ unit row-shift matrix,

$$E_N = [e_N, e_1, e_2, \cdots, e_{N-1}],$$

and where e_n is the $N \times 1$ unit vector with the nth element equal to 1 and zeros elsewhere. It is straightforward to verify that

$$E_N^{-1} = [e_2, e_3, \cdots, e_N, e_1],$$

and that E_N^k is the product of E_N (or E_N^{-1}) $|k|$ times for positive (or negative) integer k. The premultiplication of E_N (or E_N^{-1}) with a $N \times N$ matrix is equivalent to the circular up-shift (or down-shift) of its rows by one.

Then, the product of S_N, L_N and U_N is used as the desired preconditioner

$$F_N = S_N L_N U_N = E_N^{s-w} L_N U_N. \tag{2.10}$$

It inverse

$$F_N^{-1} = U_N^{-1} L_N^{-1} S_N^{-1} = U_N^{-1} L_N^{-1} E_N^{w-s}$$

can be performed effectively with $O(N)$ operations due to the special structures of S_N, L_N and U_N. The factorization (2.8) has been frequently used in the context of digital signal processing [25] to design the minimum-phase causal (or maximum-phase anti-causal) linear filter, which is by definition a system characterized by a lower (or upper) triangular matrix with a stable inverse. Thus, we call F_N defined by (2.10) the *minimum-phase LU* (MPLU) factorization preconditioner.

2.2. Spectral properties. The minimum-phase factorization procedure guarantees that $||F_N^{-1}||$ is bounded, which is proved in the following theorem.

THEOREM 1. *Let T_N be a banded Toeplitz matrix with lower bandwidth r and upper bandwidth s satisfying conditions (2.1) and (2.2), and L_N and U_N be obtained from the minimum-phase factorization (2.8)-(2.10). Then, the 1-, 2- and ∞-norms of F_N^{-1} and F_N are bounded for asymptotically large N.*

Proof. It is well known that there exists an isomorphism between the ring of the power series $G(z^{-1}) = \sum_{n=0}^{\infty} g_N z^{-n}$ and the ring of semi-infinite lower triangular Toeplitz matrices with $g_0, g_1, \cdots, g_N, \cdots$ as the first column, and the power series multiplication is isomorphic to matrix multiplication [13]. With this isomorphism, we know that L_N^{-1} is a lower triangular Toeplitz matrix whose first column $\tau_0, \tau_1, \cdots, \tau_n, \cdots$ can be obtained from the coefficients of the power series, i.e.

$$\frac{1}{L(z^{-1})} = \prod_{|z_i|<1} \frac{1}{(1 - z_i z^{-1})} = \sum_{n=0}^{\infty} \tau_n z^{-n}.$$

It is clear that $\sum_{n=0}^{\infty} |\tau_n|$ is bounded if and only if all poles of $1/L(z^{-1})$ are inside the unit circle, which is guaranteed by the minimum-phase factorization (2.8).

Condition (2.1) implies that all zeros z_i of $T(z)$ do not lie on or arbitrarily close to the unit circle, i.e.

$$|z_i| \leq 1 - \beta \quad \text{and} \quad 1 + \beta \leq |z_i| < \infty,$$

where β is a small positive number independent of N. Since

$$||L_N^{-1}||_1 = ||L_N^{-1}||_\infty = \sum_{n=0}^{N-1} |\tau_n| \leq \sum_{n=0}^{\infty} |\tau_n| \leq \prod_{|z_i|<1} \frac{1}{1 - |z_i|} \leq \beta^{-(d-w)},$$

the 2-norm of L_N is bounded by

$$||L_N^{-1}||_2 \leq (||L_N^{-1}||_1 ||L_N^{-1}||_\infty)^{1/2} = \sum_{n=0}^{N-1} |\tau_n| \leq \beta^{-(d-w)}.$$

A similar arguments can be used to prove that $||U_N^{-1}||_2 \leq \beta^{-w}$. Since $||E_N||_2 = ||E_N^{-1}||_2 = 1$, we have

$$||F_N^{-1}||_2 \leq ||L_N^{-1}||_2 ||U_N^{-1}||_2 \leq \beta^{-d},$$

which is independent of N. Besides, since $||L_N||_1 = ||L_N||_\infty < \infty$, we have

$$||L_N||_2 \leq (||L_N||_1 ||L_N||_\infty)^{1/2} < \infty.$$

Similarly, $||U_N||_2$ is bounded and $||F_N||_2 \leq ||L_N||_2 ||U_N||_2 < \infty$. □

A direct consequence of the above theorem is that preconditioner F_N is well-conditioned.

If $L(z^{-1})$ (or $U(z)$) is not chosen according to (2.8) so that there exist roots of the polynomial $L(z^{-1})$ (or $U(z)$) with magnitude greater (or less) than one, i.e. nonminimum-phase factorization, one can easily check that $||L_N^{-1}||_2$ (or $||U_N^{-1}||_2$) is unbounded for asymptotically large N. For example, if we choose

$$\tilde{L}(z^{-1}) = 1 - 3z^{-1}, \qquad \tilde{U}(z) = 2z - 0.5,$$

for L_N and U_N in Example 1, the product $L_N U_N$ leads to an ill-conditioned matrix whose smallest eigenvalue converges to zero for asymptotically large N. Thus, the minimum phase factorization is crucial for the stability of the preconditioning procedure $\mathbf{z} = F_N^{-1}\mathbf{r}$.

Next, we study the spectral properties of $F_N^{-1}T_N$. For F_N to be a good preconditioner, it is desirable that $F_N^{-1}T_N$ has clustered eigenvalues. In Theorem 2 we will prove that it has only a finite number of eigenvalues different from 1. To derive this theorem, we need two lemmas.

