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Comparison Between Correlation-Based and Cumulant-Based Approaches to the Harmonic Retrieval and Related Problems

by

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Abstract

In signal processing, we frequently encounter the problem of estimating the number of harmonics, frequencies, and amplitudes in a sum of sinusoids. The observed signals are usually corrupted by spatially and/or temporally colored noise with unknown power spectral density. It has been shown by Swami and Mendel that a cumulant-based approach to this problem is very effective. In this report, we compare the use of biased and unbiased, segmented and unsegmented estimators for both correlation and 1-D diagonal slice of the fourth-order cumulant function. We suggest using accumulated singular values to determine the number of harmonics. We compare correlation-based and cumulant-based methods for determining the number of harmonics when the amplitude of one harmonic decreases and when the frequency of one harmonic approaches the other for the case of two sinusoids measured in colored Gaussian noise. We also compare the performance of the Pisarenko, MUSIC, and minimum-norm algorithms for frequency estimation, and the performance of least square (LS), total least square (TLS), and constrained total least square (CTLS) for amplitude estimation using either correlations or cumulants. Our studies: (1) provide further support for using cumulants over correlations when measurement noise is colored and Gaussian; (2) demonstrate that one should use unbiased unsegmented correlation or cumulant estimators; (3) indicate that high-resolution results are best obtained using cumulant-based MUSIC or minimum-norm algorithms; and (4) show that LS estimates of amplitudes suffice.

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- 49 Accumulative singular values of cumulants using 4×64 (256) one realization when the fixed amplitude sinusoid at 0.1 is unity and the amplitude of the sinusoid at 0.14 is : (a) 1.02 (b) 1.015* (c) 1.01 (d) 1.005 70
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1 Introduction

The estimation of the number of harmonics and the frequencies and amplitudes of harmonics from noisy measurements is frequently encountered in several signal processing applications, such as in estimating the direction of arrival of narrow-band source signals with linear arrays, and in the retrieval of harmonics in noise problem. In this report, we are concerned with real-valued signals represented as

$$y(n) = \sum_{k=1}^p a_k \cos(n\omega_k + \phi_k) + w(n) = x(n) + w(n) \quad (1)$$

where the ϕ_k 's denote random phases which are i.i.d. and uniformly distributed over $[0, 2\pi]$, the ω_k 's are unknown deterministic frequencies and the a_k 's are unknown deterministic amplitudes. The additive noise $w(n)$ is assumed to be white or colored Gaussian noise with unknown spectral density. We will estimate the number of signals p , the angular frequencies ω_k 's, and the amplitudes a_k 's.

The autocorrelation, $r_x(\tau)$, and fourth-order cumulant, $c_{4x}(\tau_1, \tau_2, \tau_3)$, of $x(n)$ are represented as [4,5]

$$r_x(\tau) = \frac{1}{2} \sum_{k=1}^p a_k^2 \cos(\omega_k \tau) \quad (2)$$

$$\begin{aligned} c_{4x}(\tau_1, \tau_2, \tau_3) = -\frac{1}{8} \sum_{k=1}^p a_k^4 & \{ \cos\omega_k(\tau_1 - \tau_2 - \tau_3) + \cos\omega_k(\tau_2 - \tau_3 - \tau_1) \\ & + \cos\omega_k(\tau_3 - \tau_1 - \tau_2) \} \end{aligned} \quad (3)$$

The one-dimensional diagonal slice of the fourth order cumulant is given by

$$c_{4x}(\tau) = c_{4x}(\tau, \tau, \tau) = -\frac{3}{8} \sum_{k=1}^p a_k^4 \cos(\omega_k \tau) \quad (4)$$

For frequency estimation, linear prediction approaches are well explained in [4]. We describe important results from [4], here. A harmonic signal can be expressed as the output of a self-driving AR (AutoRegressive) model, i.e., let

$$x(n) = \sum_{k=1}^p a_k \exp(j\omega_k n)$$

Then, $x(n)$ satisfies the AR(p) model

$$\sum_{k=0}^p a_k x(n-k) = 0$$

where $a_0 = 1$ and the polynomial $A(z) = \sum_{k=0}^p a_k z^{-k}$ has roots at $z = e^{j\omega_k}$, $k = 1, \dots, p$. For p real sinusoids, an AR($2p$) model, whose transfer function has roots at $z = e^{\pm j\omega_k}$ is required

[9]. Using the above facts, we obtain the following basic equation for the correlation-based high-resolution eigenmethods [4]

$$\sum_{m=0}^p a(m) r_y(k-m) = \sum_{m=0}^p a(m) r_w(k-m) \quad (5)$$

For the cumulant-based methods, the basic equation is

$$\sum_{m=0}^p a(m) c_{4y}(m_1, m_2, k-m) = \sum_{m=0}^p a(m) c_{4w}(m_1, m_2, k-m) \quad (6)$$

Note that when the additive noise is Gaussian, the right-hand side of Eq. (6) equals zero, whereas the right-hand side of (5) is not zero.

Using (5) for correlations and (6) for cumulants, we can estimate the number of frequencies and frequencies of harmonics. After estimating the frequencies, we estimate the amplitudes using (2) for correlations and (4) for cumulants.

In Sections 2 and 3, we discuss estimators for correlation and the 1-D diagonal-slice of the fourth-order cumulants, respectively. The issue we focus our attention on is whether or not segmentation of the data is helpful. In Section 4, we suggest a method to estimate the number of harmonics in the signal, using accumulated singular values. In Sections 5 and 6, methods to estimate the frequencies and amplitudes are briefly described, respectively. Extensive simulation results are given in Section 7.

2 Biased and Unbiased Correlation Estimators, With or Without Data Segmentation

For estimation of the number of harmonics and their frequencies and amplitudes, first we must estimate the sample correlation and fourth-order cumulant functions. The better these estimates, the better the results we can expect. It is, therefore, important to choose the best estimators. In this section and the next, estimators for correlation and the 1-D diagonal slice of the fourth-order cumulant are studied.

Consider $\{x(n), n = 1, 2, \dots, MN\}$, with $E\{x(n)\} = 0$, $r_x(k) = E\{x(n)x(n+k)\}$, and $c_{3x}(k_1, k_2) = E\{x(n)x(n+k_1)x(n+k_2)\}$. Let $c_{4x}(k_1, k_2, k_3)$ be the fourth-order cumulant of $\{x(n)\}$, i.e.,

$$\begin{aligned} c_{4x}(k_1, k_2, k_3) &= \text{cum}(x(n), x(n+k_1), x(n+k_2), x(n+k_3)) \\ &= E\{x(n)x(n+k_1)x(n+k_2)x(n+k_3)\} - r_x(k_1)r_x(k_3-k_2) \\ &\quad - r_x(k_2)r_x(k_3-k_1) - r_x(k_3)r_x(k_2-k_1) \end{aligned}$$

We assume that the data can be divided into M segments, each segment having N samples, such that

$$x_i(n) = x(N \times (i-1) + n) \quad i = 1, 2, \dots, M \quad n = 1, 2, \dots, N$$

where $x_i(n)$ denotes the n -th sample in the i -th segment. Note that we can either assume that the data were obtained from M independent realizations, each with N samples, or, from MN samples from one realization.

Although the estimators for correlation functions are well known [3], the estimators usually use all the data from one realization without segmentation. In several recent papers [5,8,14], the concept of data segmentation for estimators was introduced. Data from one realization was divided into several segments, correlation functions were estimated for each segment, and then they were averaged to reduce the variance of estimators. In this section, we analyze biased and unbiased correlation estimators with or without segmentation. Derivations for mean and covariance functions of each estimator are given in the Appendix.

2.1 Biased Correlation Estimators

2.1.1 Estimator with Segmentation

Let $r_1^b(k)$ be the biased segmented correlation estimator,

$$r_1^b(k) = \frac{1}{M} \sum_{i=1}^M \tilde{r}_i^b(k)$$

where $\tilde{r}_i^b(k)$ denotes the estimated correlation function in the i -th segment, i.e.,

$$\tilde{r}_i^b(k) = \frac{1}{N} \sum_{n=1}^{N-|k|} x_i(n)x_i(n+k) \quad i = 1, 2, \dots, M$$

We can rewrite the estimator, $r_1^b(k)$, as

$$r_1^b(k) = \frac{1}{MN} \sum_{i=1}^M \sum_{n=1}^{N-|k|} x_i(n)x_i(n+k);$$

then(all derivations are given in the Appendix),

$$\mathbb{E}\{r_1^b(k)\} = \frac{N - |k|}{N} r(k) \tag{7}$$

and

$$\begin{aligned} \text{Cov}(r_1^b(k), r_1^b(k+v)) &= \frac{1}{MN^2} \sum_{l=-(N-k)+1}^{(N-k-v-1)} (N - \eta_N(l) - k - v) \{r(l)r(l+v) \\ &\quad + r(l+k+v)r(l-k) + c_{4x}(l, k, l+k+v)\} \\ &\quad + \frac{1}{(MN)^2} \sum_{i,j=1}^M \sum_{i \neq j}^M \sum_{n=1}^{(N-k)} \sum_{m=1}^{(N-k-v)} \mathbb{E}\{x_i(n)x_i(n+k)x_j(m)x_j(m+k+v)\} \\ &\quad - \frac{(M-1)(N-k)(N-k-v)}{MN^2} r(k)r(k+v) \end{aligned}$$

where $k \geq 0$, $k + v \geq 0$, and

$$\eta_N(l) = \begin{cases} l, & l > 0 \\ 0, & -v \leq l \leq 0 \\ -l - v, & -(N - k) + 1 \leq l \leq -v \end{cases} \quad (8)$$

When each segment of data is independent, then

$$\begin{aligned} \text{Cov} \left(r_1^b(k), r_1^b(k + v) \right) &= \frac{1}{MN^2} \sum_{l=-(N-k)+1}^{(N-k-v-1)} (N - \eta_N(l) - k - v) \{ r(l)r(l+v) \\ &\quad + r(l+k+v)r(l-k) + c_{4x}(l, k, l+k+v) \} \end{aligned} \quad (9)$$

When each segment is a part of a single realization, then

$$\begin{aligned} \text{Cov} \left(r_1^b(k), r_1^b(k + v) \right) &= \frac{1}{MN^2} \sum_{l=-(N-k)+1}^{(N-k-v-1)} \{ N - \eta_N(l) - k - v \} \{ r(l)r(l+v) \\ &\quad + r(l+k+v)r(l-k) + c_{4x}(l, k, l+k+v) \} \\ &+ \frac{1}{(MN)^2} \sum_{q=-M+1, q \neq 0}^{(M-1)} \sum_{p=-(N-k)+1}^{(N-k-v-1)} (N - \eta_N(p) - k - v) (M - |q|) \{ r(Nq+p)r(Nq+p+v) \\ &\quad + r(Nq+p+k+v)r(Nq+p-k) + c_{4x}(k, Nq+p, Nq+p+k+v) \} \end{aligned} \quad (10)$$

Note that $\mathbf{E}\{r_1^b(k)\}$ is independent of the number of segments, M . Since N is a finite constant, the estimator is not asymptotically unbiased as the number of data, and consequently M , goes to infinity.

2.1.2 Estimator without Segmentation

Now consider the estimator $r_2^b(k)$ that uses all MN data without segmentation,

$$r_2^b(k) = \frac{1}{MN} \sum_{n=1}^{MN-|k|} x(n)x(n+k)$$

Then,

$$\mathbf{E}\{r_2^b(k)\} = \frac{MN - |k|}{MN} r(k) \quad (11)$$

and

$$\begin{aligned} \text{Cov} \left(r_2^b(k), r_2^b(k + v) \right) &= \frac{1}{(MN)^2} \sum_{l=-(MN-k)+1}^{MN-k-v-1} (MN - \eta_{MN}(l) - k - v) \{ r(l)r(l+v) \\ &\quad + r(l+k+v)r(l-k) + c_{4x}(l, k, l+k+v) \} \end{aligned} \quad (12)$$

where $k \geq 0$, $k + v \geq 0$, and the function $\eta_{MN}(l)$ is defined by

$$\eta_{MN}(l) = \begin{cases} l, & l > 0 \\ 0, & -v \leq l \leq 0 \\ -l - v, & -(MN - k) + 1 \leq l \leq -v \end{cases} \quad (13)$$

Note that $r_2^b(k)$ is asymptotically unbiased.

Since, in general, $N \ll MN$, the biased unsegmented estimates have much smaller bias than those using segmentation, especially, for correlations at large lags.

2.2 Unbiased Correlation Estimators

2.2.1 Estimator with Segmentation

Let $r_1^u(k)$ be the unbiased segmented correlation estimator, i.e.,

$$r_1^u(k) = \frac{1}{M} \sum_{i=1}^M \tilde{r}_i^u(k)$$

where $\tilde{r}_i^u(k)$ denotes the estimated correlation function in the i -th segment, i.e.,

$$\tilde{r}_i^u(k) = \frac{1}{N - |k|} \sum_{n=1}^{N-|k|} x_i(n)x_i(n+k) \quad i = 1, 2, \dots, M$$

We can represent the estimator, $r_1^u(k)$, as

$$r_1^u(k) = \frac{1}{M(N - |k|)} \sum_{i=1}^M \sum_{n=1}^{N-|k|} x_i(n)x_i(n+k);$$

then,

$$\mathbb{E}\{ r_1^u(k) \} = r(k) \quad (14)$$

and

$$\begin{aligned} \text{Cov}(r_1^u(k), r_1^u(k+v)) &= \frac{1}{M(N-k)(N-k-v)} \sum_{l=-(N-k)+1}^{(N-k-v-1)} (N - \eta_N(l) - k - v) \cdot \\ &\quad \{ r(l)r(l+v) + r(l+k+v)r(l-k) + c_{4x}(l, k, l+k+v) \} \\ &+ \frac{1}{M^2(N-k)(N-k-v)} \sum_{i,j=1}^M \sum_{i \neq j}^M \sum_{n=1}^{(N-k)} \sum_{m=1}^{(N-k-v)} \mathbb{E}\{ x_i(n)x_i(n+k)x_j(m)x_j(m+k+v) \} \\ &- \frac{(M-1)}{M} r(k)r(k+v) \end{aligned}$$

where $k \geq 0$, $k + v \geq 0$, and the function $\eta_N(l)$ is defined in (8). When each segment of data is independent, then

$$\begin{aligned} \text{Cov}(r_1^u(k), r_1^u(k+v)) &= \frac{1}{M(N-k)(N-k-v)} \sum_{l=-(N-k)+1}^{(N-k-v-1)} (N - \eta_N(l) - k - v) \\ &\quad \{r(l)r(l+v) + r(l+k+v)r(l-k) + c_{4x}(l, k, l+k+v)\} \end{aligned} \quad (15)$$

When each segment is not independent, i.e., each segment is a part of a single realization, then

$$\begin{aligned} \text{Cov}(r_1^u(k), r_1^u(k+v)) &= \frac{1}{M(N-k)(N-k-v)} \sum_{l=-(N-k)+1}^{(N-k-v-1)} (N - \eta_N(l) - k - v) \cdot \\ &\quad \{r(l)r(l+v) + r(l+k+v)r(l-k) + c_{4x}(l, k, l+k+v)\} \\ &+ \frac{1}{M^2(N-k)(N-k-v)} \sum_{q=-M+1, q \neq 0}^{(M-1)} \sum_{p=-(N-k)+1}^{(N-k-v-1)} (N - \eta_N(p) - k - v)(M - |q|) \\ &\quad \{r(Nq+p)r(Nq+p+v) + r(Nq+p+k+v)r(Nq+p-k) + c_{4x}(k, Nq+p, Nq+p+k+v)\} \end{aligned} \quad (16)$$

2.2.2 Estimator without Segmentation

Now consider the unbiased unsegmented estimator,

$$r_2^u(k) = \frac{1}{MN - |k|} \sum_{n=k}^{MN - |k|} x(n)x(n+k)$$

Then,

$$\mathbb{E}\{r_2^u(k)\} = r(k) \quad (17)$$

and

$$\begin{aligned} \text{Cov}(r_2^u(k), r_2^u(k+v)) &= \frac{1}{(MN - k)(MN - k - v)} \sum_{l=-(MN-k)+1}^{MN - k - v - 1} (MN - \eta_{MN}(l) - k - v) \\ &\quad \{r(l)r(l+v) + r(l+k+v)r(l-k) + c_{4x}(l, k, l+k+v)\} \end{aligned} \quad (18)$$

where $k \geq 0$, $k + v \geq 0$, and the function $\eta_{MN}(l)$ is defined in (13).

2.3 Conclusions

It is well known that a correlation estimator that uses several independent realizations is very reliable, since doing this reduces the variance of the estimator. Additionally, most time-series analysts prefer to use the biased unsegmented correlation estimator, because the biased estimator leads to a positive semi-definite covariance matrix [3]. When we only have a single realization, the biased segmented estimator gives poorer correlation estimates than both the unbiased segmented estimator and the biased unsegmented estimator.

The biased segmented estimator is not even asymptotically unbiased, which is a serious defect. We, therefore, conclude that in the single realization case, we should use an unsegmented correlation estimator. Data segmentation for a single realization is not advantageous.

3 Estimators of the 1-D Diagonal-Slice of Fourth-Order Cumulants

The estimators for correlation functions have been analyzed in Section 2. Analyses for an unbiased estimator of the third-order cumulant are given in [6,7]. In this section we study the biased and unbiased, segmented or unsegmented estimators of the 1-D diagonal-slice of the fourth-order cumulants. Let $c_{4x}(k)$ be the 1-D diagonal slice of the fourth-order cumulant of $\{x(n)\}$ i.e.,

$$\begin{aligned} c_{4x}(k) &= \text{cum}(x(n), x(n+k), x(n+k), x(n+k)) \\ &= E\{x(n)x(n+k)x(n+k)x(n+k)\} - 3r_x(0)r_x(k) \end{aligned}$$

When we estimate $c_{4x}(k)$, it is assumed that we estimate $E\{x(n)x^3(n+k)\}$, $r_x(0)$, and $r_x(k)$, independently, i.e., we estimate each term of $c_{4x}(k)$.

3.1 Biased Estimators of the 1-D Diagonal-Slice of the Fourth-Order Cumulant

3.1.1 Estimator with Segmentation

Let $d_1^b(k)$ be the biased segmented estimator for the 1-D diagonal-slice of the fourth-order cumulant, i.e.,

$$d_1^b(k) = \frac{1}{M} \sum_{i=1}^M \tilde{d}_i^b(k)$$

where $\tilde{d}_i^b(k)$ denotes the biased cumulant estimate in the i -th segment, i.e.,

$$\tilde{d}_i^b(k) = \frac{1}{N} \sum_{n=1}^{N-|k|} x_i(n)x_i^3(n+k) - 3r_1^b(0)r_1^b(k) \quad i = 1, 2, \dots, M$$

where $r_1^b(0)$ and $r_1^b(k)$ denote the biased segmented correlation estimates (as described in Section 2). Then,

$$E\{d_1^b(k)\} = \frac{N-|k|}{N} (E\{x(n)x^3(n+k)\} - 3r_x(0) \cdot r_x(k)) \quad (19)$$

Note that the estimator for the 1-D diagonal-slice of the fourth-order cumulant is not asymptotically unbiased because N is a constant, and the mean value is independent of the number of segments, M . Covariance formulas for $d_1^b(k)$, as well as the other fourth-order cumulant estimators, are so complicated that we do not present them here. They depend on 8th-order cumulants.

3.1.2 Estimator without Segmentation

Let $d_2^b(k)$ be the biased unsegmented estimator for the 1-D diagonal-slice of the fourth-order cumulant, i.e.,

$$d_2^b(k) = \frac{1}{MN} \sum_{n=1}^{MN-|k|} x(n)x^3(n+k) - 3r_2^b(0)r_2^b(k)$$

where $r_2^b(0)$ and $r_2^b(k)$ denote the biased unsegmented correlation estimates (as described in Section 2). Then

$$E\{d_2^b(k)\} = \frac{MN - |k|}{MN} \left(E\{x(n)x^3(n+k)\} - 3r_x(0) \cdot r_x(k) \right) \quad (20)$$

Note that this estimator is asymptotically unbiased.

Since, in general, $N \ll MN$, the biased unsegmented estimates have much smaller bias than those using segmentation, especially, for cumulants at large lags.

3.2 Unbiased Estimators of the 1-D Diagonal-Slice of the Fourth-Order Cumulant

3.2.1 Estimator with Segmentation

Let $d_1^u(k)$ be the unbiased segmented estimator for the 1-D diagonal-slice of the fourth-order cumulant, i.e.,

$$d_1^u(k) = \frac{1}{M} \sum_{i=1}^M \tilde{d}_i^u(k)$$

where $\tilde{d}_i^u(k)$ denotes the unbiased estimate of the cumulant in the i -th segment, i.e.,

$$\tilde{d}_i^u(k) = \frac{1}{N - |k|} \sum_{n=1}^{N-|k|} x_i(n)x_i^3(n+k) - 3r_1^u(0)r_1^u(k) \quad i = 1, 2, \dots, M$$

where $r_1^u(0)$ and $r_1^u(k)$ denote the unbiased segmented correlation estimates (as described in Section 2). Then

$$E\{d_1^u(k)\} = c_{4x}(k) \quad (21)$$

3.2.2 Estimator without Segmentation

Let $d_2^u(k)$ be the unbiased unsegmented estimator for the 1-D diagonal-slice of the fourth-order cumulant, i.e.,

$$d_2^u(k) = \frac{1}{MN - |k|} \sum_{n=1}^{MN - |k|} x(n)x^3(n+k) - 3r_2^u(0)r_2^u(k)$$

where $r_2^u(0)$ and $r_2^u(k)$ denote the unbiased unsegmented correlation estimates (as described in Section 2). Then

$$E\{d_2^u(k)\} = c_{4x}(k)$$

3.3 Simulation Results

We should have obtained variances of the preceding estimators to compare them fully; however, since the covariance functions of the estimators are very involved (as shown in the Appendix), we have performed some simulations to show that a biased segmented estimator produces poor

estimates for the 1-D diagonal-slice of the fourth-order cumulants.

For our simulation, we chose two cosines both of whose amplitudes are unity; their frequencies are at $f_1 = 0.1$ and $f_2 = 0.2$. Colored additive Gaussian noise was generated by an ARMA(2,2) system excited by a zero-mean white Gaussian noise input. The AR coefficients of this ARMA model were [1, 1.4563, 0.81], and its MA coefficients were [1, 2, 1]. The colored noise spectrum has a strong pole at 0.4, with a damping factor of 0.9. We performed 30 independent trials. Figure 1 shows the mean values of unbiased and biased estimates for 0 dB and -3 dB local SNR's. Local SNR is defined as the ratio of local signal power to noise variance, i.e., when we have the signal in (1), local SNR corresponding to the i -th sinusoid, SNR_i , is

$$\text{SNR}_i = 10 \log \frac{a_i^2/2}{\hat{\sigma}^2} \quad i = 1, 2, \dots, p$$

where a_i denotes the amplitude of the i -th sinusoid and $\hat{\sigma}^2$ denotes the estimated noise variance obtained from output data of the ARMA model for the measurement noise. In Fig. 1, a “star” denotes the zero-noise theoretical values of the 1-D diagonal-slice of the fourth-order cumulants of the harmonics. The “dash” and “dash-dot” curves are the estimated values using one realization (4096 samples) with segmentation and without segmentation, respectively. The “solid” curve shows the estimated values when we used 64 samples of 64 independent realizations (total number of data is 4096). The figures show that the biased unsegmented estimator is much better than other biased ones, even at large lags, and, there is no difference among unbiased estimators. Although we have not shown the variance values of estimators, note that the variances of each estimate were very similar.

3.4 Conclusions

We conclude that for correlation or fourth-order cumulant estimates:

- In the case of a single realization, the biased unsegmented estimator gives better results than the biased segmented estimator.
- In the case of a single realization, the unbiased unsegmented and segmented estimators give comparable results.
- When we only have a single realization, we should not use a biased segmented estimator.

4 Estimation of the Number of Harmonics

To decide the number of sinusoids in a given signal, the singular values of a correlation or cumulant matrix can be used. In [4], after sorting the singular values and normalizing the largest singular value to unity, singular values were plotted and the number of harmonics was determined by searching for a sharp or sudden drop in the singular values. When we use this method of comparing the relative magnitudes of singular values, there are many problems, such as: how fast or sudden a “drop” of magnitude should be, and, which “small” singular values are negligible. Although a singular value of a correlation (cumulant) matrix is the energy in the direction of the corresponding singular vector (eigenvector), it seems hard to answer these questions, so, we suggest another method.

After sorting the singular values such that

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_M$$

where λ_i ’s are the singular values and M denotes the dimension of the correlation (cumulant) matrix. We define the accumulated singular values, Λ_i , as

$$\Lambda_i = \frac{\sum_{k=1}^i \lambda_k}{\sum_{k=1}^M \lambda_k} \quad i = 1, 2, \dots, M \quad (22)$$

We plot the accumulated singular values, Λ_i ’s, and then decide the number of harmonics by searching for a point where the plot has a “sharp” break that is close to unity. We illustrate such a plot in the examples described below. It is relatively easy and reliable to use.

5 Estimation of Harmonic Frequencies

In this section we briefly describe three methods to estimate the frequencies of sine waves in noise. They are the Pisarenko method [9,10], the multiple signal characterization (MUSIC) method [1,2], and the minimum-norm method [1,2]. A common feature of MUSIC and minimum-norm methods is the decomposition of a correlation (cumulant) matrix of the input signal into two orthogonal subspaces: signal and noise subspaces. For completeness, we state the three algorithms next.

Let $M \times M$ be the dimension of the correlation (cumulant) matrix, \mathbf{R} (\mathbf{C}), and p be the number of harmonics in the signal.

5.1 Pisarenko Algorithm

1. Perform the SVD on the correlation (cumulant) matrix and decide the number of sinusoids, p , in the signal.
2. Consider the principal minor of the matrix \mathbf{R} (or \mathbf{C}), whose dimension is $(2p+1) \times (2p+1)$.
3. Find the eigenvector of the $(2p+1) \times (2p+1)$ matrix corresponding to its smallest (zero) eigenvalue, and denote its components by a_0, a_1, \dots, a_{2p} .

4. Evaluate the roots, z_1, \dots, z_{2p} , of the polynomial

$$A(z) \equiv a_0 + a_1 z + \dots + a_{2p} z^{2p} = 0$$

Estimate the frequencies of the sinusoids as

$$z_k = \exp(j\omega_k) \quad -\pi \leq \omega_k \leq \pi \quad k = 1, \dots, 2p;$$

or, determine the angular frequencies as the peaks of $\left| \frac{1}{A(z)} \right|$, $z = e^{j\omega}$.

5.2 MUSIC Algorithm

1. After computing the eigenvalues and eigenvectors of the estimated correlation (cumulant) matrix, classify the eigenvalues into two groups: one consisting of the $2p$ largest eigenvalues and the other consisting of the $(M - 2p)$ smallest eigenvalues.
2. Use the eigenvectors associated with the second group to construct the $M \times (M - 2p)$ eigenvector matrix \mathbf{X}_N whose elements span the sample noise subspace.
3. Determine the angular frequencies of the sinusoids as the peaks of the sample spectrum

$$S(\omega) = \frac{1}{\mathbf{s}^H(\omega) \mathbf{X}_N \mathbf{X}_N^H \mathbf{s}(\omega)}$$

where $\mathbf{s}(\omega)$ is the $M \times 1$ frequency-searching vector, defined by

$$\mathbf{s}^T(\omega) = [1, e^{-j\omega}, \dots, e^{-j\omega(M-2p+1)}]$$

5.3 Minimum Norm Algorithm

1. Compute the eigenvalues and eigenvectors of the estimated correlation (cumulant) matrix.
2. Classify the eigenvalues into two groups: the $2p$ largest eigenvalues and the $(M - 2p)$ smallest eigenvalues. Use the eigenvectors associated with the first group to construct the $M \times 2p$ matrix \mathbf{X}_S whose elements span the sample signal subspace.
3. Partition the matrix \mathbf{X}_S as

$$\mathbf{X}_S = \begin{bmatrix} \mathbf{g}_S^T \\ \vdots \\ \mathbf{G}_S \end{bmatrix}$$

where \mathbf{g}_S^T contains the first elements of the signal subspace eigenvectors, and the $(M-1) \times 2p$ matrix \mathbf{G}_S contains the rest of the elements of \mathbf{X}_S .

