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Bispectra for Sonar

by

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BISPECTRA FOR SONAR

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Abstract

The usefulness of polyspectra, and mixed moments of both integer and rational orders, in detecting and quantifying various kinds of frequency and phase couplings is studied. Both the deterministic and random cases are considered. Several interesting issues regarding frequency coupling in deterministic signals, and detection of phase coupling in the absence of frequency coupling are addressed. In particular, the problem of estimating rational phase-coupling in sonar signal processing is considered. One approach first estimates the frequencies, amplitudes and phases, and then performs a rational phase-coupling hypothesis test on whether the estimated phases bear the same ratios as their corresponding frequencies. Although high-resolution estimates of the frequencies can be obtained using the fourth-order cumulants, the estimated phases have to be "unwrapped" owing to the modulo-2π problem. It is not clear how this can be done. Hence, this approach for unraveling rational phase-coupling seems to be infeasible. Another approach using higher-order mixed moment functions and fractional moments is introduced and studied. It is shown that this approach can detect unconventional types of frequency and phase coupling.

Contents

Abstract i

I INTRODUCTION 1

II MODEL 2

III APPROACH I: (Ngo and Mendel) 3

A Problem Statement 3

B Algorithm 3

B.1 Stage 1: Estimate frequencies ω_i 3

B.2 Stage 2: Estimate amplitude α_i and phase ϕ_i 4

B.3 Stage 3: Test for rational phase-coupling 4

C Assessment of Fourth-order Cumulant-based Methods for Harmonic Retrieval Problems 4

D Simulations 5

D.1 Frequency Estimation 5

D.2 Limitation of Bispectrum in Rational Phase Coupling 6

E Problem associated with Phase Estimation 6

IV APPROACH II: (Swami and Mendel) 7

A Introduction 7

B Bispectra of deterministic signals: frequency-coupling 7

C Bispectra of random signals: phase-coupling 10

D Rational Moment Spectra: The effects of noise 12

D.1 Independent phases 13

D.2 Integer coupling to an unobserved harmonic 13

D.3 Integer coupling 14

D.4 Rational coupling 14

D.5 Variance of sample estimates 15

E Simulations 15

V Conclusions 18

VI Summary and Recommendations 18

Acknowledgement 19

References 20

Appendix: Comparison between correlation-based and cumulant-based approaches to the harmonic retrievel and related problems

I INTRODUCTION

For several years there has been considerable interest in the automatic classification of undersea and surface sources in sonar signal processing. It is known that the radiated signals from the power plant of a vessel are composed of several spectral lines. These frequencies, which correspond to different gear ratios, are coupled among each other in a rational sense. Consequently, the issue of rational phase-coupling arises. By unraveling the phase-coupling, it is possible to determine the actual gear ratio that generated these spectral lines.

It is well-known that higher-order statistics (HOS), due to their ability to preserve phase information, can be used to detect integer-related phase-coupling which arises only among harmonically related components [4, 15]. In this report, the applicability of HOS techniques in identifying various kinds of frequency and phase coupling, is studied. In particular, algorithms based on HOS techniques are proposed to detect and quantify phase-coupling among harmonically-related lines that have ratio-of-integer relationships, such as those associated with gear ratios.

Two approaches have been pursued; the first by Ph.D. student Chiu Yeung Ngo and his advisor, P.I. Dr. Jerry M. Mendel; the second by consultant Dr. Ananthram Swami and P.I. Dr. Jerry M. Mendel. Regarding the first approach, it first estimates the frequencies, amplitudes and phases, and then performs a rational phase-coupling hypothesis test on whether the estimated phases bear the same ratios as their corresponding frequencies. Regarding the second approach, it uses higher-order mixed moment functions and fractional moments.

III APPROACH I: (Ngo and Mendel)

A Problem Statement

Although the ϕ_i are assumed to be uniformly distributed in $[0, 2\pi)$, they are fixed once they have been generated; therefore, we treat the ϕ_i as fixed parameters.

The problem considered here is to estimate the amplitude, α_i, the phase, ϕ_i, and the frequency, ω_i, from N samples of noisy measurement $\{y(1), y(2), \ldots, y(N)\}$. Once these estimates are available, we can determine whether rational phase-coupling occurs by performing a hypothesis test

$$\frac{\omega_i}{\omega_j} \stackrel{?}{=} \frac{\phi_i}{\phi_j} \qquad \forall i \neq j;\; i, j = 1, \ldots, p \tag{3}$$

B Algorithm

We propose a three-stage algorithm for identifying rational phase coupling.

B.1 Stage 1: Estimate frequencies ω_i

It is well-known that $s(n)$ obeys

$$s(k) + \sum_{i=1}^{p} a(i) s(k-i) \;=\; 0 \tag{4}$$

The coefficients $a(i)$ are such that the roots of the associated polynomial are exponentials contained in the signal. More precisely, if $A(z)$ is this corresponding polynomial, defined as

$$A(z) \;=\; 1 + \sum_{i=1}^{p} a(i) z^{-i}, \tag{5}$$

it can be expressed as

$$A(z) \;=\; \Pi_{i=1}^{p} (z^{-1} - e^{j\omega_i}), \tag{6}$$

i.e., $A(z)$ has all its zeros located on the unit circle at $e^{-j\omega_i}$.

By adding noise, $n(k)$, to the signal $s(k)$, we obtain $y(k) = s(k) + n(k)$ which leads to an ARMA(p,p) model:

$$y(k) + \sum_{i=1}^{p} a(i) y(k-i) \;=\; n(k) + \sum_{q=1}^{p} a(q) n(k-q) \tag{7}$$

The parameters, $a(i)$, can be determined via cumulant-based techniques as described in [15]. A comprehensive assessment of this cumulant-based approach has recently been studied by Shin and Mendel [9], which is discussed in the next section.

- the method works even for harmonics spaced as close as 0.04 Hz (i.e., $f_1 = 0.1$ and $f_2 = 0.14$).
- When colored noise is present, one should use a high-resolution method, such as MUSIC or Minimum Norm. The Pisarenko method provides biased estimates of the frequencies f_1 and f_2.
- Regarding amplitude restoration, because estimates of f_1 and f_2 are obtained with very high accuracy using the MUSIC or Minimum Norm methods, least-squares amplitude estimates (which are then maximum-likelihood estimates) give superior results to total least-squares or constrained total least-squares estimates.

D Simulations

We demonstrate the effectiveness of using fourth-order cumulants in estimating the frequencies using the following examples. For simplicity, we only consider a real-valued signal, $y(k)$, from p coherent sources, i.e.,

$$y(k) \;=\; \sum_{i=1}^{p} \alpha_i cos(\omega_i k + \phi_i) + n(k) \qquad . \tag{12}$$

where $n(k)$ is the corrupting zero-mean stationary Gaussian (white or colored) noise which is independent of the emitted sources, $\alpha_i \in R$ and $\phi_i \in [0, 2\pi)$ are the amplitude and phase corresponding to the frequency ω_i, respectively.

D.1 Frequency Estimation

The signal consisted of four unit-amplitude harmonics, two of which are rational phase-coupled and two of which are uncoupled. For the coupled pair, $f_1 = 0.1$ and $f_2 = 0.25$ with $\phi_2 = 2.5\phi_1$. For the uncoupled pair, $f_3 = 0.2$ and $f_4 = 0.3$ with ϕ_3 and ϕ_4 being independent and identically distributed. This signal was contaminated with colored Gaussian noise of 0dB SNR by passing white Gaussian noise through an ARMA filter with AR parameters [1, 1.4563, 0.81] and MA parameters [1, 2, 1]. The resulting noise spectrum has a strong pole around $f = 0.4$. Thirty Monte Carlo runs were performed, each of which consisted of 64 independent realizations of 64 samples. Parametric estimates were obtained using the high-resolution MUSIC and Eigenvector algorithms. For comparison, the estimates using periodogram were also included. Both fourth-order cumulant-based and correlation-based methods were used and their results are depicted in Figures 2a and 2b, respectively.

In both methods, the correct number of harmonics (p = 8) is assumed. The solid lines correspond to MUSIC; the dotted lines correspond to Eigenvector; the dashed lines correspond to periodogram. Note that the frequencies can be estimated accurately whether or not they are rationally phase coupled. Furthermore, compared with the correlation-based methods, the fourth-order cumulant-based methods give high-resolution unbiased frequency estimates. Note, also, that although the periodogram correctly locates the four frequencies, it adds a spurious peak around 0.4. The latter is due to the colored noise.

IV APPROACH II: (Swami and Mendel)

A Introduction

The second-, third- and fourth-order cumulants of zero-mean, real-valued stationary random processes are defined via,

$$C_{2x}(\tau) := E\{x(t)x(t+\tau)\} \tag{15}$$

$$C_{3x}(\tau_1,\tau_2) := E\{x(t)x(t+\tau_1)x(t+\tau_2)\} \tag{16}$$

$$\begin{aligned} C_{4x}(\tau_1,\tau_2,\tau_3) := {} & E\{x(t)x(t+\tau_1)x(t+\tau_2)x(t+\tau_3)\} \\ & -C_{2x}(\tau_1)C_{2x}(\tau_2-\tau_3) - C_{2x}(\tau_2)C_{2x}(\tau_3-\tau_1) - C_{2x}(\tau_3)C_{2x}(\tau_1-\tau_2) \end{aligned} \tag{17}$$

The bispectrum, $S_{3x}(v_1,v_2)$, and the trispectrum, $S_{4x}(v_1,v_2,v_3)$, are the 2-D and 3-D Fourier transforms (FT's) of the third- and fourth-order cumulant sequences, respectively.

The k-th order cumulant of a complex random process can be defined in 2^k different ways, not necessarily distinct, by conjugating one or more of the k terms involved; the particular definition to be used depends upon the nature of the processes involved [15].

If $x(t)$ is non-stationary, then, the cumulant statistics may not be independent of time. In this case, the notations, $C_{3x}(t;\tau_1,\tau_2)$ and $S_{3x}(t;\tau_1,\tau_2)$, etc., will be more appropriate.

B Bispectra of deterministic signals: frequency-coupling

There have been some incorrect claims in the literature regarding the bispectra and trispectra of harmonics. As is well known, the bispectrum of three quadratic phase-coupled harmonics (i.e., with frequencies $\omega_3 = \omega_1+\omega_2$ and $\phi_3 = \phi_1+\phi_2$, where ϕ_1 and ϕ_2 are independent and uniformly distributed over $[0,2\pi)$) consists of impulses at [7, 15],

$$\pm(\omega_1,\omega_2);\;\; \pm(\omega_1,-\omega_3);\;\; \pm(\omega_2,-\omega_3);\;\; \pm(\omega_2,\omega_1);\;\; \pm(-\omega_3,\omega_1);\;\; \pm(-\omega_3,\omega_2). \tag{18}$$

When the phases are random, third-order cumulants cannot be estimated from a single realization (the phases ϕ_i, are fixed for any single realization), but must be averaged across multiple realizations.

If only a single realization is available, the observed signal is[1],

$$y(t) = \sum_{k=1}^{3} \alpha_k \cos(\omega_k t + \phi_k) + g(t) \tag{19}$$

where $g(t)$ is zero-mean additive colored Gaussian noise (ACGN). Note that there is no difference between a single realization of a random phase signal, and a realization of the non-random phase signal. The effect of the harmonic terms is to make the signal non-stationary (they act as a time-varying additive mean for the noise process).

[1] For simplicity, we consider only three harmonics; the extension to cover a mixture of coupled and uncoupled harmonics is easy.

(iii) $E\{y(t_1)y(t_2)y(t_3)\} = C_{3y}(t_1,t_2,t_3);\;\; |C_{3y}| \le B,\;\; \forall t_1,t_2,t_3;$

$$\lim_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N} C_{3y}(t,t+\tau_1,t+\tau_2) = C_{3y}(\tau_1,\tau_2),\;\; \forall \tau_1,\tau_2\,.$$

In (ii) and (iii), the limits are assumed to exist; the quantities on the RHS are the limiting values.

Conditions (i) and (ii) are taken from Ljung [3]. Here, the expectation operator is w.r.t. the stochastic components of $y(t)$. A signal $y(t)$ which satisfies (i) and (ii) is called (second-order) quasi-stationary. If the stochastic component of such a signal can be represented as filtered i.i.d. noise, then, the power spectrum of $y(t)$, computed as if $y(t)$ were deterministic, coincides with probability 1, with that defined by ensemble averages [3, Theorem 2.3]. Note that this extends the concepts of ergodicity to the mixed signal case.

Condition (iii) is a natural extension of condition (ii); mimicking the proof in Ljung, it is easy to show that, [2]

$$\frac{1}{N}\sum_{t=1}^{N} y(t)y(t+\tau_1)y(t+\tau_2) \longrightarrow C_{3y}(\tau_1,\tau_2)\,,$$

where C_{3y} is the limiting value on the RHS of condition (iii). A signal consisting of harmonics with finite amplitudes and stationary noise with finite moments, meets the conditions above.

So, we must now evaluate $C_{3y}(\tau_1,\tau_2)$ for the signal $y(t) = d(t) + g(t)$. We will assume that $g(t)$ is zero-mean, symmetrically distributed, and stationary.

$$\begin{aligned} C_{3y}(t_1,t_2,t_3) &= E\{y(t_1)y(t_2)y(t_3)\} \\ &= E\{(g(t_1)+d(t_1))(g(t_2)+d(t_2))(g(t_3)+d(t_3))\} \\ &= d(t_1)d(t_2)d(t_3) + [R_g(t_1-t_2)d(t_3)]_3 \end{aligned}$$

since $E\{g(t)\} = 0$, and $E\{g(t_1)g(t_2)g(t_3)\} = 0$; $[.]_3$ indicates an additional two terms involving $d(t_1)$ and $d(t_2)$[3] Therefore,

$$\begin{aligned} C_{3y}(\tau_1,\tau_2) &= \lim_{N\to\infty}\frac{1}{N}\sum_{t=1}^{N} C_{3y}(t,t+\tau_1,t+\tau_2) \\ &= \lim_{N\to\infty}\frac{1}{N}\sum_{t=1}^{N} d(t)d(t+\tau_1)d(t+\tau_2) \\ &\quad +(R_g(\tau_1)+R_g(\tau_2)+R_g(\tau_1-\tau_2))\mu_d \\ &= C_{3d}(\tau_1,\tau_2) + (R_g(\tau_1)+R_g(\tau_2)+R_g(\tau_1-\tau_2))\mu_d \end{aligned}$$

[3] We stress that

(a) the Expectation operator is w.r.t. the random component, $g(t)$.

(b) $y(t)$ is non-stationary; it is implicitly assumed that multiple realizations are available to do the ensemble averaging.

(c) Signal alignment across ensembles is also implicitly assumed; i.e., if k denotes the realization number, $y_k(t) = d(t)+g_k(t)$.

(d) If the additive noise is white, the signal can be recovered from $C_{3y}(m_o,n_o,t) = d(m_o)d(n_o)d(t)$, $m_o \ne n_o \ne t$, where m_o and n_o should be chosen such that $d(m_o) \ne 0$, $d(n_o) \ne 0$.

(e) If the noise is Gaussian (not necessarily white), then, the ML estimate of the deterministic signal is simply the mean, i.e., $\hat{d}_{ML} = \frac{1}{K}\sum_{k=1}^{K} y_k(t)$. Under the preceding assumptions and conditions, it is sub-optimal, and expensive, to use cumulant-based approaches. The latter are suggested when signal alignment problems occur, and it can further be assumed that the signal is completely contained within each record.

For non-stationary signals, the obvious estimate is

$$\hat{C}_{3y}(t,m,n) := \frac{1}{K}\sum_{k=1}^{K} y_k(t)y_k(m)y_k(n) \tag{28}$$

The corresponding bispectrum will be indexed by time t.

Let ϕ be uniformly distributed over $[0, 2\pi)$; then, it follows that

$$E\{\exp(j\beta\phi)\} = \begin{cases} \frac{\exp(j2\pi\beta)-1}{j\beta} & \text{if } \beta \text{ is non-integer} \\ 1 & \text{if } \beta = 0 \\ 0 & \text{if } \beta = \pm1, \pm2, \ldots \end{cases} \tag{29}$$

From (29), (24) and (25), we note that the bispectrum will be non-zero if and only if both frequency coupling of the form given in (23) and the corresponding phase coupling exist, i.e.,

$$\phi_2 = 2\phi_1; \quad \phi_3 = 2\phi_1; \quad \phi_3 = 2\phi_2; \quad \text{or} \quad \phi_3 = \phi_1 + \phi_2 \tag{30}$$

What happens if we have phase coupling, but no frequency coupling? Assume that the observed signal is,

$$y(t) = \sum_{k=1}^{3} \alpha_k \cos(\omega_k t + \phi_k) + g(t) \tag{31}$$

$$f_3 \neq f_1 + f_2 \tag{32}$$

$$\phi_3 = \phi_1 + \phi_2 \tag{33}$$

This is a non-stationary signal; the third-order cumulant is now given by,

$$C_{3y}(t;\tau_1,\tau_2) := E\{y(t)y(t+\tau_1)y(t+\tau_2)\} \tag{34}$$

$$= \frac{\alpha_1\alpha_2\alpha_3}{4}[\cos(\omega_o t + d_1\tau_1 + d_2\tau_2)]_6 \tag{35}$$

where $\omega_o := \omega_1 + \omega_2 - \omega_3$ and $[.]_6$ are the six terms obtained by choosing d_1 and d_2 from $(\omega_1, \omega_2, -\omega_3)$. In the absence of frequency coupling, $\omega_o \neq 0$; hence, temporal averaging (over t, along the lines of eq (20)) will cause the third-order cumulants to vanish (see, eq. (21)).