LEMMA 1. *Let T_N be a banded Toeplitz matrix with lower bandwidth r and upper bandwidth s, where $r + s = d < N$, generated by $T(z)$ which has w roots outside the unit circle. Then, for L_N and U_N obtained by the minimum-phase factorization (2.8) and (2.9), $L_N U_N$ is a banded Toeplitz matrix generated by $z^{w-s}T(z)$ with lower bandwidth $d - w$ and upper bandwidth w except its northwest $(d-w) \times w$ block.*

Proof. This lemma can be proved with definitions and direct matrix multiplication. □

Lemma 1 basically says that the product $L_N U_N$ is a nearly banded Toeplitz matrix. Despite that T_N and $L_N U_N$ have the same total bandwidth d, they do not have the same lower bandwidth and upper bandwidth unless $w = s$. By shifting the rows of $L_N U_N$ circularly, we are able to construct another nearly banded Toeplitz $F_N = E_N^{s-w} L_N U_N$ which has the same lower and upper bandwidthes as T_N.

LEMMA 2. *Let T_N be a banded Toeplitz matrix with lower bandwidth r and upper bandwidth s, where $r+s=d<N$, generated by $T(z)$ which has w roots outside the unit circle. Then, the matrix $F_N = E_N^{s-w} L_N U_N$ defined in (2.10) is a nearly banded Toeplitz matrix. Elements of matrices T_N and F_N are identical except the following:*
1) the northwest $r \times s$ block when $s = w$;
2) the northwest $r \times w$ block and the northeast $(w-s) \times r$ block when $s < w$;
3) the northwest $r \times w$ block, the southwest $(s-w) \times s$ block and the southeast $(s-w) \times (d-w)$ block when $s > w$.

Proof. When $s = w$, it can be directly verified that $F_N = L_N U_N$ is a banded Toeplitz generated by $T(z)$ with lower bandwidth r and upper bandwidth s except the northwest $r \times s$ block. When $s < w$, recall that the rows of $F_N = E_N^{s-w} L_N U_N$ are obtained from those of $L_N U_N$ with circularly downward-shift $w-s$ rows so that the last $w-s$ rows in $L_N U_N$ become the the first $w-s$ rows of F_N and the first $N-(w-s)$ rows in $L_N U_N$ become the last $N-(w-s)$ rows of F_N. By using Lemma 1, we can clearly see that F_N is a banded Toeplitz with lower bandwidth r and upper bandwidth s generated by $T(z)$ except the northwest $r \times w$ block and the northeast $(w-s) \times r$ block. Similarly, one can prove the case $s > w$. □

Lemma 2 tells us that $\Delta T_N = T_N - F_N$ is a zero matrix except at most three small blocks. Based on this lemma, we characterize the spectral properties of $F_N^{-1} T_N$ in Theorem 2.

THEOREM 2. *Let T_N be a banded Toeplitz matrix with lower bandwidth r and upper bandwidth s, where $r+s=d<N$, generated by $T(z)$ which has w roots outside the unit circle. Then, there are at most α_F eigenvalues of $F_N^{-1}T_N$ not equal to 1, where*

$$\alpha_F = \begin{cases} \min(r,s), & s = w, \\ \min(r, 2w-s), & s < w, \\ \min(d-w, s), & s > w. \end{cases} \tag{2.11}$$

Proof. Since we have

$$F_N^{-1} T_N = (T_N + \Delta T_N)^{-1} T_N = (I_N + T_N^{-1} \Delta T_N)^{-1},$$

where I_N denotes the $N \times N$ identity matrix, the eigenvalue 1 of $F_N^{-1} T_N$ corresponds to the eigenvalue 0 of $T_N^{-1} \Delta T_N$, and the number of eigenvalues of $F_N^{-1} T_N$ not equal to 1 is determined by the rank of ΔT_N. The rank of a matrix is bounded by the number of nonzero rows or columns, and the rank of the sum of two matrices is bounded by the sum of their individual ranks. All nonzero elements in ΔT_N are inside the blocks given by Lemma 2. When $s = w$, since all nonzero elements of ΔT_N are in the first r rows or the first s columns, the rank of ΔT_N is bounded by $\min(r,s)$. When $s < w$, we have $w - s \le d - s = r$. Since all nonzero elements of ΔT_N are either in the first r rows or in the union of the first w columns and the first $w-s$ rows, the rank of ΔT_N is bounded by $\min(r, 2w-s)$. When $s > w$, since all nonzero elements of ΔT_N are either in the union of the first r and the last $s-w$ rows or in the union of the first w columns and the last $s-w$ rows, the rank of ΔT_N is bounded by $\min(d-w, s)$. The proof is completed. □

We use an example to illustrate the above theorem.

Example 2: Consider the following $N \times N$ banded Toeplitz matrices with $N \ge 4$,

$$\begin{array}{lll} T_{N,1} & [(r,s)=(3,0)]: & t_3 = 2,\ t_2 = -5,\ t_1 = 6,\ t_0 = -2, \\ T_{N,2} & [(r,s)=(2,1)]: & t_2 = 2,\ t_1 = -5,\ t_0 = 6,\ t_{-1} = -2, \\ T_{N,3} & [(r,s)=(1,2)]: & t_1 = 2,\ t_0 = -5,\ t_{-1} = 6,\ t_{-2} = -2, \\ T_{N,4} & [(r,s)=(0,3)]: & t_0 = 2,\ t_{-1} = -5,\ t_{-2} = 6,\ t_{-3} = -2. \end{array}$$

$T(z)$ has roots $0.5+0.5i$, $0.5-0.5i$ and 2 so that $w = 1$. For these matrices, the MPLU factorization results in the same L_N and U_N defined by the generating sequences

$$\begin{array}{llll} l_0 = 1, & l_1 = -1, & l_2 = 0.5, & l_n = 0 \quad n \ne 0,1,2, \\ u_0 = 4, & u_{-1} = -2, & & u_n = 0 \quad n \ne 0,-1. \end{array}$$

	d	r	s	w	α_F
$T_{N,1}$	3	3	0	1	2
$T_{N,2}$	3	2	1	1	1
$T_{N,3}$	3	1	2	1	2
$T_{N,4}$	3	0	3	1	2

TABLE 1
An example to illustrate Theorem 2.