4. Compute the minimum-norm value of the $M \times 1$ vector, \mathbf{a} :

$$\mathbf{a} = \begin{bmatrix} 1 \\ \dots \\ -(1 - \mathbf{g}_S^H \mathbf{g}_S)^{-1} \mathbf{G}_S^* \mathbf{g}_S \end{bmatrix}$$

5. Determine the angular frequencies of the sinusoids as the peaks of the sample spectrum

$$S(\omega) = \frac{1}{|\mathbf{a}^H \mathbf{s}(\omega)|^2}$$

where $\mathbf{s}(\omega)$ is the $M \times 1$ frequency-searching vector defined by

$$\mathbf{s}^T(\omega) = [1, e^{-j\omega}, \dots, e^{-j(M-1)\omega}]$$

6 Estimation of Harmonic Amplitudes

After estimating the frequencies of harmonics, we estimate the amplitudes corresponding to the estimated frequencies using Eq. (2) for correlations and Eq. (4) for the fourth-order cumulants. Usually the number of these equations is larger than the number of unknowns, and we have to solve overdetermined linear equations. To do this we can use least squares (LS), total least squares (TLS), or constrained total least squares (CTLS).

We can represent the amplitude-estimation problems as

$$\mathbf{A} \mathbf{x} = \mathbf{b} \quad (23)$$

where [see (2) or (4)],

$$\mathbf{A} = \begin{bmatrix} \cos(\hat{\omega}_1 \cdot 0) & \cos(\hat{\omega}_2 \cdot 0) & \cdots & \cos(\hat{\omega}_p \cdot 0) \\ \cos(\hat{\omega}_1 \cdot 1) & \cos(\hat{\omega}_2 \cdot 1) & \cdots & \cos(\hat{\omega}_p \cdot 1) \\ \vdots & \vdots & \vdots & \vdots \\ \cos(\hat{\omega}_1 \cdot (M-1)) & \cos(\hat{\omega}_2 \cdot (M-1)) & \cdots & \cos(\hat{\omega}_p \cdot (M-1)) \end{bmatrix} \quad (24)$$

$$\mathbf{b} = \begin{cases} -2 [\hat{r}_x(0) \ \hat{r}_x(1) \ \cdots \ \hat{r}_x(M-1)]^T & \text{for correlation} \\ -\frac{8}{3} [\hat{c}_{4x}(0) \ \hat{c}_{4x}(1) \ \cdots \ \hat{c}_{4x}(M-1)]^T & \text{for cumulant} \end{cases} \quad (25)$$

$$\mathbf{x} = \begin{cases} [a_1^2 \ a_2^2 \ \cdots \ a_p^2]^T & \text{for correlation} \\ [a_1^4 \ a_2^4 \ \cdots \ a_p^4]^T & \text{for cumulant} \end{cases} \quad (26)$$

in which M and p are numbers of equations and unknowns, respectively, $\hat{\omega}_i$'s denote estimated frequencies, and, $\hat{r}_x(i)$'s and $\hat{c}_{4x}(i)$'s denote estimated correlations and 1-D diagonal-slice fourth-order cumulants, respectively.

6.1 Least Squares Method

In the least squares (LS) technique, we assume that noise is present in the data vector \mathbf{b} . The LS solution for \mathbf{x} is obtained as:

$$\begin{aligned} & \text{Min}_{\mathbf{x}, \Delta \mathbf{b}} \|\Delta \mathbf{b}\| \\ & \text{subject to} \quad \mathbf{A} \mathbf{x} = \mathbf{b} + \Delta \mathbf{b} \end{aligned}$$

in which we try to keep the correction term $\Delta\mathbf{b}$ as small as possible while simultaneously compensating for the noise present in \mathbf{b} , by forcing $\mathbf{A}\mathbf{x} = \mathbf{b} + \Delta\mathbf{b}$. When the noise in \mathbf{A} is zero and the noise in \mathbf{b} is zero-mean Gaussian, the LS solution, \mathbf{x}_{LS} , is identical to the maximum likelihood solution, i.e.,

$$\mathbf{x}_{LS} = (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{b}$$

6.2 Total Least Squares Method

When \mathbf{A} is also noisy, \mathbf{x}_{LS} is no longer optimal from a statistical point of view; it suffers from bias and increased covariance due to the accumulation of noise errors in $\mathbf{A}^H \mathbf{A}$. To alleviate this problem, a generalization of the LS solution was formally introduced by Golub and Van Loan [11], called total least squares (TLS). TLS attempts to remove the noise in \mathbf{A} and \mathbf{b} using a perturbation on \mathbf{A} and \mathbf{b} of smallest 2-norm which makes the system of overdetermined linear equations consistent. The TLS solution for \mathbf{x} is obtained as

$$\begin{aligned} & \text{Min } \|[\Delta\mathbf{A} : \Delta\mathbf{b}]\| \\ & \text{subject to } (\mathbf{A} + \Delta\mathbf{A})\mathbf{x} = \mathbf{b} + \Delta\mathbf{b} \end{aligned}$$

The TLS solution can be expressed algebraically as

$$\mathbf{x}_{TLS} = (\mathbf{A}^H \mathbf{A} - \sigma^2 \mathbf{I})^{-1} \mathbf{A}^H \mathbf{b}$$

where σ^2 is the minimum eigenvalue(s) of $[\mathbf{A} : \mathbf{b}]^H [\mathbf{A} : \mathbf{b}]$. It is also true that the TLS solution can be obtained explicitly from the right singular vector that corresponds to the smallest singular value of the singular value decomposition (SVD) of $\mathbf{C} = [\mathbf{A} : \mathbf{b}]$. From a statistical point of view, TLS operates under the assumption that the noise components of \mathbf{A} and \mathbf{b} are zero mean and identically independently distributed.

6.3 Constrained Total Least Squares Method

If there is a linear dependence among the noise components in \mathbf{A} and \mathbf{b} , then the TLS problem must be reformulated to take into account the reduced dimensionality of the noise entries. Abatzoglou and Mendel [12,13] discuss a reformulation of the TLS method, which they call constrained total least squares (CTLS), that accounts for the linear algebraic relations among the noise entries of \mathbf{A} and \mathbf{b} .

The CTLS solution for \mathbf{x} is defined by

$$\text{Min}_{\mathbf{v}, \mathbf{x}} \| \Delta\mathbf{C} \| \quad \text{where } (\mathbf{C} + \Delta\mathbf{C}) \begin{bmatrix} \mathbf{x} \\ -1 \end{bmatrix} = 0$$

and

$$\begin{aligned} \Delta\mathbf{C} &= \left[\Delta\mathbf{A} : \Delta\mathbf{b} \right] \\ &= \left[\mathbf{F}_1 \mathbf{v} : \dots : \mathbf{F}_{p+1} \mathbf{v} \right] \end{aligned}$$

in which $\mathbf{C} = [\mathbf{A} : \mathbf{b}]$ and $\Delta\mathbf{C} = [\Delta\mathbf{A} : \Delta\mathbf{b}]$. Let $\mathbf{v} = [v_1, v_2, \dots, v_K]^T$ be the minimal algebraic set of linearly independent random variables such that

$$\Delta\mathbf{C}_i = \mathbf{F}_i \mathbf{v} \quad i = 1, \dots, p+1$$

where the \mathbf{F}_i 's are $M \times K$ matrices. Since \mathbf{v} is not a white random vector, we can perform whitening by using its Cholesky factorization,

$$\mathbf{E}\{\mathbf{v}\mathbf{v}^H\} = \mathbf{P}\mathbf{P}^H$$

We define a white noise vector \mathbf{u} as

$$\mathbf{u} = \mathbf{P}^{-1}\mathbf{v};$$

so $\Delta\mathbf{C}$ can be expressed as

$$\Delta\mathbf{C}_i = \mathbf{F}_i \mathbf{P} \mathbf{u} = \mathbf{G}_i \mathbf{u} \quad i = 1, \dots, p+1$$

where $\mathbf{G}_i = \mathbf{F}_i \mathbf{P}$. When $\text{rank}(\mathbf{H}_x) = \text{full}$ and $M \leq K$, the CTLS solution can be obtained as the \mathbf{x} that minimizes the function $F(\mathbf{x})$,

$$F(\mathbf{x}) = \begin{bmatrix} \mathbf{x} \\ -1 \end{bmatrix}^H \mathbf{C}^H (\mathbf{H}_x \mathbf{H}_x^H)^{-1} \mathbf{C} \begin{bmatrix} \mathbf{x} \\ -1 \end{bmatrix}$$

where

$$\mathbf{H}_x = \sum_{i=1}^p x_i \mathbf{G}_i - \mathbf{G}_{p+1}$$

The Newton iteration to solve this nonlinear problem is given in [13], as

$$\mathbf{x} = \mathbf{x}_0 + (\bar{\mathbf{A}} \mathbf{B}^{-1} \mathbf{A} - \bar{\mathbf{B}})^{-1} (\bar{\mathbf{a}} - \bar{\mathbf{A}} \mathbf{B}^{-1} \mathbf{a})$$

where

$$\mathbf{a} = (\mathbf{u}^H \tilde{\mathbf{B}})^T$$

$$\mathbf{A} = -\tilde{\mathbf{G}}^H \mathbf{H}_x^H (\mathbf{H}_x \mathbf{H}_x^H)^{-1} \tilde{\mathbf{B}} - (\tilde{\mathbf{G}}^H \mathbf{H}_x^H (\mathbf{H}_x \mathbf{H}_x^H)^{-1} \tilde{\mathbf{B}})^T$$

$$\mathbf{B} = \left(\tilde{\mathbf{B}}^H (\mathbf{H}_x \mathbf{H}_x^H)^{-1} \tilde{\mathbf{B}} \right)^T + \tilde{\mathbf{G}}^H \left(\mathbf{H}_x^H (\mathbf{H}_x \mathbf{H}_x^H)^{-1} \mathbf{H}_x - \mathbf{I} \right) \tilde{\mathbf{G}}$$

$$\mathbf{u} = (\mathbf{H}_x \mathbf{H}_x^H)^{-1} \mathbf{C} \begin{bmatrix} \mathbf{x} \\ -1 \end{bmatrix}$$

$$\tilde{\mathbf{B}} = \mathbf{C} \mathbf{I}_{p+1,p} - \left[\mathbf{G}_1 \mathbf{H}_x^H \mathbf{u} \quad ; \quad \dots \quad ; \quad \mathbf{G}_p \mathbf{H}_x^H \mathbf{u} \right]$$

and

$$\tilde{\mathbf{G}} = \left[\mathbf{G}_1^H \mathbf{u} \quad ; \quad \dots \quad ; \quad \mathbf{G}_p^H \mathbf{u} \right]$$

7 Simulations

In this section, we compare the correlation-based and cumulant-based harmonic retrieval approaches through simulations. Our goal is to learn where each approach breaks down as certain experimental conditions are changed.

Throughout the following simulations, colored additive Gaussian noise was generated through an ARMA(2,2) system excited by a zero-mean white Gaussian noise input. The AR coefficients of this ARMA model equal [1, 1.4563, 0.81], and its MA coefficients equal [1, 2, 1], as in [4]. The resulting colored Gaussian noise spectrum has a strong pole around 0.4. For each simulation, we performed 30 Monte Carlo trials.

The signal consisted of two harmonics whose frequencies are $f_1 = 0.1$ and $f_2 = 0.2$, and whose amplitudes are both unity.

We used either one realization having 64×64 (4096) samples, or 64 independent realizations, each with 64 samples. In the case of a single realization, the data were divided into 64 segments, each segment with 64 samples. For estimated correlations, $\hat{r}_x(k)$, and estimated fourth-order cumulant, $\hat{c}_{4x}(k)$, $k = 0, 1, \dots, 15$, we used the unbiased segmented estimator, i.e.,

$$\hat{r}_x(k) = \frac{1}{M} \sum_{i=1}^M \left(\frac{1}{N - |k|} \sum_{n=1}^{N-|k|} x_i(n)x_i(n+k) \right) \quad (27)$$

and

$$\hat{c}_{4x}(k) = \frac{1}{M} \sum_{i=1}^M \left(\frac{1}{N - |k|} \sum_{n=1}^{N-|k|} x_i(n)x_i^3(n+k) - 3\hat{r}_x(0)\hat{r}_x(k) \right) \quad (28)$$

where M and N denote the number of segments and the number of samples in one segment, respectively. Although $c_{4x}(k) = c_{4x}(-k)$ for a stationary process, the estimates in Eq. (28) lose the symmetry property; hence, we used the averaged value $\frac{\hat{c}_{4x}(k) + \hat{c}_{4x}(-k)}{2}$ for the fourth-order cumulants at lags k and $-k$.

7.1 Variable: Amplitude

After fixing all other parameters and one of the sinusoidal amplitudes (at unity), we varied the amplitude of the second sinusoid ($f_2 = 0.2$) in order to determine where the correlation-based and cumulant-based methods break down. Figures 2 and 3 show the mean values of accumulative singular values of cumulants and correlations, respectively, using one realization with segmentation, when the local SNR (which is defined in Section 3.3), of the fixed sinusoid is 5 dB. Observe that better results are obtained for cumulants.

Figures 4 and 5 show the mean values of accumulative singular values of cumulants using either one realization with segmentation or independent realizations, respectively, when the local SNR of the fixed sinusoid is 0 dB. Note that no appreciable differences are visible.

Figures 6 and 7 show the mean values of accumulative singular values of correlations using one realization with segmentation and independent realizations, respectively, when the local SNR of the fixed sinusoid is 0 dB. Again, no appreciable differences are observed.

Values of means and standard deviations for the 0 dB case are given in Tables 1 and 2 for cumulants and in Tables 3 and 4 for correlations.

Figures 4-7 and Tables 1-4 show that there are no differences between performances of the unbiased estimators using either one realization with segmentation or several independent realizations, when data is long enough. These results also show the superiority of using cumulants, when one sinusoid has a small amplitude. When we used cumulants, the singular-value graphs show that for a fixed local SNR of 5 dB (Fig. 2) we still detect the correct number of harmonics even when the amplitude of the second sinusoid is -0.8dB; and, that for a fixed local SNR of 0 dB (Fig. 4) we obtain an incorrect estimate of the number of sinusoids if the amplitude of the second harmonic is less than around -5 dB. In the case of correlations, we estimate an incorrect number of harmonics even when the varying local SNR is 5 dB (Fig. 3-(a)).

7.2 Variable: Frequency

Next, we fixed all other parameters (i.e., the amplitudes of the two sinusoids equal unity) and varied the frequency f_2 (originally at 0.2), letting it approach the other frequency, f_1 , at 0.1. Figures 8 and 9 show the mean values of accumulative singular values of cumulants and correlations, respectively, when the local SNR's of both sinusoids are 5 dB.

Figures 10 and 11 show the mean values of accumulative singular values of cumulants using either one realization with segmentation or independent realizations, respectively, when each local SNR is 0 dB.

Figures 12 and 13 show the mean values of accumulative singular values of correlations using either one realization with segmentation or independent realizations, respectively, when each local SNR is 0 dB. Values of means and standard deviations for the 0 dB case are given in Tables 5 and 6 for cumulants and in Tables 7 and 8 for correlations.

Figures 10-13 and Tables 5-8 show that there are essentially no differences between performances of the unbiased estimators using either one realization with segmentation or several independent realizations, when data is long enough. These results also show the superiority of using cumulants over correlations when frequencies are very close to one another. When we used cumulants, the singular-value graphs show that for 5 dB local SNR (Fig. 8) we still detect the correct number of harmonics even when the varying frequency is 0.12; and, that for 0 dB local SNR (Fig. 10) we obtain an incorrect estimate of the number of sinusoids if the varying frequency is less than 0.13. For correlation-based estimates, we estimate an incorrect number of harmonics even when the spacing between the two frequencies is 0.1 and each local SNR is 5 dB (Fig. 9-(a)).

7.3 Amplitude Estimation

To study the performance of the amplitude estimation algorithms, three examples are given in this section. Figures 14-17 show the mean values of accumulative singular values and the estimated frequencies using the Pisarenko, MUSIC, and minimum-norm methods for the problem of two harmonics in colored Gaussian noise. The “solid” curves denote the results obtained using cumulants, and the “dash” and “dash-dot” curves denote the results obtained using correlations. In the following examples, note that only the estimated frequencies using MUSIC were used to estimate amplitudes and that the estimated amplitudes using LS were used as initial values for the Newton iteration using CTLS. The estimated frequencies using MUSIC and minimum-norm methods were very similar [4]. In the CTLS method, the value to stop the Newton iterations was chosen as 10^{-8} .

We assumed frequency estimation errors are independent and white, i.e.,

$$\hat{\omega}_i = \omega_i + \delta\omega_i \quad i = 1, \dots, p$$

where ω_i 's denote the true frequencies, $\delta\omega_i$'s denote the independent estimation errors, and p is the estimated number of sinusoids (see Figs. 14-17). Since each element of matrix \mathbf{A} is a nonlinear function of $\hat{\omega}_i$'s (see Eq. (24)), we can linearize the elements as follows : when $\delta\omega_i \cdot m$, $m = 0, 1, \dots, 15$, are very small, $\cos(\delta\omega_i \cdot m) \approx 1$ and $\sin(\delta\omega_i \cdot m) \approx m \cdot \delta\omega_i$; hence,

$$\cos(m(\omega_i + \delta\omega_i)) = \cos(m\omega_i) \cdot \cos(m\delta\omega_i) - \sin(m\omega_i) \cdot \sin(m\delta\omega_i) \quad (29)$$

$$\approx \cos(m\omega_i) - m\delta\omega_i \sin(m\omega_i) \quad (30)$$

and

$$\sin(m(\omega_i + \delta\omega_i)) = \sin(m\omega_i) \cdot \cos(m\delta\omega_i) + \cos(m\omega_i) \cdot \sin(m\delta\omega_i) \quad (31)$$

$$\approx \sin(m\omega_i) + m\delta\omega_i \cos(m\omega_i) \quad (32)$$

Consequently, we can approximate $\cos(m(\omega_i + \delta\omega_i))$ as follows;

$$\cos(m(\omega_i + \delta\omega_i)) \approx \cos(m\omega_i) - m \sin(m(\omega_i + \delta\omega_i)) \cdot \delta\omega_i + m^2 \cos(m\omega_i) \cdot (\delta\omega_i)^2$$

Since the third term on the right-hand side of this equation is negligible,

$$\cos(m\hat{\omega}_i) \approx \cos(m\omega_i) - m \sin(m\hat{\omega}_i) \cdot \delta\omega_i \quad (33)$$

where $i = 1, \dots, p$ and $m = 0, 1, \dots, 15$. Note that this equation provides a computable variation of $\cos(m\omega_i)$ due to errors in $\hat{\omega}_i$, whereas the Eq. (30) does not. Although the estimation errors of correlations (cumulants) are correlated, we assumed for simplicity they are independent of each other and are white, i.e.,

$$\hat{r}_x(k) = r_x(k) + \delta r_x(k) \quad \text{and} \quad \hat{c}_{4x}(k) = c_{4x}(k) + \delta c_{4x}(k)$$

$$k = 0, 1, 2, \dots, 15$$

where $\delta r_x(k)$ and $\delta c_{4x}(k)$ denote the correlation and cumulant estimation errors, respectively. We define the white noise vector \mathbf{u} as

$$\mathbf{u} = \begin{cases} [\delta\omega_1, \dots, \delta\omega_p, \delta r_x(0), \dots, \delta r_x(15)]^T & \text{for correlations} \\ [\delta\omega_1, \dots, \delta\omega_p, \delta c_{4x}(0), \dots, \delta c_{4x}(15)]^T & \text{for cumulants} \end{cases}$$

such that

$$\Delta \mathbf{C}_i = \mathbf{G}_i \mathbf{u} \quad i = 1, \dots, p+1$$

where (from $\Delta \mathbf{C} = [\Delta \mathbf{A} : \Delta \mathbf{b}]$, (24), (25), and (33))

$$\Delta \mathbf{C}_i = [0, -\sin(\hat{\omega}_i) \cdot \delta\omega_i, \dots, -15 \sin(15\hat{\omega}_i) \cdot \delta\omega_i]^T \quad i = 1, \dots, p$$

and

$$\Delta \mathbf{C}_{p+1} = \begin{cases} 2 [\delta r_x(0), \dots, \delta r_x(15)]^T & \text{for correlations} \\ -\frac{8}{3} [\delta c_{4x}(0), \dots, \delta c_{4x}(15)]^T & \text{for cumulants} \end{cases}$$

Now we obtain the \mathbf{G}_i 's and then solve the CTLS problem using the Newton iteration (described in Section 6.3).

7.3.1 Example 1: Local SNR's Equal 5 dB and 0 dB

We set the two amplitudes to 1 and 0.6 at the frequencies $f_1 = 0.1$ and $f_2 = 0.2$, respectively. The local SNR's were 5 dB and 0 dB, respectively. The mean values of accumulative singular values and estimated frequencies using Pisarenko, MUSIC, and minimum-norm methods are shown in Fig. 14. To obtain these values we used 30 independent trials. Table 9 shows the means and standard deviations of estimated amplitudes using LS, TLS, and CTLS. Conclusions are summarized in Paragraph 7.3.4 for all three examples.

7.3.2 Example 2: Local SNR's Both Equal 0 dB

We set both amplitudes to unity at the frequencies $f_1 = 0.1$ and $f_2 = 0.2$. The local SNR's were both 0 dB. Again, 30 independent trials were used. Figure 15 shows the mean values of the accumulative singular values and estimated frequencies using Pisarenko, MUSIC, and minimum-norm methods, and Table 10 shows the means and standard deviations of estimated amplitudes using LS, TLS, and CTLS when unbiased segmented estimates from a single realization were used. Figure 16 and Table 11 show the results using 64 independent realizations. These results demonstrate that similar behaviors are obtained for unbiased estimates using one segmented realization and independent realizations.

7.3.3 Example 3: Local SNR's Both Equal -3 dB

We set both amplitudes to unity at the frequencies $f_1 = 0.1$ and $f_2 = 0.2$. In this example, both local SNR's were -3 dB. Again, 30 independent trials were performed. Figure 17 shows the mean values of the accumulative singular values and estimated frequencies using Pisarenko, MUSIC, and minimum-norm methods. Table 12 shows the means and standard deviations of estimated amplitudes using LS, TLS, and CTLS.

7.3.4 Conclusions from Examples

From these three examples, we conclude that;

- the resulting estimated frequencies using MUSIC and minimum-norm methods are very similar and very good.
- the Pisarenko method using correlation gives poor results when noise is colored. It also can give biased results for low SNR's when cumulants are used.
- the resulting estimated amplitudes using cumulants are better than those obtained using correlations.
- the results using LS are usually better than those obtained using TLS or CTLS because of good estimated frequencies.
- although we expected the superiority of CTLS, there is no great advantage to using CTLS because of very good estimation of frequencies by all the methods.
- the assumption of independency of the correlation (cumulant) errors may have caused the poor performance of CTLS results.
- using several independent realizations gives similar results to using one realization in the case of high SNR for large amounts of data.

7.4 Variable: Amplitude, Frequency, and Data Length

Throughout the above several simulations and examples, we showed the superiority of using cumulants to using correlations in the harmonic retrieval problem. In this subsection we find the points where the cumulant-based methods for harmonic retrieval break down as a function of amplitude, frequency of one harmonic, and length of data.

Our signal consists of two real sinusoids measured in colored Gaussian noise. The first sinusoid has a fixed local SNR of 0 dB (corresponding magnitude is unity) at $f_1 = 0.1$. The second sinusoid has varying amplitude at $f_2 = 0.2, 0.18, 0.16$, or 0.14 . The additive noise was the colored Gaussian noise generated by the ARMA system described in Section 7.

In order to determine where the cumulant-based methods break down, we varied the amplitude of the second sinusoid at a fixed f_2 , and also varied the data length. We used one realization divided into segments of 64 samples. We used 8 different data lengths: 64×64 (4096), 50×64 (3200), 40×64 (2560), 32×64 (2048), 25×64 (1600), 16×64 (1024), 8×64 (512), and 4×64 (256). For each simulation, we performed 30 Monte Carlo trials.

Figures 18-25 showed the accumulated singular values of cumulants when the second frequency is $f_2 = 0.2$. The values labeled with * were judged (albeit, subjectively) to be the smallest amplitudes of the second sinusoid when we can visually still estimate the correct number of harmonics. Figures 26-33, 34-41, and 42-49 are for $f_2 = 0.18, 0.16$, and 0.14 , respectively. Note that

the amplitudes stated in the captions of Figs. 18-49 are not local SNR's.

Results from Figures 18-49 have been summarized in Fig. 50, where we observe that:

- as the length of data decreases, the minimum local SNR for correctly estimating the number of harmonics increases.
- the larger the spacing between the two harmonics, the smaller the data length or local SNR is needed to estimate the correct number of harmonics.
- as the length of data increases, the pattern of decreasing the minimum local SNR for correctly estimating the number of harmonics is approximately exponential. Perhaps, there is a theoretical explanation for this.
- when any two of the variables (local SNR, data length, or spacing between the two real sinusoids) are fixed, we can obtain the minimum value of the third variable using Fig. 50.

8 Conclusions

When data is available from only a single realization, one should not use a data segmentation technique and a biased estimator. Since biased segmented estimates have large bias, one must use a biased unsegmented estimator. An unbiased estimator does not seem to be affected by data segmentation. These claims were verified via simulations and some analyses.

When the number of harmonics is estimated using singular values, we suggest using accumulative singular values instead of comparing relative magnitudes of singular values.

When the amplitude of one harmonic decreases or when the frequencies of one harmonic approaches the other, for the case of two sinusoids measured in colored Gaussian noise, the accumulated singular values of the cumulant matrix give much better results than those of the correlation matrix. It is known, however, that a cumulant-based method needs relatively longer data to produce an acceptable result.

In estimation of the frequencies of harmonics, MUSIC and minimum-norm methods are much better than the Pisarenko method, for additive colored Gaussian noise. When we do use the Pisarenko method for colored noise, the cumulant-based technique is better than the correlation-based technique.

We have studied the performances of LS, TLS, and CTLS for estimation of amplitudes, by means of three examples. In the examples, LS and CTLS results are better than TLS. Actually, the LS method gave the best results. The reason for this is that the estimated frequencies obtained using MUSIC and minimum-norm are very close to the true frequencies. This implies that there is almost no noise in matrix \mathbf{A} in the overdetermined system, $\mathbf{A} \mathbf{x} = \mathbf{b}$. Since there is no noise in matrix \mathbf{A} and the estimation errors for correlations and cumulants are asymptotically

normal, the LS solution is identical to the maximum likelihood solution [13].

Finally, we obtained a plot of minimum local SNR vs. different lengths of data and several f_2 where we can still determine the correct number of harmonics by looking at breaks in plots of accumulated singular values. For each f_2 the minimum local SNR decreases approximately exponentially as the length of data increases. Using Fig. 50 we can estimate the minimum value of the third variable when any two of the three variables are fixed.

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References

- [1] S. Haykin, *Adaptive Filter Theory*, Prentice-Hall, New Jersey, 1986.
- [2] S. Haykin, *Modern Filters*, Macmillan Pub. Comp., New York, 1989.
- [3] M. B. Priestley, *Spectral Analysis and Time Series*, Vol.1, Academic-Press, Florida, 1981.
- [4] A. Swami and J. M. Mendel, "Cumulant-Based Approach to the Harmonic Retrieval and Related Problems", accepted for publication in *IEEE Trans. ASSP*, May, 1991.
- [5] A. Swami and J. M. Mendel, "Cumulant-Based Approach to the Harmonic Retrieval Problem", *Proc. IEEE ICASSP-88*, New York, NY, April, 1988, pp. 2264-2267.
- [6] B. Porat and B. Friedlander, "Performance Analysis of Parameter Estimation Algorithms Based on High-Order Moments", *Int. J. Adaptive Control and Signal Processing*, Vol.3, 1989, pp. 191-229.
- [7] B. Porat and B. Friedlander, "The Asymptotic Second Order Moments of the Estimated Third Order Cumulants of Stationary Processes", *Proc. American Control Conference*, Atlanta, GA, June 1988, pp. 2108-2113.
- [8] G. B. Giannakis and J. M. Mendel, "Identification of Non-Minimum Phase Systems Using Higher-Order Statistics", *IEEE Trans. ASSP*, Vol.37, No.3, March 1989, pp. 360-377.
- [9] S. M. Kay and S. L. Marple, "Spectrum Analysis - A Modern Perspective", *Proc. IEEE*, Vol.69, No.11, Nov. 1981, pp. 1380-1419.
- [10] V. F. Pisarenko, "The Retrieval of Harmonics from a Covariance Function", *Geophys. J. R. Astr. Soc.*, Vol.33, 1973. pp. 347-366.
- [11] G. H. Golub and C. F. Van Loan, "An Analysis of the Total Least Squares Problem", *SIAM J. Numer. Anal.*, Vol.17, No.6, Dec. 1980, pp. 883-893.
- [12] T. J. Abatzoglou and J. M. Mendel, "Constrained Total Least Squares", *Proc. IEEE ICASSP-87*, Dallas, Texas, April. 1987, pp. 1485-1488.
- [13] T. J. Abatzoglou , J. M. Mendel and G. A. Harada, "The Constrained Total Least Squares Technique and its Applications to Harmonic Superresolution", accepted for publication in *IEEE Trans. ASSP*, May, 1991.
- [14] M. R. Raghuveer and C. L. Nikias, "Bispectrum Estimator : A Parametric Approach", *IEEE Trans. ASSP*, Vol.33, No.4, Oct. 1985, pp. 1213-1230.