Since the statistic is periodic with period $2\pi/\omega_o$, periodic temporal averaging is okay (the phases are coherent), but this assumes that ω_o is known. It is easy to estimate ω_o, for example, by fixing τ_1 and τ_2, and looking at the FT of $C_{3y}(t;\tau_1,\tau_2)$, w.r.t. t, or fitting an AR model to it. Note also that it suffices to estimate $C_{3y}(t;\tau_1,\tau_2)$, at only one t.

An alternative approach is to use the estimate

$$B(v_1,v_2) := \frac{1}{T}\sum_{t=1}^{T} |\mathcal{F}(C_{3y}(t;\tau_1,\tau_2))| \tag{36}$$

where the Fourier transform is w.r.t. the lags τ_1 and τ_2. For example, the term $\cos(\omega_o t + d_1\tau_1 + d_2\tau_2)$, leads to impulses at $\pm(d_1, d_2)$, with amplitude factors, $\exp(\pm j\omega_o t)$. Direct temporal averaging causes the bispectrum to go to zero. By using only absolute values, the $\omega_o t$ term is suppressed. An alternative approach is the use of Third-Order Wigner Distributions (TWD's); see [13] and [12].

where α_i's are non-random, and ϕ_i's are random. We will consider only the off-diagonal ($t_1 \neq t_2$) region; hence, we can ignore the noise. We have,

$$\begin{aligned} R^s_{p,q}(t_1,t_2) &= \sum_{m=0}^{p}\sum_{n=0}^{q}\binom{p}{m}\binom{q}{n}\alpha_1^m\bar{\alpha}_1^n\alpha_2^{p-m}\bar{\alpha}_2^{q-n}E\{\exp(j(m-n)\phi_1+j(p-q+n-m)\phi_2)\} \\ &\quad \times\ \exp(j(m\omega_1t_1-n\omega_1t_2))\exp(j((p-m)\omega_2t_1-(q-n)\omega_2t_2)) \end{aligned} \tag{42}$$

Note that the key term to be evaluated is

$$f(\phi_1,\phi_2) := E\{\exp(j(m-n)\phi_1+j(p-q+n-m)\phi_2)\} \tag{43}$$

where the dependence upon m, n, p and q is not explicitly shown. We will consider several cases.

D.1 Independent phases

If ϕ_1 and ϕ_2 are independent and uniformly distributed over $[0,2\pi)$, we have,

$$f(\phi_1,\phi_2) = \delta(m-n)\delta(p-q)\ .$$

Hence,

$$R^s_{p,p}(t_1,t_2) = \sum_{m=0}^{p}\binom{p}{m}\binom{p}{m}|\alpha_1|^{2m}|\alpha_2|^{2p-2m}e^{j(m\omega_1t_1-m\omega_1t_2)}e^{j((p-m)\omega_2t_1-(p-m)\omega_2t_2)} \tag{44}$$

Note that the statistic is a function only of (t_1-t_2); hence, it is stationary;

$$\begin{aligned} R^s_{p,p}(\tau) &:= R^s_{p,p}(t_1,t_1-\tau) \\ &= \sum_{m=0}^{p}\binom{p}{m}\binom{p}{m}|\alpha_1|^{2m}|\alpha_2|^{2p-2m}\exp(jm\omega_1\tau+j(p-m)\omega_2\tau) \end{aligned}$$

A closed-form solution exists for the special case when $\alpha_1=\alpha_2=\alpha$, and the observations are noise-free,

$$R^s_{p,p}(0) = \binom{2p}{p}|\alpha|^{2p}\ .$$

The normalized statistic is,

$$\Gamma^s_{p,p}(0) = \binom{2p}{p}/2^p\ .$$

D.2 Integer coupling to an unobserved harmonic

Assume that $\phi_1 = k_1\phi_0$, and $\phi_2 = k_2\phi_0$, where ϕ_0 is $U[0,2\pi)$, and k_1 and k_2 are integers. We have,

$$\begin{aligned} f(\phi_1,\phi_2) &:= E\{\exp(j(m-n)\phi_1+j(p-q+n-m)\phi_2)\} \\ &= E\{\exp(jk_1(m-n)\phi_0+jk_2(p-q+n-m)\phi_0)\} \\ &= \delta((m-n)(k_1-k_2)+k_2(p-q)) \end{aligned} \tag{45}$$

Note that the $p=q$ case leads to the same results as in the case of independent phases.

With this (standard) notation, we have (a standard result),

$$(1+x)^p = \sum_{k=0}^{\infty} \binom{p}{k} x^k .$$

Note that for non-integer p, the series converges absolutely only if $|x| < 1$. Additionally, the interpretation of z^p, for z complex is no longer unique[5]. The same problem is encountered if we work directly in the moment-spectrum or polyspectrum domain (recall that polyspectra can be interpreted as Fourier transforms of cumulants, or as cumulants of Fourier transforms - a consequence of the multi-linearity properties of cumulants).

D.5 Variance of sample estimates

An obvious estimate of $R^s_{p,q}(t_1, t_2)$ is its sample mean. Under boundedness assumptions, it is easy to show that the expressions for the variance of sample estimates of k-th order moments and cumulants are dominated by the summation of corresponding moments of order $2k$.

E Simulations

Several of the ideas discussed so far are illustrated by simulations. Since the accompanying figures give full details of the simulations, we will only provide an overview here.

Figure 5 depicts the non-parametric bispectrum for the standard quadratic phase coupling problem, where both the phases and the frequencies are coupled, i.e., $f_3 = f_1 + f_2$, and $\phi_3 = \phi_1 + \phi_2$, with $f_1 = 0.1$, $f_2 = 0.2$ and $f_3 = 0.3$.

Figures 6a-6c address the deterministic case, where there is frequency coupling. The data consists of three harmonics with $f_1 = 0.1$, $f_2 = 2f_1 = 0.2$, and $f_3 = 0.35$. The non-parametric bispectrum was estimated using the indirect method. The major conclusion is that for the deterministic problem, record segmentation must be done with care: unbiased estimates must be used (unless the segment lengths are large enough), so as not to degrade the value of the estimated bicoherence. Additionally, record segmentation *may* artificially cause the data to appear as if they were independent realizations of the random process problem. However, phase coupling will be induced only if the starting phases, the frequencies and the segment lengths are in the right relationships with one another.

Figures 7a-7c address the deterministic case, where there is no frequency coupling. The data consists of three harmonics with $f_1 = 0.1$, $f_2 = 0.15$, and $f_3 = 0.35$. The non-parametric bispectrum was estimated using the indirect method. The major conclusion is that for the deterministic problem, record segmentation must be done with care: unbiased estimates must be used (unless the segment lengths are large enough), so as not to degrade the value of the estimated bicoherence. Additionally, record segmentation *may* artificially cause the data to appear as if there were some structure in the data, such as due to frequency or phase coupling. If record segmentation must be done, try different segment sizes and see if the resulting estimates are consistent with one another.

[5] If $p = n/d$, where n and d are co-prime integers, we have d possible solutions; if p is irrational, the number of solutions is infinite.

Table 1: Performance of the $R_{3,5}(0)$ statistic-based test for the $(3,5)$ case.

SNR		real-m	imag-m	real-s	imag-s	%error
∞	C	1.0023	0.0080	0.1339	0.1276	0
	U	0.0275	-0.0077	0.1481	0.1630	0
20 dB	C	1.0631	0.1012	0.4833	0.3919	12
	U	0.0594	-0.0197	0.4648	0.4143	16
10 dB	C	0.7761	-0.0551	1.8163	1.9536	48
	U	0.0033	0.0312	1.8837	1.7219	36
0 dB	C	2.6917	7.5508	48.4320	52.0910	48
	U	4.5166	12.6212	52.8066	59.7258	57

64 independent records, each consisting of 64 samples were generated. The parameters were $f_o = 0.05$ $k_1 = 3$ $k_2 = 5$ $m = 3$ $n = 5$ $\alpha_1 = \alpha_2 = 1$. The statistic $R_{p,q}(0)$ was estimated for each realization. In the absence of noise, we have $R_{3,5}(0) = 1$, if coupling is present, and $R_{3,5}(0) = 0$, if no coupling is present. Hence, the test was to compare the absolute value of $R_{3,5}(0)$ with the threshold value of 0.5. The above table summarizes the results of a 100 realization study. The columns show the SNR, whether or not the harmonics were coupled (C for coupling), the mean value of the real and imaginary parts of $R_{3,5}(0)$, and the corresponding variances. The last column shows the percentage error in classification.

Table 2: Performance of the $R_{4,7}(0)$ statistic-based test for the $(3,5)$ case.

SNR		real-m	imag-m	real-s	imag-s
∞	C	0.0821	0.1350	1.1200	0.9872
	U	0.0184	0.0782	1.1489	0.9161
20 dB	C	0.0522	0.5632	4.2573	3.9408
	U	0.7175	0.0012	4.5495	4.6820
10 dB	C	5.2126	3.2088	28.9038	28.4556
	U	5.6866	3.8653	27.0223	28.2427
0 dB	C	108.9	-486.5	3751.9	3470.1
	U	-365.4	273.0	4049.0	3740.4

The $R_{4,7}(0)$ statistic was tested on the same data that were used to generate Table 1. Note that the expected value of this statistic is zero for both the coupled (3,5) and uncoupled (3,5) cases. This test, which does not take into account noise, degrades rapidly below 20 dB SNR.

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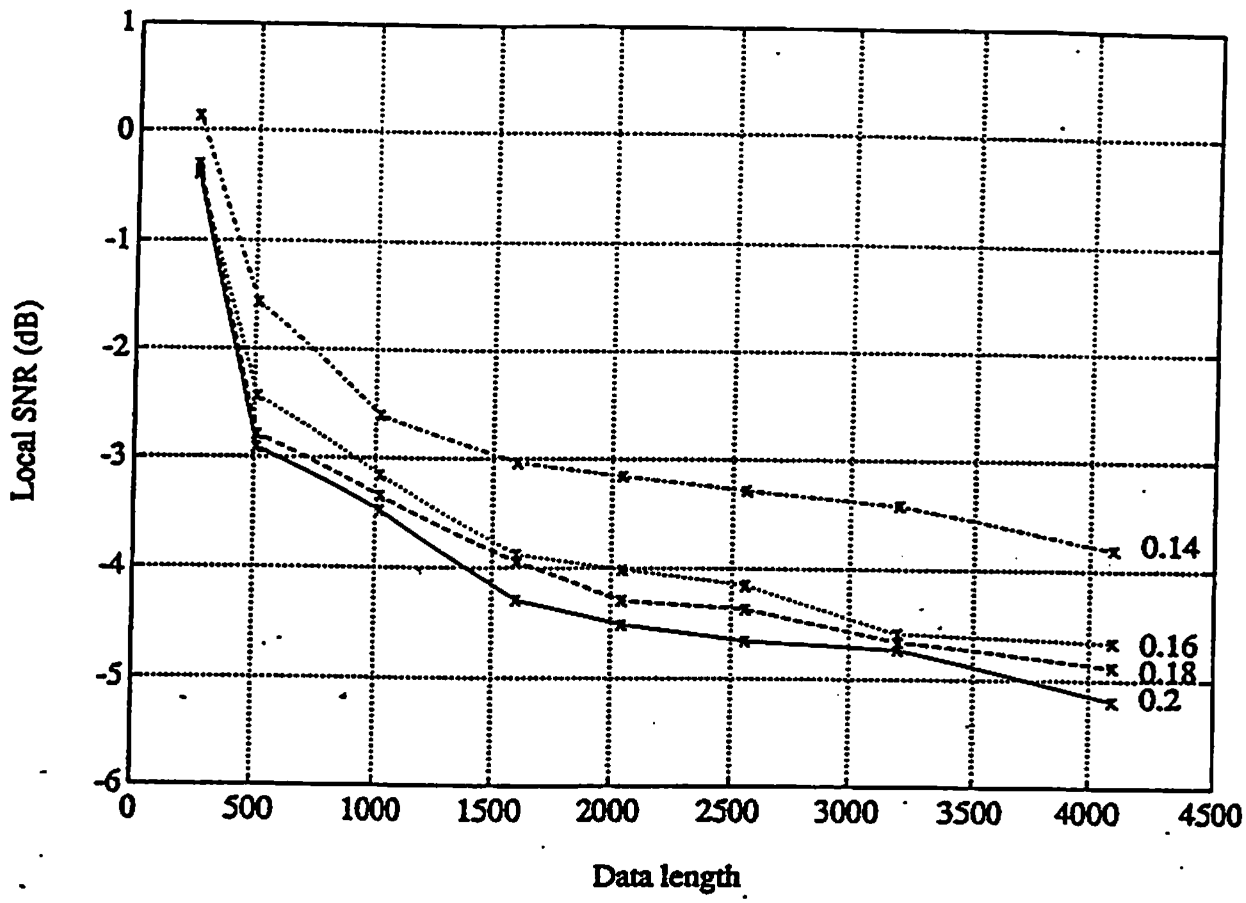

Figure 1: Minimum local SNR (dB) of the amplitude of the second harmonic at f_2 when we can still determine the correct number of harmonics. The fixed amplitude at $f_1 = 0.1$ has a local SNR of 0 dB. The crossed points denote the values that were obtained from simulations. The solid line is for $f_2 = 0.2$, and the dashed, dotted, and dash-dotted lines are for $f_2 = 0.18$, 0.16, *and* 0.14, respectively.

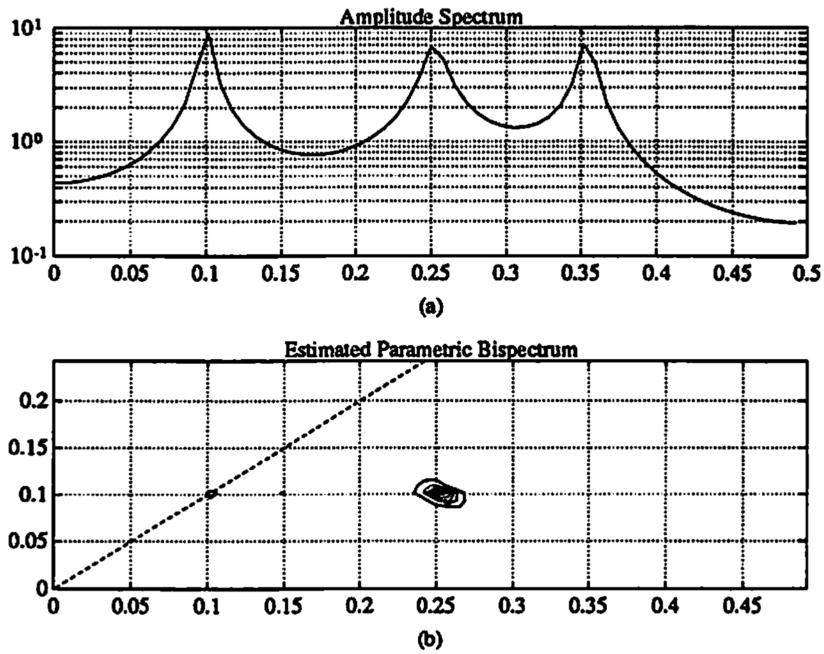

Figure 3: Detection of quadratic phase-coupling using bispectrum: (a) Amplitude spectrum, (b) Contour of bispectrum with peak value = 437.3 at ($f_1 = 0.1$, $f_2 = 0.25$).

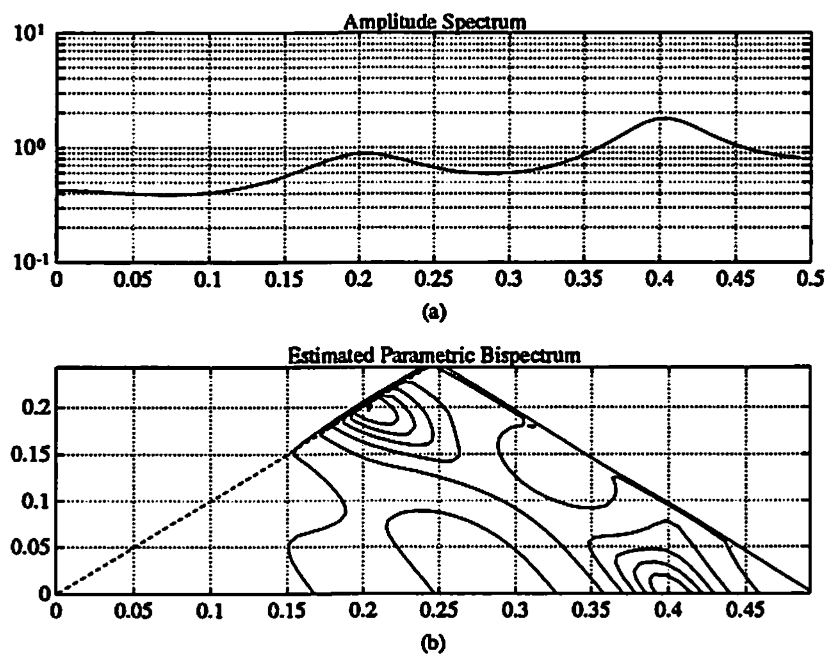

Figure 4: Detection of rational phase-coupling using bispectrum: (a) Amplitude spectrum, (b) Contour of bispectrum with peak value = 1.43 at ($f_1 = 0.20$, $f_2 = 0.20$).