To illustrate Theorem 2, we list values of d, s, r, w and α_F in Table 1.

Since $F_N^{-1}T_N$ has only at most $\alpha_F + 1$ distinct eigenvalues, appropriate preconditioned iterative methods, such as GMRES and CGS, converge in at most $\alpha_F + 1$ iterations with exact arithmetic (see Test Problems 1 and 4 in §5).

3. Preconditioning full Toeplitz matrices. In this section, we generalize the MPLU preconditioning technique to full Toeplitz matrices. The basic idea is to approximate the full Toeplitz with a rational Toeplitz, transform the rational Toeplitz to a nearly banded Toeplitz, and then construct the MPLU preconditioner for the nearly banded Toeplitz.

3.1. Rational Toeplitz. Toeplitz matrices with a rational generating function can be transformed to banded ones [13]. We describe the transformation briefly as follows. Let the generating function of T_N be of the form

$$T(z) = \frac{A(z^{-1})}{B(z^{-1})} + \frac{C(z)}{D(z)}, \tag{3.1}$$

where $A(z)$, $B(z)$, $C(z)$ and $D(z)$ are polynomials in z with orders p_1, q_1, p_2 and q_2, respectively. Note that a special case of (3.1) is $A(z) = C(z)$ and $B(z) = D(z)$, which leads to a symmetric rational Toeplitz of order (p, q) with $p_1 = p_2 = p$ and $q_1 = q_2 = q$. By applying the isomorphism between the ring of the power series and the ring of semi-infinite triangular Toeplitz matrices, we have the following relationship

$$T_N = L_a L_b^{-1} + U_c U_d^{-1},$$

where L_a (or L_b) is an $N \times N$ lower triangular Toeplitz matrix with the first N coefficients in $A(z)$ (or $B(z)$) as its first column and U_c (or U_d) is an $N \times N$ upper triangular Toeplitz matrix with the first N coefficients in $C(z)$ (or $D(z)$) as its first row. Since power series multiplication is commutative, we have

$$\tilde{T}_N = L_b T_N U_d = L_a U_d + L_b U_c. \tag{3.2}$$

where $\tilde{T}_N$ is banded and nearly Toeplitz characterized by the following lemma.

LEMMA 3. *Let T_N be the $N \times N$ Toeplitz matrix generated by $T(z)$ in (3.1), the corresponding $\tilde{T}_N$ obtained from (3.2) is a banded Toeplitz with lower bandwidth $r = \max(p_1, q_1)$ and upper bandwidth $s = \max(p_2, q_2)$ generated by*

$$\tilde{T}(z) = A(z^{-1})D(z) + B(z^{-1})C(z), \tag{3.3}$$

except the northwest $r \times s$ block.

Proof. Consider $N \times N$ Toeplitz matrices L_a and U_d, where L_a is lower triangular with lower bandwidth p_1 generated by $A(z^{-1})$, U_d is upper triangular with upper bandwidth q_2 generated by $D(z)$. One can verify that the product $L_a U_d$ is banded Toeplitz generated by $A(z^{-1})D(z)$, except its northwest $p_1 \times q_2$ block. This result can be easily generalized to the sum of two such products, i.e. $\tilde{T}_N = L_a U_d + L_b U_c$, and the proof is completed. □

Through (3.2), the system $T_N \mathbf{x} = \mathbf{b}$ is transformed to an equivalent system

$$\tilde{T}_N \tilde{\mathbf{x}} = \tilde{\mathbf{b}},$$

where $\mathbf{x} = U_d \tilde{\mathbf{x}}$ and $\tilde{\mathbf{b}} = L_b \mathbf{b}$. We then use the procedure described in §2.1 to construct the MPLU preconditioner $\tilde{F}_N$ for $\tilde{T}_N$ as if it were an exact banded Toeplitz. The following theorem characterizes the spectral properties of $\tilde{F}_N^{-1}\tilde{T}_N$.

THEOREM 3. *Let T_N be the $N \times N$ rational Toeplitz matrix generated by $T(z)$ in (3.1), and $\tilde{F}_N$ the MPLU preconditioner constructed with respect to $\tilde{T}(z)$ in (3.3). In addition,* $r = \max(p_1, q_1)$, $s = \max(p_2, q_2)$ *and w denotes the number of roots of $\tilde{T}(z)$ outside the unit circle. Then, when $r + s = d < N$, there are at most α_F eigenvalues of $\tilde{F}_N^{-1}\tilde{T}_N$ not equal to* 1, *where*

$$\alpha_F = \begin{cases} \min(r, s), & s = w, \\ \min(r, 2w - s), & s < w, \\ \min(d - w, 2s - w), & s > w. \end{cases}$$

Proof. By Lemma 3, $\tilde{T}_N$ is a banded Toeplitz matrix with generating function $\tilde{T}(z)$ except the northwest $r \times s$ block. The $\tilde{F}_N$ is a banded Toeplitz matrix with generating function $\tilde{T}(z)$ except the blocks described in Lemma 2. Define $\Delta\tilde{T}_N = \tilde{F}_N - \tilde{T}_N$. We can use arguments similar to those in proving Theorem 2 to determine the bound of the rank of $\Delta\tilde{T}_N$ and, hence, the number of eigenvalues of $\tilde{F}_N^{-1}\tilde{T}_N$ not equal to 1. □

Since $\tilde{F}_N^{-1}\tilde{T}_N$ has only at most $\alpha_F + 1$ distinct eigenvalues, appropriate preconditioned iterative methods converge in at most $\alpha_F + 1$ iterations with exact arithmetic (see Test Problems 2 and 5 in §5). Note that $\tilde{F}_N$ is a preconditioner for $\tilde{T}_N$ rather than T_N. However, since $\tilde{T}_N$ is related to T_N via (3.2), the equivalent preconditioner for matrix T_N is

$$F_N = L_b^{-1} \tilde{F}_N U_d^{-1} = L_b^{-1} E_N^{\tilde{s}-\tilde{w}} \tilde{L}_N \tilde{U}_N U_d^{-1},$$

where $\tilde{s}$, $\tilde{w}$, $\tilde{L}_N$ and $\tilde{U}_N$ are obtained with respect to $\tilde{T}(z)$ $(= A(z^{-1})D(z) + B(z^{-1})C(z))$. Thus, the preconditioning step can be implemented as

$$F_N^{-1}\mathbf{r} = U_d \tilde{U}_N^{-1} \tilde{L}_N^{-1} E_N^{\tilde{w}-\tilde{s}} L_b \mathbf{r},$$

for arbitrary $\mathbf{r}$ with $O(N)$ operations.