A Appendix

In this appendix, the following results will be used : when $\mathbf{E}\{x(n)\} = 0$, for all n ,

$$\begin{aligned} \mathbf{E}\{x(n)x(n+k)x(m)x(m+k+v)\} &= r(k)r(k+v) + r(m-n)r(m-n+v) \\ &\quad + r(m-n+k+v)r(m-n+k) + c_{4x}(k, m-n, m-n+k+v) \end{aligned} \quad (\text{A-1})$$

where $\text{cum}(x(n), x(n+k), x(m), x(m+k+v)) = c_{4x}(k, m-n, m-n+k+v)$. Note that $c_{4x}(r, s-t, s-t+r+v) = c_{4x}(s-t, r, s-t+r+v)$. Now consider

$$\sum_{n=1}^{(N-k)} \sum_{m=1}^{(N-k-v)} \{r(m-n)r(m-n+v) + r(m+k+v-n)r(m-n-k) + c_{4x}(m-n, k, m-n+k+v)\}$$

where $k \geq 0$ and $k+v \geq 0$. We make a change of variables from m and n to $l = (m-n)$ and n . The summand depends only on l , and a careful examination of the limits of n gives [3],

$$\begin{aligned} &\sum_{n=1}^{(N-k)} \sum_{m=1}^{(N-k-v)} \{r(m-n)r(m-n+v) + r(m+k+v-n)r(m-n-k) + c_{4x}(m-n, k, m-n+k+v)\} \\ &= \sum_{l=-(N-k)+1}^{(N-k-v-1)} (N - \eta_N(l) - k - v) \{r(l)r(l+v) + r(l+k+v)r(l-k) + c_{4x}(l, k, l+k+v)\} \end{aligned} \quad (\text{A-2})$$

where $\eta_N(l)$, which is a function of N as well as l , is defined as

$$\eta_N(l) = \begin{cases} l, & l > 0 \\ 0, & -v \leq l \leq 0 \\ -l - v, & -(N-k) + 1 \leq l \leq -v \end{cases} \quad (\text{A-3})$$

A.1 Biased Estimators of Correlation

In this section, we derive the mean and covariance functions of the biased segmented estimator, $r_b^1(k)$, and the biased unsegmented estimator, $r_b^2(k)$.

A.1.1 Estimator with Segmentation

$$r_b^1(k) = \frac{1}{MN} \sum_{i=1}^M \sum_{n=1}^{N-|k|} x_i(n)x_i(n+k)$$

Then,

$$\mathbf{E}\{r_b^1(k)\} = \frac{1}{MN} \sum_{i=1}^M \sum_{n=1}^{N-|k|} \mathbf{E}\{x_i(n)x_i(n+k)\}$$

Using stationarity, we obtain

$$\mathbf{E}\{r_b^1(k)\} = \frac{N-|k|}{N} r(k)$$

For $k \geq 0$ and $k + v \geq 0$,

$$\begin{aligned}
& \text{Cov}(r_1^b(k), r_1^b(k+v)) = \mathbb{E}\{r_1^b(k)r_1^b(k+v)\} - \mathbb{E}\{r_1^b(k)\}\mathbb{E}\{r_1^b(k+v)\} \\
&= \frac{1}{(MN)^2} \sum_{i=1}^M \sum_{n=1}^{(N-k)} \sum_{j=1}^M \sum_{m=1}^{(N-k-v)} \mathbb{E}\{x_i(n)x_i(n+k)x_j(m)x_j(m+k+v)\} - \frac{N-k}{N} \frac{N-k-v}{N} r(k)r(k+v) \\
&= \frac{1}{(MN)^2} \sum_{i=1}^M \sum_{n=1}^{(N-k)} \sum_{m=1}^{(N-k-v)} \mathbb{E}\{x(n)x(n+k)x(m)x(m+k+v)\} + \frac{1}{(MN)^2} \sum_{i,j=1}^M \sum_{i \neq j} \sum_{n=1}^{(N-k)} \sum_{m=1}^{(N-k-v)} \mathbb{E}\{x_i(n) \\
&\quad x_i(n+k)x_j(m)x_j(m+k+v)\} - \frac{(N-k)(N-k-v)}{N^2} r(k)r(k+v)
\end{aligned}$$

We now apply Eq. (A-1) to this last result, to obtain

$$\begin{aligned}
& \text{Cov}(r_1^b(k), r_1^b(k+v)) = \frac{1}{(MN)^2} \sum_{i=1}^M \sum_{n=1}^{(N-k)} \sum_{m=1}^{(N-k-v)} \{r(m-n)r(m-n+v) + r(m+k+v-n)r(m-n-k) + \\
&\quad c_{4x}(m-n, k, m-n+k+v)\} + \frac{(N-k)(N-k-v)}{MN^2} r(k)r(k+v) \\
&+ \frac{1}{(MN)^2} \sum_{i,j=1}^M \sum_{i \neq j} \sum_{n=1}^{(N-k)} \sum_{m=1}^{(N-k-v)} \mathbb{E}\{x_i(n)x_i(n+k)x_j(m)x_j(m+k+v)\} - \frac{(N-k)(N-k-v)}{N^2} r(k)r(k+v)
\end{aligned}$$

Using Eq. (A-2), we simplify the covariance function to :

$$\begin{aligned}
& \text{Cov}(r_1^b(k), r_1^b(k+v)) = \frac{1}{MN^2} \sum_{l=-(N-k)+1}^{(N-k-v-1)} (N - \eta_N(l) - k - v) \{r(l)r(l+v) + r(l+k+v)r(l-k) + \\
&\quad c_{4x}(l, k, l+k+v)\} + \frac{1}{(MN)^2} \sum_{i,j=1}^M \sum_{i \neq j} \sum_{n=1}^{(N-k)} \sum_{m=1}^{(N-k-v)} \mathbb{E}\{x_i(n)x_i(n+k)x_j(m)x_j(m+k+v)\} \\
&\quad - \frac{(M-1)(N-k)(N-k-v)}{MN^2} r(k)r(k+v)
\end{aligned}$$

where the function $\eta_N(l)$ is defined in Eq. (A-3).

When each segment of data is independent, then, for $i \neq j$,

$$\mathbb{E}\{x_i(n)x_i(n+k)x_j(m)x_j(m+k+v)\} = r(k)r(k+v)$$

Using this fact, the second term of $\text{Cov}(r_1^b(k), r_1^b(k+v))$ becomes

$$\frac{(M-1)(N-k)(N-k-v)}{MN^2} r(k)r(k+v);$$

thus,

$$\begin{aligned} \text{Cov}(r_1^b(k), r_1^b(k+v)) &= \frac{1}{MN^2} \sum_{l=-(N-k)+1}^{(N-k-v-1)} (N - \eta_N(l) - k - v) \{ r(l)r(l+v) \\ &\quad + r(l+k+v)r(l-k) + c_{4x}(l, k, l+k+v) \} \end{aligned} \quad (\text{A-4})$$

When each segment is a part of a single realization, i.e.,

$$\begin{aligned} & \mathbb{E}\{x_i(n)x_i(n+k)x_j(m)x_j(m+k+v)\} \\ &= \mathbb{E}\{x(N(i-1)+n)x(N(i-1)+n+k)x(N(j-1)+m)x(N(j-1)+m+k+v)\} \\ &= r(k)r(k+v) + r(N(j-i)+m-n)r(N(j-i)+m-n+v) + r(N(j-i)+m-n+k+v) \\ &\quad r(N(j-i)+m-n-k) + c_{4x}(k, N(j-i)+m-n, N(j-i)+m-n+k+v) \end{aligned}$$

where we have used Eq. (A-1) to obtain the last line; thus,

$$\begin{aligned} \text{Cov}(r_1^b(k), r_1^b(k+v)) &= \frac{1}{MN^2} \sum_{l=-(N-k)+1}^{(N-k-v-1)} (N - \eta_N(l) - k - v) \{ r(l)r(l+v) + r(l+k+v)r(l-k) + \\ &\quad c_{4x}(l, k, l+k+v) \} + \frac{1}{(MN)^2} \sum_{i,j=1}^M \sum_{i \neq j}^M \sum_{n=1}^{(N-k)} \sum_{m=1}^{(N-k-v)} \{ r(N(j-i)+m-n)r(N(j-i)+m-n+v) \\ &+ r(N(j-i)+m-n+k+v)r(N(j-i)+m-n-k) + c_{4x}(k, N(j-i)+m-n, N(j-i)+m-n+k+v) \} \end{aligned}$$

As in Eq. (A-2), letting $p = m - n$ and $q = j - i$ in the last term of the preceding equation gives

$$\begin{aligned} \text{Cov}(r_1^b(k), r_1^b(k+v)) &= \frac{1}{MN^2} \sum_{l=-(N-k)+1}^{(N-k-v-1)} (N - \eta_N(l) - k - v) \{ r(l)r(l+v) + r(l+k+v)r(l-k) \\ &\quad + c_{4x}(l, k, l+k+v) \} + \frac{1}{(MN)^2} \sum_{q=-M+1, q \neq 0}^{(M-1)} \sum_{p=-(N-k)+1}^{(N-k-v-1)} (N - \eta_N(p) - k - v) (M - |q|) \{ r(Nq+p) \\ &\quad r(Nq+p+v) + r(Nq+p+k+v)r(Nq+p-k) + c_{4x}(k, Nq+p, Nq+p+k+v) \} \end{aligned} \quad (\text{A-5})$$

A.1.2 Estimator without Segmentation

$$r_2^b(k) = \frac{1}{MN} \sum_{n=1}^{MN-|k|} x(n)x(n+k)$$

Then

$$\mathbb{E}\{r_2^b(k)\} = \frac{1}{MN} \sum_{n=1}^{MN-|k|} \mathbb{E}\{x(n)x(n+k)\} = \frac{MN-|k|}{MN} r(k)$$

Without loss of generality, let $k \geq 0, k+v \geq 0$; then

$$\begin{aligned} & \text{Cov}(r_2^b(k), r_2^b(k+v)) = \mathbb{E}\{r_2^b(k)r_2^b(k+v)\} - \mathbb{E}\{r_2^b(k)\} \mathbb{E}\{r_2^b(k+v)\} \\ &= \frac{1}{(MN)^2} \sum_{n=1}^{(MN-k)(MN-k-v)} \mathbb{E}\{x(n)x(n+k)x(m)x(m+k+v)\} - \frac{MN-k}{MN} \frac{MN-k-v}{MN} r(k)r(k+v) \end{aligned}$$

Using Eq. (A-1),

$$\text{Cov}(r_2^b(k), r_2^b(k+v)) = \frac{1}{(MN)^2} \sum_{n=1}^{MN-k} \sum_{m=1}^{MN-k-v} \{r(m-n)r(m-n+v) + r(m+k+v-n)r(m-n-k) + c_{4x}(m-n, k, m-n+k+v)\}$$

and applying Eq. (A-2), we obtain

$$\begin{aligned} \text{Cov}(r_2^b(k), r_2^b(k+v)) &= \frac{1}{(MN)^2} \sum_{l=-(MN-k)+1}^{MN-k-v-1} (MN - \eta_{MN}(l) - k - v) \{r(l)r(l+v) + \\ &\quad r(l+k+v)r(l-k) + c_{4x}(l, k, l+k+v)\} \end{aligned} \quad (\text{A-6})$$

where the function $\eta_{MN}(l)$ is defined by Eq. (A-3).

A.2 Unbiased Estimators of Correlation

In this section, we derive the mean and covariance functions of the unbiased segmented estimator, $r_1^u(k)$, and the unbiased unsegmented estimator, $r_2^u(k)$.

A.2.1 Estimator with Segmentation

$$r_1^u(k) = \frac{1}{M(N-|k|)} \sum_{i=1}^M \sum_{n=1}^{N-|k|} x_i(n)x_i(n+k)$$

Then,

$$\mathbb{E}\{r_1^u(k)\} = \frac{1}{M(N-|k|)} \sum_{i=1}^M \sum_{n=1}^{N-|k|} \mathbb{E}\{x_i(n)x_i(n+k)\}$$

Using stationarity, we obtain

$$\mathbb{E}\{r_1^u(k)\} = r(k)$$

Assume $k \geq 0$ and $k+v \geq 0$, then

$$\begin{aligned} \text{Cov}(r_1^u(k), r_1^u(k+v)) &= \mathbb{E}\{r_1^u(k)r_1^u(k+v)\} - \mathbb{E}\{r_1^u(k)\}\mathbb{E}\{r_1^u(k+v)\} \\ &= \frac{1}{M^2(N-k)(N-k-v)} \sum_{i=1}^M \sum_{n=1}^{(N-k)} \sum_{j=1}^M \sum_{m=1}^{(N-k-v)} \mathbb{E}\{x_i(n)x_i(n+k)x_j(m)x_j(m+k+v)\} - r(k)r(k+v) \\ &= \frac{1}{M^2(N-k)(N-k-v)} \sum_{i=1}^M \sum_{n=1}^{(N-k)} \sum_{m=1}^{(N-k-v)} \mathbb{E}\{x(n)x(n+k)x(m)x(m+k+v)\} \\ &+ \frac{1}{M^2(N-k)(N-k-v)} \sum_{i,j=1}^M \sum_{\substack{n=1 \\ i \neq j}}^{(N-k)} \sum_{m=1}^{(N-k-v)} \mathbb{E}\{x_i(n)x_i(n+k)x_j(m)x_j(m+k+v)\} - r(k)r(k+v) \end{aligned}$$

Using the fourth-order cumulant Eq. (A-1), we get

$$\begin{aligned}\text{Cov}(r_1^u(k), r_1^u(k+v)) &= \frac{1}{M^2(N-k)(N-k-v)} \sum_{i=1}^M \sum_{n=1}^{(N-k)(N-k-v)} \{r(m-n)r(m-n+v) + \\ &\quad r(m+k+v-n)r(m-n-k) + c_{4x}(m-n, k, m-n+k+v)\} + \frac{1}{M}r(k)r(k+v) \\ &+ \frac{1}{M^2(N-k)(N-k-v)} \sum_{i,j=1}^M \sum_{i \neq j}^{(N-k)(N-k-v)} \sum_{n=1}^M \sum_{m=1}^{(N-k)(N-k-v)} \mathbb{E}\{x_i(n)x_i(n+k)x_j(m)x_j(m+k+v)\} - r(k)r(k+v)\end{aligned}$$

Using Eq. (A-2) gives

$$\begin{aligned}\text{Cov}(r_1^u(k), r_1^u(k+v)) &= \frac{1}{M(N-k)(N-k-v)} \sum_{l=-N-k+1}^{(N-k-v-1)} (N - \eta_N(l) - k - v) \\ &\quad \{r(l)r(l+v) + r(l+k+v)r(l-k) + c_{4x}(l, k, l+k+v)\} \\ &+ \frac{1}{M^2(N-k)(N-k-v)} \sum_{i,j=1}^M \sum_{i \neq j}^{(N-k)(N-k-v)} \sum_{n=1}^M \sum_{m=1}^{(N-k)(N-k-v)} \mathbb{E}\{x_i(n)x_i(n+k)x_j(m)x_j(m+k+v)\} \\ &\quad - \frac{(M-1)}{M}r(k)r(k+v)\end{aligned}$$

where $\eta_N(l)$ is defined in Eq. (A-3).

When each segment of data is independent,

$$\mathbb{E}\{x_i(n)x_i(n+k)x_j(m)x_j(m+k+v)\} = r(k)r(k+v), \quad \text{for } i \neq j$$

Using this fact, the second term of $\text{Cov}(r_1^u(k), r_1^u(k+v))$ becomes $\frac{(M-1)}{M}r(k)r(k+v)$; thus,

$$\begin{aligned}\text{Cov}(r_1^u(k), r_1^u(k+v)) &= \frac{1}{M(N-k)(N-k-v)} \sum_{l=-N-k+1}^{(N-k-v-1)} (N - \eta_N(l) - k - v) \\ &\quad \{r(l)r(l+v) + r(l+k+v)r(l-k) + c_{4x}(l, k, l+k+v)\} \quad (\text{A-7})\end{aligned}$$

When each segment is a part of a single realization, i.e.,

$$\begin{aligned}&\mathbb{E}\{x_i(n)x_i(n+k)x_j(m)x_j(m+k+v)\} \\ &= \mathbb{E}\{x(N(i-1)+n)x(N(i-1)+n+k)x(N(j-1)+m)x(N(j-1)+m+k+v)\} \\ &= r(k)r(k+v) + r(N(j-i)+m-n)r(N(j-i)+m-n+v) + r(N(j-i)+m-n+k+v) \\ &\quad r(N(j-i)+m-n-k) + c_{4x}(k, N(j-i)+m-n, N(j-i)+m-n+k+v)\end{aligned}$$

where we have used Eq. (A-1) to obtain the last line; thus, the covariance function can be rewritten as

$$\begin{aligned}
\text{Cov}(r_1^u(k), r_1^u(k+v)) &= \frac{1}{M(N-k)(N-k-v)} \sum_{l=-N-k+1}^{(N-k-v-1)} (N - \eta_N(l) - k - v) \\
&\quad \{r(l)r(l+v) + r(l+k+v)r(l-k) + c_{4x}(l, k, l+k+v)\} \\
&+ \frac{1}{M^2(N-k)(N-k-v)} \sum_{i,j=1}^M \sum_{i \neq j}^{M-k} \sum_{n=1}^{(N-k)} \sum_{m=1}^{(N-k-v)} \{r(N(j-i)+m-n)r(N(j-i)+m-n+v) + \\
&r(N(j-i)+m-n+k+v)r(N(j-i)+m-n-k) + c_{4x}(k, N(j-i)+m-n, N(j-i)+m-n+k+v)\}
\end{aligned}$$

As in Eq. (A-2), letting $p = m - n$ and $q = j - i$ gives

$$\begin{aligned}
\text{Cov}(r_1^u(k), r_1^u(k+v)) &= \frac{1}{M(N-k)(N-k-v)} \sum_{l=-N-k+1}^{(N-k-v-1)} (N - \eta_N(l) - k - v) \\
&\quad \{r(l)r(l+v) + r(l+k+v)r(l-k) + c_{4x}(l, k, l+k+v)\} \\
&+ \frac{1}{M^2(N-k)(N-k-v)} \sum_{q=-M+1, q \neq 0}^{(M-1)} \sum_{p=-N-k+1}^{(N-k-v-1)} (N - \eta_N(p) - k - v) (M - |q|) \{r(Nq+p) \\
&r(Nq+p+v) + r(Nq+p+k+v)r(Nq+p-k) + c_{4x}(k, Nq+p, Nq+p+k+v)\} \quad (\text{A-8})
\end{aligned}$$

A.2.2 Estimator without Segmentation

$$r_2^u(k) = \frac{1}{MN - |k|} \sum_{n=k}^{MN-|k|} x(n)x(n+k)$$

Then

$$\mathbb{E}\{r_2^u(k)\} = \frac{1}{MN - |k|} \sum_{n=1}^{MN-|k|} \mathbb{E}\{x(n)x(n+k)\} = r(k)$$

Assume that $k \geq 0$ and $k + v \geq 0$, then

$$\begin{aligned}
\text{Cov}(r_2^u(k), r_2^u(k+v)) &= \mathbb{E}\{r_2^u(k)r_2^u(k+v)\} - \mathbb{E}\{r_2^u(k)\} \mathbb{E}\{r_2^u(k+v)\} \\
&= \frac{1}{(MN-k)(MN-k-v)} \sum_{n=1}^{(MN-k)} \sum_{m=1}^{(MN-k-v)} \mathbb{E}\{x(n)x(n+k)x(m)x(m+k+v)\} - r(k)r(k+v)
\end{aligned}$$

Using Eq. (A-1), we obtain

$$\begin{aligned}
\text{Cov}(r_2^u(k), r_2^u(k+v)) &= \frac{1}{(MN-k)(MN-k-v)} \sum_{n=1}^{(MN-k)} \sum_{m=1}^{(MN-k-v)} \{r(m-n) \cdot \\
&r(m-n+v) + r(m-n+k+v)r(m-n-k) + c_{4x}(m-n, k, m-n+k+v)\}
\end{aligned}$$

and applying Eq. (A-2) gives

$$\text{Cov}(r_2^u(k), r_2^u(k+v)) = \frac{1}{(MN-k)(MN-k-v)} \sum_{l=-(MN-k)+1}^{MN-k-v-1} \{MN - \eta_{MN}(l) - k - v\} \\ \{r(l)r(l+v) + r(l+k+v)r(l-k) + c_{4x}(l, k, l+k+v)\} \quad (\text{A-9})$$

where $\eta_{MN}(l)$ is defined in Eq. (A-3).

A.3 Biased Estimators of the 1-D Diagonal-Slice of the Fourth-Order Cumulant

In this section, we derive the mean functions of the biased segmented cumulant estimator, $d_1^b(k)$, and the biased unsegmented estimator, $d_2^b(k)$.

A.3.1 Estimator with Segmentation

$$d_1^b(k) = \frac{1}{M} \sum_{i=1}^M \tilde{d}_i^b(k)$$

where $\tilde{d}_i^b(k)$ denotes the estimated cumulant in the i -th segment, i.e.,

$$\tilde{d}_i^b(k) = \frac{1}{N} \sum_{n=1}^{N-|k|} x_i(n)x_i^3(n+k) - 3r_1^b(0)r_1^b(k) \quad i = 1, 2, \dots, M$$

where $r_1^b(0)$ and $r_1^b(k)$ denote the biased segmented correlation estimates as in Section A.1.1. Then,

$$\begin{aligned} \mathbb{E}\{d_1^b(k)\} &= \frac{1}{M} \sum_{i=1}^M \frac{1}{N} \sum_{n=1}^{N-|k|} \mathbb{E}\{x_i(n)x_i^3(n+k)\} - 3r(0) \left(\frac{N-|k|}{N} r(k) \right) \\ &= \frac{N-|k|}{N} (\mathbb{E}\{x(n)x^3(n+k)\} - 3r(0) \cdot r(k)) \end{aligned}$$

A.3.2 Estimator without Segmentation

$$d_2^b(k) = \frac{1}{MN} \sum_{n=1}^{MN-|k|} x(n)x^3(n+k) - 3r_2^b(0)r_2^b(k)$$

where $r_2^b(0)$ and $r_2^b(k)$ denote the biased unsegmented correlation estimates as in Section A.1.2. Then

$$\begin{aligned} \mathbb{E}\{d_2^b(k)\} &= \frac{1}{MN} \sum_{n=1}^{MN-|k|} \mathbb{E}\{x(n)x^3(n+k)\} - 3r(0) \left(\frac{MN-|k|}{MN} r(k) \right) \\ &= \frac{MN-|k|}{MN} (\mathbb{E}\{x(n)x^3(n+k)\} - 3r(0) \cdot r(k)) \end{aligned}$$

A.4 Unbiased Estimators of the 1-D Diagonal-Slice of the Fourth-Order Cumulant

In this section, we derive the mean and covariance functions of the unbiased segmented cumulant estimator, $d_1^u(k)$, and the unbiased unsegmented estimator, $d_2^u(k)$.

A.4.1 Estimator with Segmentation

$$d_1^u(k) = \frac{1}{M} \sum_{i=1}^M \tilde{d}_i^u(k)$$

where $\tilde{d}_i^u(k)$ denotes the unbiased cumulant estimate in the i -th segment, i.e.,

$$\tilde{d}_i^u(k) = \frac{1}{N - |k|} \sum_{n=1}^{N-|k|} x_i(n)x_i^3(n+k) - 3r_i^u(0)r_i^u(k) \quad i = 1, 2, \dots, M$$

where $r_i^u(0)$ and $r_i^u(k)$ denote the unbiased segmented correlation estimates as in Section A.2.1. Then

$$\begin{aligned} \mathbb{E}\{d_1^u(k)\} &= \frac{1}{M} \sum_{i=1}^M \frac{1}{N - |k|} \sum_{n=1}^{N-|k|} \mathbb{E}\{x_i(n)x_i^3(n+k)\} - 3r(0) \cdot r(k) \\ &= \mathbb{E}\{x(n)x^3(n+k)\} - 3r(0) \cdot r(k) = c4(k) \end{aligned}$$

Assume that $k \geq 0, k + v \geq 0$, then

$$\begin{aligned} \text{Cov}(d_1^u(k), d_1^u(k+v)) &= \mathbb{E}\{d_1^u(k)d_1^u(k+v)\} - \mathbb{E}\{d_1^u(k)\}\mathbb{E}\{d_1^u(k+v)\} \\ &= \frac{1}{M^2(N-k)(N-k-v)} \sum_{i=1}^M \sum_{j=1}^M \sum_{n=1}^{(N-k)} \sum_{m=1}^{(N-k-v)} \mathbb{E}\{x_i(n)x_i^3(n+k)x_j(m)x_j^3(m+k+v)\} \\ &\quad - \frac{9}{M^3(N-k)N(N-k-v)} \sum_{i=1}^M \sum_{j=1}^M \sum_{p=1}^M \sum_{n=1}^{(N-k)} \sum_{m=1}^N \sum_{q=1}^{(N-k-v)} \mathbb{E}\{x_i(n)x_i^3(n+k)x_j^2(m)x_p(q)x_p(q+k+v)\} \\ &\quad - \frac{3}{M^3(N-k-v)N(N-k)} \sum_{i=1}^M \sum_{j=1}^M \sum_{p=1}^M \sum_{n=1}^{(N-k-v)} \sum_{m=1}^N \sum_{q=1}^{(N-k)} \mathbb{E}\{x_i(n)x_i^3(n+k+v)x_j^2(m)x_p(q)x_p(q+k)\} \\ &\quad + \frac{9}{M^4(N-k-v)N^2(N-k)} \sum_{i=1}^M \sum_{j=1}^M \sum_{p=1}^M \sum_{r=1}^M \sum_{n=1}^{(N-k-v)} \sum_{m=1}^N \sum_{q=1}^N \sum_{s=1}^N \mathbb{E}\{x_i(n)x_i(n+k+v)x_j(m)x_j(m+k) \\ &\quad \quad \quad x_p^2(q)x_r^2(s)\} - c4(k)c4(k+v) \end{aligned}$$

A.4.2 Estimator without Segmentation

$$d_2^u(k) = \frac{1}{MN - |k|} \sum_{n=1}^{MN-|k|} x(n)x^3(n+k) - 3r_2^u(0)r_2^u(k)$$

where $r_2^u(0)$ and $r_2^u(k)$ denote the unbiased unsegmented correlation estimates. Then

$$\begin{aligned}\mathbb{E}\{d_2^u(k)\} &= \frac{1}{MN - |k|} \sum_{n=1}^{MN-|k|} \mathbb{E}\{x(n)x^3(n+k)\} - 3r(0) \cdot r(k) \\ &= \mathbb{E}\{x(n)x^3(n+k)\} - 3r(0) \cdot r(k) = c4(k)\end{aligned}$$