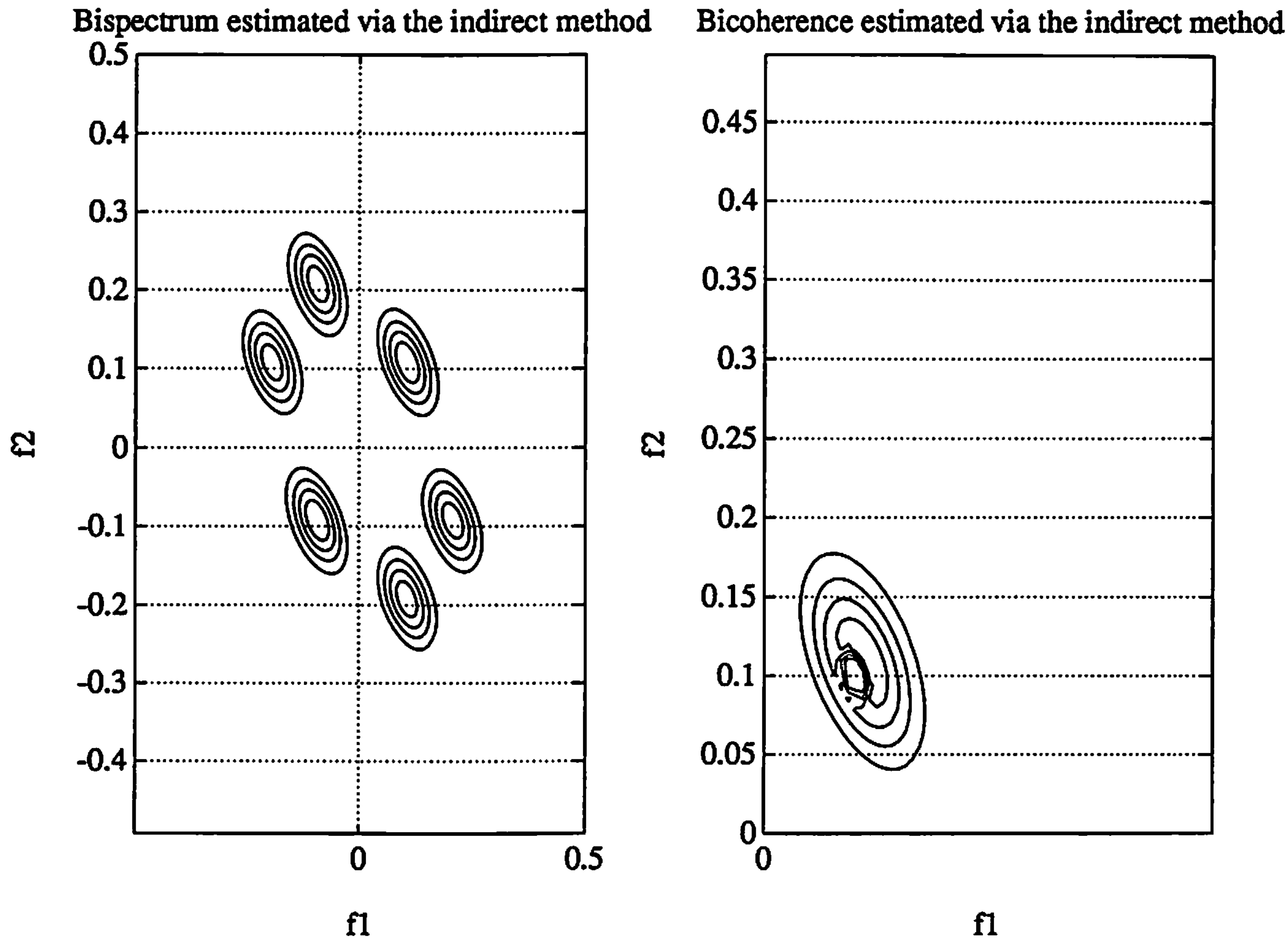

Figure 6a: The observed signal is $z(n) = \cos(.2\pi n) + \cos(.4\pi n) + \cos(.7\pi n)$, $n = 1, ..., 8192$. Biased estimates of third-order cumulants, $C(m,n)$, were computed for $|m|, |n| < 21$. The bispectrum was then estimated using a 128-point 2-D FFT. No record segmentation was done. The bicoherence has a peak value of 0.9222 at $(0.11720, .1094)$.

The signal is deterministic, and we have frequency coupling, since $f_1 = 0.1$, $f_2 = 0.2 = 2 * f_1$, $f_3 = 0.35$. Since $f_2 = 2f_1$, we obtain six peaks; see (24).

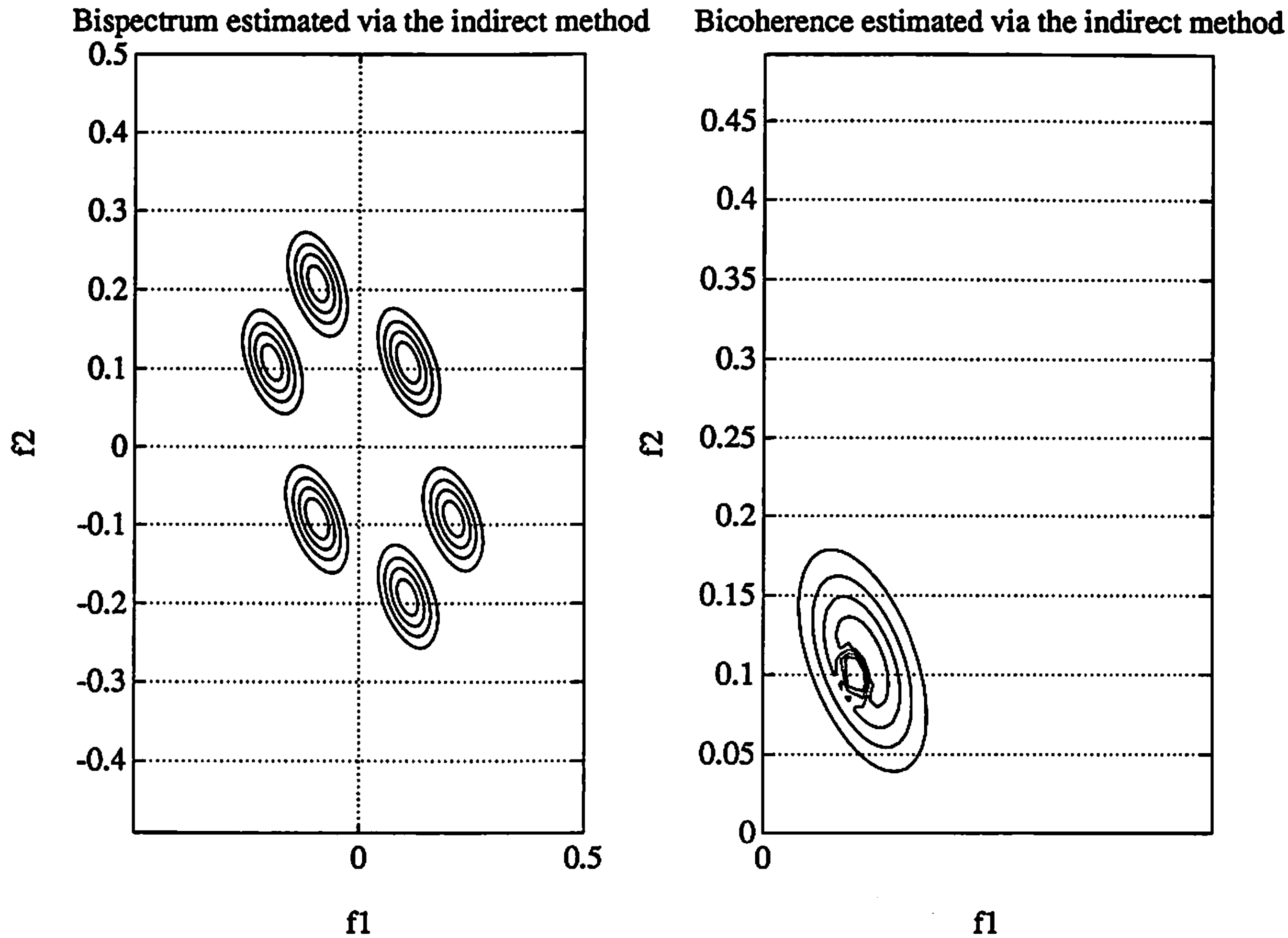

Figure 6c: The observed signal is $z(n) = \cos(.2\pi n) + \cos(.4\pi n) + \cos(.7\pi n)$, $n = 1, ..., 8192$. Biased estimates of third-order cumulants, $C(m,n)$, were computed for $|m|, |n| < 21$. The bispectrum was then estimated using a 128-point 2-D FFT. Data were segmented into records of length 120 for computing cumulants. The bicoherence has a peak value of 0.8038 at (0.11720, .1094).

Record segmentation did not appreciably change the shape of the bispectrum or the bicoherence. The decrease in the value of the bicoherence is due to the use of biased estimates. If record segmentation must be done, it is better to use unbiased estimates.

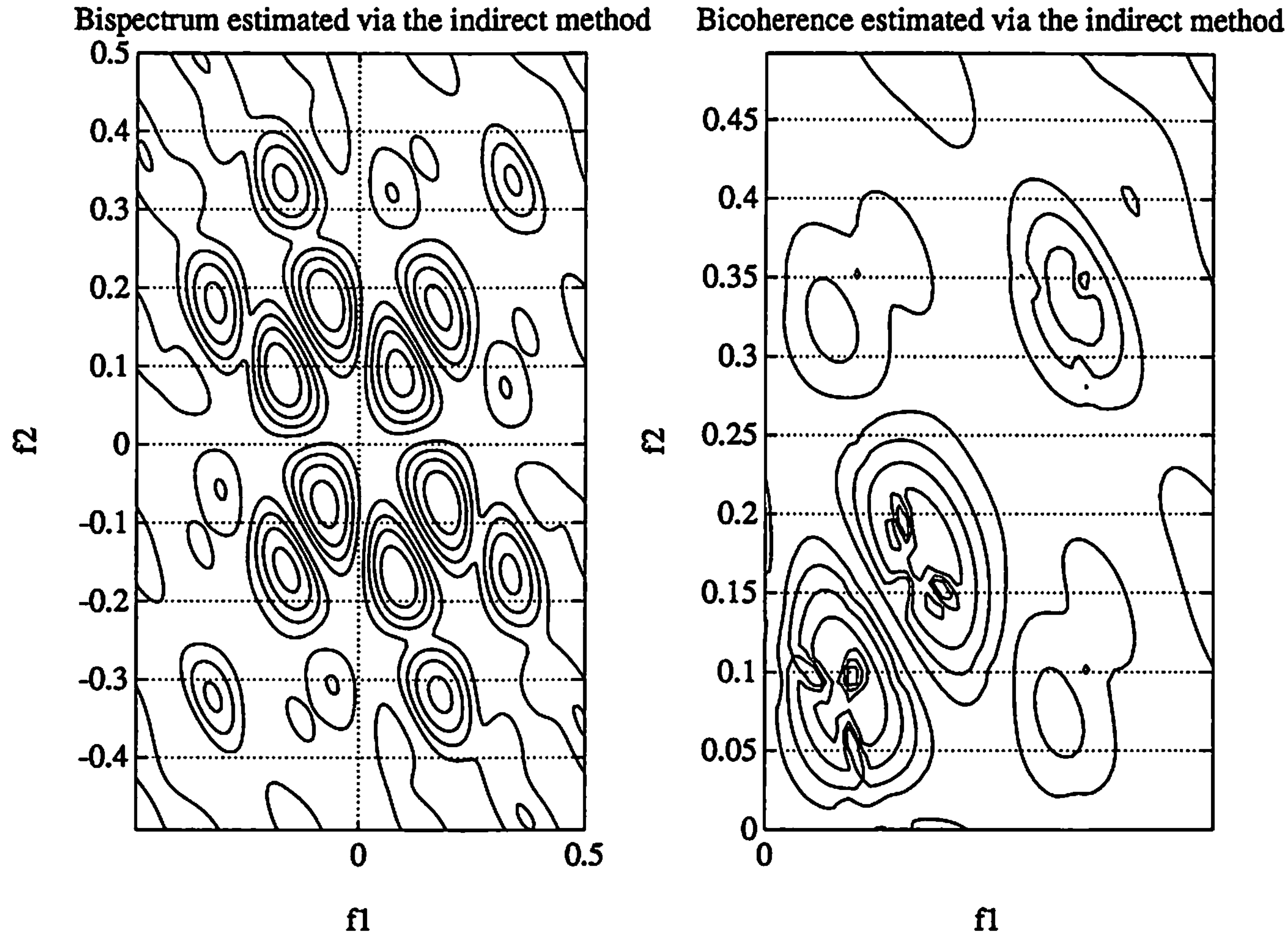

Figure 7b: The observed signal is $z(n) = \cos(.2\pi n) + \cos(.3\pi n) + \cos(.7\pi n)$, $n = 1, ..., 8192$. Biased estimates of third-order cumulants, $C(m, n)$, were computed for $|m|, |n| < 21$. The bispectrum was then estimated using a 128-point 2-D FFT. Data were segmented into records of 128 samples each for computing cumulants. The bicoherence has a peak value of 0.1819 at (0.1797, .1797).

The signal is deterministic and does not satisfy any of the frequency coupling conditions in (23). The estimated bispectrum shows apparent structure; this is a consequence of end effects due to record segmentation (and relatively small record sizes).

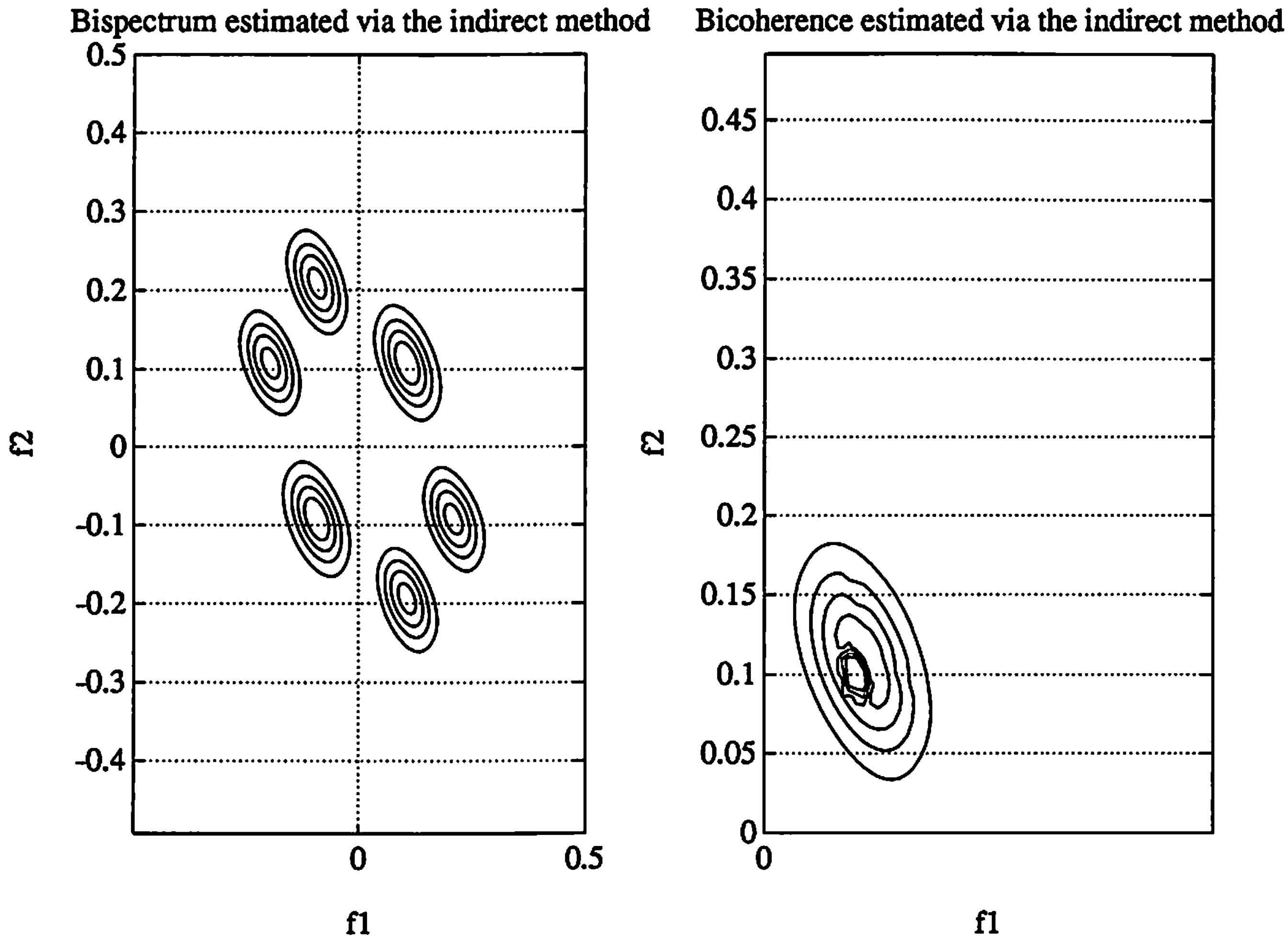

Figure 8a: The observed signal is $z(n) = \cos(.2\pi n + \phi_1) + \cos(.4\pi n + \phi_2) + \cos(.5\pi n + \phi_1 + \phi_2)$, $n = 1, ..., 128$; the phases ϕ_1 and ϕ_2 were independent and uniformly distributed over $[0, 2\pi]$. 64 independent realizations were generated. Note that we have phase-coupling but not frequency-coupling. Biased estimates of third-order cumulants, $C(m, n)$, were computed for $|m|, |n| < 21$. The bispectrum was then estimated using a 128-point 2-D FFT. The bicoherence has a peak value of 0.3196 at (0.1172, .1094).

Compare with Figure 6a. In particular, note that the third harmonic at $f_3 = 0.25$, is not evident in the plots. See discussion following (34).