3.2. Nonrational Toeplitz. When T_N is generated by a nonrational function $T(z)$, we use the *Padé* approximation [4], [14], to approximate $T(z)$ with a certain rational function

$$T'(z) = \frac{A'(z^{-1})}{B'(z^{-1})} + \frac{C'(z)}{D'(z)},$$

where $A'(z)$, $B'(z)$, $C'(z)$ and $D'(z)$ are polynomials in z with orders p_1, q_1, p_2 and q_2, respectively. The coefficients of $A'(z)$, $B'(z)$, $C'(z)$ and $D'(z)$ are chosen such that

$$\begin{aligned} T_+(z^{-1})B(z^{-1}) - A(z^{-1}) &= O(z^{-(p1+q1+1)}), \\ T_-(z)D(z) - C(z) &= O(z^{p2+q2+1}), \end{aligned}$$

where

$$\begin{aligned} T_+(z^{-1}) &= ct_0 + \sum_{n=1}^{\infty} t_n z^{-n}, \\ T_-(z) &= (1-c)t_0 + \sum_{n=1}^{\infty} t_{-n} z^n, \end{aligned}$$

with given c. We then construct the preconditioner $\tilde{F}'_N$ with respect to $\tilde{T}'(z)$ $(= A'(z^{-1})D'(z) + B'(z^{-1})C'(z))$ or, equivalently, use F'_N $(= (L'_b)^{-1}\tilde{F}'_N(U'_d)^{-1})$ as preconditioner for T_N. Since $T(z) \neq T'(z)$, the eigenvalues of $(F'_N)^{-1}T_N$ are not repeated at but clustered around 1 (see Test Problems 3 and 6 in §5).

4. Comparison of factorization and circulant preconditioners. Various preconditioners in circulant form have been proposed for symmetric Toeplitz matrices [6], [8], [17], [19], [29]. All these preconditioners can be inverted effectively via fast transform algorithms with $O(N \log N)$ operations. This preconditioning technique can be easily generalized to nonsymmetric Toeplitz matrices. In the following, we discuss the generalization of the preconditioner $K_{1,N}$ [19] proposed by the authors to the nonsymmetric case.

Let T_N be an $N \times N$ Toeplitz matrix,

$$T_N = \begin{bmatrix} t_0 & t_{-1} & \cdot & t_{-(N-2)} & t_{-(N-1)} \\ t_1 & t_0 & t_{-1} & \cdot & t_{-(N-2)} \\ \cdot & t_1 & t_0 & \cdot & \cdot \\ t_{N-2} & \cdot & \cdot & \cdot & t_{-1} \\ t_{N-1} & t_{N-2} & \cdot & t_1 & t_0 \end{bmatrix}.$$

We define a $2N \times 2N$ circulant matrix using elements of T_N as

$$R_{2N} = \begin{bmatrix} T_N & \Delta T_N \\ \Delta T_N & T_N \end{bmatrix}, \tag{4.1}$$

where

$$\Delta T_N = \begin{bmatrix} t_N & t_{N-1} & \cdot & t_2 & t_1 \\ t_{-(N-1)} & t_N & t_{N-1} & \cdot & t_2 \\ \cdot & t_{-(N-1)} & t_N & \cdot & \cdot \\ t_{-2} & \cdot & \cdot & \cdot & t_{N-1} \\ t_{-1} & t_{-2} & \cdot & t_{-(N-1)} & t_N \end{bmatrix}.$$

Since the augmented circulant system

$$\begin{bmatrix} T_N & \Delta T_N \\ \Delta T_N & T_N \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{x} \end{bmatrix} = \begin{bmatrix} \mathbf{b} \\ \mathbf{b} \end{bmatrix},$$

is equivalent to

$$(T_N + \Delta T_N)\mathbf{x} = \mathbf{b},$$

the $(T_N + \Delta T_N)^{-1}\mathbf{b}$ can be computed efficiently via FFT so that

$$K_N = T_N + \Delta T_N$$

can be used as a preconditioner for T_N. Note that K_N is also a circulant matrix.

When T_N is a symmetric Toeplitz matrix generated by a positive function in the Wiener class, it can be proved [7], [19] that the eigenvalues of the circulant-preconditioned Toeplitz are clustered around 1 except a finite number of outliers. When T_N is additionally rational of order (p, q), the eigenvalues of $K_N^{-1}T_N$ are clustered between $(1-\epsilon_K, 1+\epsilon_K)$ except $\alpha_K = 2\max(p,q)$ outliers, where $\epsilon_K = O(|t_N|)$ [21]. A special case of Theorem 3 is that when T_N is a symmetric rational matrix, we have $r = s = w$ and $\alpha_F = \max(p,q) = \frac{1}{2}\alpha_K$. By generalizing the proofs in [21], we are able to obtain the following more general result applicable to the nonsymmetric case.