Assume that $k \geq 0, k + v \geq 0$, then

$$\begin{aligned}\text{Cov}(d_2^u(k), d_2^u(k+v)) &= \mathbb{E}\{d_2^u(k)d_2^u(k+v)\} - \mathbb{E}\{d_2^u(k)\}\mathbb{E}\{d_2^u(k+v)\} \\ &= \frac{1}{(MN-k)(MN-k-v)} \sum_{n=1}^{(MN-k)(MN-k-v)} \sum_{m=1}^{(MN-k)(MN-k-v)} \mathbb{E}\{x(n)x^3(n+k)x(m)x^3(m+k+v)\} \\ &\quad - \frac{3}{(MN-k)(MN)(MN-k-v)} \sum_{n=1}^{(MN-k)(MN)(MN-k-v)} \sum_{i=1}^{(MN-k)(MN)(MN-k-v)} \mathbb{E}\{x(n)x^3(n+k)x^2(i)x(m)x^3(m+k+v)\} \\ &\quad - \frac{3}{(MN-k-v)(MN)(MN-k)} \sum_{n=1}^{(MN-k-v)(MN)(MN-k)} \sum_{i=1}^{(MN-k-v)(MN)(MN-k)} \mathbb{E}\{x(n)x^3(n+k+v)x^2(i)x(m)x^3(m+k)\} \\ &\quad + \frac{9}{(MN)^2(MN-k)(MN-k-v)} \sum_{i=1}^{(MN)(MN)(MN-k)(MN-k-v)} \sum_{n=1}^{(MN)(MN)(MN-k)(MN-k-v)} \mathbb{E}\{x^2(i)x(n)x^3(n+k)x^2(j)x(m) \\ &\quad \quad \quad x^3(m+k+v)\} - c4(k)c4(k+v)\end{aligned}$$

Table 1: Means and standard deviations (std) of accumulative singular values for fourth-order cumulants using one realization (Fig. 4)¹

Amplitudes	Number of singular value						
	1	2	3	4	5	6	
[1(0) 1(0)]	mean	0.2882	0.5539	0.7447	0.9152	0.9345	0.9504
	std	0.0099	0.0159	0.0213	0.0252	0.0191	0.0140
[1(0) 0.7(-3)]	mean	0.3816	0.7161	0.8071	0.8847	0.9123	0.9359
	std	0.0176	0.0332	0.0301	0.0336	0.0247	0.0189
[1(0) 0.65(-3.7)]	mean	0.4095	0.7691	0.8389	0.8982	0.9206	0.9400
	std	0.0180	0.0335	0.0278	0.0294	0.0226	0.0183
[1(0) 0.625(-4)]	mean	0.4021	0.7549	0.8238	0.8828	0.9082	0.9306
	std	0.0197	0.0370	0.0320	0.0314	0.0263	0.0224
[1(0) 0.6(-4.4)]	mean	0.4164	0.7831	0.8424	0.8932	0.9159	0.9362
	std	0.0141	0.0263	0.0219	0.0235	0.0200	0.0179
[1(0) 0.575(-4.8)]	mean	0.4255	0.7994	0.8488	0.8914	0.9137	0.9338
	std	0.0216	0.0396	0.0324	0.0294	0.0239	0.0201
[1(0) 0.55(-5.2)]	mean	0.4314	0.8108	0.8572	0.8977	0.9199	0.9394
	std	0.0145	0.0274	0.0217	0.0199	0.0169	0.0152
[1(0) 0.525(-5.6)]	mean	0.4352	0.8184	0.8588	0.8935	0.9153	0.9345
	std	0.0177	0.0335	0.0282	0.0265	0.0208	0.0170

1 : In tables 1-4, [a(b) c(d)] denotes amplitudes and their associated local SNR's (in parentheses) for first sinusoid [a(b)] and second sinusoid [c(d)]

Table 2: Means and standard deviations (std) of accumulative singular values for fourth-order cumulants using independent realizations (Fig. 5)

Amplitudes	Number of singular value						
	1	2	3	4	5	6	
[1(0) 1(0)]	mean	0.2831	0.5478	0.7369	0.9081	0.9286	0.9465
	std	0.0106	0.0180	0.0242	0.0299	0.0221	0.0157
[1(0) 0.7(-3)]	mean	0.3848	0.7220	0.8088	0.8832	0.9079	0.9303
	std	0.0193	0.0356	0.0293	0.0300	0.0217	0.0156
[1(0) 0.65(-3.7)]	mean	0.4045	0.7597	0.8287	0.8884	0.9132	0.9358
	std	0.0239	0.0439	0.0356	0.0336	0.0244	0.0167
[1(0) 0.625(-4)]	mean	0.4090	0.7688	0.8307	0.8845	0.9095	0.9316
	std	0.0222	0.0409	0.0342	0.0330	0.0249	0.0196
[1(0) 0.6(-4.4)]	mean	0.4213	0.7919	0.8446	0.8909	0.9136	0.9336
	std	0.0240	0.0445	0.0359	0.0318	0.0248	0.0197
[1(0) 0.575(-4.8)]	mean	0.4254	0.7996	0.8459	0.8866	0.9100	0.9310
	std	0.0217	0.0398	0.0321	0.0288	0.0218	0.0167
[1(0) 0.55(-5.2)]	mean	0.4336	0.8153	0.8578	0.8954	0.9181	0.9384
	std	0.0247	0.0453	0.0348	0.0288	0.0220	0.0169
[1(0) 0.525(-5.6)]	mean	0.4353	0.8188	0.8598	0.8961	0.9167	0.9356
	std	0.0216	0.0396	0.0301	0.0243	0.0204	0.0175

Table 3: Means and standard deviations (std) of accumulative singular values for correlations using one realization (Fig. 6)

Amplitudes	Number of singular value								
		1	2	3	4	5	6	7	8
[1(0) 1(0)]	mean	0.2214	0.4292	0.5827	0.7238	0.7898	0.8485	0.8754	0.9018
	std	0.0027	0.0043	0.0061	0.0072	0.0051	0.0042	0.0033	0.0027
[1(0) 0.7(-3)]	mean	0.2385	0.4480	0.5657	0.6685	0.7471	0.8168	0.8496	0.8818
	std	0.0049	0.0092	0.0092	0.0093	0.0072	0.0062	0.0050	0.0040
[1(0) 0.65(-3.7)]	mean	0.2429	0.4563	0.5636	0.6582	0.7399	0.8118	0.8450	0.8777
	std	0.0045	0.0086	0.0082	0.0079	0.0051	0.0045	0.0037	0.0032
[1(0) 0.625(-4)]	mean	0.2444	0.4593	0.5623	0.6538	0.7370	0.8096	0.8433	0.8763
	std	0.0043	0.0082	0.0086	0.0086	0.0060	0.0052	0.0043	0.0036
[1(0) 0.6(-4.4)]	mean	0.2481	0.4664	0.5640	0.6517	0.7345	0.8066	0.8408	0.8745
	std	0.0050	0.0096	0.0091	0.0084	0.0059	0.0054	0.0044	0.0037
[1(0) 0.575(-4.8)]	mean	0.2501	0.4705	0.5634	0.6486	0.7321	0.8044	0.8388	0.8726
	std	0.0063	0.0118	0.0103	0.0080	0.0055	0.0043	0.0036	0.0030
[1(0) 0.55(-5.2)]	mean	0.2531	0.4761	0.5667	0.6503	0.7324	0.8034	0.8382	0.8723
	std	0.0060	0.0113	0.0095	0.0080	0.0070	0.0063	0.0054	0.0047
[1(0) 0.525(-5.6)]	mean	0.2552	0.4803	0.5691	0.6513	0.7316	0.8008	0.8362	0.8709
	std	0.0048	0.0092	0.0073	0.0066	0.0060	0.0058	0.0045	0.0033

Table 4: Means and standard deviations (std) of accumulative singular values for correlations using independent realizations (Fig. 7)

Amplitudes	Number of singular value								
		1	2	3	4	5	6	7	8
[1(0) 1(0)]	mean	0.2235	0.4340	0.5890	0.7317	0.7946	0.8506	0.8772	0.9027
	std	0.0026	0.0058	0.0073	0.0090	0.0066	0.0052	0.0041	0.0032
[1(0) 0.7(-3)]	mean	0.2409	0.4524	0.5727	0.6776	0.7528	0.8196	0.8517	0.8826
	std	0.0040	0.0075	0.0090	0.0107	0.0066	0.0044	0.0035	0.0029
[1(0) 0.65(-3.7)]	mean	0.2461	0.4623	0.5721	0.6687	0.7462	0.8148	0.8478	0.8796
	std	0.0049	0.0093	0.0096	0.0104	0.0070	0.0053	0.0041	0.0032
[1(0) 0.625(-4)]	mean	0.2483	0.4667	0.5714	0.6641	0.7426	0.8117	0.8453	0.8775
	std	0.0040	0.0076	0.0087	0.0101	0.0065	0.0049	0.0039	0.0032
[1(0) 0.6(-4.4)]	mean	0.2509	0.4718	0.5712	0.6599	0.7398	0.8096	0.8435	0.8760
	std	0.0040	0.0077	0.0082	0.0089	0.0064	0.0058	0.0045	0.0035
[1(0) 0.575(-4.8)]	mean	0.2536	0.4770	0.5719	0.6576	0.7375	0.8071	0.8414	0.8744
	std	0.0044	0.0084	0.0093	0.0092	0.0057	0.0045	0.0036	0.0030
[1(0) 0.55(-5.2)]	mean	0.2564	0.4823	0.5727	0.6556	0.7357	0.8050	0.8398	0.8732
	std	0.0052	0.0100	0.0091	0.0078	0.0059	0.0055	0.0042	0.0032
[1(0) 0.525(-5.6)]	mean	0.2586	0.4866	0.5742	0.6551	0.7340	0.8021	0.8373	0.8712
	std	0.0043	0.0081	0.0067	0.0057	0.0048	0.0049	0.0039	0.0033

Table 5: Means and standard deviations (std) of accumulative singular values for fourth-order cumulants using one realization (Fig. 10)

Frequencies	Number of singular value						
		1	2	3	4	5	6
[0.1 0.2]	mean	0.2882	0.5539	0.7447	0.9152	0.9345	0.9504
	std	0.0099	0.0159	0.0213	0.0252	0.0191	0.0140
[0.1 0.14]	mean	0.3396	0.6629	0.7981	0.9092	0.9313	0.9516
	std	0.0120	0.0235	0.0273	0.0318	0.0224	0.0155
[0.1 0.13]	mean	0.3995	0.7569	0.8479	0.9086	0.9310	0.9507
	std	0.0128	0.0238	0.0254	0.0265	0.0192	0.0156
[0.1 0.1275]	mean	0.4100	0.7716	0.8482	0.8989	0.9258	0.9495
	std	0.0178	0.0341	0.0344	0.0316	0.0197	0.0133
[0.1 0.125]	mean	0.4232	0.7921	0.8610	0.90442	0.9282	0.9491
	std	0.0170	0.0312	0.0320	0.0312	0.0221	0.0157
[0.1 0.1225]	mean	0.4393	0.8161	0.8738	0.9090	0.9303	0.9487
	std	0.0089	0.0163	0.0189	0.0185	0.0138	0.0128
[0.1 0.12]	mean	0.4469	0.8260	0.8735	0.9059	0.9284	0.9469
	std	0.0142	0.0266	0.0262	0.0230	0.0181	0.0167
[0.1 0.1175]	mean	0.4579	0.8430	0.8820	0.9098	0.9315	0.9482
	std	0.0193	0.0352	0.0284	0.0219	0.0164	0.0133

Table 6: Means and standard deviations (std) of accumulative singular values for fourth-order cumulants using independent realizations (Fig. 11)

Frequencies	Number of singular value						
		1	2	3	4	5	6
[0.1 0.2]	mean	0.2831	0.5478	0.7369	0.9081	0.9286	0.9465
	std	0.0106	0.0180	0.0242	0.0299	0.0221	0.0157
[0.1 0.14]	mean	0.3387	0.6636	0.7976	0.9092	0.9311	0.9508
	std	0.0099	0.0196	0.0226	0.0264	0.0176	0.0141
[0.1 0.13]	mean	0.3959	0.7522	0.8453	0.9085	0.9300	0.9490
	std	0.0160	0.0292	0.0273	0.0265	0.0187	0.0136
[0.1 0.1275]	mean	0.4091	0.7706	0.8507	0.9018	0.9250	0.9457
	std	0.0161	0.0300	0.0292	0.0290	0.0205	0.0148
[0.1 0.125]	mean	0.4219	0.7889	0.8568	0.9003	0.9260	0.9487
	std	0.0171	0.0317	0.0331	0.0327	0.0234	0.0169
[0.1 0.1225]	mean	0.4349	0.8083	0.8665	0.9047	0.9279	0.9474
	std	0.0177	0.0324	0.0318	0.0281	0.0203	0.0175
[0.1 0.12]	mean	0.4449	0.8231	0.8742	0.9076	0.9286	0.9462
	std	0.0168	0.0311	0.0277	0.0238	0.0183	0.0158
[0.1 0.1175]	mean	0.4527	0.8341	0.8771	0.9076	0.9278	0.9448
	std	0.0183	0.0343	0.0301	0.0239	0.0199	0.0173

Table 7: Means and standard deviations (std) of accumulative singular values for correlations using one realization (Fig. 12)

Frequencies	Number of singular value								
		1	2	3	4	5	6	7	8
[0.1 0.2]	mean	0.2214	0.4292	0.5827	0.7238	0.7898	0.8485	0.8754	0.9018
	std	0.0027	0.0043	0.0061	0.0072	0.0051	0.0042	0.0033	0.0027
[0.1 0.14]	mean	0.2616	0.5122	0.6249	0.7233	0.7891	0.8463	0.8735	0.9001
	std	0.0041	0.0084	0.0101	0.0109	0.0078	0.0061	0.0051	0.0043
[0.1 0.13]	mean	0.3052	0.5808	0.6623	0.7290	0.7940	0.8462	0.8732	0.8999
	std	0.0040	0.0075	0.0088	0.0072	0.0057	0.0054	0.0046	0.0040
[0.1 0.1275]	mean	0.3153	0.5956	0.6695	0.7351	0.7985	0.8467	0.8735	0.8997
	std	0.0048	0.0089	0.0091	0.0070	0.0055	0.0055	0.0047	0.0039
[0.1 0.125]	mean	0.3260	0.6117	0.6801	0.7430	0.8039	0.8469	0.8737	0.9001
	std	0.0054	0.0097	0.0085	0.0074	0.0064	0.0061	0.0053	0.0045
[0.1 0.1225]	mean	0.3356	0.6257	0.6923	0.7529	0.8084	0.8461	0.8731	0.8996
	std	0.0046	0.0087	0.0064	0.0054	0.0055	0.0055	0.0047	0.0041
[0.1 0.12]	mean	0.3449	0.6397	0.7061	0.7663	0.8143	0.8472	0.8738	0.9000
	std	0.0049	0.0090	0.0068	0.0056	0.0056	0.0055	0.0047	0.0040
[0.1 0.1175]	mean	0.3515	0.6496	0.7164	0.7768	0.8183	0.8473	0.8737	0.8999
	std	0.0048	0.0089	0.0066	0.0052	0.0054	0.0054	0.0047	0.0041

Table 8: Means and standard deviations (std) of accumulative singular values for correlations using independent realizations (Fig. 13)

Frequencies	Number of singular value								
		1	2	3	4	5	6	7	8
[0.1 0.2]	mean	0.2235	0.4340	0.5890	0.7317	0.7946	0.8506	0.8772	0.9027
	std	0.0026	0.0058	0.0073	0.0090	0.0066	0.0052	0.0041	0.0032
[0.1 0.14]	mean	0.2650	0.5193	0.6333	0.7327	0.7954	0.8501	0.8767	0.9021
	std	0.0036	0.0073	0.0085	0.0091	0.0068	0.0058	0.0047	0.0037
[0.1 0.13]	mean	0.3100	0.5902	0.6720	0.7374	0.7996	0.8503	0.8768	0.9023
	std	0.0047	0.0086	0.0088	0.0074	0.0050	0.0043	0.0036	0.0031
[0.1 0.1275]	mean	0.3209	0.6063	0.6799	0.7422	0.8030	0.8502	0.8767	0.9022
	std	0.0044	0.0081	0.0083	0.0067	0.0058	0.0055	0.0044	0.0037
[0.1 0.125]	mean	0.3312	0.6213	0.6881	0.7489	0.8078	0.8505	0.8768	0.9022
	std	0.0045	0.0084	0.0081	0.0068	0.0061	0.0059	0.0048	0.0039
[0.1 0.1225]	mean	0.3404	0.6347	0.6983	0.7566	0.8119	0.8498	0.8762	0.9017
	std	0.0050	0.0093	0.0071	0.0058	0.0054	0.0054	0.0046	0.0039
[0.1 0.12]	mean	0.3493	0.6479	0.7109	0.7684	0.8170	0.8500	0.8763	0.9018
	std	0.0053	0.0099	0.0071	0.0056	0.0058	0.0058	0.0047	0.0038
[0.1 0.1175]	mean	0.3573	0.6601	0.7230	0.7802	0.8214	0.8501	0.8762	0.9018
	std	0.0039	0.0072	0.0056	0.0056	0.0055	0.0055	0.0046	0.0039

Table 9: Means and standard deviations of estimated amplitudes for Example 1¹

True Amplitude	1.0	0.6	
Correlation (2 harmonics)			
LS	1.0173 (0.0069)	0.6289 (0.0051)	
TLS	1.0211 (0.0068)	0.6313 (0.0052)	
CTLS	1.0168 (0.0068)	0.6275 (0.0052)	
Correlations (3 harmonics)			
LS	1.0144 (0.0069)	0.6248 (0.0051)	0.3207 (0.0108)
TLS	1.0159 (0.0069)	0.6257 (0.0051)	0.3208 (0.0108)
CTLS	1.0144 (0.0068)	0.6232 (0.0052)	0.3205 (0.0108)
Cumulants			
LS	1.0036 (0.0100)	0.6095 (0.0236)	
TLS	1.0038 (0.0101)	0.6096 (0.0236)	
CTLS	1.0036 (0.0108)	0.6078 (0.0237)	

1 : In tables 9-12, $m(s)$ denotes mean value(m) and standard deviation(s) of estimated amplitude

Table 10: Means and standard deviations of estimated amplitudes using one realization for Example 2

True Amplitude	1.0	1.0	
Correlation (2 harmonics)			
LS	1.0469 (0.0115)	1.0489 (0.0089)	
TLS	1.0721 (0.0118)	1.0743 (0.0097)	
CTLS	1.0486 (0.0113)	1.0520 (0.0089)	
Correlations (3 harmonics)			
LS	1.0404 (0.0117)	1.0447 (0.0088)	0.5683 (0.0190)
TLS	1.0495 (0.0118)	1.0538 (0.0088)	0.5715 (0.0193)
CTLS	1.0425 (0.0113)	1.0437 (0.0090)	0.5677 (0.0193)
Cumulants			
LS	1.0132 (0.0307)	1.0109 (0.0257)	
TLS	1.0143 (0.0307)	1.0119 (0.0257)	
CTLS	1.0127 (0.0304)	1.0104 (0.0255)	

Table 11: Means and standard deviations of estimated amplitudes using independent realizations for Example 2

True Amplitude	1.0	1.0	
Correlation (2 harmonics)			
LS	1.0435 (0.0093)	1.0506 (0.0132)	
TLS	1.0662 (0.0095)	1.0734 (0.0126)	
CTLS	1.0434 (0.0093)	1.0504 (0.0133)	
Correlations (3 harmonics)			
LS	1.0331 (0.0096)	1.0452 (0.0134)	0.5548 (0.0174)
TLS	1.0415 (0.0099)	1.0537 (0.0132)	0.5578 (0.0176)
CTLS	1.0351 (0.0092)	1.0438 (0.0133)	0.5533 (0.0173)
Cumulants			
LS	0.9865 (0.0319)	0.9899 (0.0283)	
TLS	0.9879 (0.0315)	0.9914 (0.0281)	
CTLS	0.9885 (0.0314)	0.9895 (0.0282)	

Table 12: Means and standard deviations of estimated amplitudes for Example 3

True Amplitude	1.0	1.0	
Correlation (2 harmonics)			
LS	1.0899 (0.0189)	1.0991 (0.0215)	
TLS	1.1800 (0.0197)	1.1896 (0.0217)	
CTLS	1.0909 (0.0190)	1.0997 (0.0215)	
Correlations (3 harmonics)			
LS	1.0664 (0.0191)	1.0851 (0.0217)	0.7996 (0.0234)
TLS	1.0975 (0.0183)	1.1175 (0.0216)	0.8201 (0.0240)
CTLS	1.0739 (0.0190)	1.0859 (0.0216)	0.7969 (0.0234)
Cumulants			
LS	0.9928 (0.0534)	0.9985 (0.0696)	
TLS	1.0024 (0.0475)	0.9979 (0.0631)	
CTLS	0.9887 (0.0542)	0.9887 (0.0697)	

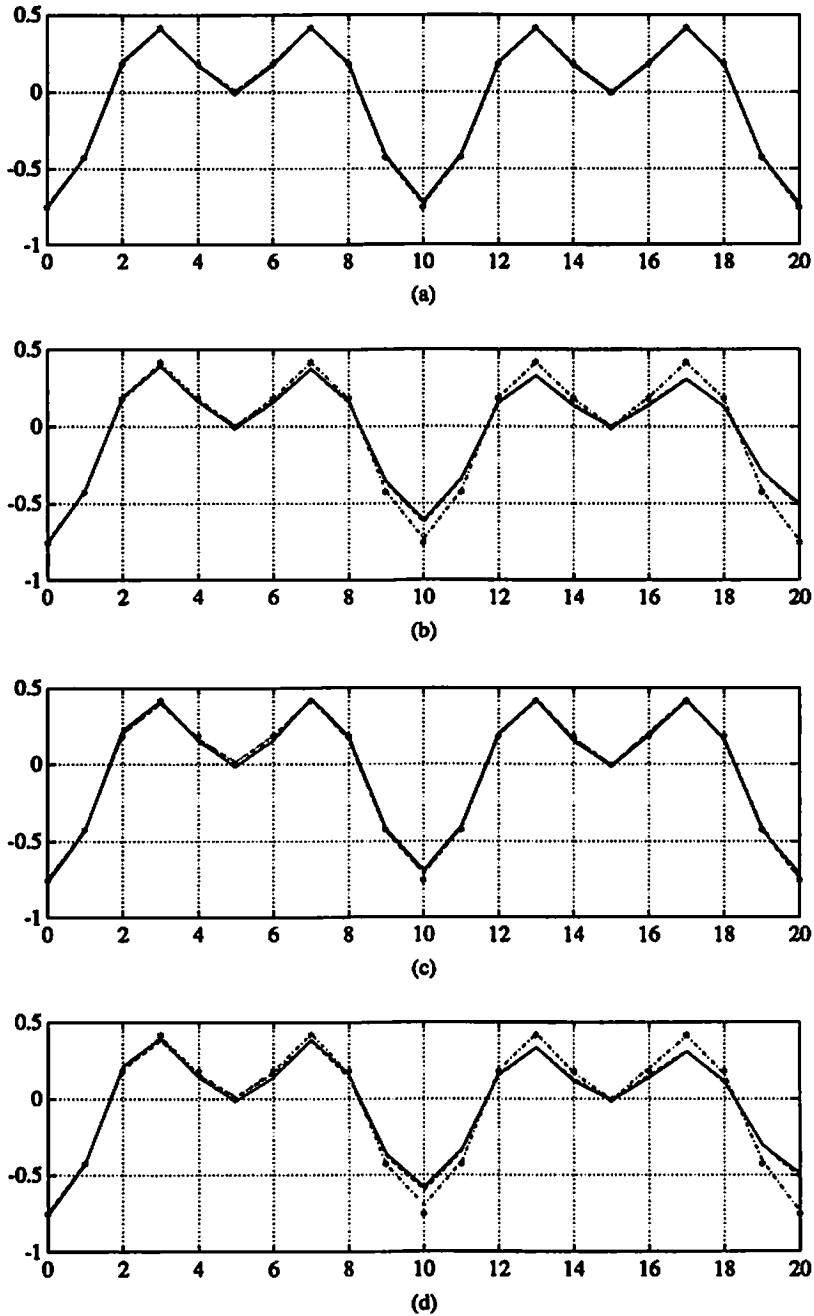


Figure 1: Unbiased and biased estimates of 1-D diagonal slice fourth-order cumulants : (a) unbiased estimates for 0 dB local SNR, (b) biased estimates for 0 dB local SNR, (c) unbiased estimates for -3 dB local SNR, (d) biased estimates for -3 dB local SNR. In panels (a)-(d), the starred points correspond to the no-noise theoretical cumulants, the dashed and dash-dotted lines correspond to estimates with and without segmentation in one realization, and the solid line corresponds to estimates using independent realizations.

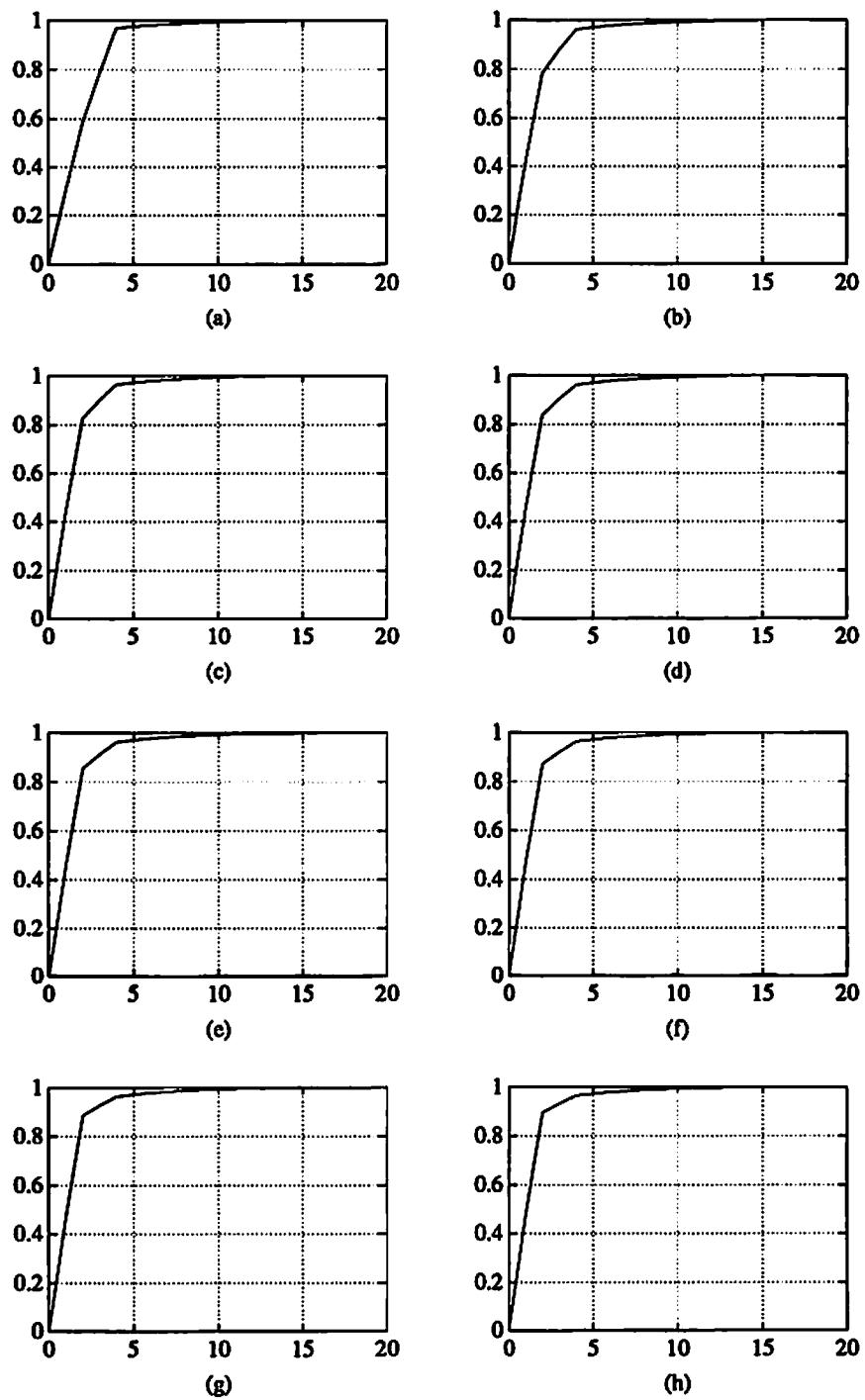


Figure 2: Accumulative singular values of cumulants using one realization when the fixed amplitude sinusoid at 0.1 has a local SNR of 5 dB and the amplitude of the sinusoid at 0.2 has a local SNR (dB) of : (a) 5 (b) 1.7 (c) 1 (d) 0.7 (e) 0.3 (f) 0 (g) -0.4 (h) -0.8