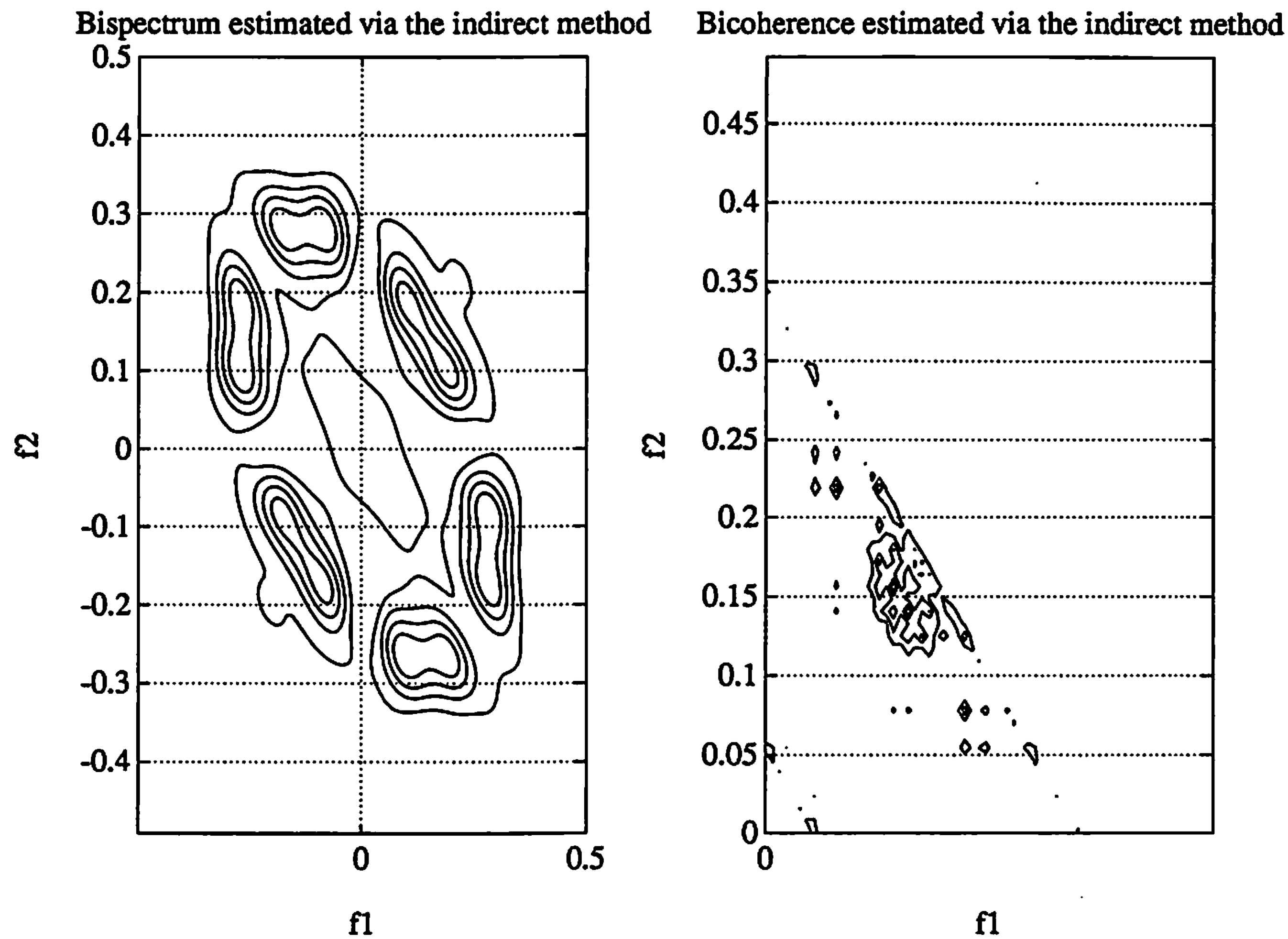

Figure 8c: The observed signal is $z(n) = \cos(.2\pi n + \phi_1) + \cos(.4\pi n + \phi_2) + \cos(.5\pi n + \phi_1 + \phi_2)$, $n = 1, ..., 128$; the phases ϕ_1 and ϕ_2 were independent and uniformly distributed over $[0, 2\pi]$. 64 independent realizations were generated. Note that we have phase-coupling but not frequency-coupling. Biased estimates of third-order cumulants, $C(0; m, n)$, were computed for $|m|, |n| < 21$. The bispectrum was then estimated using a 128-point 2-D FFT. The bicoherence peaks at (0.1562, .1406).

The estimates are rather poor because only the $t = 0$ slice was used.

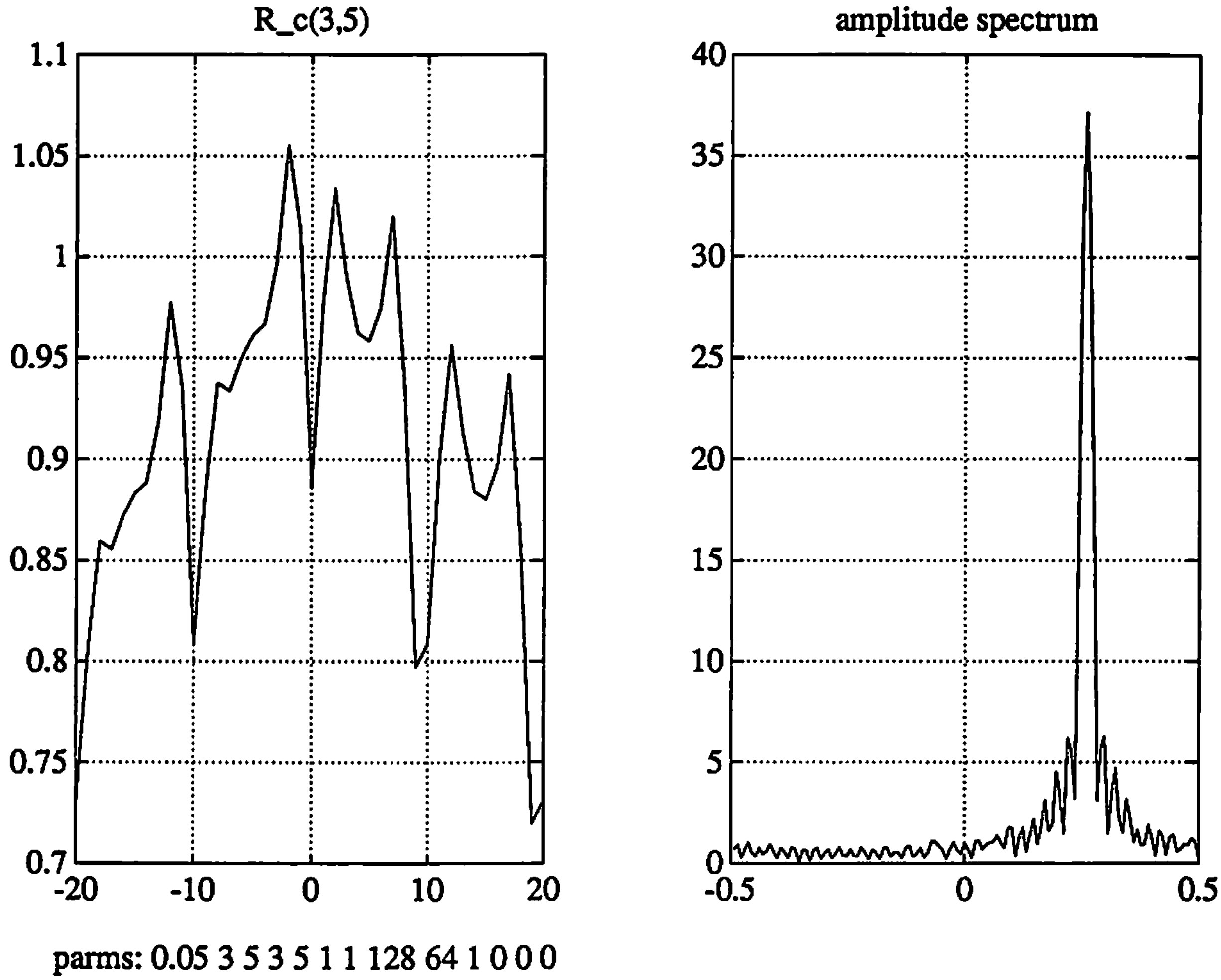

Figure 9a: This is the $R_{3,5}$ statistic for the $(3,5)$ coupled case. Notice the peak at 0.25. With $k_1 = 3$, $k_2 = 5$, $f_o = 0.05$, the expected peak is at $-k_1k_2f_o = -0.75$[7]; because of aliasing, the peak appears at 0.25. Since the signal variance is 2, the normalized peak value of the spectrum is approximately $40/\sqrt{2}^{3+5} = 2.5$.

[7] See eq (46) and discussions related to it.

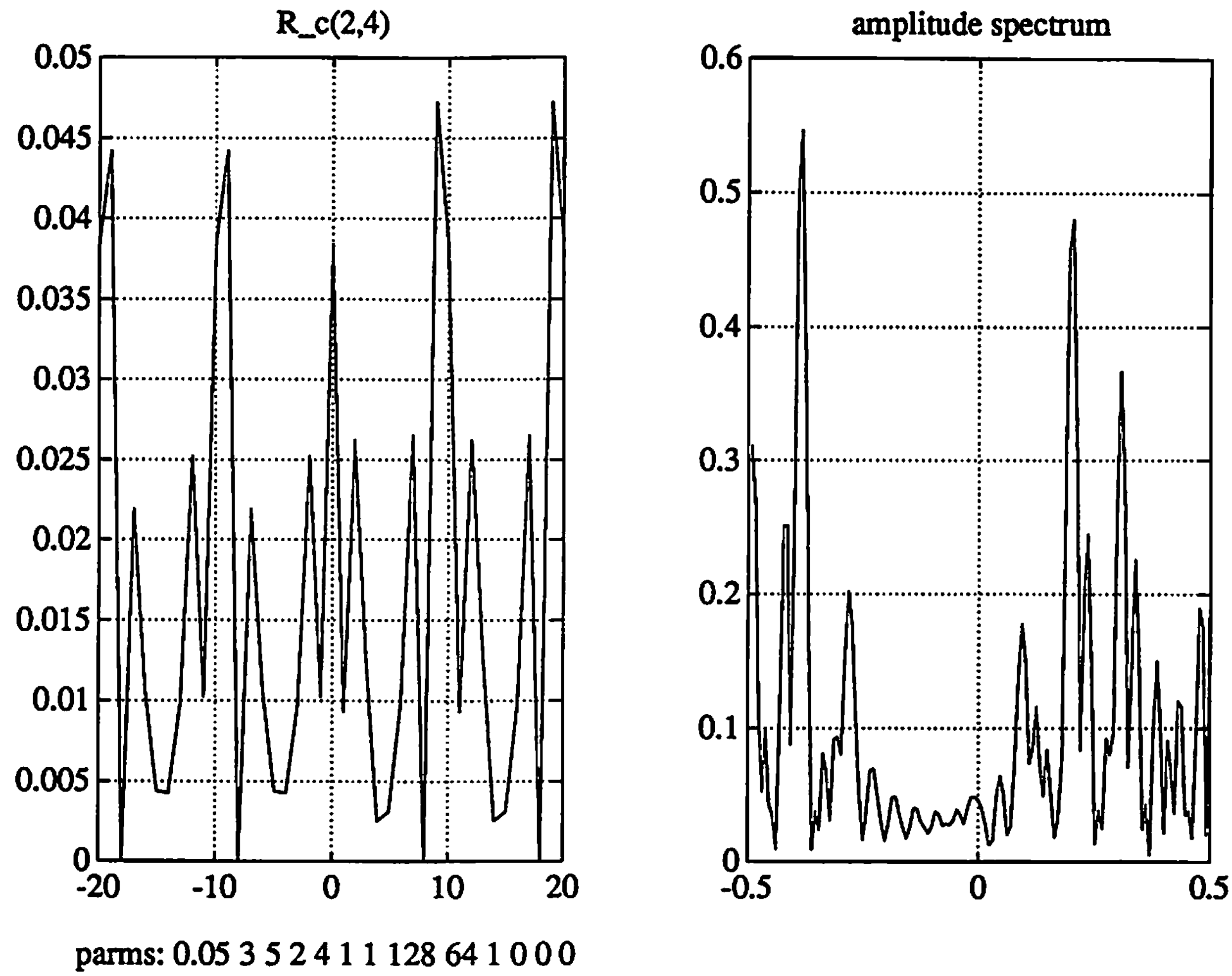

Figure 10: This is the $R_{2,4}$ statistic for the (3, 5) coupled case. Moments of order (2,4) should be blind to coupling of order (3,5). The normalized peak value of the spectrum is $0.6/\sqrt{2}^6 = 0.075$ is close to zero, as expected.

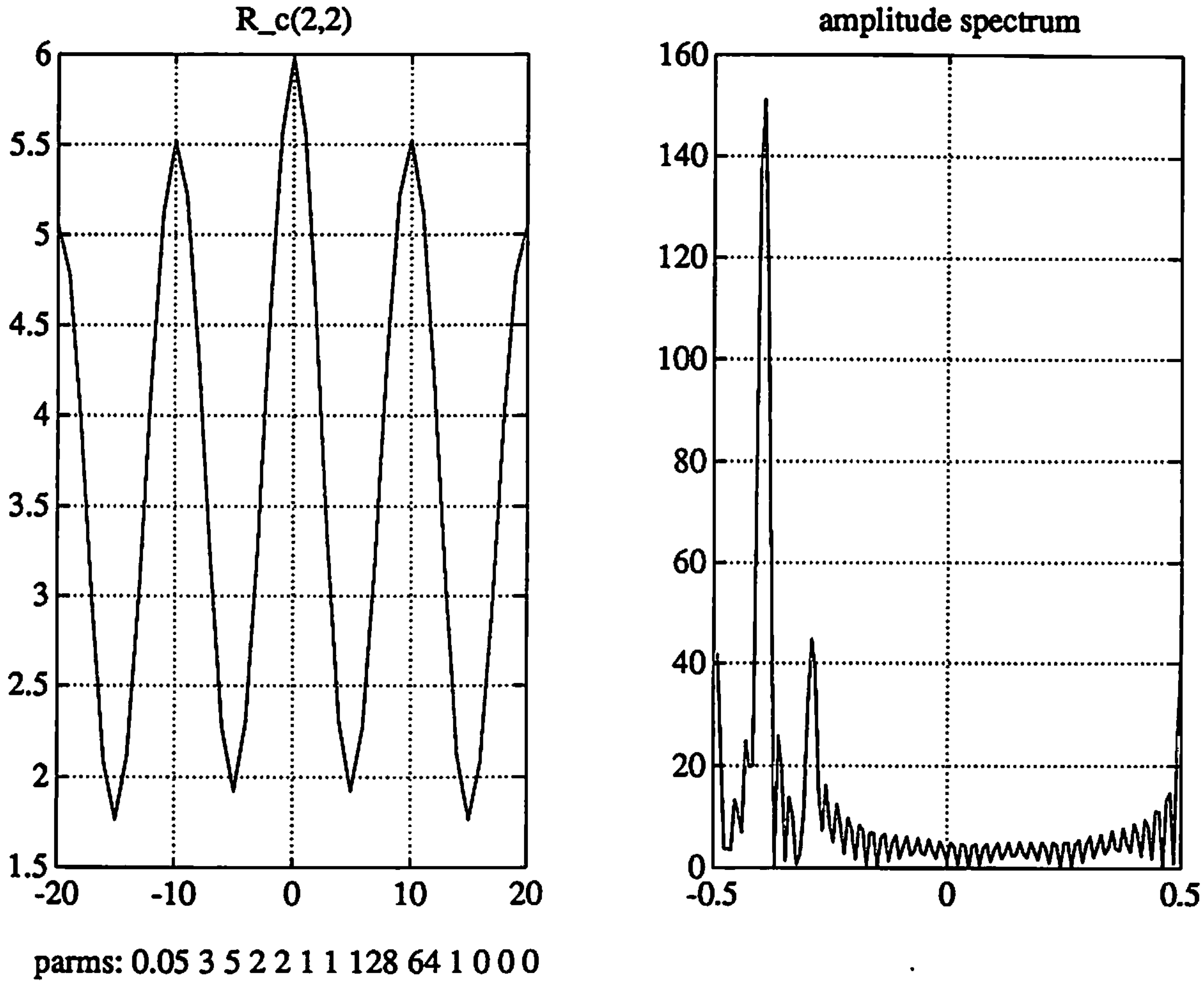

Figure 12: This is the $R_{2,2}$ statistic for the $(3,5)$ coupled case. Recall that the $R_{2,2}$ statistic is the autocorrelation of the square of the observed signal. Theory predicts peaks at frequencies $-2f_1$, $-2f_2$ and $-f_1 - f_2$, with amplitudes in the ratio $1:1:4$[10]. Since $f_1 = .15$ and $f_2 = 0.25$, the three peaks should be at -0.3, -0.5 and -0.4.

[10]See eqns. (42), (43) and (45).