THEOREM 4. *Let T_N be a rational Toeplitz matrix satisfying (2.1) and (2.2) and generated by $T(z)$ of order (p_1, q_1, p_2, q_2) as given by (3.1). Then, the eigenvalues of $K_N^{-1}T_N$ are clustered between $(1-\epsilon_K, 1+\epsilon_K)$ except at most $\alpha_K = \max(p1, q1) + \max(p2, q2)$ outliers, where $\epsilon_K = O(|t_N| + |t_{-N}|)$.*

Proof. See [22] □

The spectral properties of $F_N^{-1}T_N$ and $K_N^{-1}T_N$ for rational T_N are compared as follows. One difference is that the eigenvalues except outliers are exactly repeated at 1 for $F_N^{-1}T_N$ but only clustered around 1 for $K_N^{-1}T_N$, i.e. $\epsilon_K \geq \epsilon_F = 0$. Another difference is the number of outliers which are by definition the eigenvalues not converging to 1 for asymptotically large N. Comparing Theorems 3 and 4, we know that $\alpha_K \geq \alpha_F$. The CGS (or GMRES) method with preconditioners F_N and K_N

converges asymptotically in at most $\alpha_F + 1$ and $\alpha_K + 1$ iterations, respectively. Base on the above theoretical study, we can conclude that the MPLU preconditioner F_N should outperform the circulant preconditioner K_N for rational T_N. It is observed that F_N performs better than K_N for both rational and nonrational T_N in our numerical experiments.

The preconditioning step $F_N^{-1}\mathbf{r}$ can be accomplished with $O(N)$ operations by permutation, forward- and back-substitution, since F_N is a product of a shift matrix, lower- and upper-triangular banded Toeplitz matrices. In comparison, the preconditioning step $K_N^{-1}\mathbf{r}$ requires $O(N \log N)$ operations via FFT. Hence, in terms of computational complexity per iteration, preconditioner F_N is slightly better. However, note that $F_N^{-1}\mathbf{r}$ has to be implemented sequentially whereas $K_N^{-1}\mathbf{r}$ can be easily parallelized via the parallelism provided by FFT.

5. Numerical results. Our numerical experiments include both symmetric positive-definite (SPD) and nonsymmetric Toeplitz with banded, rational and nonrational generating sequences. The SPD problems are solved by the PCG method. For nonsymmetric systems, there exist numerous iterative algorithms for their solution [2], [26]. As suggested by [24], we applied the preconditioned version of three iterative methods, i.e. CGNR, GMRES and CGS, for our numerical experiments. We observed that GMRES and CGS converge faster than CGNR and that CGS outperform GMRES by a factor of 1 to 2 for all test problems. Since our focus is on the preconditioners rather than the iterative methods, only results solved by the CGS iteration are reported. All experiments are performed with $N = 32$, $\mathbf{b} = (1, \cdots, 1)^T$, and zero initial guess.

Test Problem 1: symmetric banded Toeplitz.

The generating function is

$$T(z) = z^{-4} + 3z^{-3} + 4z^{-2} + 7z^{-1} + 11 + 7z + 4z^2 + 3z^3 + z^4.$$

The convergence history of the PCG method with preconditioners F_N and K_N is plotted in Figure 1. We clearly see that the 2-norm of the residual is significantly reduced in 4 iterations for both F_N and K_N and that F_N performs slightly better than K_N. We want to point out that $F_N^{-1}T_N$ and $K_N^{-1}T_N$ have 4 and 8 outliers, respectively. However, for this test problem, the outliers of $K_N^{-1}T_N$ are related in pairs and it takes only $\frac{1}{2}\alpha_K$ iterations to eliminate these α_K outliers. A similar kind of convergence behavior for $K_N^{-1}T_N$ was reported in [19]. In general, preconditioners F_N and K_N have a similar performance for symmetric banded Toeplitz matrices.

Test Problem 2: symmetric rational Toeplitz.

The generating function is

$$T(z) = \frac{(1-0.2z^{-1})(1+0.3z^{-1})(1-0.5z^{-1})}{(1-0.3z^{-1})(1+0.5z^{-1})(1-0.7z^{-1})} + \frac{(1-0.2z)(1+0.3z)(1-0.5z)}{(1-0.3z)(1+0.5z)(1-0.7z)}.$$

Since T_N is symmetric ($r = s = w = 3$), the eigenvalues of $F_N^{-1}T_N$ are repeated at 1 except 3 outliers (see Theorem 3), and the eigenvalues of $K_N^{-1}T_N$ are clustered around 1 except 6 outliers (see Theorem 4). The convergence history of the PCG method with preconditioners F_N and K_N is plotted in Figure 2. Since $F_N^{-1}T_N$ has 4 distinct eigenvalues, the PCG method with preconditioner F_N converges in 4 iterations. However, although $K_N^{-1}T_N$ has 6 outliers, it only requires 3 iterations to eliminate the outliers. The convergence rate after the first 3 iterations depends on the clustering radius ϵ_K. It is clear that preconditioner F_N performs better than preconditioner K_N.

Test Problem 3: symmetric nonrational Toeplitz.

The generating sequence is

$$t_n = \begin{cases} 2, & n = 0, \\ 1/(1+|n|), & n \neq 0, \end{cases}$$

and the corresponding generating function is

$$T(z) = T_+(z^{-1}) + T_+(z),$$

where

$$T_+(z^{-1}) = \sum_{n=0}^{\infty} \frac{z^{-n}}{1+n}.$$

Consider the *Padé* approximant of order (p,q), i.e. $A'_p(z^{-1})/B'_q(z^{-1})$, to $T_+(z^{-1})$. Preconditioner $F_{p,q,N}$ are then constructed with respect to

$$T'_{p,q}(z) = \frac{A'_p(z^{-1})}{B'_q(z^{-1})} + \frac{A'_p(z)}{B'_q(z)}.$$

In our experiment, (p,q) is chosen to be $(3,3)$ and $(4,4)$. The convergence history of the PCG method with preconditioners $F_{3,3,N}$, $F_{4,4,N}$ and K_N is plotted in Figure 3. All these preconditioners converge in a similar rate.

Test Problem 4: nonsymmetric banded Toeplitz.