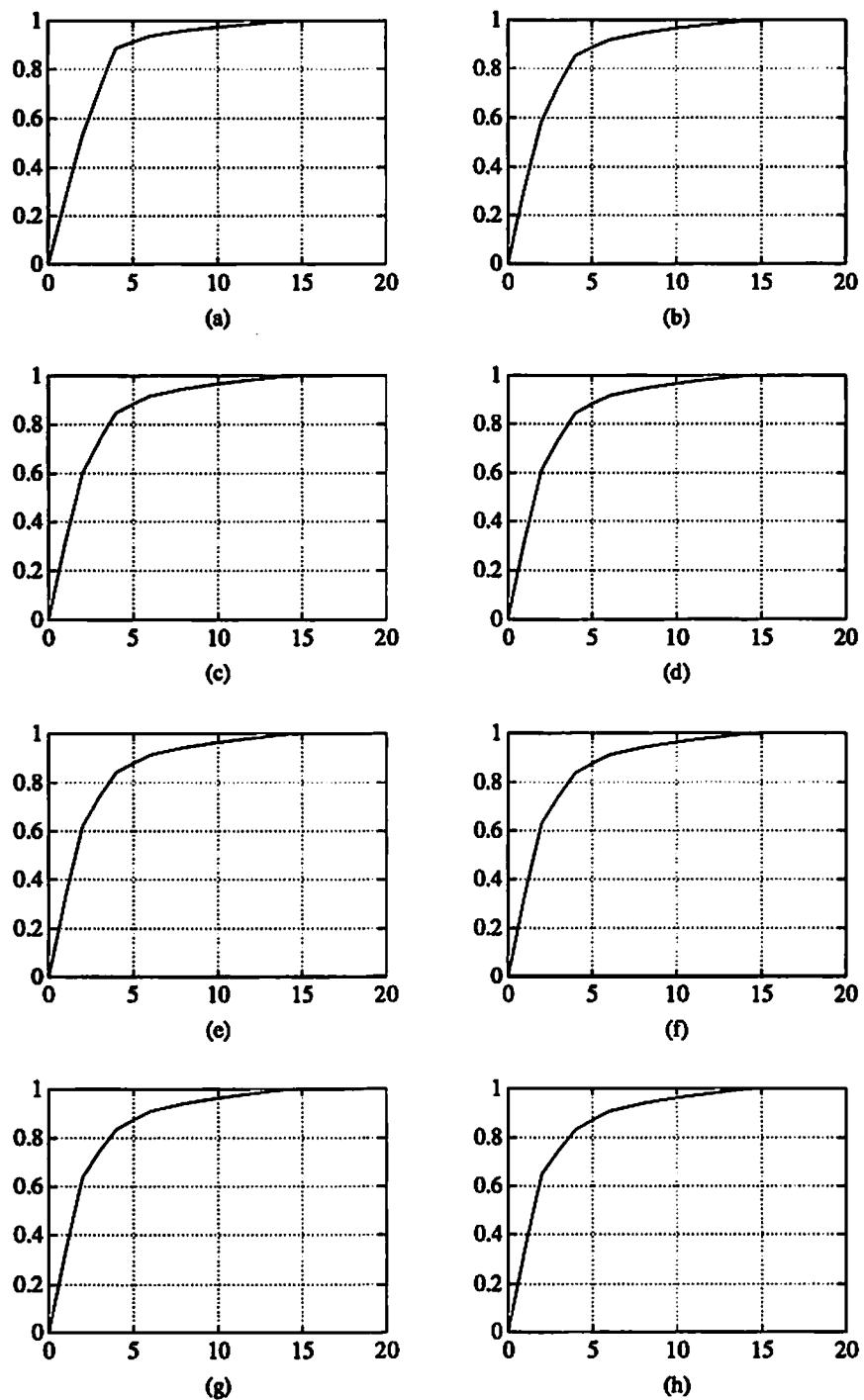


Figure 3: Accumulative singular values of correlations using one realization when the fixed amplitude sinusoid at 0.1 has a local SNR of 5 dB and the amplitude of the sinusoid at 0.2 has a local SNR (dB) of : (a) 5 (b) 1.7 (c) 1 (d) 0.7 (e) 0.3 (f) 0 (g) -0.4 (h) -0.8

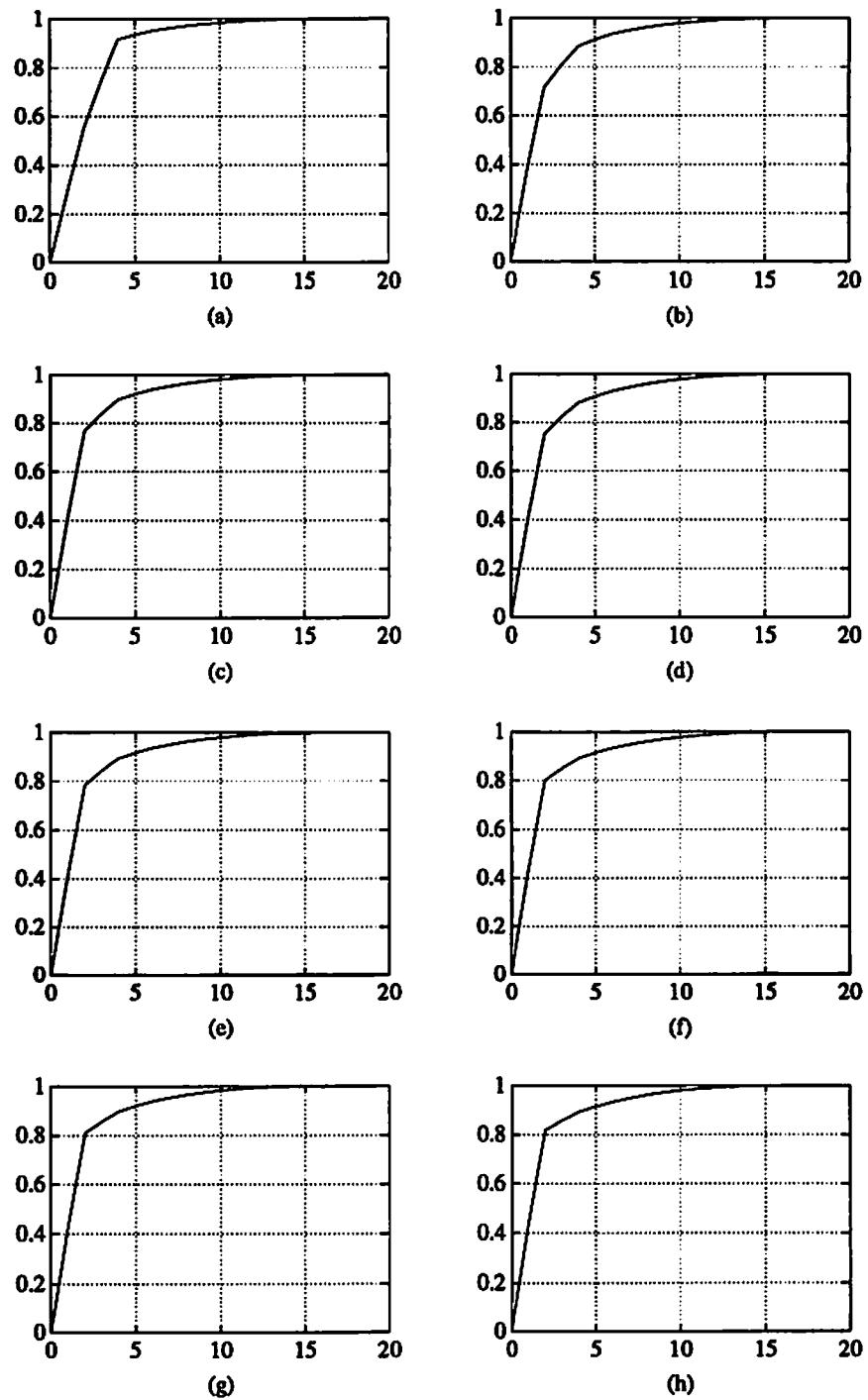


Figure 4: Accumulative singular values of cumulants using one realization when the fixed amplitude sinusoid at 0.1 has a local SNR of 0 dB and the amplitude of the sinusoid at 0.2 has a local SNR (dB) of : (a) 0 (b) -3 (c) -3.7 (d) -4 (e) -4.4 (f) -4.8 (g) -5.2 (h) -5.6

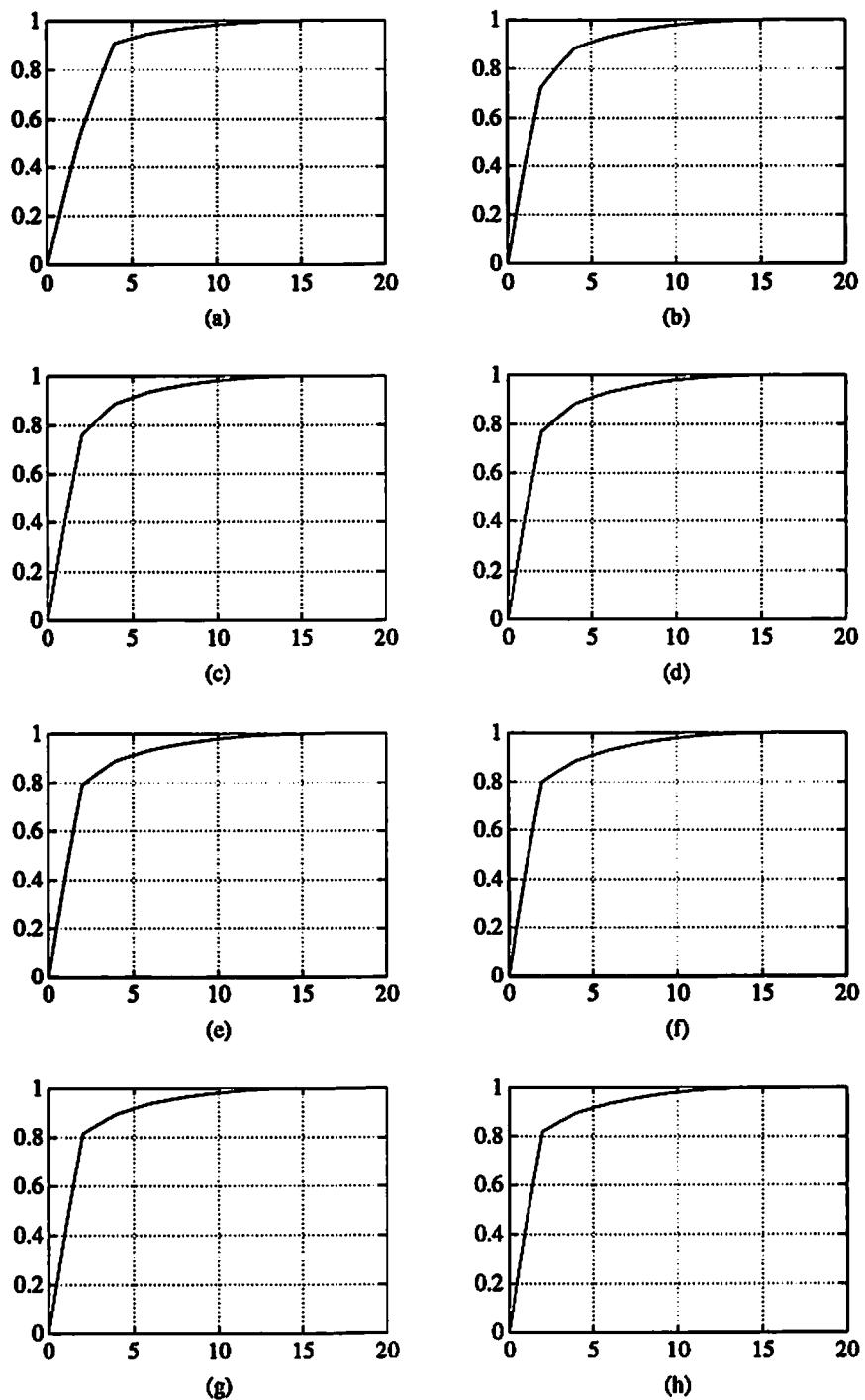


Figure 5: Accumulative singular values of cumulants using independent realizations when the fixed amplitude sinusoid at 0.1 has a local SNR of 0 dB and the amplitude of the sinusoid at 0.2 has a local SNR (dB) of : (a) 0 (b) -3 (c) -3.7 (d) -4 (e) -4.4 (f) -4.8 (g) -5.2 (h) -5.6

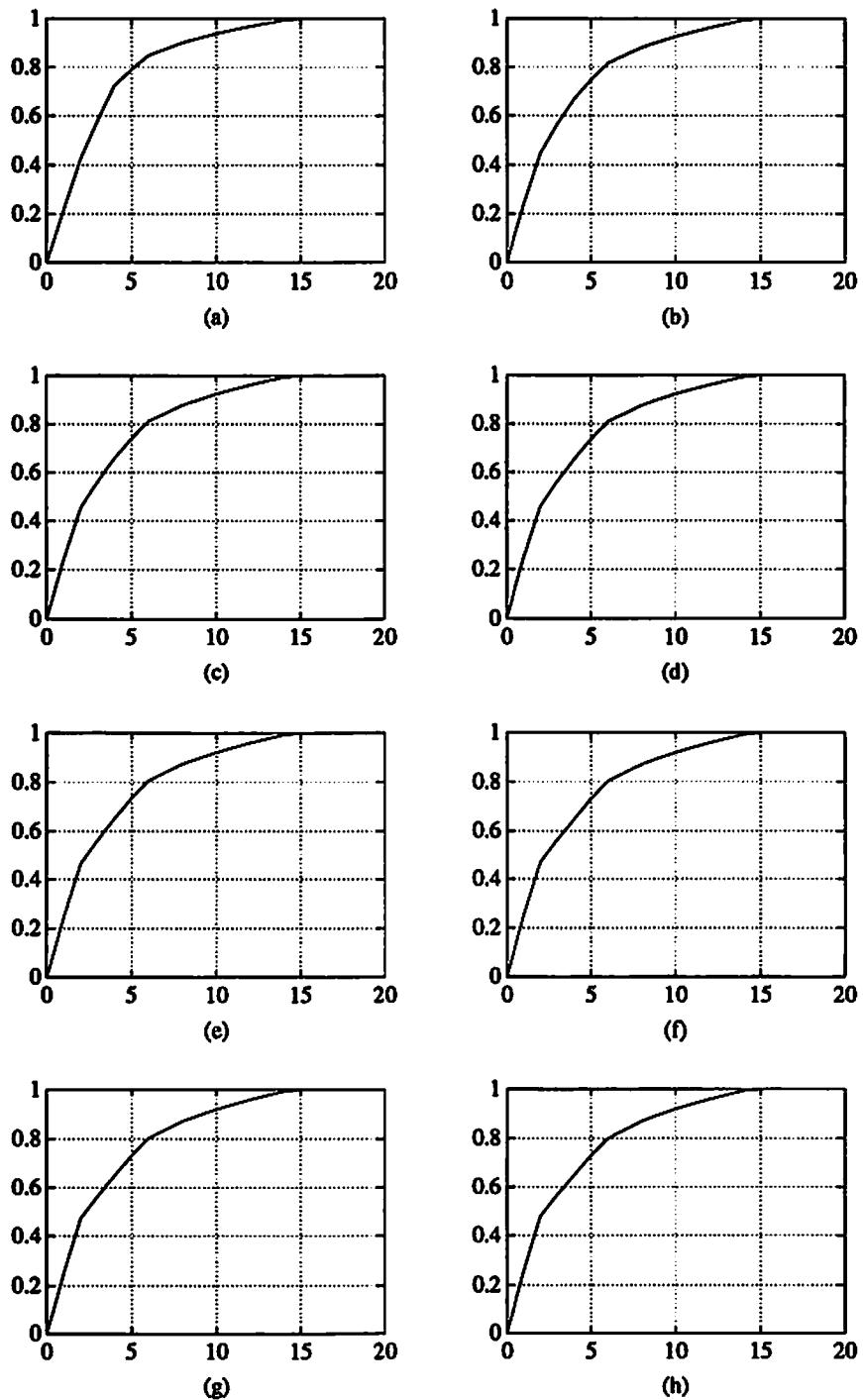


Figure 6: Accumulative singular values of correlations using one realization when the fixed amplitude sinusoid at 0.1 has a local SNR of 0 dB and the amplitude of the sinusoid at 0.2 has a local SNR (dB) of : (a) 0 (b) -3 (c) -3.7 (d) -4 (e) -4.4 (f) -4.8 (g) -5.2 (h) -5.6

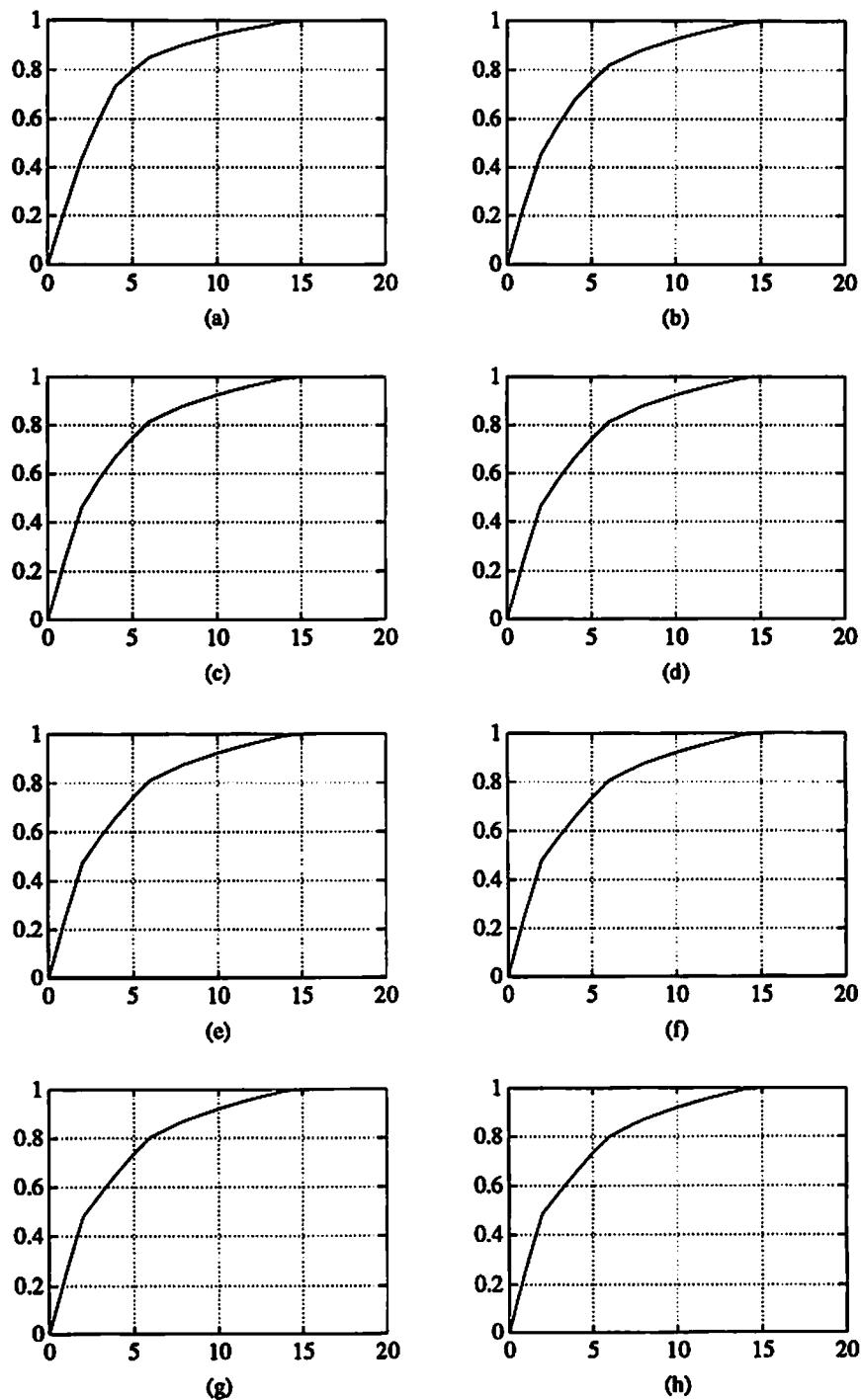


Figure 7: Accumulative singular values of correlations using independent realizations when the fixed amplitude sinusoid at 0.1 has a local SNR of 0 dB and the amplitude of the sinusoid at 0.2 has a local SNR (dB) of : (a) 0 (b) -3 (c) -3.7 (d) -4 (e) -4.4 (f) -4.8 (g) -5.2 (h) -5.6

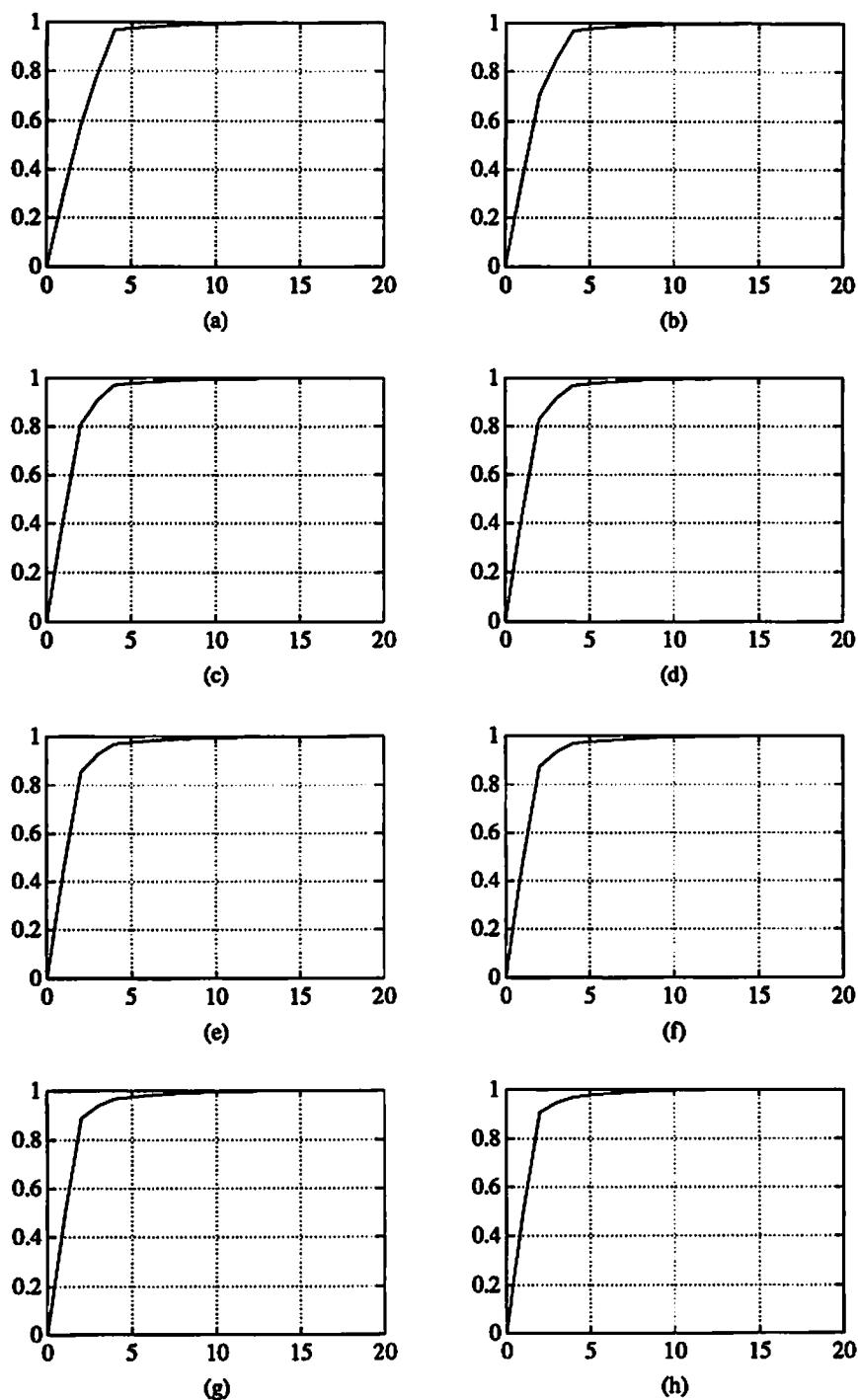


Figure 8: Accumulative singular values of cumulants using one realization when both amplitudes of sinusoids have local SNR's of 5 dB at $f_1 = 0.1$ and $f_2 = :$ (a) 0.2 (b) 0.14 (c) 0.13 (d) 0.1275 (e) 0.125 (f) 0.1225 (g) 0.12 (h) 0.1175

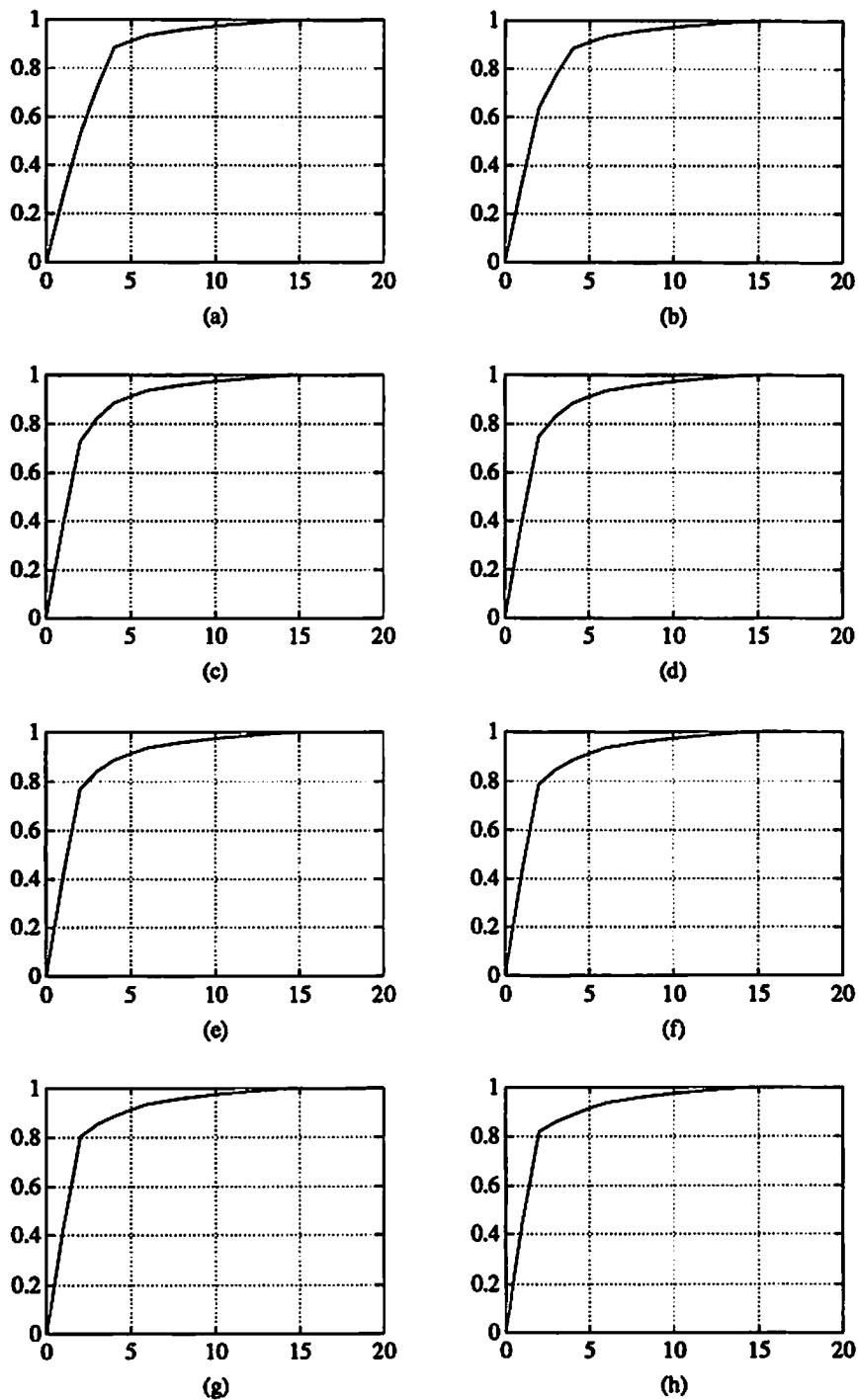


Figure 9: Accumulative singular values of correlations using one realization when both amplitudes of sinusoids have local SNR's of 5 dB at $f_1 = 0.1$ and $f_2 = :$ (a) 0.2 (b) 0.14 (c) 0.13 (d) 0.1275 (e) 0.125 (f) 0.1225 (g) 0.12 (h) 0.1175