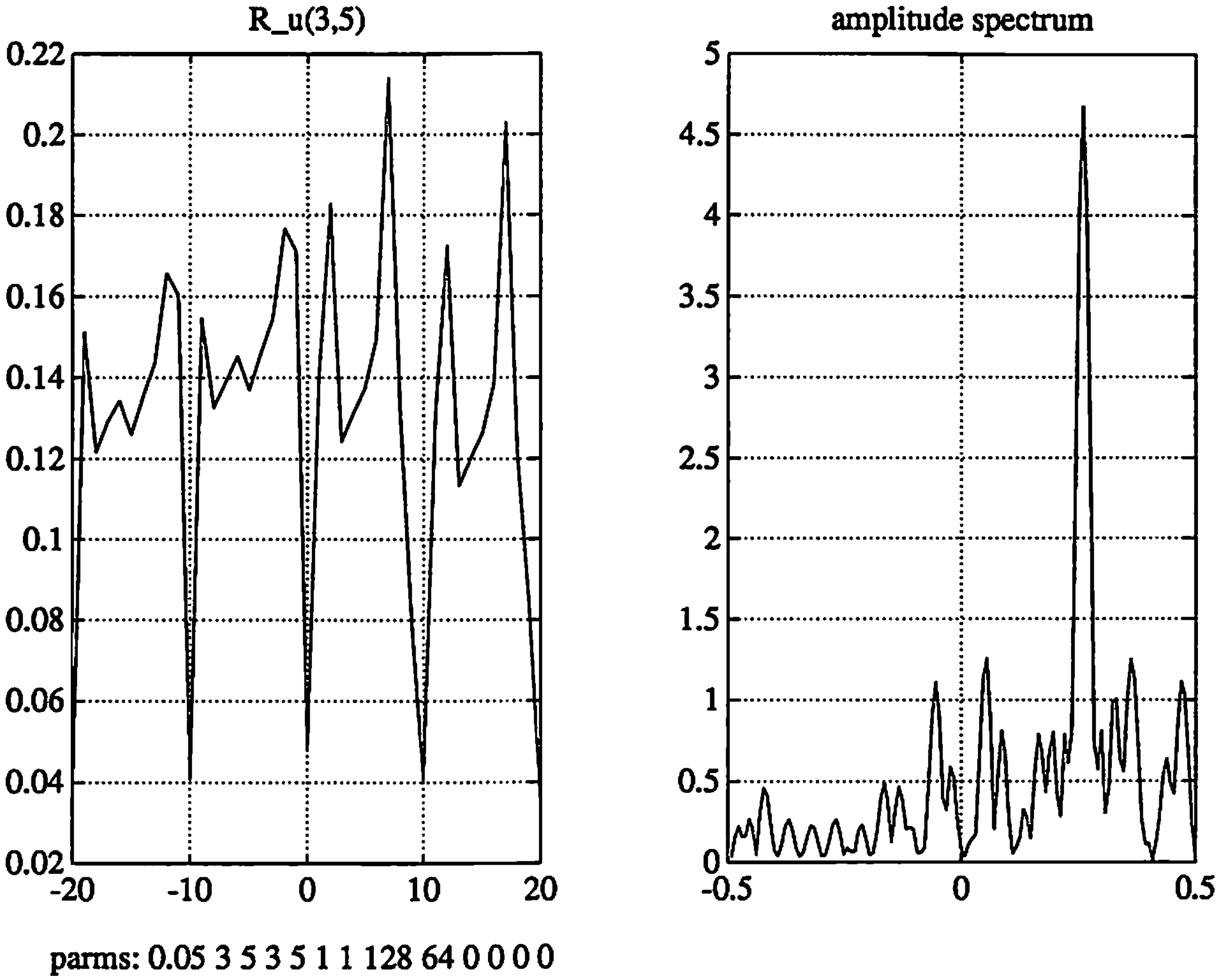

Figure 14: This is the $R_{3,5}$ statistic for the $(3,5)$ uncoupled case. Note that the normalized peak value of the spectrum is $5/16 \approx 0.3$. Compare with Figure 9a, which shows the corresponding coupled case. These figures verify that the R_{k_1,k_2} statistic may be used to distinguish between the coupled and uncoupled cases, with $f_1 = k_1 f_o$ and $f_2 = k_2 f_o$.

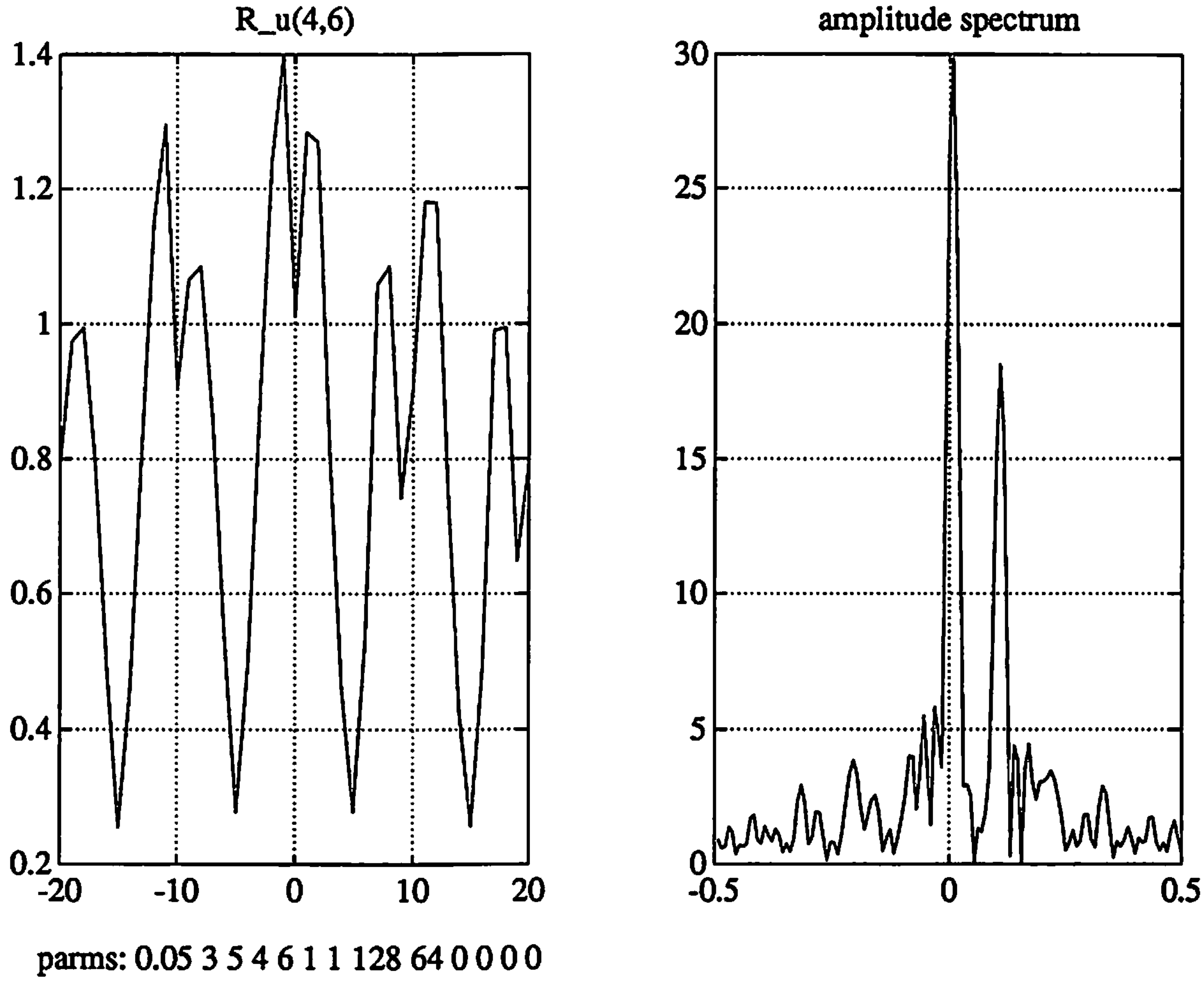

Figure 16: This is the $R_{4,6}$ statistic for the $(3,5)$ uncoupled case. Note that the peak value of the spectrum is $30/2^5 \approx 0.94$. Compare with Figure 11, which shows the corresponding coupled case.

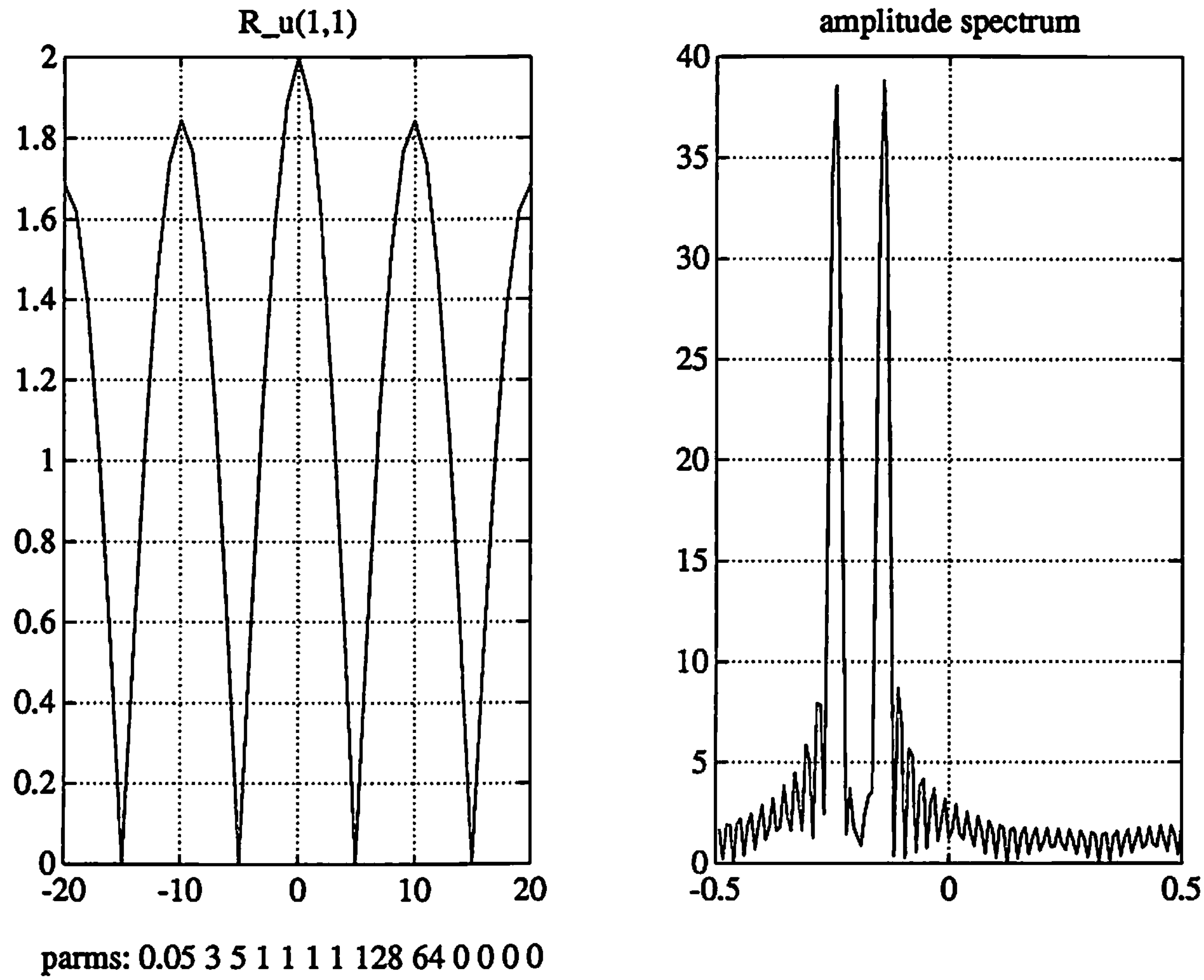

Figure 18: This is the $R_{1,1}$ statistic (conventional auto-correlation) for the $(3, 5)$ uncoupled case. This figure is virtually identical with Figure 13, which shows the same statistic for the coupled case.

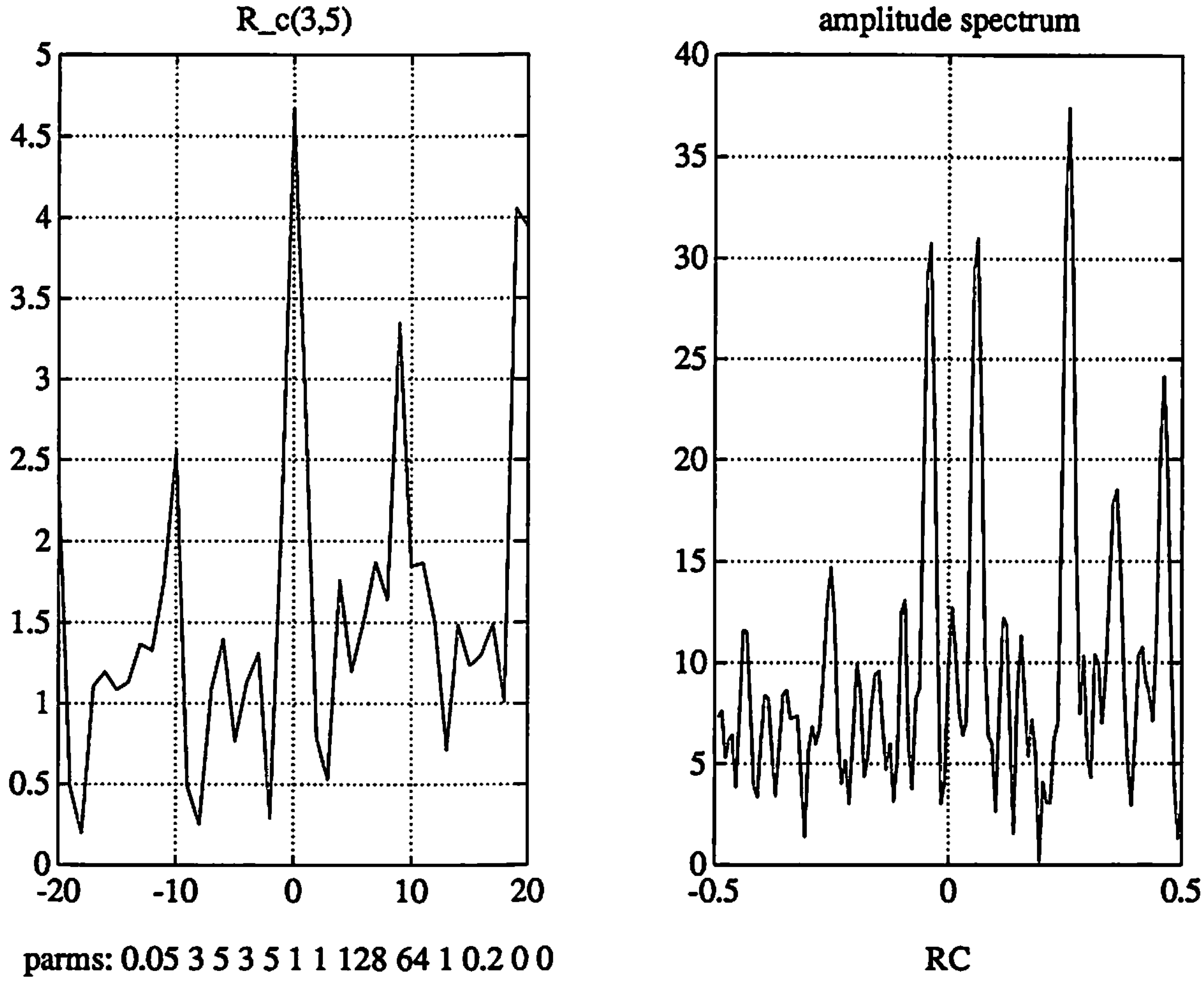

Figure 19b: This is the $R_{3,5}$ statistic for the $(3, 5)$ coupled case, with $f_2/f_1 = \dot{\phi}_2/\dot{\phi}_1 = k_2/k_1 = 5/3$. Additive white noise with a variance of 0.2 was added to obtain a SNR of 10 dB. Notice the strong peak at the zero lag of the statistic. Compare the statistic and its FT with those in Figure 19a. As in Figure 19a, a strong peak is seen at the correct frequency of 0.25 Hz. Notice the peak at 0.25. With $f_1 = .15$, the expected peak is at $-k_2 f_1 = -0.75$[12]; because of aliasing, the peak appears at 0.25. Since the signal variance is 2, the normalized peak value of the spectrum is approximately $40/\sqrt{2}^{3+5} = 2.5$. Also compare with Figure 9.

[12]See eq (46) and discussions related to it.

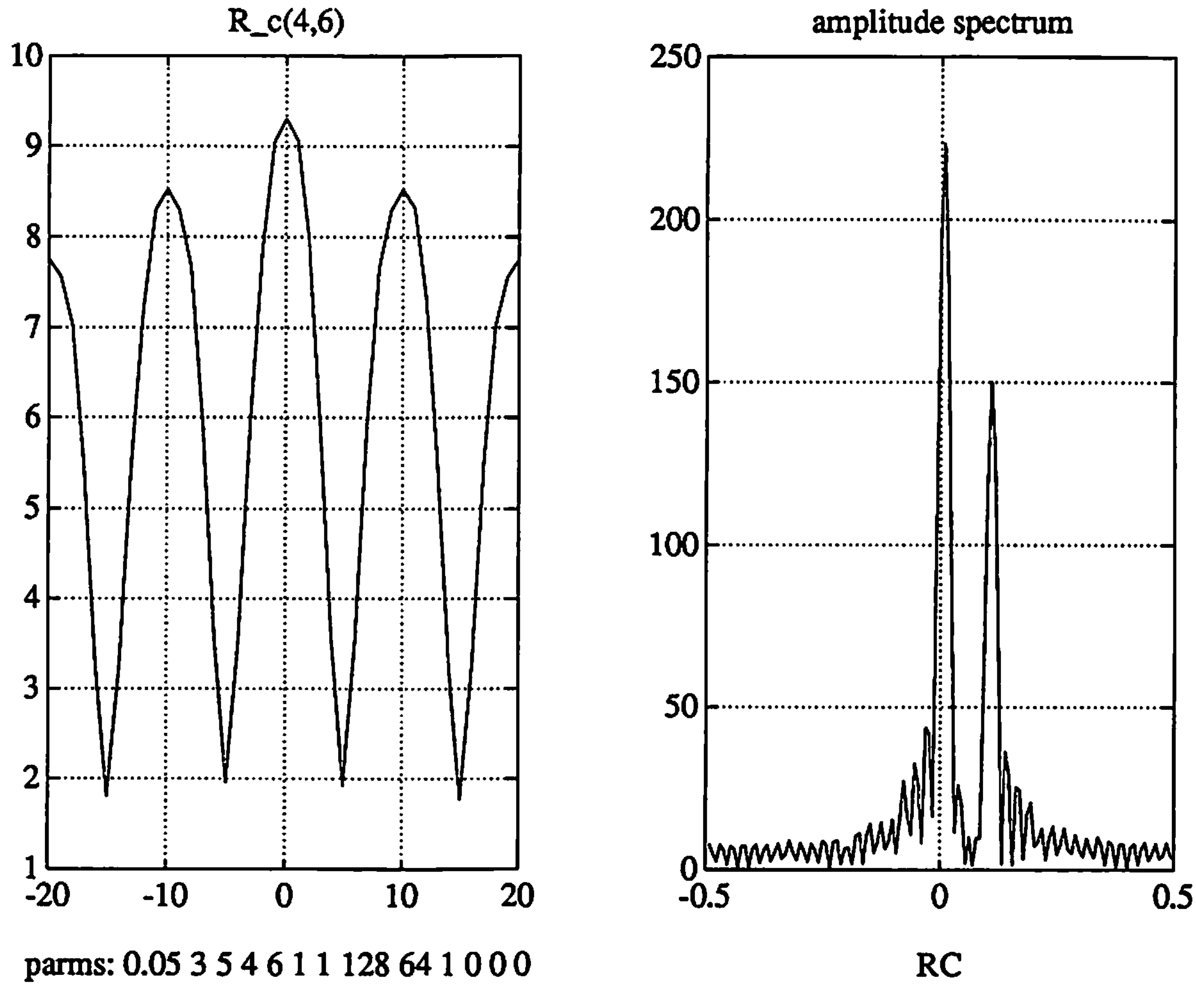

Figure 21: This is the $R_{4,6}$ statistic for the $(3,5)$ coupled case, with $f_2/f_1 = \phi_2/\phi_1 = k_2/k_1 = 5/3$. Theory predicts two peaks at $6f_1$ and $4f_2$, corresponding to $m = 0, n = 5$, and $m = 1, n = 6$[13]. Since $f_1 = 0.15$, we see the aliased versions at 0 and 0.1. The normalized peak value of the spectrum is approximately $250/2^5 \approx 8$. Compare with Figure 11.