The generating function is

$$T(z) = -z^{-3} + 2z^{-2} + 9z^{-1} + 4 - 2z - 3z^2 + z^3,$$

so that T_N is a banded Toeplitz with lower bandwidth $r = 3$ and upper bandwidth $s = 3$. Note also that $T(z)$ has $w = 4$ roots outside the unit circle. The convergence history of the CGS method with preconditioners F_N and K_N is plotted in Figure 4. According to Theorem 2, $F_N^{-1}T_N$ has 3 eigenvalues different from 1 and, consequently, the CGS method with preconditioner F_N converges in 4 iterations. According to Theorem 4, $K_N^{-1}T_N$ has 6 eigenvalues not equal to 1 so that the CGS method with preconditioner K_N converges in 7 iterations. We see clearly that the CGS method with preconditioner F_N converges faster.

Test Problem 5: nonsymmetric rational Toeplitz.

The generating function is

$$T(z) = \frac{(1-0.2z^{-1})(1+0.3z^{-1})(1-0.5z^{-1})}{(1-0.7z^{-1})(1+0.5z^{-1})} + \frac{1+2z}{(1.5-z)(2+z)(2-z)}.$$

We can transform T_N into a banded matrix with $r = s = w = 3$. From Theorems 3 and 4, we know that $F_N^{-1}T_N$ has only 3 eigenvalues not equal to 1 and the eigenvalues of $K_N^{-1}T_N$ are clustered around 1 except 6 outliers. The convergence history of the CGS method with preconditioners F_N and K_N is plotted in Figure 5. The CGS method with preconditioner F_N performs better.

Test Problem 6: nonsymmetric nonrational Toeplitz.

Let T_N be a nonsymmetric Toeplitz matrix with generating sequence

$$t_n = \begin{cases} 1/\log(2-n), & n \le -1, \\ 1/\log(2-n) + 1/(1+n), & n = 0, \\ 1/(1+n), & n \ge 1. \end{cases}$$

The corresponding causal and anti-causal generating functions can be written as

$$\begin{aligned} T_+(z^{-1}) &= \sum_{n=0}^{\infty} \frac{z^{-n}}{1+n}, \\ T_-(z) &= \sum_{n=0}^{\infty} \frac{z^n}{\log(2+n)}. \end{aligned}$$

Let the *Padé* approximants of order (p,q), to $T_+(z^{-1})$ and $T_-(z)$ be $A'_p(z^{-1})/B'_q(z^{-1})$ and $C'_p(z)/D'_q(z)$, respectively. We construct preconditioner $F'_{p,q,N}$ for

$$T'_{p,q}(z) = \frac{A'_p(z^{-1})}{B'_p(z^{-1})} + \frac{C'_q(z)}{D'_q(z)},$$

N	$F_{2,2,N}$	$F_{3,3,N}$	$F_{4,4,N}$	K_N
32	6	5	5	11
64	8	7	6	11
128	9	8	7	13

TABLE 2
The numbers of iterations required for the CGS method.

with $p = q = 2, 3, 4$. The convergence history of the CGS method with preconditioners $F_{p,q,N}$ and K_N is plotted in Figure 6. In order to understand the asymptotical behavior of the preconditioned CGS method, we also perform experiments for this test problem with $N = 64, 128$. The numbers of iterations required with preconditioners $F_{p,q,N}$, $p = q = 2, 3, 4$, and K_N satisfying $||\mathbf{b} - T_N\mathbf{x}||_2 < 10^{-15}$ are summarized in Table 2 for different N. Note that the numbers of iterations required for all preconditioners increase slightly as N becomes larger. However, preconditioners $F_{p,q,N}$, $p = q = 2, 3, 4$, perform better than preconditioner K_N.

6. Conclusion. In this paper, we applied the minimum-phase factorization technique to Toeplitz generating functions and obtain a new Toeplitz preconditioner called the MPLU preconditioner. This preconditioning technique is applicable to both banded and full Toeplitz matrices. We characterized the spectral properties of the MPLU preconditioned Toeplitz matrices and showed that most of their eigenvalues are repeated exactly at unity for rational Toeplitz. Thus, an $N \times N$ rational Toeplitz system can be solved by preconditioned iterative methods with $O(N)$ complexity. We also demonstrate the superior performance of the MPLU preconditioner over another Toeplitz preconditioner in circulant form with several numerical examples, including both rational and nonrational cases.

Although our discussion on the MPLU factorization preconditioner has primarily focused on real nonsymmetric Toeplitz systems, its application to complex nonhermitian Toeplitz systems can be generalized in a straightforward way. However, the MPLU factorization preconditioning technique cannot be easily extended to higher-dimensional Toeplitz systems such as block Toeplitz matrices. This is due to the absence of the fundamental theorem of algebra for multivariate polynomials. In contrast, higher-dimensional Toeplitz matrices can be preconditioned with higher-dimensional circulant matrices. See [20] for the two-dimensional case. Another limitation of the MPLU preconditioner is that it is not as easily parallelizable as the preconditioners in circulant form.

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Figure Captions

Figure 1: The convergence history of the PCG method with preconditioners F_N and K_N for Test Problem 1.

Figure 2: The convergence history of the PCG method with preconditioners F_N and K_N for Test Problem 2.

Figure 3: The convergence history of the PCG method with preconditioners $F_{3,3,N}$, $F_{4,4,N}$ and K_N for Test Problem 3.

Figure 4: The convergence history of the CGS method with preconditioners F_N and K_N for Test Problem 4.

Figure 5: The convergence history of the CGS method with preconditioners F_N and K_N for Test Problem 5.

Figure 6: The convergence history of the CGS method with preconditioners $P_{p,q,N}$, $p = q = 2, 3, 4$, and K_N for Test Problem 6.