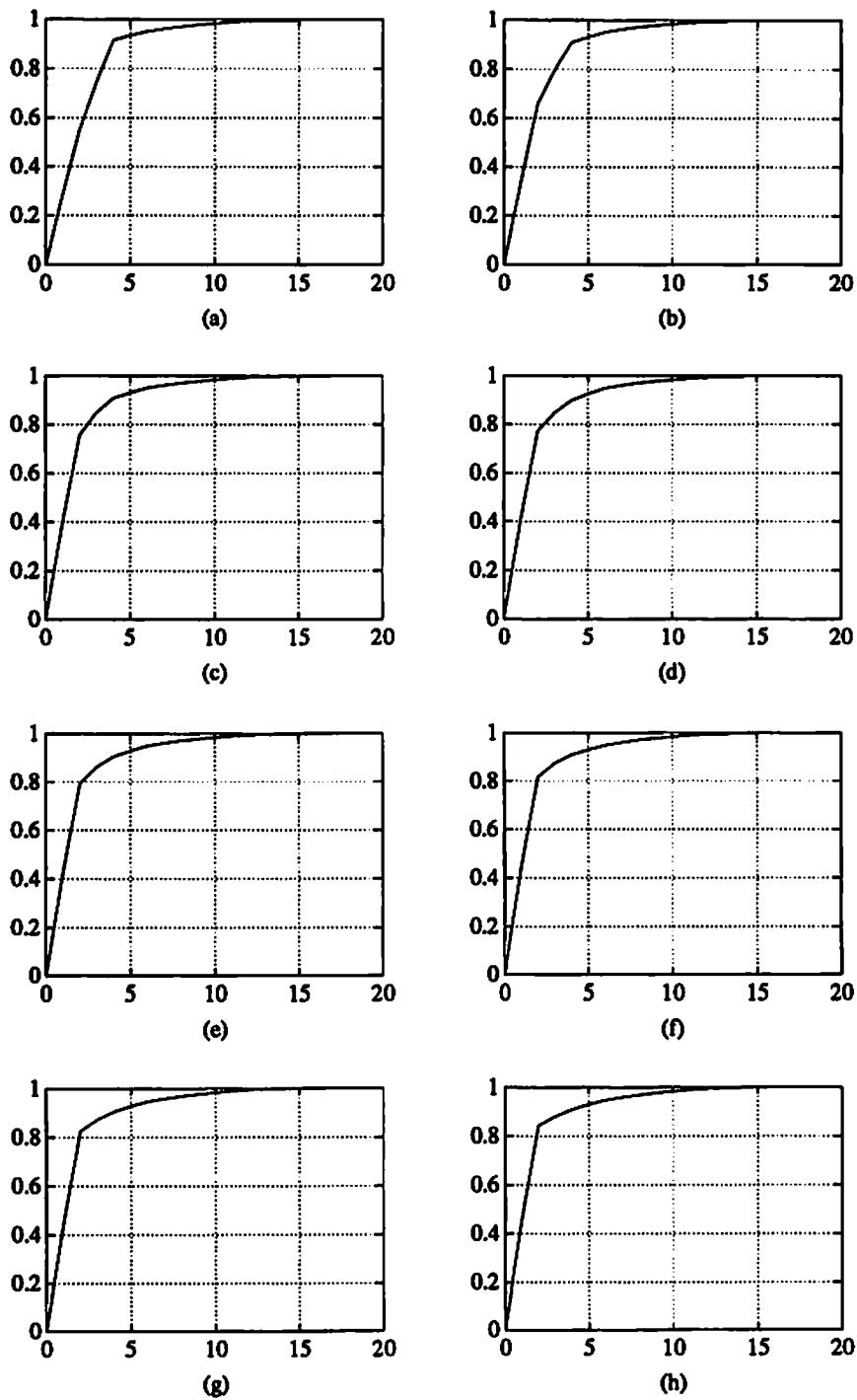


Figure 10: Accumulative singular values of cumulants using one realization when both amplitudes of sinusoids have local SNR's of 0 dB at $f_1 = 0.1$ and $f_2 = :$ (a) 0.2 (b) 0.14 (c) 0.13 (d) 0.1275 (e) 0.125 (f) 0.1225 (g) 0.12 (h) 0.1175

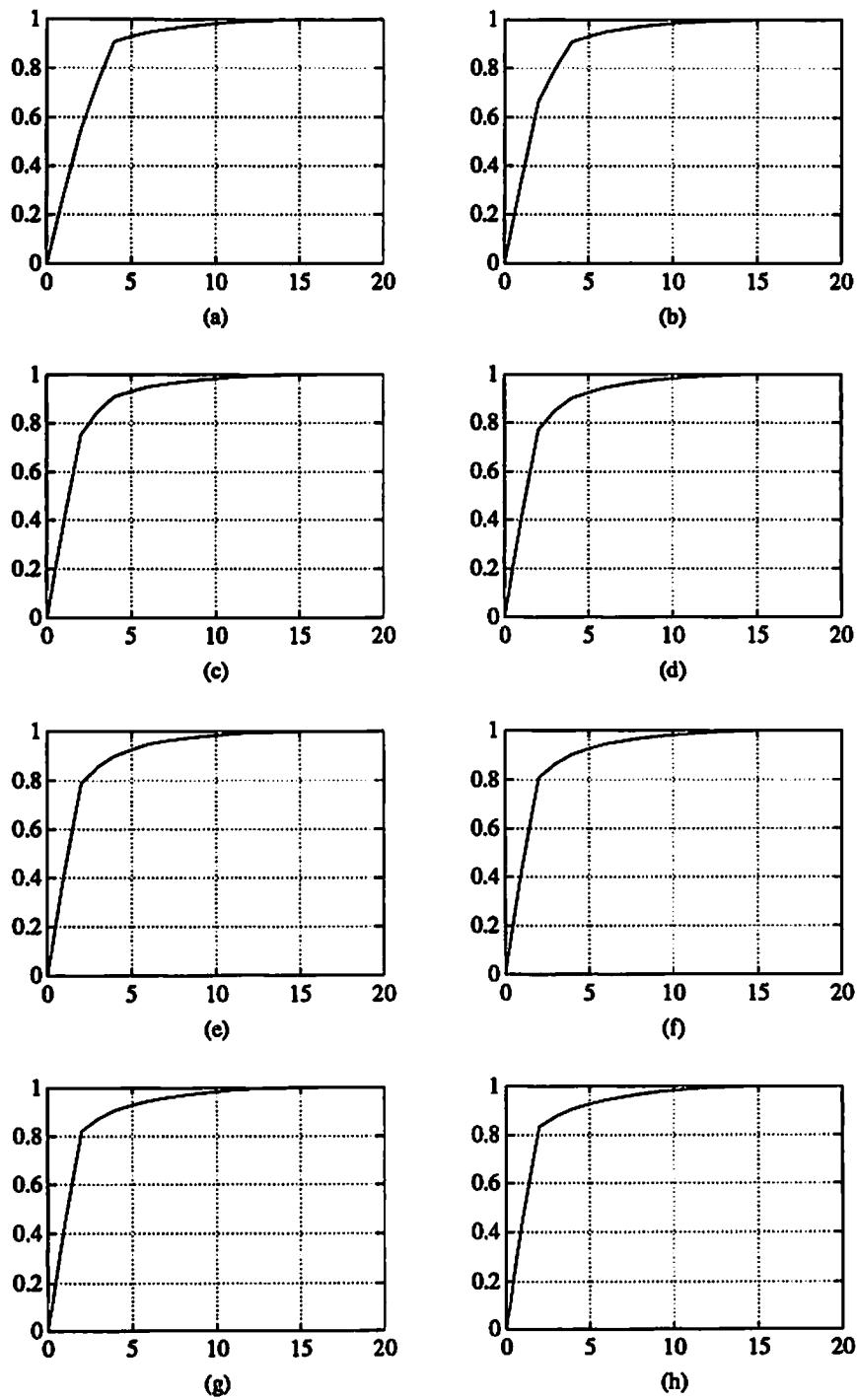


Figure 11: Accumulative singular values of cumulants using independent realizations when both amplitudes of sinusoids have local SNR's of 0 dB at $f_1 = 0.1$ and $f_2 = :$ (a) 0.2 (b) 0.14 (c) 0.13 (d) 0.1275 (e) 0.125 (f) 0.1225 (g) 0.12 (h) 0.1175

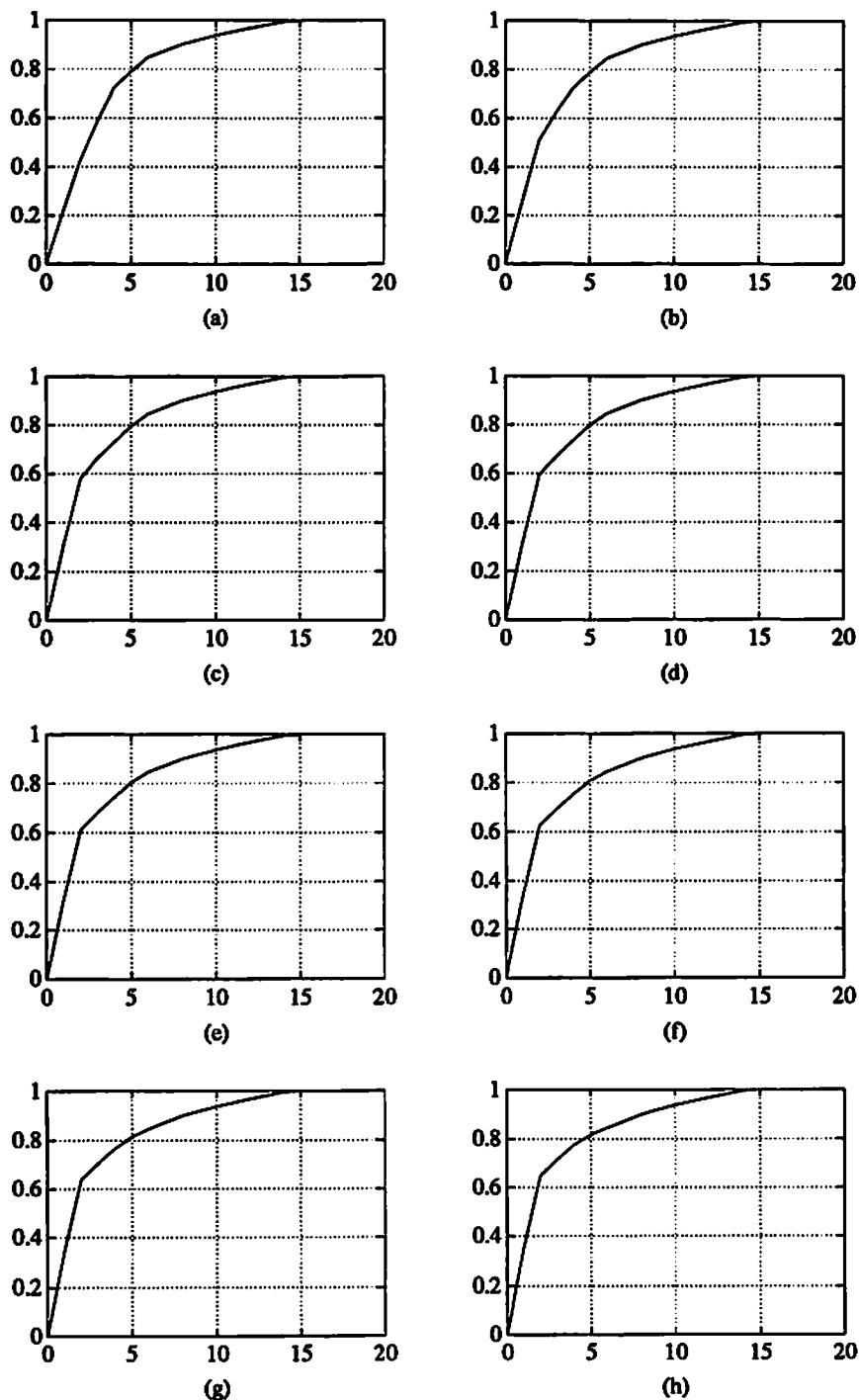


Figure 12: Accumulative singular values of correlations using one realization when both amplitudes of sinusoids have local SNR's of 0 dB at $f_1 = 0.1$ and $f_2 = :$ (a) 0.2 (b) 0.14 (c) 0.13 (d) 0.1275 (e) 0.125 (f) 0.1225 (g) 0.12 (h) 0.1175

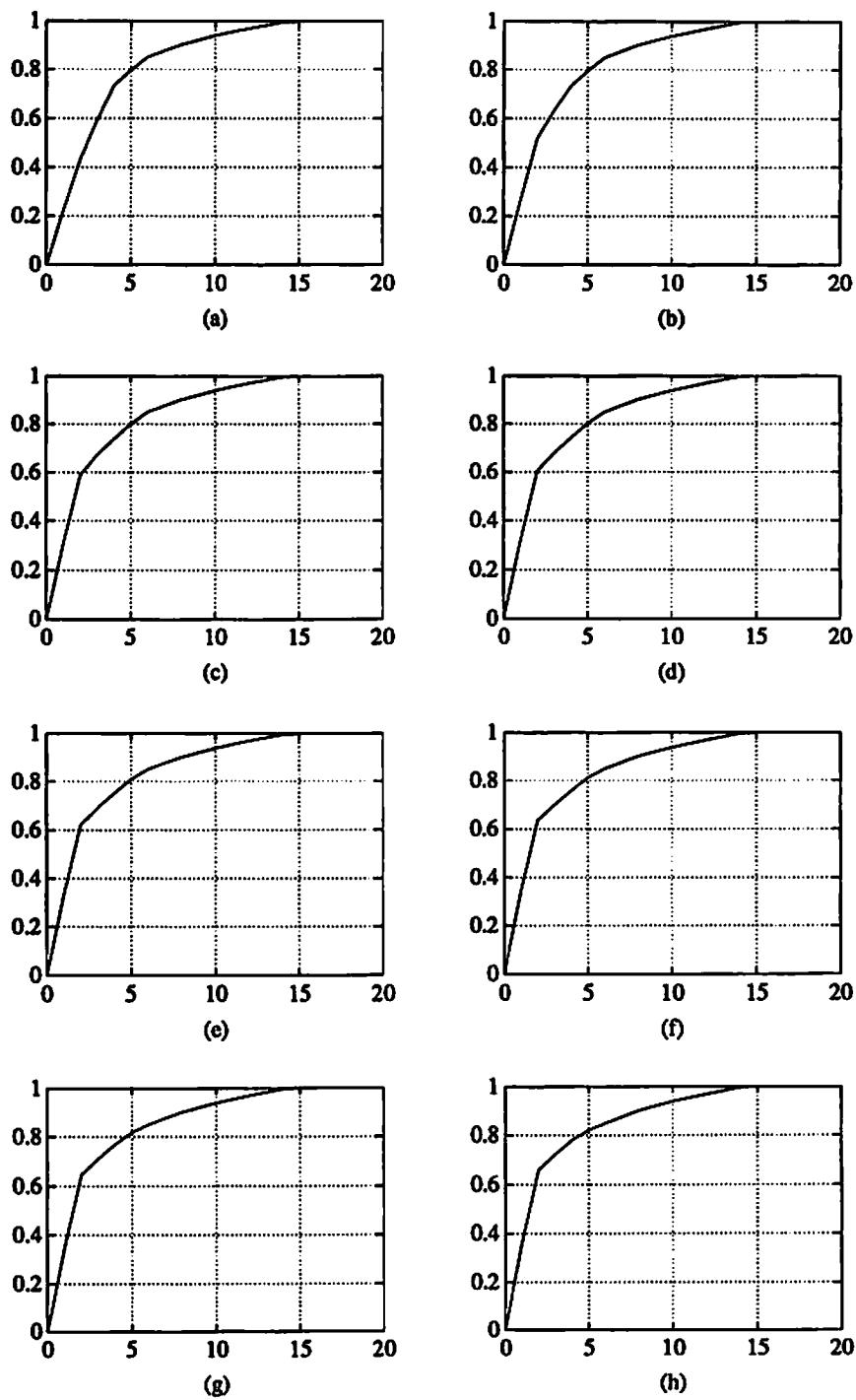


Figure 13: Accumulative singular values of correlations using independent realizations when both amplitudes of sinusoids have local SNR's of 0 dB at $f_1 = 0.1$ and $f_2 = :$ (a) 0.2 (b) 0.14 (c) 0.13 (d) 0.1275 (e) 0.125 (f) 0.1225 (g) 0.12 (h) 0.1175

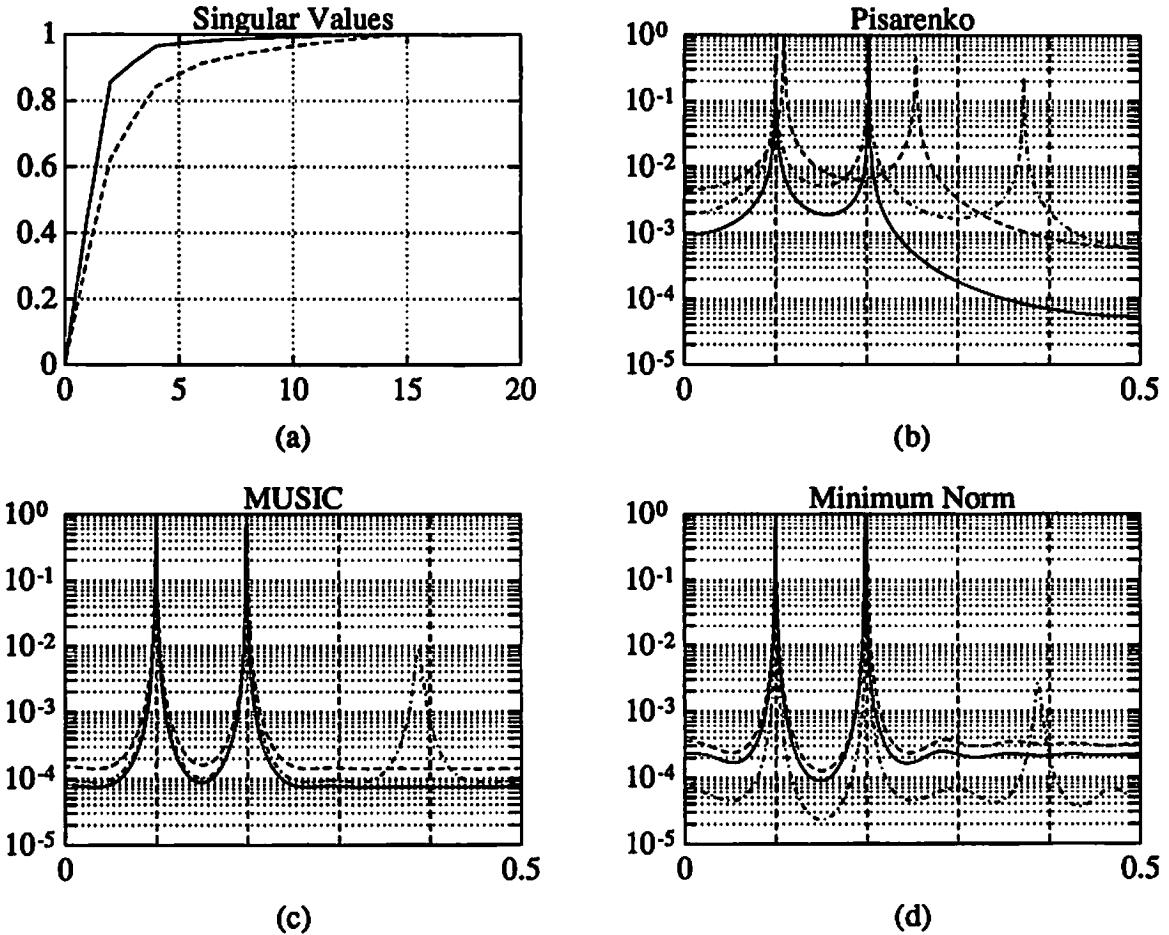


Figure 14: Results of estimated frequencies for Example 1: (a) mean of accumulative singular values (solid curve is for cumulants, dashed curve is for correlations); mean estimated spectra using (b) Pisarenko method, (c) MUSIC method, and (d) minimum norm method. In panels (b)-(d), the solid lines correspond to cumulant-based estimates, with $p=2$, and the dashed and dash-dotted lines correspond to correlation-based estimates with $p=2$ and $p=3$, respectively.

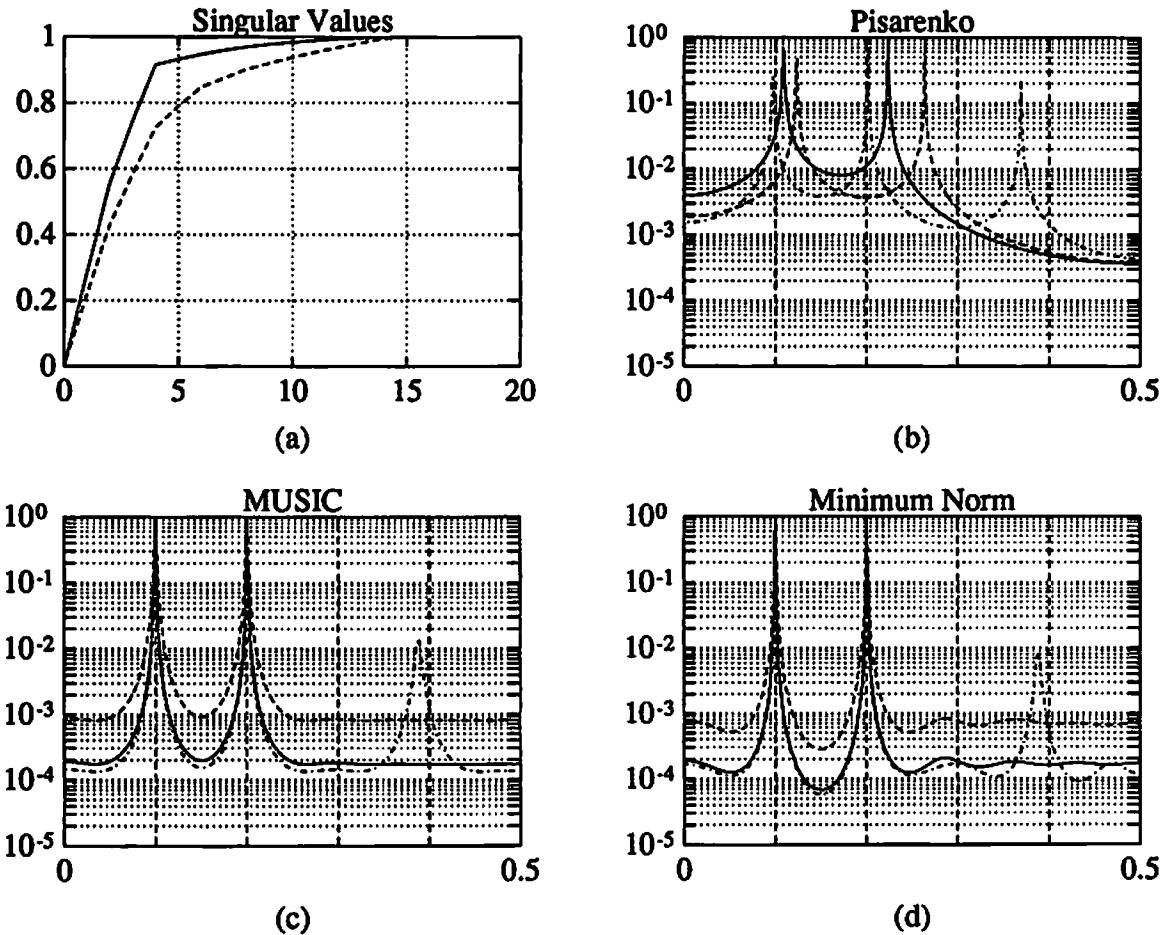


Figure 15: Results of estimated frequencies using one segmented realization for Example 2: (a) mean of accumulative singular values (solid curve is for cumulants, dashed curve is for correlations); mean estimated spectra using (b) Pisarenko method, (c) MUSIC method, and (d) minimum norm method. In panels (b)-(d), the solid lines correspond to cumulant-based estimates, with $p=2$, and the dashed and dash-dotted lines correspond to correlation-based estimates with $p=2$ and $p=3$, respectively.

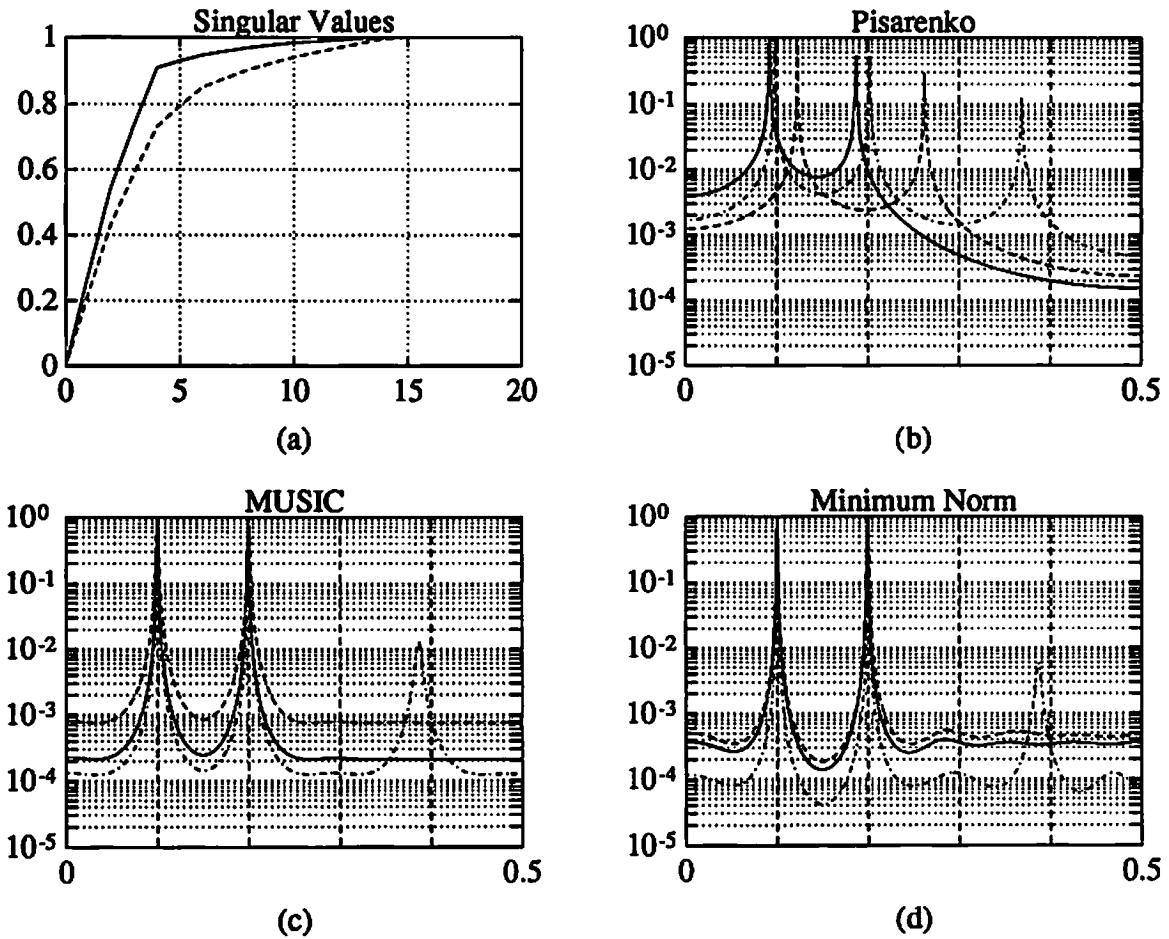


Figure 16: Results of estimated frequencies using independent realizations for Example 2: (a) mean of accumulative singular values (solid curve is for cumulants, dashed curve is for correlations); mean estimated spectra using (b) Pisarenko method, (c) MUSIC method, and (d) minimum norm method. In panels (b)-(d), the solid lines correspond to cumulant-based estimates, with $p=2$, and the dashed and dash-dotted lines correspond to correlation-based estimates with $p=2$ and $p=3$, respectively.