[13]See eqns. (42), (43) and (45).

APPENDIX

Comparison between Correlation-Based and Cumulant-Based Approaches to the Harmonic Retrieval and Related Problems

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Abstract

In signal processing, we frequently encounter the problem of estimating the number of harmonics, frequencies, and amplitudes in a sum of sinusoids. The observed signals are usually corrupted by spatially and/or temporally colored noise with unknown power spectral density. It has been shown by Swami and Mendel that a cumulant-based approach to this problem is very effective. In this report, we compare the use of biased and unbiased, segmented and unsegmented estimators for both correlation and 1-D diagonal slice of the fourth-order cumulant function. We suggest using accumulated singular values to determine the number of harmonics. We compare correlation-based and cumulant-based methods for determining the number of harmonics when the amplitude of one harmonic decreases and when the frequency of one harmonic approaches the other for the case of two sinusoids measured in colored Gaussian noise. We also compare the performance of the Pisarenko, MUSIC, and minimum-norm algorithms for frequency estimation, and the perfomance of least square (LS), total least square (TLS), and constrained total least square (CTLS) for amplitude estimation using either correlations or cumulants. Our studies: (1) provide further support for using cumulants over correlations when measurement noise is colored and Gaussian; (2) demonstrate that one should use unbiased unsegmented correlation or cumulant estimators; (3) indicate that high-resolution results are best obtained using cumulant-based MUSIC or minimum-norm algorithms; and (4) show that LS estimates of amplitudes suffice.

Contents

1 **Introduction** 1

2 **Biased and Unbiased Correlation Estimators, With or Without Data Segmentation** 2

2.1 Biased Correlation Estimators 3

2.1.1 Estimator with Segmentation 3

2.1.2 Estimator without Segmentation 4

2.2 Unbiased Correlation Estimators 5

2.2.1 Estimator with Segmentation 5

2.2.2 Estimator without Segmentation 6

2.3 Conclusions 6

3 **Estimators of the 1-D Diagonal-Slice of Fourth-Order Cumulants** 7

3.1 Biased Estimators of the 1-D Diagonal-Slice of the Fourth-Order Cumulant 7

3.1.1 Estimator with Segmentation 7

3.1.2 Estimator without Segmentation 7

3.2 Unbiased Estimators of the 1-D Diagonal-Slice of the Fourth-Order Cumulant . . . 8

3.2.1 Estimator with Segmentation 8

3.2.2 Estimator without Segmentation 8

3.3 Simulation Results 8

3.4 Conclusions 9

4 **Estimation of the Number of Harmonics** 10

5 **Estimation of Harmonic Frequencies** 10

5.1 Pisarenko Algorithm 10

5.2 MUSIC Algorithm 11

5.3 Minimum Norm Algorithm 11

6 **Estimation of Harmonic Amplitudes** 12

6.1 Least Squares Method 12

6.2 Total Least Squares Method 13

6.3 Constrained Total Least Squares Method 13

7 **Simulations** 15

7.1 Variable: Amplitude 15

7.2 Variable: Frequency 16

7.3 Amplitude Estimation 17

7.3.1 Example 1: Local SNR's Equal 5 dB and 0 dB 18

7.3.2 Example 2: Local SNR's Both Equal 0 dB 18

7.3.3 Example 3: Local SNR's Both Equal -3 dB 18

7.3.4 Conclusions from Examples 19

7.4 Variable: Amplitude, Frequency, and Data Length 19

List of Tables

1 Means and standard deviations (std) of accumulative singular values for fourth-order cumulants using one realization (Fig. 4)[1] 32
2 Means and standard deviations (std) of accumulative singular values for fourth-order cumulants using independent realizations (Fig. 5) 32
3 Means and standard deviations (std) of accumulative singular values for correlations using one realization (Fig. 6) 33
4 Means and standard deviations (std) of accumulative singular values for correlations using independent realizations (Fig. 7) 33
5 Means and standard deviations (std) of accumulative singular values for fourth-order cumulants using one realization (Fig. 10) 34
6 Means and standard deviations (std) of accumulative singular values for fourth-order cumulants using independent realizations (Fig. 11) 34
7 Means and standard deviations (std) of accumulative singular values for correlations using one realization (Fig. 12) 35
8 Means and standard deviations (std) of accumulative singular values for correlations using independent realizations (Fig. 13) 35
9 Means and standard deviations of estimated amplitudes for Example 1[1] 36
10 Means and standard deviations of estimated amplitudes using one realization for Example 2 36
11 Means and standard deviations of estimated amplitudes using independent realizations for Example 2 37
12 Means and standard deviations of estimated amplitudes for Example 3 37

11 Accumulative singular values of cumulants using independent realizations when both amplitudes of sinusoids have local SNR's of 0 dB at $f_1 = 0.1$ and $f_2 = :$ (a) 0.2 (b) 0.14 (c) 0.13 (d) 0.1275 (e) 0.125 (f) 0.1225 (g) 0.12 (h) 0.1175 48
12 Accumulative singular values of correlations using one realization when both amplitudes of sinusoids have local SNR's of 0 dB at $f_1 = 0.1$ and $f_2 = :$ (a) 0.2 (b) 0.14 (c) 0.13 (d) 0.1275 (e) 0.125 (f) 0.1225 (g) 0.12 (h) 0.1175 49
13 Accumulative singular values of correlations using independent realizations when both amplitudes of sinusoids have local SNR's of 0 dB at $f_1 = 0.1$ and $f_2 = :$ (a) 0.2 (b) 0.14 (c) 0.13 (d) 0.1275 (e) 0.125 (f) 0.1225 (g) 0.12 (h) 0.1175 50
14 Results of estimated frequencies for Example 1: (a) mean of accumulative singular values (solid curve is for cumulants, dashed curve is for correlations); mean estimated spectra using (b) Pisarenko method, (c) MUSIC method, and (d) minimum norm method. In panels (b)-(d), the solid lines correspond to cumulant-based estimates, with p=2, and the dashed and dash-dotted lines correspond to correlation-based estimates with p=2 and p=3, respectively. 51
15 Results of estimated frequencies using one segmented realization for Example 2: (a) mean of accumulative singular values (solid curve is for cumulants, dashed curve is for correlations); mean estimated spectra using (b) Pisarenko method, (c) MUSIC method, and (d) minimum norm method. In panels (b)-(d), the solid lines correspond to cumulant-based estimates, with p=2, and the dashed and dash-dotted lines correspond to correlation-based estimates with p=2 and p=3, respectively. 52
16 Results of estimated frequencies using independent realizations for Example 2: (a) mean of accumulative singular values (solid curve is for cumulants, dashed curve is for correlations); mean estimated spectra using (b) Pisarenko method, (c) MUSIC method, and (d) minimum norm method. In panels (b)-(d), the solid lines correspond to cumulant-based estimates, with p=2, and the dashed and dash-dotted lines correspond to correlation-based estimates with p=2 and p=3, respectively. . . 53
17 Results of estimated frequencies for Example 3: (a) mean of accumulative singular values (solid curve is for cumulants, dashed curve is for correlations); mean estimated spectra using (b) Pisarenko method, (c) MUSIC method, and (d) minimum norm method. In panels (b)-(d), the solid lines correspond to cumulant-based estimates, with p=2, and the dashed and dash-dotted lines correspond to correlation-based estimates with p=2 and p=3, respectively. 54
18 Accumulative singular values of cumulants using 64×64 (4096) one realization when the fixed amplitude sinusoid at 0.1 is unity and the amplitude of the sinusoid at 0.2 is : (a) 0.55^* (b) 0.545 (c) 0.54 (d) 0.535 55
19 Accumulative singular values of cumulants using 50×64 (3200) one realization when the fixed amplitude sinusoid at 0.1 is unity and the amplitude of the sinusoid at 0.2 is : (a) 0.585 (b) 0.58^* (c) 0.575 (d) 0.57 55
20 Accumulative singular values of cumulants using 40×64 (2560) one realization when the fixed amplitude sinusoid at 0.1 is unity and the amplitude of the sinusoid at 0.2 is : (a) 0.595 (b) 0.59 (c) 0.585^* (d) 0.58 56

35 Accumulative singular values of cumulants using 50×64 (3200) one realization when the fixed amplitude sinusoid at 0.1 is unity and the amplitude of the sinusoid at 0.16 is : (a) 0.595 (b) 0.59* (c) 0.585 (d) 0.58 63
36 Accumulative singular values of cumulants using 40×64 (2560) one realization when the fixed amplitude sinusoid at 0.1 is unity and the amplitude of the sinusoid at 0.16 is : (a) 0.625 (b) 0.62* (c) 0.615 (d) 0.61 64
37 Accumulative singular values of cumulants using 32×64 (2048) one realization when the fixed amplitude sinusoid at 0.1 is unity and the amplitude of the sinusoid at 0.16 is : (a) 0.635 (b) 0.63* (c) 0.625 (d) 0.62 64
38 Accumulative singular values of cumulants using 25×64 (1600) one realization when the fixed amplitude sinusoid at 0.1 is unity and the amplitude of the sinusoid at 0.16 is : (a) 0.65 (b) 0.645 (c) 0.64* (d) 0.635 65
39 Accumulative singular values of cumulants using 16×64 (1024) one realization when the fixed amplitude sinusoid at 0.1 is unity and the amplitude of the sinusoid at 0.16 is : (a) 0.7 (b) 0.695* (c) 0.69 (d) 0.685 65
40 Accumulative singular values of cumulants using 8×64 (512) one realization when the fixed amplitude sinusoid at 0.1 is unity and the amplitude of the sinusoid at 0.16 is : (a) 0.765 (b) 0.76 (c) 0.755* (d) 0.75 66
41 Accumulative singular values of cumulants using 4×64 (256) one realization when the fixed amplitude sinusoid at 0.1 is unity and the amplitude of the sinusoid at 0.16 is : (a) 0.97 (b) 0.965 (c) 0.96* (d) 0.955 66
42 Accumulative singular values of cumulants using 64×64 (4096) one realization when the fixed amplitude sinusoid at 0.1 is unity and the amplitude of the sinusoid at 0.14 is : (a) 0.655 (b) 0.65 (c) 0.645* (d) 0.64 67
43 Accumulative singular values of cumulants using 50×64 (3200) one realization when the fixed amplitude sinusoid at 0.1 is unity and the amplitude of the sinusoid at 0.14 is : (a) 0.68 (b) 0.675* (c) 0.67 (d) 0.665 67
44 Accumulative singular values of cumulants using 40×64 (2560) one realization when the fixed amplitude sinusoid at 0.1 is unity and the amplitude of the sinusoid at 0.14 is : (a) 0.69 (b) 0.685* (c) 0.68 (d) 0.675 68
45 Accumulative singular values of cumulants using 32×64 (2048) one realization when the fixed amplitude sinusoid at 0.1 is unity and the amplitude of the sinusoid at 0.14 is : (a) 0.695* (b) 0.69 (c) 0.685 (d) 0.68 68
46 Accumulative singular values of cumulants using 25×64 (1600) one realization when the fixed amplitude sinusoid at 0.1 is unity and the amplitude of the sinusoid at 0.14 is : (a) 0.71 (b) 0.705* (c) 0.7 (d) 0.695 69
47 Accumulative singular values of cumulants using 16×64 (1024) one realization when the fixed amplitude sinusoid at 0.1 is unity and the amplitude of the sinusoid at 0.14 is : (a) 0.74* (b) 0.735 (c) 0.73 (d) 0.725 69
48 Accumulative singular values of cumulants using 8×64 (512) one realization when the fixed amplitude sinusoid at 0.1 is unity and the amplitude of the sinusoid at 0.14 is : (a) 0.835* (b) 0.83 (c) 0.825 (d) 0.82 70

1 Introduction

The estimation of the number of harmonics and the frequencies and amplitudes of harmonics from noisy measurements is frequently encountered in several signal processing applications, such as in estimating the direction of arrival of narrow-band source signals with linear arrays, and in the retrieval of harmonics in noise problem. In this report, we are concerned with real-valued signals represented as

$$y(n) = \sum_{k=1}^{p} a_k \cos(n\omega_k + \phi_k) + w(n) = x(n) + w(n) \tag{1}$$

where the ϕ_k's denote random phases which are i.i.d. and uniformly distributed over $[0, 2\pi]$, the ω_k's are unknown deterministic frequencies and the a_k's are unknown deterministic amplitudess. The additive noise $w(n)$ is assumed to be white or colored Gaussian noise with unknown spectral density. We will estimate the number of signals p, the angular frequencies ω_k's, and the amplitudes a_k's.

The autocorrelation, $r_x(\tau)$, and fourth-order cumulant, $c_{4x}(\tau_1, \tau_2, \tau_3)$, of $x(n)$ are represented as [4,5]

$$r_x(\tau) = \frac{1}{2} \sum_{k=1}^{p} a_k^2 \cos(\omega_k \tau) \tag{2}$$

$$\begin{aligned} c_{4x}(\tau_1, \tau_2, \tau_3) = -\frac{1}{8} \sum_{k=1}^{p} a_k^4 \; \{&\cos\omega_k(\tau_1 - \tau_2 - \tau_3) + \cos\omega_k(\tau_2 - \tau_3 - \tau_1) \\ &+ \cos\omega_k(\tau_3 - \tau_1 - \tau_2)\} \end{aligned} \tag{3}$$

The one-dimensional diagonal slice of the fourth order cumulant is given by

$$c_{4x}(\tau) = c_{4x}(\tau, \tau, \tau) = -\frac{3}{8} \sum_{k=1}^{p} a_k^4 \cos(\omega_k \tau) \tag{4}$$

For frequency estimation, linear prediction approaches are well explained in [4]. We describe important results from [4], here. A harmonic signal can be expressed as the output of a self-driving AR (AutoRegressive) model, i.e., let

$$x(n) = \sum_{k=1}^{p} a_k \exp(j\omega_k n)$$

Then, $x(n)$ satisfies the AR(p) model

$$\sum_{k=0}^{p} a_k \, x(n-k) = 0$$

where $a_0 = 1$ and the polynomial $A(z) = \sum_{k=0}^{p} a_k \, z^{-k}$ has roots at $z = e^{j\omega_k}$, $k = 1, \ldots, p$. For p real sinusoids, an AR($2p$) model, whose transfer function has roots at $z = e^{\pm j\omega_k}$ is required where $x_i(n)$ denotes the n-th sample in the i-th segment. Note that we can either assume that the data were obtained from M independent realizations, each with N samples, or, from MN samples from one realization.

Although the estimators for correlation functions are well known [3], the estimators usually use all the data from one realization without segmentation. In several recent papers [5,8,14], the concept of data segmentation for estimators was introduced. Data from one realization was divided into several segments, correlation functions were estimated for each segment, and then they were averagd to reduce the variance of estimators. In this section, we analyze biased and unbiased correlation estimators with or without segmentation. Derivations for mean and covariance functions of each estimator are given in the Appendix.

2.1 Biased Correlation Estimators

2.1.1 Estimator with Segmentation

Let $r_1^b(k)$ be the biased segmented correlation estimator,

$$r_1^b(k) = \frac{1}{M}\sum_{i=1}^{M} \tilde{r}_i^b(k)$$

where $\tilde{r}_i^b(k)$ denotes the estimated correlation function in the i-th segment, i.e.,

$$\tilde{r}_i^b(k) = \frac{1}{N}\sum_{n=1}^{N-|k|} x_i(n)x_i(n+k) \qquad i = 1, 2, \ldots, M$$

We can rewrite the estimator, $r_1^b(k)$, as

$$r_1^b(k) = \frac{1}{MN}\sum_{i=1}^{M}\sum_{n=1}^{N-|k|} x_i(n)x_i(n+k);$$

then(all derivations are given in the Appendix),

$$\mathrm{E}\{\, r_1^b(k)\,\} = \frac{N-|k|}{N}\, r(k) \tag{7}$$

and

$$\begin{aligned}\mathrm{Cov}\left(r_1^b(k), r_1^b(k+v)\right) = {} & \frac{1}{MN^2}\sum_{l=-(N-k)+1}^{(N-k-v-1)} (N-\eta_N(l)-k-v)\,\{r(l)r(l+v) \\ & + r(l+k+v)r(l-k) + c_{4x}(l,k,l+k+v)\} \\ & + \frac{1}{(MN)^2}\sum_{i,j=1}^{M}\sum_{i\neq j}^{M}\sum_{n=1}^{(N-k)}\sum_{m=1}^{(N-k-v)} \mathrm{E}\{x_i(n)x_i(n+k)x_j(m)x_j(m+k+v)\} \\ & - \frac{(M-1)(N-k)(N-k-v)}{MN^2} r(k)r(k+v)\end{aligned}$$

where $k \geq 0$, $k+v \geq 0$, and the function $\eta_{MN}(l)$ is defined by

$$\eta_{MN}(l) = \begin{cases} l, & l > 0 \\ 0, & -v \leq l \leq 0 \\ -l-v, & -(MN-k)+1 \leq l \leq -v \end{cases} \tag{13}$$

Note that $r_2^b(k)$ is asymptotically unbiased.