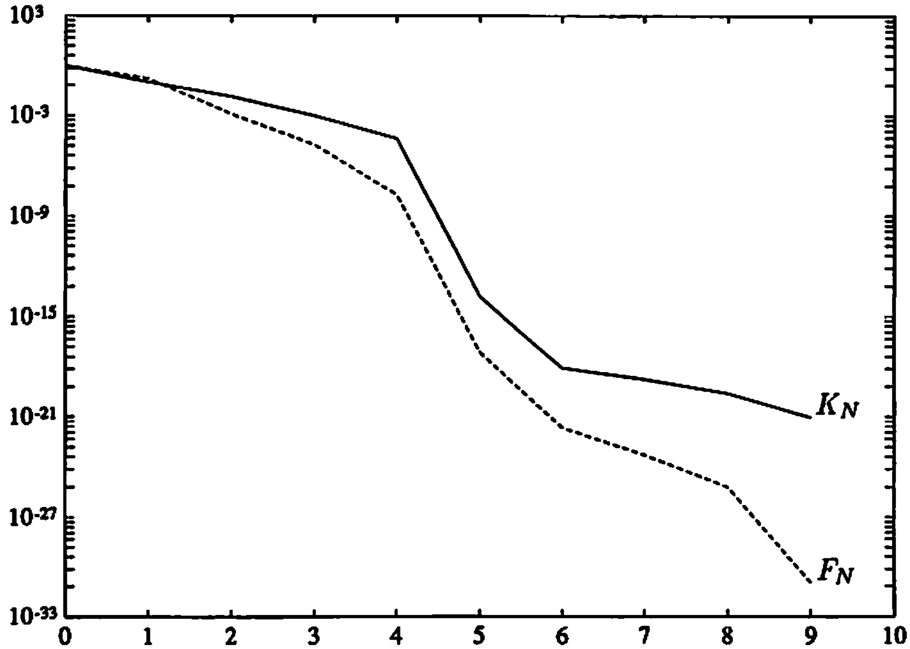

FIG. 1. *The convergence history of the PCG method with preconditioners F_N and K_N for Test Problem 1.*

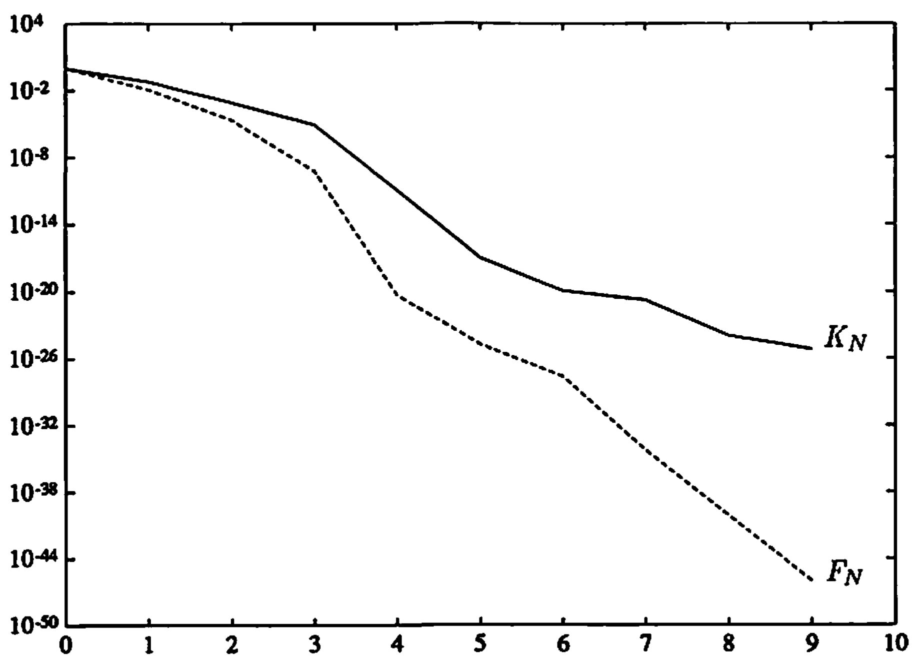

FIG. 2. *The convergence history of the PCG method with preconditioners F_N and K_N for Test Problem 2.*

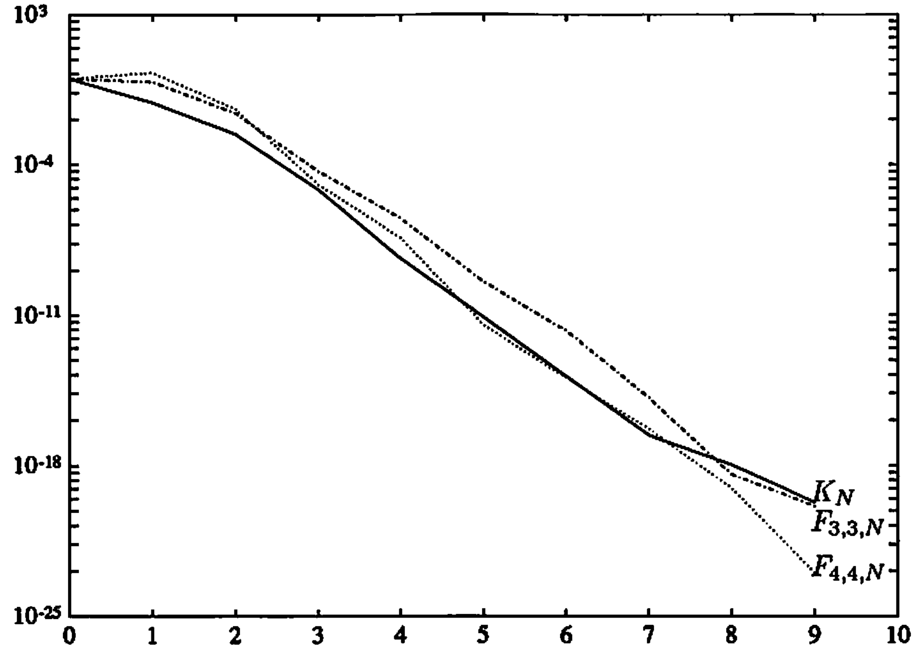

FIG. 3. *The convergence history of the PCG method with preconditioners $F_{3,3,N}$, $F_{4,4,N}$ and K_N for Test Problem 3.*