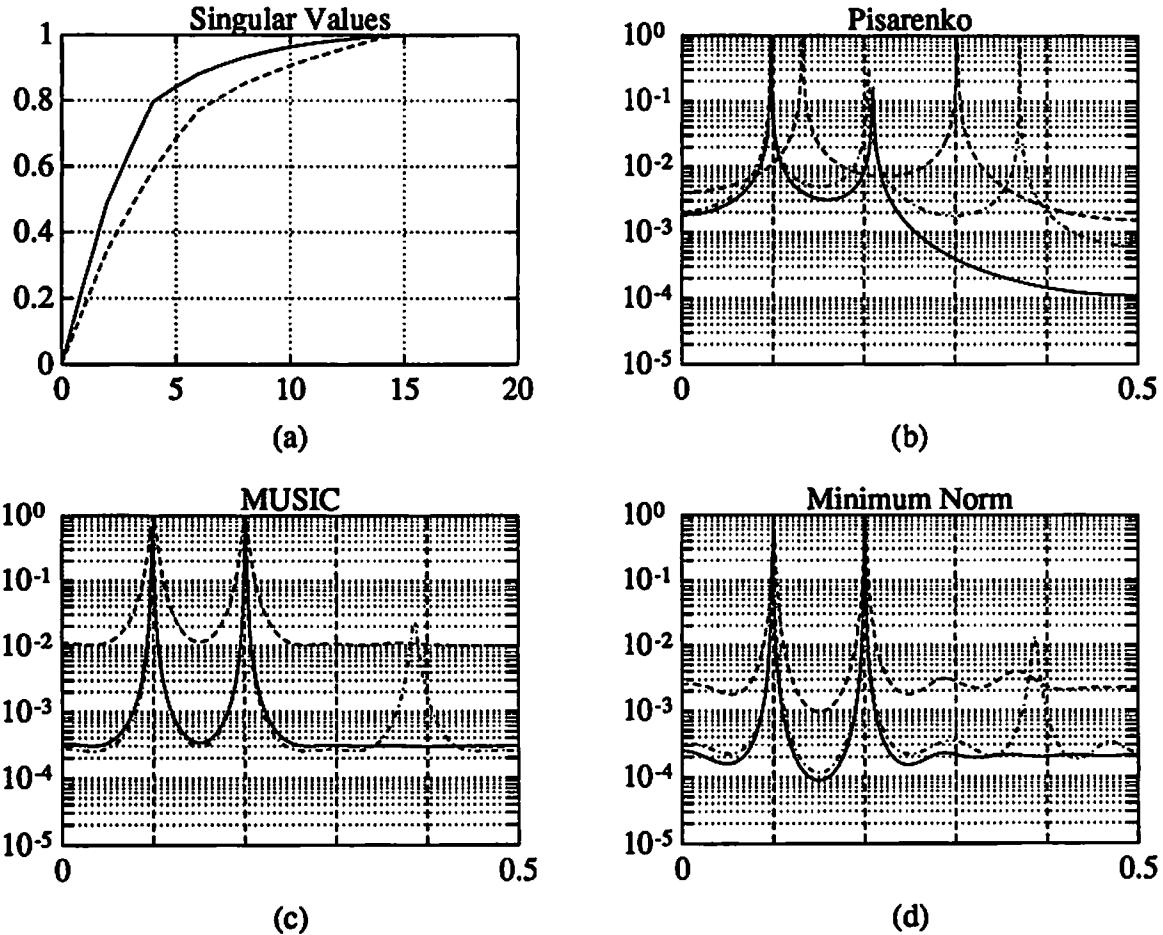


Figure 17: Results of estimated frequencies for Example 3: (a) mean of accumulative singular values (solid curve is for cumulants, dashed curve is for correlations); mean estimated spectra using (b) Pisarenko method, (c) MUSIC method, and (d) minimum norm method. In panels (b)-(d), the solid lines correspond to cumulant-based estimates, with $p=2$, and the dashed and dash-dotted lines correspond to correlation-based estimates with $p=2$ and $p=3$, respectively.

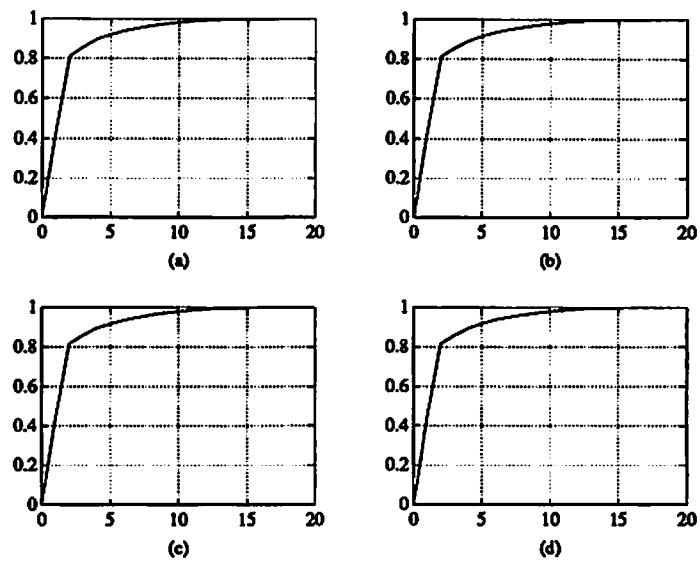


Figure 18: Accumulative singular values of cumulants using 64×64 (4096) one realization when the fixed amplitude sinusoid at 0.1 is unity and the amplitude of the sinusoid at 0.2 is : (a) 0.55* (b) 0.545 (c) 0.54 (d) 0.535

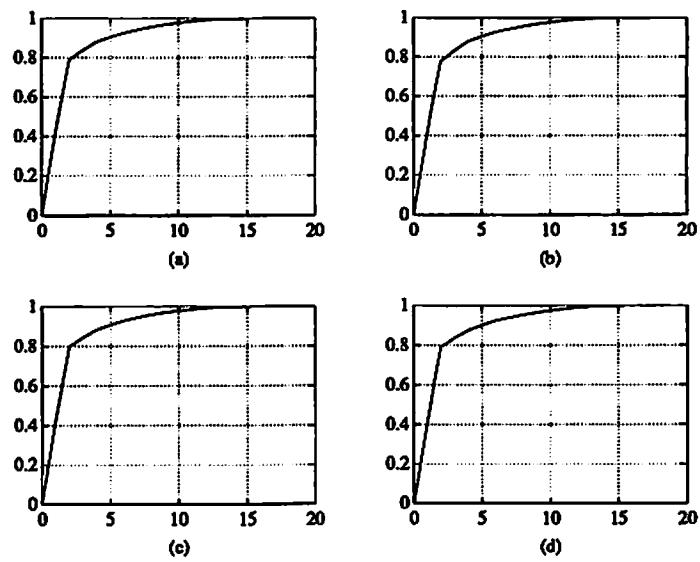


Figure 19: Accumulative singular values of cumulants using 50×64 (3200) one realization when the fixed amplitude sinusoid at 0.1 is unity and the amplitude of the sinusoid at 0.2 is : (a) 0.585 (b) 0.58* (c) 0.575 (d) 0.57

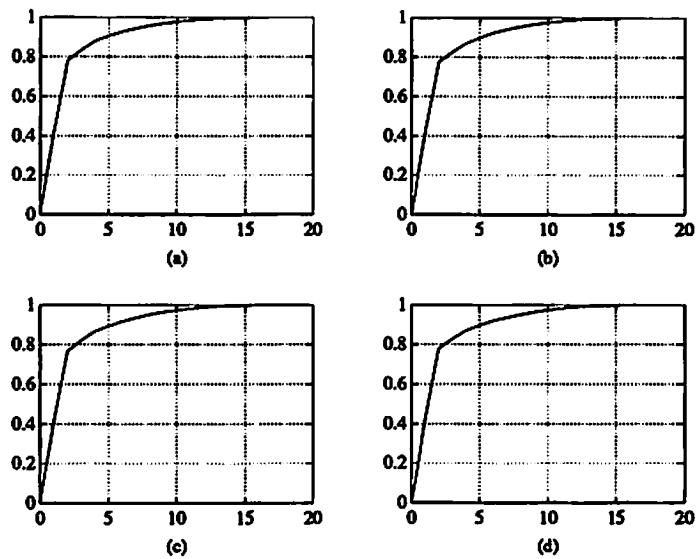


Figure 20: Accumulative singular values of cumulants using 40×64 (2560) one realization when the fixed amplitude sinusoid at 0.1 is unity and the amplitude of the sinusoid at 0.2 is : (a) 0.595 (b) 0.59 (c) 0.585* (d) 0.58

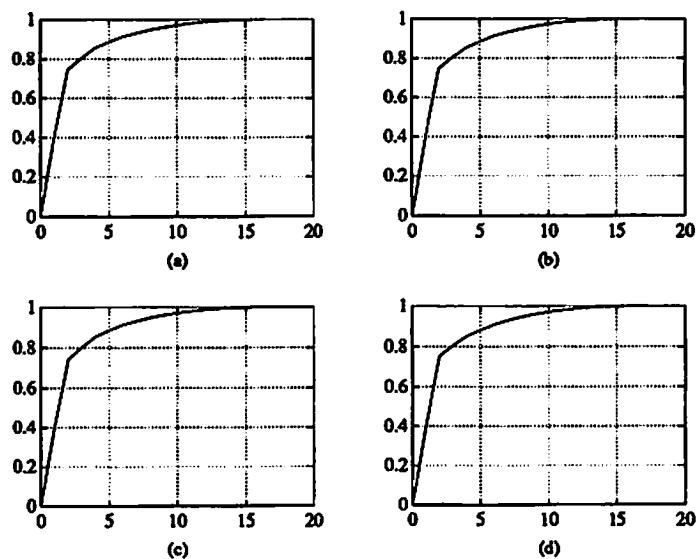


Figure 21: Accumulative singular values of cumulants using 32×64 (2048) one realization when the fixed amplitude sinusoid at 0.1 is unity and the amplitude of the sinusoid at 0.2 is : (a) 0.605 (b) 0.6 (c) 0.595* (d) 0.59

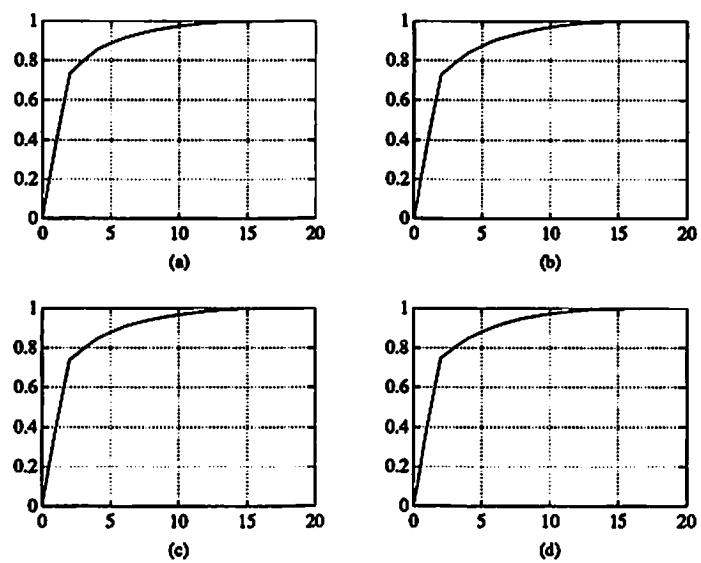


Figure 22: Accumulative singular values of cumulants using 25×64 (1600) one realization when the fixed amplitude sinusoid at 0.1 is unity and the amplitude of the sinusoid at 0.2 is : (a) 0.61* (b) 0.605 (c) 0.6 (d) 0.595

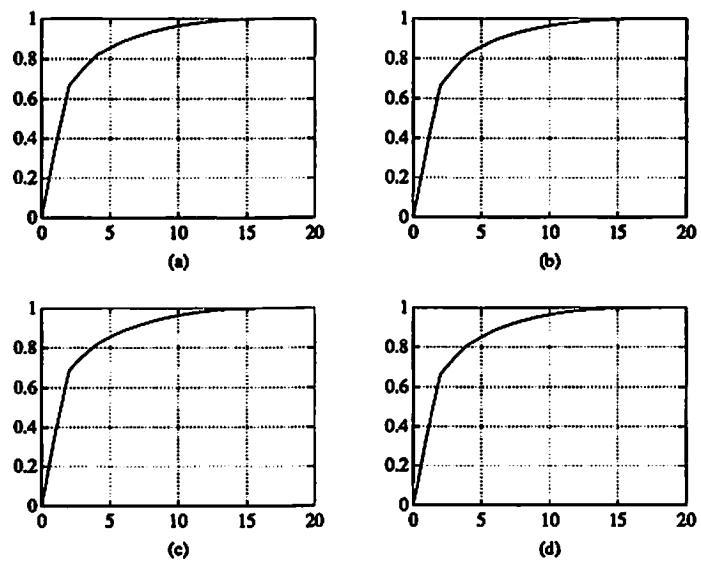


Figure 23: Accumulative singular values of cumulants using 16×64 (1024) one realization when the fixed amplitude sinusoid at 0.1 is unity and the amplitude of the sinusoid at 0.2 is : (a) 0.675 (b) 0.67* (c) 0.665 (d) 0.66

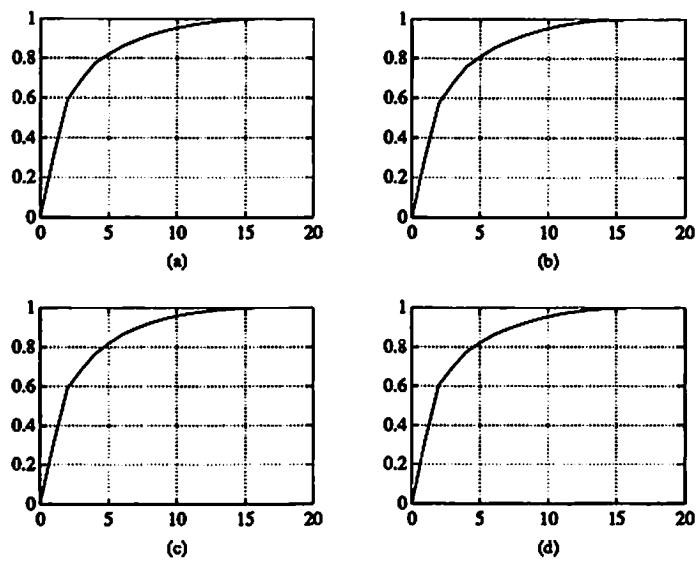


Figure 24: Accumulative singular values of cumulants using 8×64 (512) one realization when the fixed amplitude sinusoid at 0.1 is unity and the amplitude of the sinusoid at 0.2 is : (a) 0.72 (b) 0.715* (c) 0.71 (d) 0.705

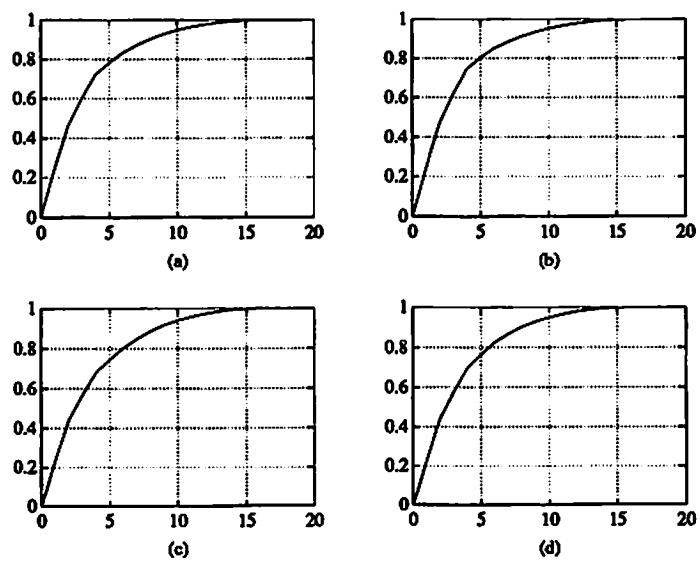


Figure 25: Accumulative singular values of cumulants using 4×64 (256) one realization when the fixed amplitude sinusoid at 0.1 is unity and the amplitude of the sinusoid at 0.2 is : (a) 0.96 (b) 0.955* (c) 0.95 (d) 0.945

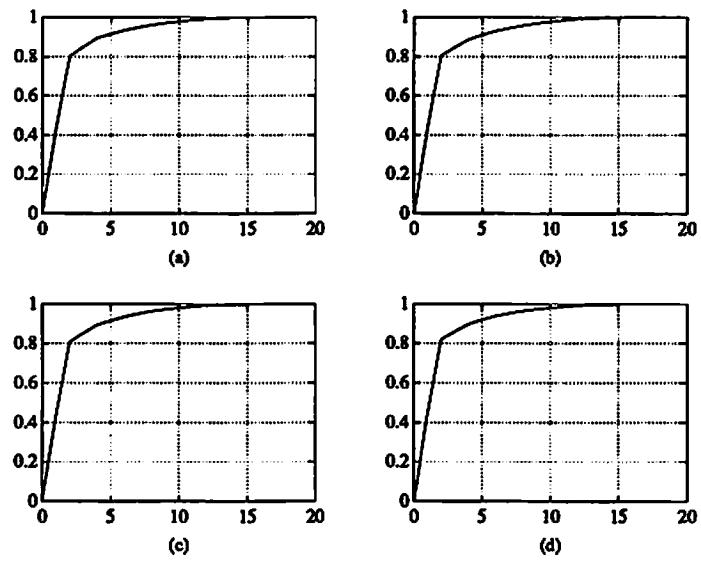


Figure 26: Accumulative singular values of cumulants using 64×64 (4096) one realization when the fixed amplitude sinusoid at 0.1 is unity and the amplitude of the sinusoid at 0.18 is : (a) 0.58 (b) 0.575 (c) 0.57* (d) 0.565

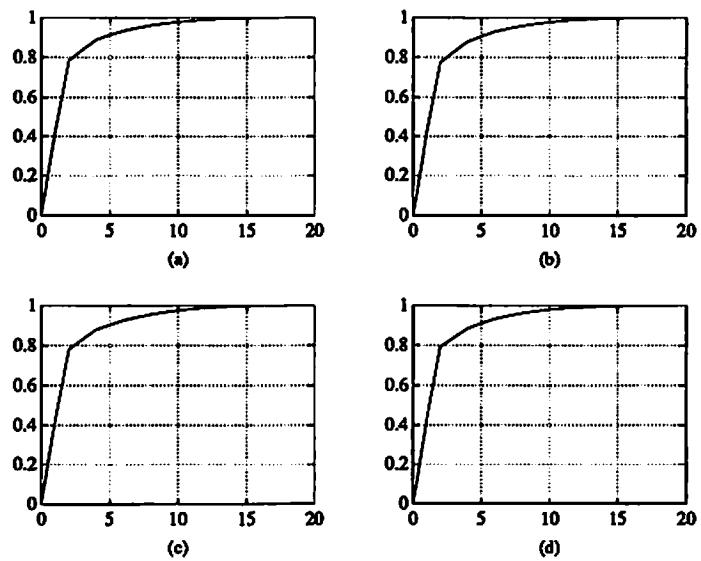


Figure 27: Accumulative singular values of cumulants using 50×64 (3200) one realization when the fixed amplitude sinusoid at 0.1 is unity and the amplitude of the sinusoid at 0.18 is : (a) 0.595 (b) 0.59 (c) 0.585* (d) 0.58

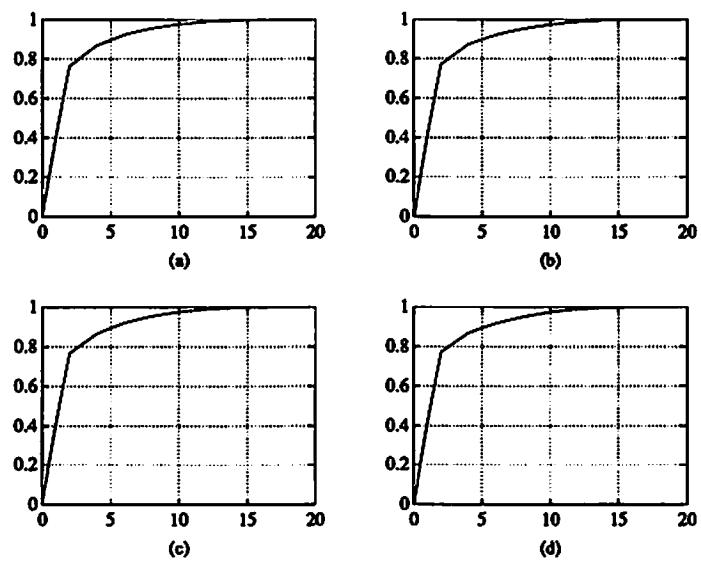


Figure 28: Accumulative singular values of cumulants using 40×64 (2560) one realization when the fixed amplitude sinusoid at 0.1 is unity and the amplitude of the sinusoid at 0.18 is : (a) 0.61 (b) 0.605* (c) 0.6 (d) 0.595

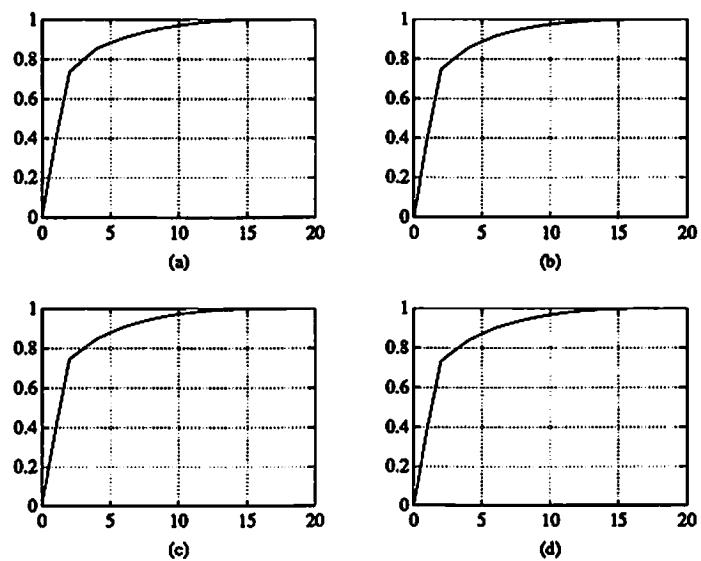


Figure 29: Accumulative singular values of cumulants using 32×64 (2048) one realization when the fixed amplitude sinusoid at 0.1 is unity and the amplitude of the sinusoid at 0.18 is : (a) 0.61* (b) 0.605 (c) 0.6 (d) 0.595

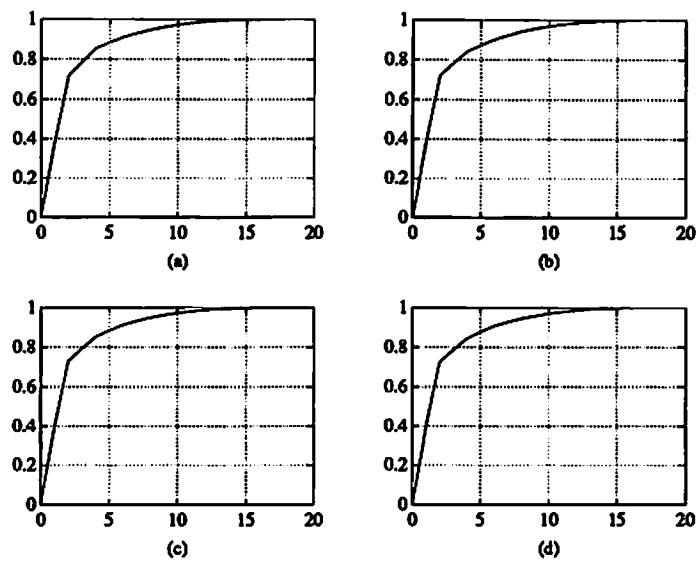


Figure 30: Accumulative singular values of cumulants using 25×64 (1600) one realization when the fixed amplitude sinusoid at 0.1 is unity and the amplitude of the sinusoid at 0.18 is : (a) 0.64 (b) 0.635* (c) 0.63 (d) 0.625

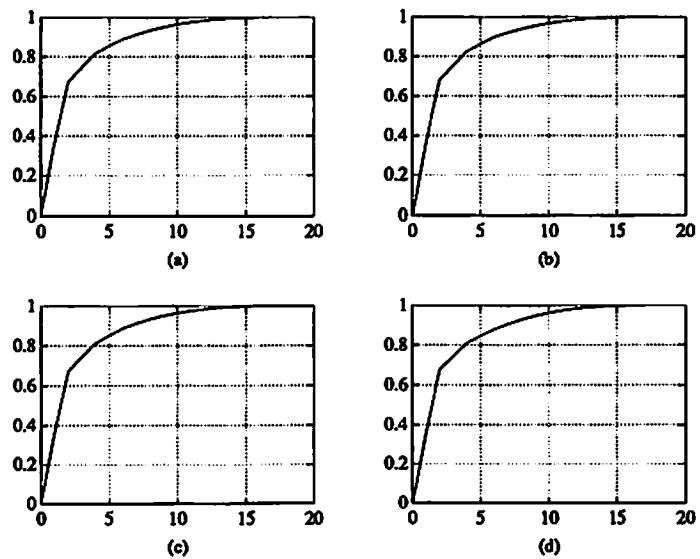


Figure 31: Accumulative singular values of cumulants using 16×64 (1024) one realization when the fixed amplitude sinusoid at 0.1 is unity and the amplitude of the sinusoid at 0.18 is : (a) 0.685 (b) 0.68* (c) 0.675 (d) 0.67

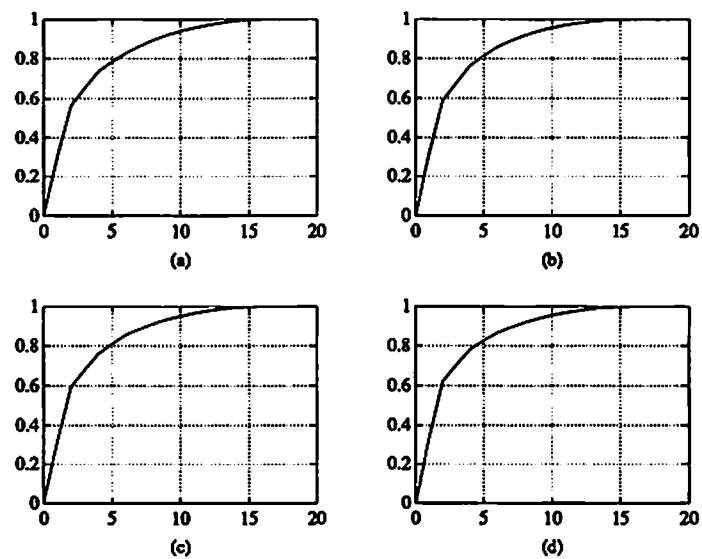


Figure 32: Accumulative singular values of cumulants using 8×64 (512) one realization when the fixed amplitude sinusoid at 0.1 is unity and the amplitude of the sinusoid at 0.18 is : (a) 0.73 (b) 0.725* (c) 0.72 (d) 0.715

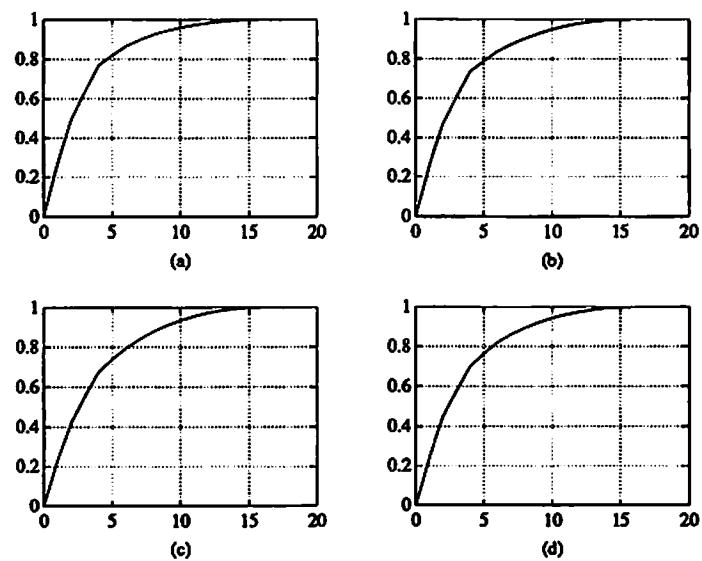


Figure 33: Accumulative singular values of cumulants using 4×64 (256) one realization when the fixed amplitude sinusoid at 0.1 is unity and the amplitude of the sinusoid at 0.18 is : (a) 0.97 (b) 0.965* (c) 0.96 (d) 0.955