Since, in general, $N << MN$, the biased unsegmented estimates have much smaller bias than those using segmentation, especially, for correlations at large lags.

2.2 Unbiased Correlation Estimators

2.2.1 Estimator with Segmentation

Let $r_1^u(k)$ be the unbiased segmented correlation estimator, i.e.,

$$r_1^u(k) = \frac{1}{M}\sum_{i=1}^{M} \tilde{r}_i^u(k)$$

where $\tilde{r}_i^u(k)$ denotes the estimated correlation function in the i-th segment, i.e.,

$$\tilde{r}_i^u(k) = \frac{1}{N-|k|}\sum_{n=1}^{N-|k|} x_i(n)x_i(n+k) \qquad i = 1, 2, \ldots, M$$

We can represent the estimator, $r_1^u(k)$, as

$$r_1^u(k) = \frac{1}{M(N-|k|)}\sum_{i=1}^{M}\sum_{n=1}^{N-|k|} x_i(n)x_i(n+k);$$

then,

$$\mathrm{E}\{\, r_1^u(k) \,\} = r(k) \tag{14}$$

and

$$\begin{aligned}\mathrm{Cov}\,(r_1^u(k), r_1^u(k+v)) = {} & \frac{1}{M(N-k)(N-k-v)}\sum_{l=-(N-k)+1}^{(N-k-v-1)} (N - \eta_N(l) - k - v)\cdot \\ & \{r(l)r(l+v) + r(l+k+v)r(l-k) + c_{4x}(l,k,l+k+v)\} \\ + {} & \frac{1}{M^2(N-k)(N-k-v)}\sum_{i,j=1}^{M}\sum_{i\neq j}^{M}\sum_{n=1}^{(N-k)}\sum_{m=1}^{(N-k-v)} \mathrm{E}\{x_i(n)x_i(n+k)x_j(m)x_j(m+k+v)\} \\ & - \frac{(M-1)}{M} r(k)r(k+v)\end{aligned}$$

3 Estimators of the 1-D Diagonal-Slice of Fourth-Order Cumulants

The estimators for correlation functions have been analyzed in Section 2. Analyses for an unbiased estimator of the third-order cumulant are given in [6,7]. In this section we study the biased and unbiased, segmented or unsegmented estimators of the 1-D diagonal-slice of the fourth-order cumulants. Let $c_{4x}(k)$ be the 1-D diagonal slice of the fourth-order cumulant of $\{x(n)\}$ i.e.,

$$\begin{aligned} c_{4x}(k) &= \mathrm{cum}\,(x(n), x(n+k), x(n+k), x(n+k)) \\ &= \mathrm{E}\{x(n)x(n+k)x(n+k)x(n+k)\} \; - \; 3\, r_x(0) r_x(k) \end{aligned}$$

When we estimate $c_{4x}(k)$, it is assumed that we estimate $\mathrm{E}\{x(n)x^3(n+k)\}, r_x(0)$, and $r_x(k)$, independently, i.e., we estimate each term of $c_{4x}(k)$.

3.1 Biased Estimators of the 1-D Diagonal-Slice of the Fourth-Order Cumulant

3.1.1 Estimator with Segmentation

Let $d_1^b(k)$ be the biased segmented estimator for the 1-D diagonal-slice of the fourth-order cumulant, i.e.,

$$d_1^b(k) = \frac{1}{M} \sum_{i=1}^{M} \bar{d}_i^b(k)$$

where $\bar{d}_i^b(k)$ denotes the biased cumulant estimate in the i-th segment, i.e.,

$$\bar{d}_i^b(k) = \frac{1}{N} \sum_{n=1}^{N-|k|} x_i(n) x_i^3(n+k) \; - \; 3 r_1^b(0) r_1^b(k) \qquad i = 1, 2, \ldots, M$$

where $r_1^b(0)$ and $r_1^b(k)$ denote the biased segmented correlation estimates (as described in Section 2). Then,

$$\mathrm{E}\{d_1^b(k)\} = \frac{N-|k|}{N} \Big(\mathrm{E}\{\, x(n)x^3(n+k)\,\} \; - \; 3\, r_x(0) \cdot r_x(k) \Big) \tag{19}$$

Note that the estimator for the 1-D diagonal-slice of the fourth-order cumulant is not asymptotically unbiased because N is a constant, and the mean value is independent of the number of segments, M. Covariance formulas for $d_1^b(k)$, as well as the other fourth-order cumulant estimators, are so complicated that we do not present them here. They depend on 8th-order cumulants.

3.1.2 Estimator without Segmentation

Let $d_2^b(k)$ be the biased unsegmented estimator for the 1-D diagonal-slice of the fourth-order cumulant, i.e.,

$$d_2^b(k) = \frac{1}{MN} \sum_{n=1}^{MN-|k|} x(n) x^3(n+k) \; - \; 3 r_2^b(0) r_2^b(k)$$

estimates for the 1-D diagonal-slice of the fourth-order cumulants.

For our simulation, we chose two cosines both of whose amplitudes are unity; their frequencies are at $f_1 = 0.1$ and $f_2 = 0.2$. Colored additive Gaussian noise was generated by an ARMA(2,2) system excited by a zero-mean white Gaussian noise input. The AR coefficients of this ARMA model were [1, 1.4563, 0.81], and its MA coefficients were [1, 2, 1]. The colored noise spectrum has a strong pole at 0.4, with a damping factor of 0.9. We performed 30 independent trials. Figure 1 shows the mean values of unbiased and biased estimates for 0 dB and -3 dB local SNR's. Local SNR is defined as the ratio of local signal power to noise variance, i.e., when we have the signal in (1), local SNR corresponding to the i−th sinusoid, SNR_i, is

$$\mathrm{SNR}_i = 10 \log \frac{a_i^2/2}{\hat{\sigma}^2} \qquad i = 1, 2, \ldots, p$$

where a_i denotes the amplitude of the i−th sinusoid and $\hat{\sigma}^2$ denotes the estimated noise variance obtained from output data of the ARMA model for the measurement noise. In Fig. 1, a "star" denotes the zero-noise theoretical values of the 1-D diagonal-slice of the fourth-order cumulants of the harmonics. The "dash" and "dash-dot" curves are the estimated values using one realization (4096 samples) with segmentation and without segmentation, respectively. The "solid" curve shows the estimated values when we used 64 samples of 64 independent realizations (total number of data is 4096). The figures show that the biased unsegmented estimator is much better than other biased ones, even at large lags, and, there is no difference among unbiased estimators. Although we have not shown the variance values of estimators, note that the variances of each estimate were very similar.

3.4 Conclusions

We conclude that for correlation or fourth-order cumulant estimates:

- In the case of a single realization, the biased unsegmented estimator gives better results than the biased segmented estimator.
- In the case of a single realization, the unbiased unsegmented and segmented estimators give comparable results.
- When we only have a single realization, we should not use a biased segmented estimator.

4. Evaluate the roots, $z_1, \ldots, z_{2p}$, of the polynomial

$$A(z) \equiv a_0 + a_1 z + \cdots + a_{2p} z^{2p} = 0$$

Estimate the frequencies of the sinusoids as

$$z_k = \exp(j\omega_k) \qquad -\pi \leq \omega_k \leq \pi \quad k = 1, \ldots, 2p;$$

or, determine the angular frequencies as the peaks of $\left|\frac{1}{A(z)}\right|$, $z = e^{j\omega}$.

5.2 MUSIC Algorithm

1. After computing the eigenvalues and eigenvectors of the estimated correlation (cumulant) matrix, classify the eigenvalues into two groups: one consisting of the $2p$ largest eigenvalues and the other consisting of the $(M - 2p)$ smallest eigenvalues.

2. Use the eigenvectors associated with the second group to construct the $M \times (M - 2p)$ eigenvector matrix $\mathbf{X}_N$ whose elements span the sample noise subspace.

3. Determine the angular frequencies of the sinusoids as the peaks of the sample spectrum

$$S(\omega) = \frac{1}{\mathbf{s}^H(\omega)\mathbf{X}_N\mathbf{X}_N^H\mathbf{s}(\omega)}$$

where $\mathbf{s}(\omega)$ is the $M \times 1$ frequency-searching vector, defined by

$$\mathbf{s}^T(\omega) = \left[1, e^{-j\omega}, \ldots, e^{-j\omega(M-2p+1)}\right]$$

5.3 Minimum Norm Algorithm

1. Compute the eigenvalues and eigenvectors of the estimated correlation (cumulant) matrix.

2. Classify the eigenvalues into two groups: the $2p$ largest eigenvalues and the $(M - 2p)$ smallest eigenvalues. Use the eigenvectors associated with the first group to construct the $M \times 2p$ matrix $\mathbf{X}_S$ whose elements span the sample signal subspace.

3. Partition the matrix $\mathbf{X}_S$ as

$$\mathbf{X}_S = \begin{bmatrix} \mathbf{g}_S^T \\ \cdots \\ \mathbf{G}_S \end{bmatrix}$$

where $\mathbf{g}_S^T$ contains the first elements of the signal subspace eigenvectors, and the $(M-1)\times 2p$ matrix $\mathbf{G}_S$ contains the rest of the elements of $\mathbf{X}_S$.

4. Compute the minimum-norm value of the $M \times 1$ vector, $\mathbf{a}$:

$$\mathbf{a} = \begin{bmatrix} 1 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ -(1 - \mathbf{g}_S^H\mathbf{g}_S)^{-1}\mathbf{G}_S^*\mathbf{g}_S \end{bmatrix}$$

in which we try to keep the correction term $\Delta\mathbf{b}$ as small as possible while simultaneously compensating for the noise present in $\mathbf{b}$, by forcing $\mathbf{Ax} = \mathbf{b} + \Delta\mathbf{b}$. When the noise in $\mathbf{A}$ is zero and the noise in $\mathbf{b}$ is zero-mean Gaussian, the LS solution, $\mathbf{x}_{LS}$, is identical to the maximum likelihood solution, i.e.,

$$\mathbf{x}_{\mathrm{LS}} = \left(\mathbf{A}^H\mathbf{A}\right)^{-1}\mathbf{A}^H\mathbf{b}$$

6.2 Total Least Squares Method

When $\mathbf{A}$ is also noisy, $\mathbf{x}_{LS}$ is no longer optimal from a statistical point of view; it suffers from bias and increased covariance due to the accumulation of noise errors in $\mathbf{A}^H\mathbf{A}$. To alleviate this problem, a generalization of the LS solution was formally introduced by Golub and Van Loan [11], called total least squares (TLS). TLS attempts to remove the noise in $\mathbf{A}$ and $\mathbf{b}$ using a perturbation on $\mathbf{A}$ and $\mathbf{b}$ of smallest 2-norm which makes the system of overdetermined linear equations consistent. The TLS solution for $\mathbf{x}$ is obtained as

$$\mathrm{Min}\ \|[\Delta\mathbf{A} \vdots \Delta\mathbf{b}]\|$$

$$\text{subject to} \qquad (\mathbf{A} + \Delta\mathbf{A})\mathbf{x} = \mathbf{b} + \Delta\mathbf{b}$$

The TLS solution can be expressed algebraically as

$$\mathbf{x}_{\mathrm{TLS}} = \left(\mathbf{A}^H\mathbf{A} - \sigma^2\mathbf{I}\right)^{-1}\mathbf{A}^H\mathbf{b}$$

where σ^2 is the minimum eigenvalue(s) of $[\mathbf{A} \vdots \mathbf{b}]^H[\mathbf{A} \vdots \mathbf{b}]$. It is also true that the TLS solution can be obtained explicitly from the right singular vector that corresponds to the smallest singular value of the singular value decomposition (SVD) of $\mathbf{C} = [\mathbf{A} \vdots \mathbf{b}]$. From a statistical point of view, TLS operates under the assumption that the noise components of $\mathbf{A}$ and $\mathbf{b}$ are zero mean and identically independently distributed.

6.3 Constrained Total Least Squares Method

If there is a linear dependence among the noise components in $\mathbf{A}$ and $\mathbf{b}$, then the TLS problem must be reformulated to take into account the reduced dimensionality of the noise entries. Abatzoglou and Mendel [12,13] discuss a reformulation of the TLS method, which they call constrained total least squares (CTLS), that accounts for the linear algebraic relations among the noise entries of $\mathbf{A}$ and $\mathbf{b}$.

The CTLS solution for $\mathbf{x}$ is defined by

$$\mathrm{Min}_{\mathbf{v},\mathbf{x}}\ \|\,\Delta\mathbf{C}\,\| \qquad \text{where} \quad (\mathbf{C} + \Delta\mathbf{C})\begin{bmatrix} \mathbf{x} \\ -1 \end{bmatrix} = 0$$

and

$$\begin{aligned} \Delta\mathbf{C} &= \left[\ \Delta\mathbf{A} \ \vdots \ \Delta\mathbf{b}\ \right] \\ &= \left[\ \mathbf{F}_1\mathbf{v} \ \vdots \ \cdots \ \vdots \ \mathbf{F}_{p+1}\mathbf{v}\ \right] \end{aligned}$$

7 Simulations

In this section, we compare the correlation-based and cumulant-based harmonic retrieval approaches through simulations. Our goal is to learn where each approach breaks down as certain experimental conditions are changed.

Throughout the following simulations, colored additive Gaussian noise was generated through an ARMA(2,2) system excited by a zero-mean white Gaussian noise input. The AR coefficients of this ARMA model equal [1, 1.4563, 0.81], and its MA coefficients equal [1, 2, 1], as in [4]. The resulting colored Gaussian noise spectrum has a strong pole around 0.4. For each simulation, we performed 30 Monte Carlo trials.

The signal consisted of two harmonics whose frequencies are $f_1 = 0.1$ and $f_2 = 0.2$, and whose amplitudes are both unity.

We used either one realization having 64×64 (4096) samples, or 64 independent realizations, each with 64 samples. In the case of a single realization, the data were divided into 64 segments, each segment with 64 samples. For estimated correlations, $\hat{r}_x(k)$, and estimated fourth-order cumulant, $\hat{c}_{4x}(k)$, $k = 0, 1, \ldots, 15$, we used the unbiased segmented estimator, i.e.,

$$\hat{r}_x(k) = \frac{1}{M}\sum_{i=1}^{M}\left(\frac{1}{N-|k|}\sum_{n=1}^{N-|k|} x_i(n)x_i(n+k)\right) \tag{27}$$

and

$$\hat{c}_{4x}(k) = \frac{1}{M}\sum_{i=1}^{M}\left(\frac{1}{N-|k|}\sum_{n=1}^{N-|k|} x_i(n)x_i^3(n+k) - 3\,\hat{r}_x(0)\hat{r}_x(k)\right) \tag{28}$$

where M and N denote the number of segments and the number of samples in one segment, respectively. Although $c_{4x}(k) = c_{4x}(-k)$ for a stationary process, the estimates in Eq. (28) lose the symmetry property; hence, we used the averaged value $\frac{\hat{c}_{4x}(k)+\hat{c}_{4x}(-k)}{2}$ for the fourth-order cumulants at lags k and $-k$.

7.1 Variable: Amplitude

After fixing all other parameters and one of the sinusoidal amplitudes (at unity), we varied the amplitude of the second sinusoid ($f_2 = 0.2$) in order to determine where the correlation-based and cumulant-based methods break down. Figures 2 and 3 show the mean values of accumulative singular values of cumulants and correlations, respectively, using one realization with segmentation, when the local SNR (which is defined in Section 3.3), of the fixed sinusoid is 5 dB. Observe that better results are obtained for cumulants.

Figures 4 and 5 show the mean values of accumulative singular values of cumulants using either one realization with segmentation or independent realizations, respectively, when the local SNR of the fixed sinusoid is 0 dB. Note that no appreciable differences are visible.

7.3 Amplitude Estimation

To study the performance of the amplitude estimation algorithms, three examples are given in this section. Figures 14-17 show the mean values of accumulative singular values and the estimated frequencies using the Pisarenko, MUSIC, and minimum-norm methods for the problem of two harmonics in colored Gaussian noise. The "solid" curves denote the results obtained using cumulants, and the "dash" and "dash-dot" curves denote the results obtained using correlations. In the following examples, note that only the estimated frequencies using MUSIC were used to estimate amplitudes and that the estimated amplitudes using LS were used as initial values for the Newton iteration using CTLS. The estimated frequencies using MUSIC and minimum-norm methods were very similar [4]. In the CTLS method, the value to stop the Newton iterations was chosen as 10^{-8}.