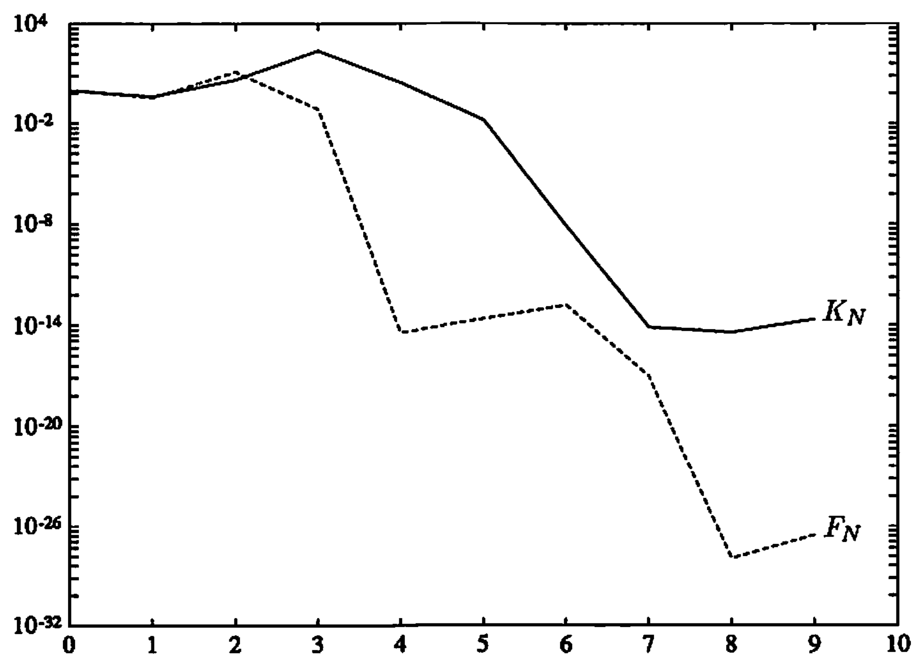

FIG. 4. *The convergence history of the CGS method with preconditioners F_N and K_N for Test Problem 4.*

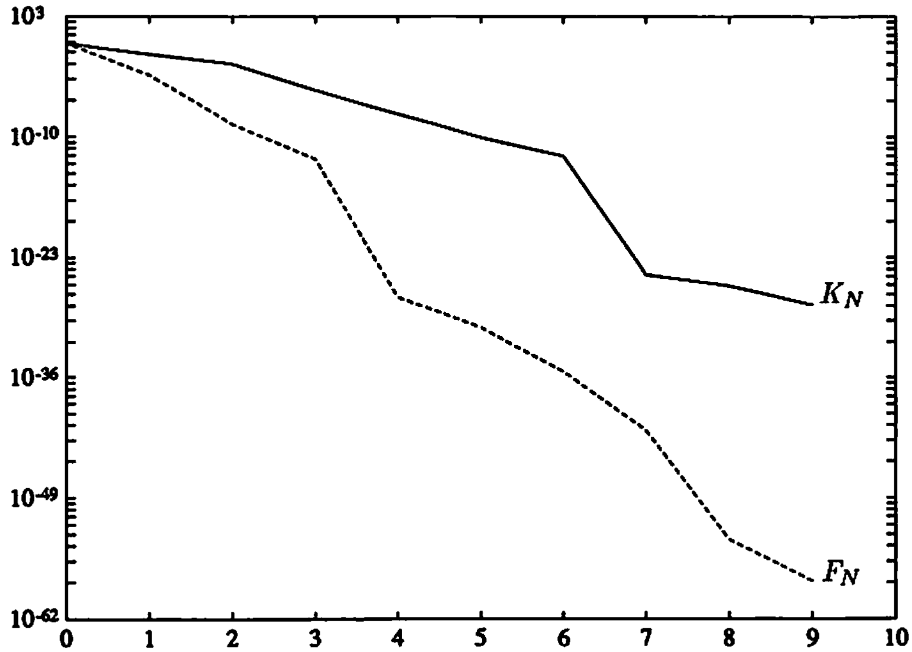

FIG. 5. *The convergence history of the CGS method with preconditioners F_N and K_N for Test Problem 5.*

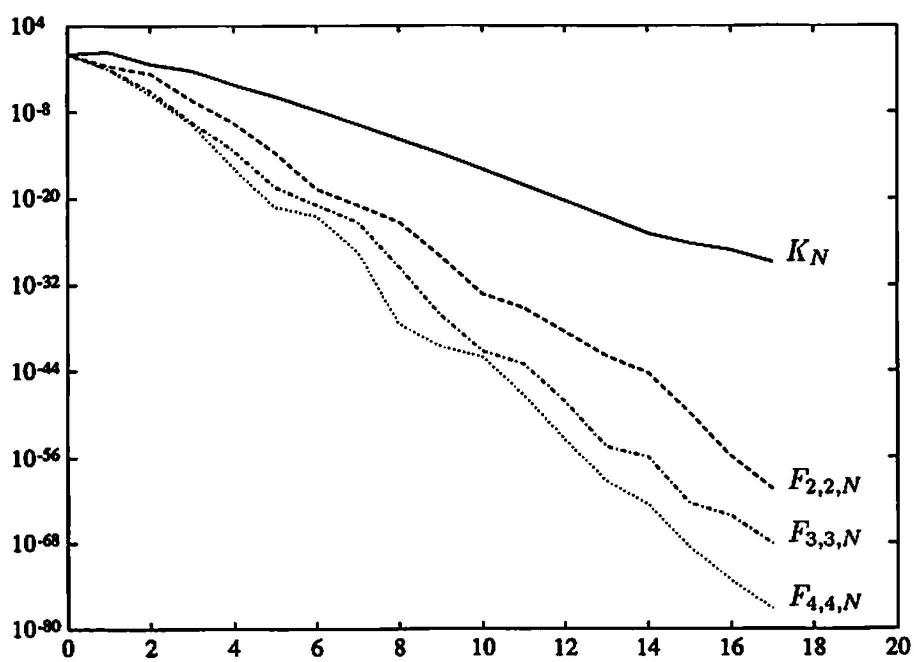

FIG. 6. *The convergence history of the CGS method with preconditioners $P_{p,q,N}$, $p = q = 2,3,4$, and K_N for Test Problem 6.*