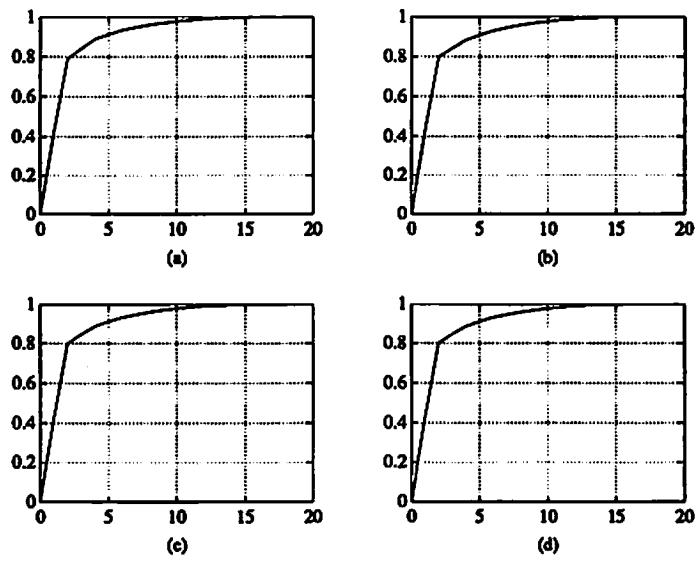


Figure 34: Accumulative singular values of cumulants using 64×64 (4096) one realization when the fixed amplitude sinusoid at 0.1 is unity and the amplitude of the sinusoid at 0.16 is : (a) 0.585* (b) 0.58 (c) 0.575 (d) 0.57

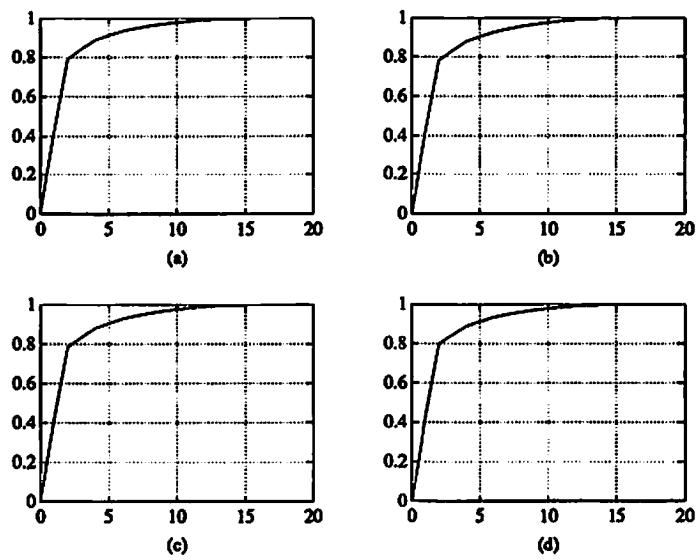


Figure 35: Accumulative singular values of cumulants using 50×64 (3200) one realization when the fixed amplitude sinusoid at 0.1 is unity and the amplitude of the sinusoid at 0.16 is : (a) 0.595 (b) 0.59* (c) 0.585 (d) 0.58

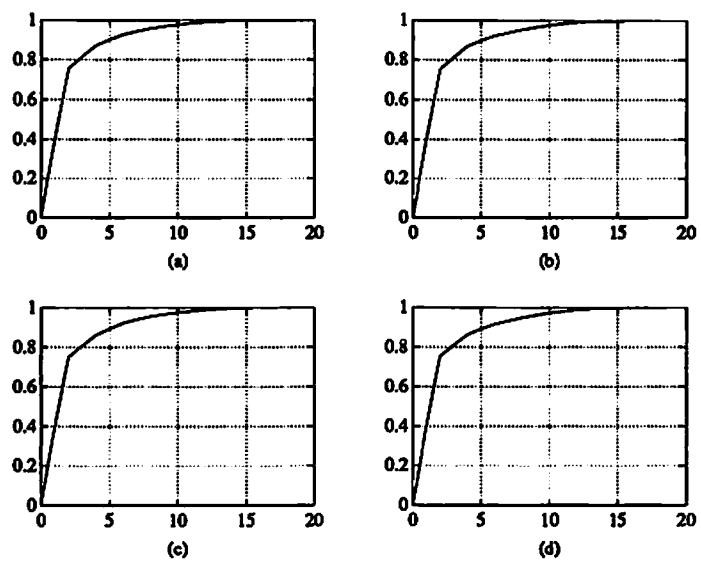


Figure 36: Accumulative singular values of cumulants using 40×64 (2560) one realization when the fixed amplitude sinusoid at 0.1 is unity and the amplitude of the sinusoid at 0.16 is : (a) 0.625 (b) 0.62* (c) 0.615 (d) 0.61

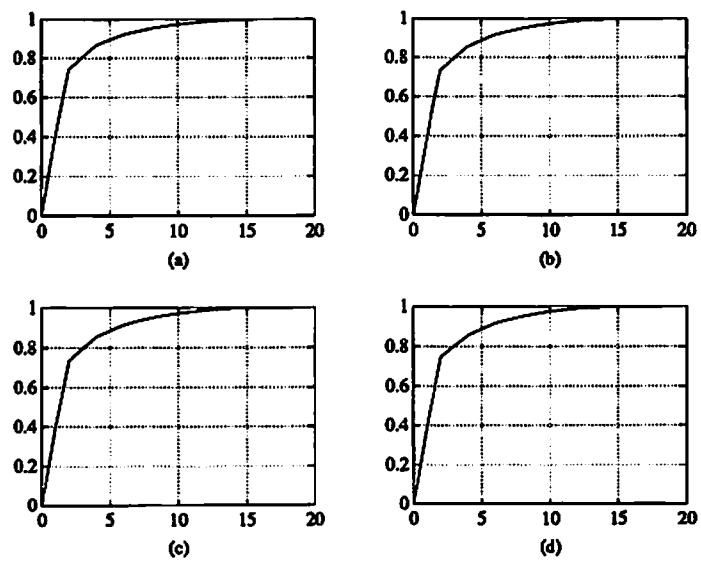


Figure 37: Accumulative singular values of cumulants using 32×64 (2048) one realization when the fixed amplitude sinusoid at 0.1 is unity and the amplitude of the sinusoid at 0.16 is : (a) 0.635 (b) 0.63* (c) 0.625 (d) 0.62

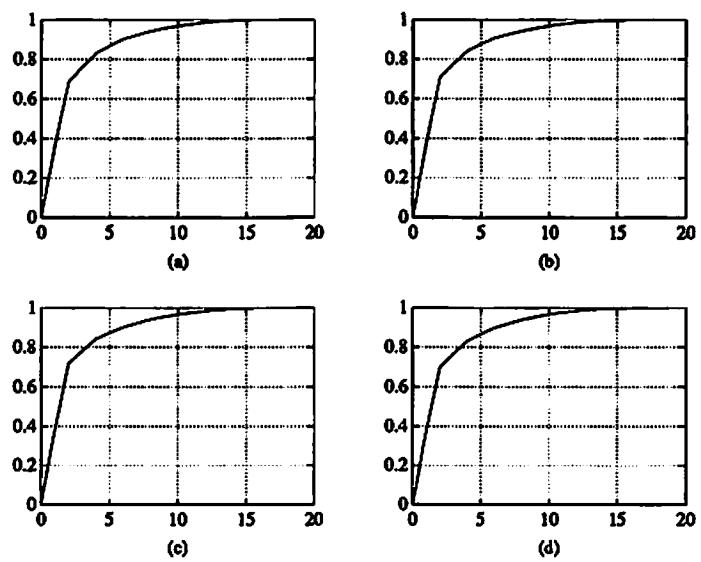


Figure 38: Accumulative singular values of cumulants using 25×64 (1600) one realization when the fixed amplitude sinusoid at 0.1 is unity and the amplitude of the sinusoid at 0.16 is : (a) 0.65 (b) 0.645 (c) 0.64* (d) 0.635

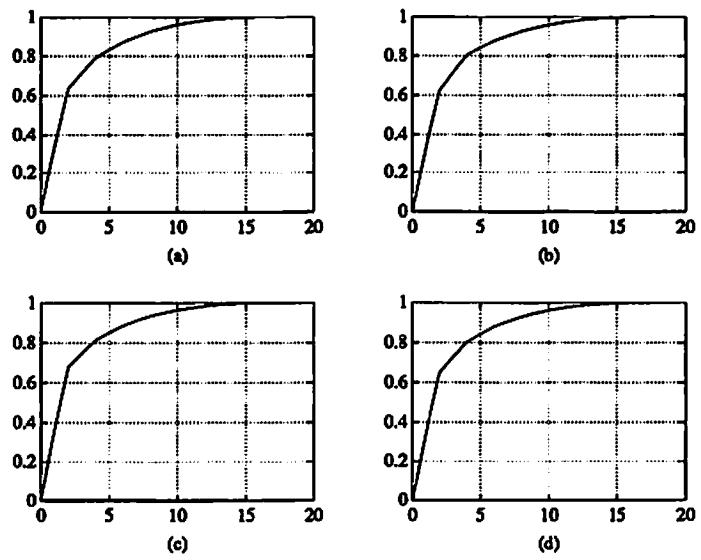


Figure 39: Accumulative singular values of cumulants using 16×64 (1024) one realization when the fixed amplitude sinusoid at 0.1 is unity and the amplitude of the sinusoid at 0.16 is : (a) 0.7 (b) 0.695* (c) 0.69 (d) 0.685

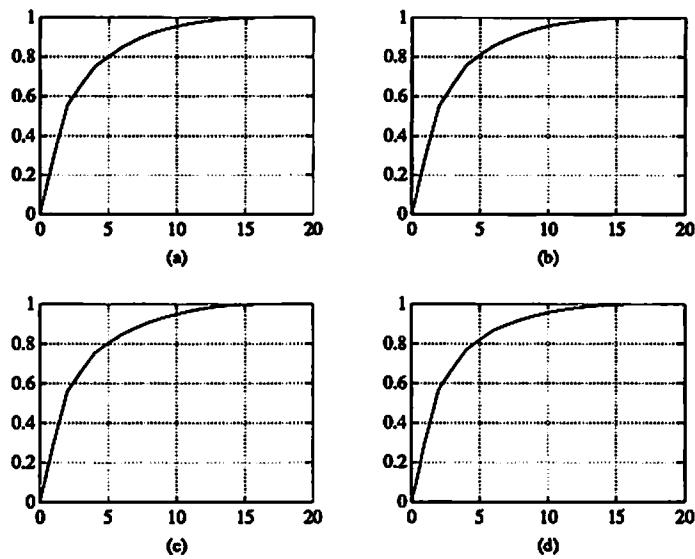


Figure 40: Accumulative singular values of cumulants using 8×64 (512) one realization when the fixed amplitude sinusoid at 0.1 is unity and the amplitude of the sinusoid at 0.16 is : (a) 0.765 (b) 0.76 (c) 0.755* (d) 0.75

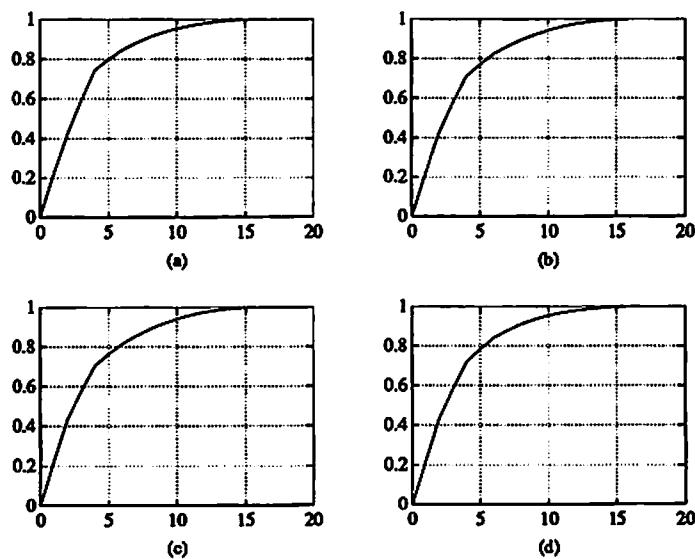


Figure 41: Accumulative singular values of cumulants using 4×64 (256) one realization when the fixed amplitude sinusoid at 0.1 is unity and the amplitude of the sinusoid at 0.16 is : (a) 0.97 (b) 0.965 (c) 0.96* (d) 0.955

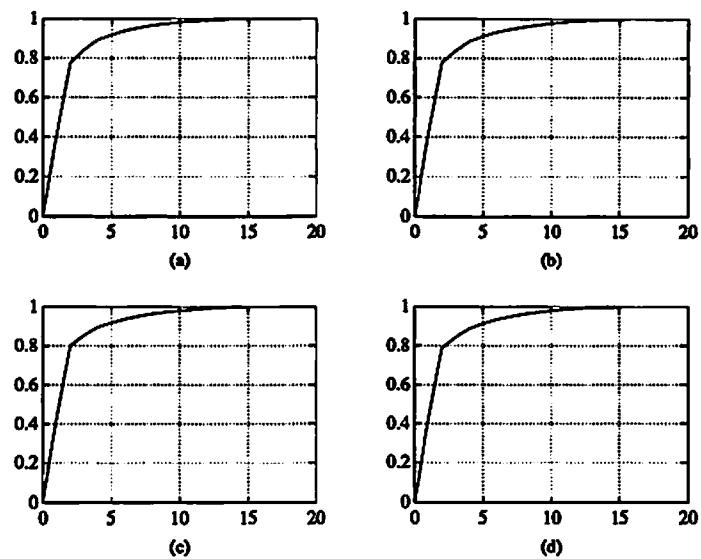


Figure 42: Accumulative singular values of cumulants using 64×64 (4096) one realization when the fixed amplitude sinusoid at 0.1 is unity and the amplitude of the sinusoid at 0.14 is : (a) 0.655 (b) 0.65 (c) 0.645* (d) 0.64

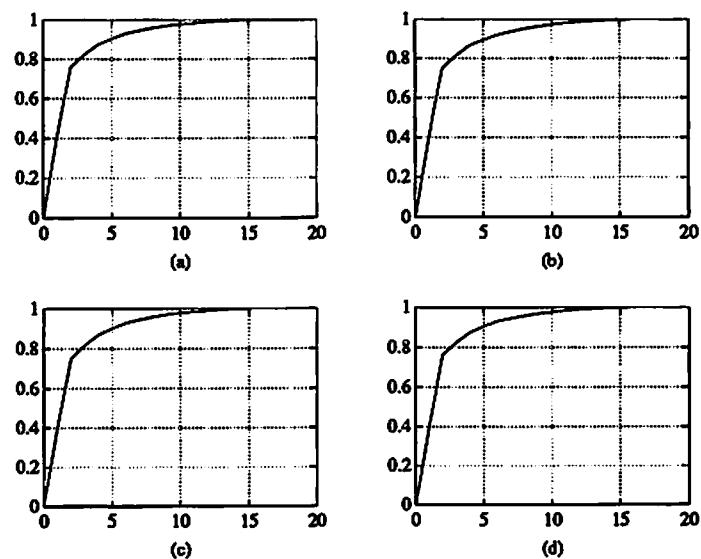


Figure 43: Accumulative singular values of cumulants using 50×64 (3200) one realization when the fixed amplitude sinusoid at 0.1 is unity and the amplitude of the sinusoid at 0.14 is : (a) 0.68 (b) 0.675* (c) 0.67 (d) 0.665

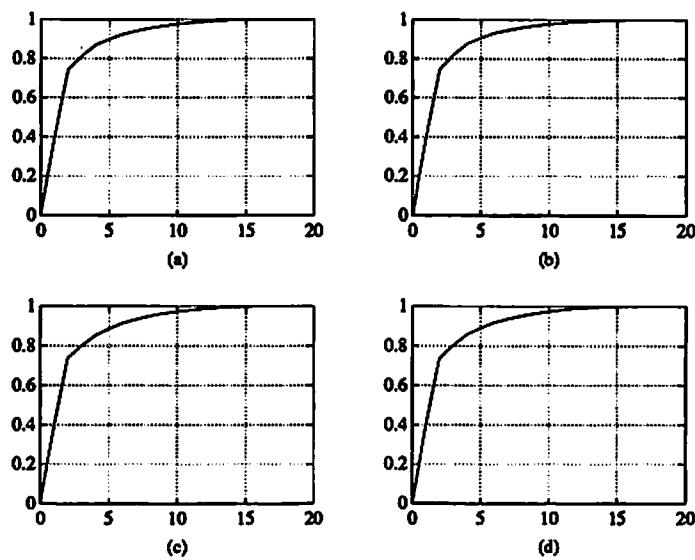


Figure 44: Accumulative singular values of cumulants using 40×64 (2560) one realization when the fixed amplitude sinusoid at 0.1 is unity and the amplitude of the sinusoid at 0.14 is : (a) 0.69 (b) 0.685* (c) 0.68 (d) 0.675

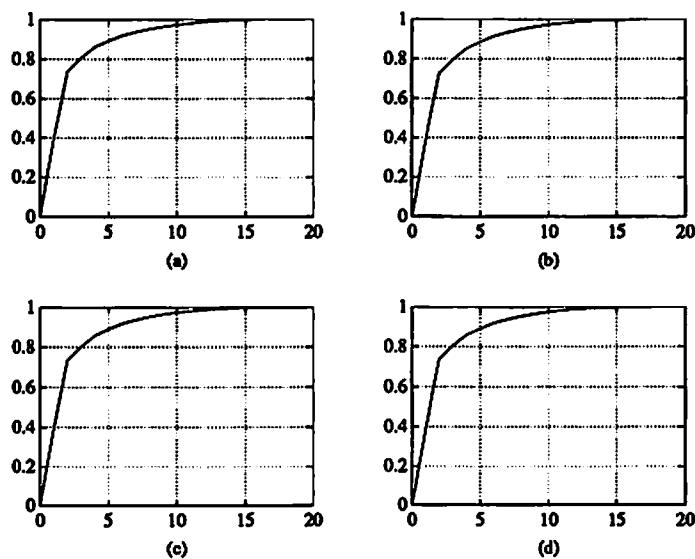


Figure 45: Accumulative singular values of cumulants using 32×64 (2048) one realization when the fixed amplitude sinusoid at 0.1 is unity and the amplitude of the sinusoid at 0.14 is : (a) 0.695* (b) 0.69 (c) 0.685 (d) 0.68

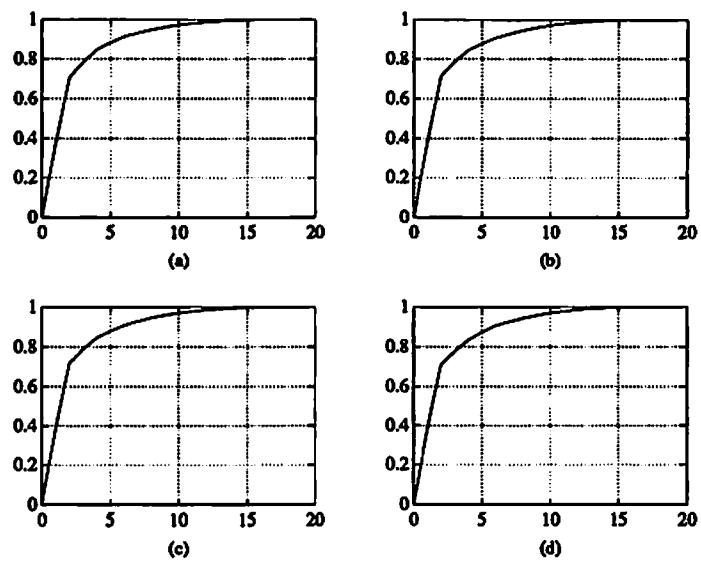


Figure 46: Accumulative singular values of cumulants using 25×64 (1600) one realization when the fixed amplitude sinusoid at 0.1 is unity and the amplitude of the sinusoid at 0.14 is : (a) 0.71 (b) 0.705* (c) 0.7 (d) 0.695

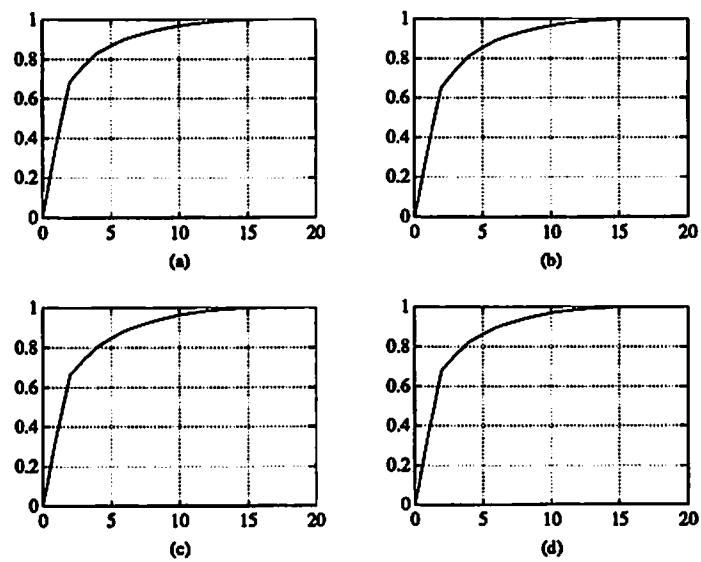


Figure 47: Accumulative singular values of cumulants using 16×64 (1024) one realization when the fixed amplitude sinusoid at 0.1 is unity and the amplitude of the sinusoid at 0.14 is : (a) 0.74* (b) 0.735 (c) 0.73 (d) 0.725

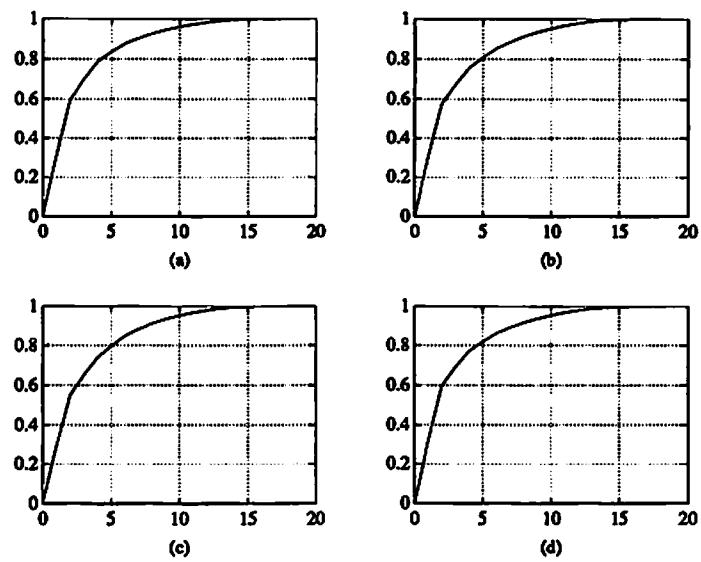


Figure 48: Accumulative singular values of cumulants using 8×64 (512) one realization when the fixed amplitude sinusoid at 0.1 is unity and the amplitude of the sinusoid at 0.14 is : (a) 0.835*
(b) 0.83 (c) 0.825 (d) 0.82

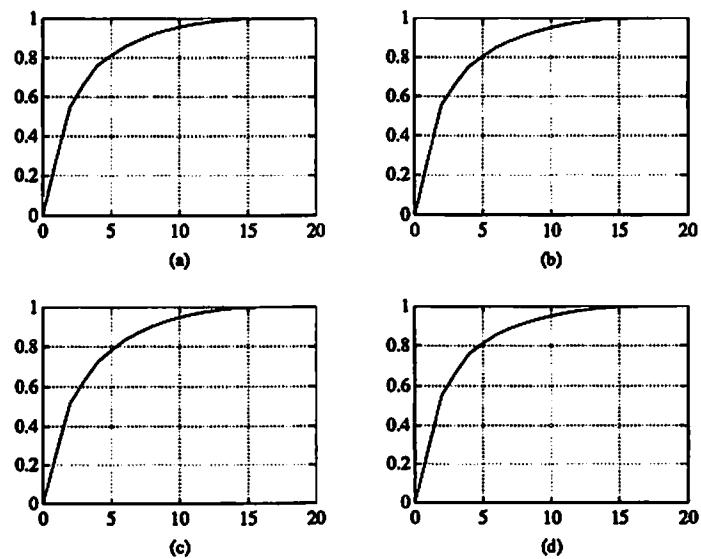


Figure 49: Accumulative singular values of cumulants using 4×64 (256) one realization when the fixed amplitude sinusoid at 0.1 is unity and the amplitude of the sinusoid at 0.14 is : (a) 1.02
(b) 1.015* (c) 1.01 (d) 1.005

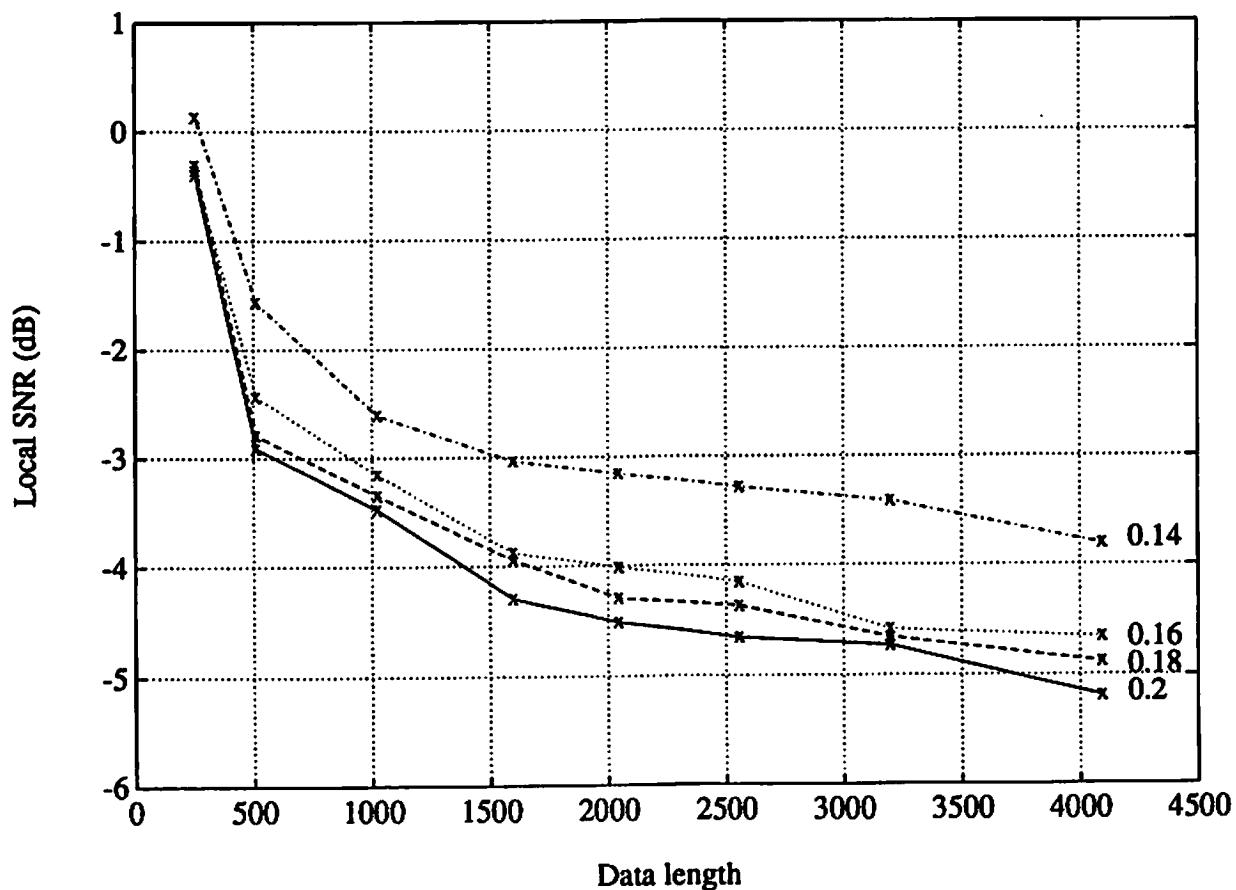


Figure 50: Minimum local SNR (dB) of the amplitude of the second sinusoid at f_2 when we can still determine the correct number of harmonics. The fixed amplitude at $f_1 = 0.1$ has local SNR of 0 dB. The crossed points denote the values that we got through simulations. The solid line is for $f_2 = 0.2$, and the dashed, dotted, and dash-dotted lines are for $f_2 = 0.18, 0.16$, and 0.14 , respectively.