We assumed frequency estimation errors are independent and white, i.e.,

$$\hat{\omega}_i = \omega_i + \delta\omega_i \qquad i = 1, \ldots, p$$

where ω_i's denote the true frequencies, $\delta\omega_i$'s denote the independent estimation errors, and p is the estimated number of sinusoids (see Figs. 14-17). Since each element of matrix $\mathbf{A}$ is a nonlinear function of $\hat{\omega}_i$'s (see Eq. (24)), we can linearize the elements as follows : when $\delta\omega_i \cdot m, \;\; m = 0, 1, \ldots, 15$, are very small, $\cos(\delta\omega_i \cdot m) \approx 1$ and $\sin(\delta\omega_i \cdot m) \approx m \cdot \delta\omega_i$; hence,

$$\begin{aligned} \cos(\, m\,(\omega_i + \delta\omega_i)\,) &= \cos(m\,\omega_i)\cdot\cos(m\,\delta\omega_i) - \sin(m\,\omega_i)\cdot\sin(m\,\delta\omega_i) && (29)\\ &\approx \cos(m\,\omega_i) - m\,\delta\omega_i\,\sin(m\,\omega_i) && (30) \end{aligned}$$

and

$$\begin{aligned} \sin(\, m\,(\omega_i + \delta\omega_i)\,) &= \sin(m\,\omega_i)\cdot\cos(m\,\delta\omega_i) + \cos(m\,\omega_i)\cdot\sin(m\,\delta\omega_i) && (31)\\ &\approx \sin(m\,\omega_i) + m\,\delta\omega_i\,\cos(m\,\omega_i) && (32) \end{aligned}$$

Consequently, we can approximate $\cos(m(\omega_i + \delta\omega_i))$ as follows;

$$\cos(m(\omega_i + \delta\omega_i)) \approx \cos(m\,\omega_i) - m\,\sin(m(\omega_i + \delta\omega_i))\cdot\delta\omega_i + m^2\cos(m\,\omega_i)\cdot(\delta\omega_i)^2$$

Since the third term on the right-hand side of this equation is negligible,

$$\cos(m\,\hat{\omega}_i) \approx \cos(m\,\omega_i) - m\,\sin(m\,\hat{\omega}_i)\cdot\delta\omega_i \qquad (33)$$

where $i = 1, \ldots, p$ and $m = 0, 1, \ldots, 15$. Note that this equation provides a computable variation of $\cos(m\,\omega_i)$ due to errors in $\hat{\omega}_i$, whereas the Eq. (30) does not. Although the estimation errors of correlations (cumulants) are correlated, we assumed for simplicity they are independent of each other and are white, i.e.,

$$\hat{r}_x(k) = r_x(k) + \delta r_x(k) \;\;\text{and}\;\; \hat{c}_{4x}(k) = c_{4x}(k) + \delta c_{4x}(k)$$

$$k = 0, 1, 2, \ldots, 15$$

7.3.4 Conclusions from Examples

From these three examples, we conclude that;

- the resulting estimated frequencies using MUSIC and minimum-norm methods are very similar and very good.
- the Pisarenko method using correlation gives poor results when noise is colored. It also can give biased results for low SNR's when cumulants are used.
- the resulting estimated amplitudes using cumulants are better than those obtained using correlations.
- the results using LS are usually better than those obtained using TLS or CTLS because of good estimated frequencies.
- although we expected the superiority of CTLS, there is no great advantage to using CTLS because of very good estimation of frequencies by all the methods.
- the assumption of independency of the correlation (cumulant) errors may have caused the poor performance of CTLS results.
- using several independent realizations gives similar results to using one realization in the case of high SNR for large amounts of data.

7.4 Variable: Amplitude, Frequency, and Data Length

Throughout the above several simulations and examples, we showed the superiority of using cumulants to using correlations in the harmonic retrieval problem. In this subsection we find the points where the cumulant-based methods for harmonic retrieval break down as a function of amplitude, frequency of one harmonic, and length of data.

Our signal consists of two real sinusoids measured in colored Gaussian noise. The first sinusoid has a fixed local SNR of 0 dB (corresponding magnitude is unity) at $f_1 = 0.1$. The second sinusoid has varying amplitude at $f_2 = 0.2, 0.18, 0.16,$ or 0.14. The additive noise was the colored Gaussian noise generated by the ARMA system described in Section 7.

In order to determine where the cumulant-based methods break down, we varied the amplitude of the second sinusoid at a fixed f_2, and also varied the data length. We used one realization divided into segments of 64 samples. We used 8 different data lengths: 64×64 (4096), 50×64 (3200), 40×64 (2560), 32×64 (2048), 25×64 (1600), 16×64 (1024), 8×64 (512), and 4×64 (256). For each simulation, we performed 30 Monte Carlo trials.

Figures 18-25 showed the accumulated singular values of cumulants when the second frequency is $f_2 = 0.2$. The values labeled with $*$ were judged (albeit, subjectively) to be the smallest amplitudes of the second sinusoid when we can visually still estimate the correct number of harmonics. Figures 26-33, 34-41, and 42-49 are for $f_2 = 0.18, 0.16,$ and 0.14, respectively. Note that normal, the LS solution is identical to the maximum likelihood solution [13].

Finally, we obtained a plot of minimum local SNR vs. different lengths of data and several f_2 where we can still determine the correct number of harmonics by looking at breaks in plots of accumulated singular values. For each f_2 the minimum local SNR decreases approximately exponentially as the length of data increases. Using Fig. 50 we can estimate the minimum value of the third variable when any two of the three variables are fixed.

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A Appendix

In this appendix, the following results will be used : when $\mathrm{E}\{x(n)\} = 0$, for all n,

$$\mathrm{E}\{x(n)x(n+k)x(m)x(m+k+v)\} = r(k)r(k+v) + r(m-n)r(m-n+v)$$

$$+ r(m-n+k+v)r(m-n+k) + c_{4x}(k, m-n, m-n+k+v) \qquad \text{(A-1)}$$

where $\mathrm{cum}(x(n), x(n+k), x(m), x(m+k+v) = c_{4x}(k, m-n, m-n+k+v)$. Note that $c_{4x}(r, s-t, s-t+r+v) = c_{4x}(s-t, r, s-t+r+v)$. Now consider

$$\sum_{n=1}^{(N-k)}\sum_{m=1}^{(N-k-v)} \{r(m-n)r(m-n+v) + r(m+k+v-n)r(m-n-k) + c_{4x}(m-n, k, m-n+k+v)\}$$

where $k \geq 0$ and $k+v \geq 0$. We make a change of variables from m and n to $l = (m-n)$ and n. The summand depends only on l, and a careful examination of the limits of n gives [3],

$$\sum_{n=1}^{(N-k)}\sum_{m=1}^{(N-k-v)} \{r(m-n)r(m-n+v) + r(m+k+v-n)r(m-n-k) + c_{4x}(m-n, k, m-n+k+v)\}$$

$$= \sum_{l=-(N-k)+1}^{(N-k-v-1)} (N - \eta_N(l) - k - v)\{r(l)r(l+v) + r(l+k+v)r(l-k) + c_{4x}(l, k, l+k+v)\} \qquad \text{(A-2)}$$

where $\eta_N(l)$, which is a function of N as well as l, is defined as

$$\eta_N(l) = \begin{cases} l, & l > 0 \\ 0, & -v \leq l \leq 0 \\ -l-v, & -(N-k)+1 \leq l \leq -v \end{cases} \qquad \text{(A-3)}$$

A.1 Biased Estimators of Correlation

In this section, we derive the mean and covariance functions of the biased segmented estimator, $r_b^1(k)$, and the biased unsegmented estimator, $r_b^2(k)$.

A.1.1 Estimator with Segmentation

$$r_1^b(k) = \frac{1}{MN}\sum_{i=1}^{M}\sum_{n=1}^{N-|k|} x_i(n)x_i(n+k)$$

Then,

$$\mathrm{E}\{\ r_1^b(k)\ \} = \frac{1}{MN}\sum_{i=1}^{M}\sum_{n=1}^{N-|k|} \mathrm{E}\{\ x_i(n)x_i(n+k)\ \}$$

Using stationarity, we obtain

$$\mathrm{E}\{\ r_1^b(k)\ \} = \frac{N-|k|}{N}\, r(k)$$

$$\operatorname{Cov}\left(r_1^b(k), r_1^b(k+v)\right) = \frac{1}{MN^2} \sum_{l=-(N-k)+1}^{(N-k-v-1)} (N - \eta_N(l) - k - v) \{r(l)r(l+v)$$
$$+ r(l+k+v)r(l-k) + c_{4x}(l,k,l+k+v)\} \qquad \text{(A-4)}$$

When each segment is a part of a single realization, i.e.,

$$\begin{aligned}
&\mathrm{E}\{x_i(n)\, x_i(n+k)\, x_j(m)\, x_j(m+k+v)\} \\
&= \mathrm{E}\{x(N(i-1)+n)\, x(N(i-1)+n+k)\, x(N(j-1)+m)\, x(N(j-1)+m+k+v)\} \\
&= r(k)r(k+v) + r(N(j-i)+m-n)r(N(j-i)+m-n+v) + r(N(j-i)+m-n+k+v) \\
&\quad r(N(j-i)+m-n-k) + c_{4x}(k, N(j-i)+m-n, N(j-i)+m-n+k+v)
\end{aligned}$$

where we have used Eq. (A-1) to obtain the last line; thus,

$$\operatorname{Cov}\left(r_1^b(k), r_1^b(k+v)\right) = \frac{1}{MN^2} \sum_{l=-(N-k)+1}^{(N-k-v-1)} (N - \eta_N(l) - k - v) \{r(l)r(l+v) + r(l+k+v)r(l-k)+$$
$$c_{4x}(l,k,l+k+v)\} + \frac{1}{(MN)^2} \sum_{i,j=1}^{M} \sum_{i\neq j}^{M} \sum_{n=1}^{(N-k)} \sum_{m=1}^{(N-k-v)} \{r(N(j-i)+m-n)r(N(j-i)+m-n+v)$$
$$+ r(N(j-i)+m-n+k+v)r(N(j-i)+m-n-k) + c_{4x}(k, N(j-i)+m-n, N(j-i)+m-n+k+v)\}$$

As in Eq. (A-2), letting $p = m - n$ and $q = j - i$ in the last term of the preceding equation gives

$$\operatorname{Cov}\left(r_1^b(k), r_1^b(k+v)\right) = \frac{1}{MN^2} \sum_{l=-(N-k)+1}^{(N-k-v-1)} (N - \eta_N(l) - k - v) \{r(l)r(l+v) + r(l+k+v)r(l-k)$$
$$+c_{4x}(l,k,l+k+v)\} + \frac{1}{(MN)^2} \sum_{q=-M+1, q\neq 0}^{(M-1)} \sum_{p=-(N-k)+1}^{(N-k-v-1)} (N - \eta_N(p) - k - v)\,(M - |q|)\,\{r(Nq+p)$$
$$r(Nq+p+v) + r(Nq+p+k+v)r(Nq+p-k) + c_{4x}(k, Nq+p, Nq+p+k+v)\} \qquad \text{(A-5)}$$

A.1.2 Estimator without Segmentation

$$r_2^b(k) = \frac{1}{MN} \sum_{n=1}^{MN-|k|} x(n)x(n+k)$$

Then

$$\mathrm{E}\{\, r_2^b(k)\,\} = \frac{1}{MN} \sum_{n=1}^{MN-|k|} \mathrm{E}\{\, x(n)x(n+k)\,\} = \frac{MN-|k|}{MN}\, r(k)$$

Without loss of generality, let $k \geq 0, k+v \geq 0$; then

$$\operatorname{Cov}\left(r_2^b(k), r_2^b(k+v)\right) = \mathrm{E}\left\{r_2^b(k) r_2^b(k+v)\right\} - \mathrm{E}\left\{r_2^b(k)\right\}\mathrm{E}\left\{r_2^b(k+v)\right\}$$
$$= \frac{1}{(MN)^2} \sum_{n=1}^{(MN-k)} \sum_{m=1}^{(MN-k-v)} \mathrm{E}\{\, x(n)x(n+k)x(m)x(m+k+v)\,\} - \frac{MN-k}{MN}\,\frac{MN-k-v}{MN} r(k)r(k+v)$$

Using the fourth-order cumulant Eq. (A-1), we get

$$\mathrm{Cov}\,(r_1^u(k), r_1^u(k+v)) = \frac{1}{M^2(N-k)(N-k-v)} \sum_{i=1}^{M} \sum_{n=1}^{(N-k)} \sum_{m=1}^{(N-k-v)} \{r(m-n)r(m-n+v)+$$

$$r(m+k+v-n)r(m-n-k) + c_{4x}(m-n, k, m-n+k+v)\} + \frac{1}{M} r(k)r(k+v)$$

$$+ \frac{1}{M^2(N-k)(N-k-v)} \sum_{i,j=1}^{M} \sum_{i \neq j}^{M} \sum_{n=1}^{(N-k)} \sum_{m=1}^{(N-k-v)} \mathrm{E}\{x_i(n)x_i(n+k)x_j(m)x_j(m+k+v)\} - r(k)r(k+v)$$

Using Eq. (A-2) gives

$$\mathrm{Cov}\,(r_1^u(k), r_1^u(k+v)) = \frac{1}{M(N-k)(N-k-v)} \sum_{l=-(N-k)+1}^{(N-k-v-1)} (N - \eta_N(l) - k - v)$$

$$\{r(l)r(l+v) + r(l+k+v)r(l-k) + c_{4x}(l, k, l+k+v)\}$$

$$+ \frac{1}{M^2(N-k)(N-k-v)} \sum_{i,j=1}^{M} \sum_{i \neq j}^{M} \sum_{n=1}^{(N-k)} \sum_{m=1}^{(N-k-v)} \mathrm{E}\{x_i(n)x_i(n+k)x_j(m)x_j(m+k+v)\}$$

$$- \frac{(M-1)}{M} r(k)r(k+v)$$

where $\eta_N(l)$ is defined in Eq. (A-3).

When each segment of data is independent,

$$\mathrm{E}\{x_i(n)x_i(n+k)x_j(m)x_j(m+k+v)\} = r(k)r(k+v), \qquad \text{for } i \neq j$$

Using this fact, the second term of $\mathrm{Cov}(r_1^u(k), r_1^u(k+v))$ becomes $\frac{(M-1)}{M} r(k)r(k+v)$; thus,

$$\mathrm{Cov}\,(r_1^u(k), r_1^u(k+v)) = \frac{1}{M(N-k)(N-k-v)} \sum_{l=-(N-k)+1}^{(N-k-v-1)} (N - \eta_N(l) - k - v)$$

$$\{r(l)r(l+v) + r(l+k+v)r(l-k) + c_{4x}(l, k, l+k+v)\} \tag{A-7}$$

When each segment is a part of a single realization, i.e.,

$$\mathrm{E}\{x_i(n)\; x_i(n+k)\; x_j(m)\; x_j(m+k+v)\}$$

$$= \mathrm{E}\{x(N(i-1)+n)\; x(N(i-1)+n+k)\; x(N(j-1)+m)\; x(N(j-1)+m+k+v)\}$$

$$= r(k)r(k+v) + r(N(j-i)+m-n)r(N(j-i)+m-n+v) + r(N(j-i)+m-n+k+v)$$

$$r(N(j-i)+m-n-k) + c_{4x}(k, N(j-i)+m-n, N(j-i)+m-n+k+v)$$

where we have used Eq. (A-1) to obtain the last line; thus, the covariance function can be rewritten as

$$\mathrm{Cov}\left(r_2^u(k), r_2^u(k+v)\right) = \frac{1}{(MN-k)(MN-k-v)} \sum_{l=-(MN-k)+1}^{MN-k-v-1} \{MN - \eta_{MN}(l) - k - v\}$$

$$\{r(l)r(l+v)+r(l+k+v)r(l-k)+c_{4x}(l,k,l+k+v)\} \qquad \text{(A-9)}$$

where $\eta_{MN}(l)$ is defined in Eq. (A-3).

A.3 Biased Estimators of the 1-D Diagonal-Slice of the Fourth-Order Cumulant

In this section, we derive the mean functions of the biased segmented cumulant estimator, $d_b^1(k)$, and the biased unsegmented estimator, $d_b^2(k)$.

A.3.1 Estimator with Segmentation

$$d_1^b(k) = \frac{1}{M} \sum_{i=1}^{M} \tilde{d}_i^b(k)$$

where $\tilde{d}_i^b(k)$ denotes the estimated cumulant in the i-th segment, i.e.,

$$\tilde{d}_i^b(k) = \frac{1}{N} \sum_{n=1}^{N-|k|} x_i(n)x_i^3(n+k) \;-\; 3r_1^b(0)r_1^b(k) \qquad i = 1, 2, \ldots, M$$

where $r_1^b(0)$ and $r_1^b(k)$ denote the biased segmented correlation estimates as in Section A.1.1. Then,

$$\begin{aligned} \mathrm{E}\{d_1^b(k)\} &= \frac{1}{M} \sum_{i=1}^{M} \frac{1}{N} \sum_{n=1}^{N-|k|} \mathrm{E}\{x_i(n)x_i^3(n+k)\} - 3\,r(0)\,\left(\frac{N-|k|}{N} r(k)\right) \\ &= \frac{N-|k|}{N} \left(\mathrm{E}\{x(n)x^3(n+k)\} \;-\; 3\,r(0)\cdot r(k)\right) \end{aligned}$$

A.3.2 Estimator without Segmentation

$$d_2^b(k) = \frac{1}{MN} \sum_{n=1}^{MN-|k|} x(n)x^3(n+k) \;-\; 3r_2^b(0)r_2^b(k)$$

where $r_2^b(0)$ and $r_2^b(k)$ denote the biased unsegmented correlation estimates as in Section A.1.2. Then

$$\begin{aligned} \mathrm{E}\{d_2^b(k)\} &= \frac{1}{MN} \sum_{n=1}^{MN-|k|} \mathrm{E}\{x(n)x^3(n+k)\} - 3\,r(0)\,\left(\frac{MN-|k|}{MN} r(k)\right) \\ &= \frac{MN-|k|}{MN} \left(\mathrm{E}\{x(n)x^3(n+k)\} \;-\; 3\,r(0)\cdot r(k)\right) \end{aligned}$$

where $r_2^u(0)$ and $r_2^u(k)$ denote the unbiased unsegmented correlation estimates. Then

$$\mathbf{E}\{d_2^u(k)\} = \frac{1}{MN-|k|} \sum_{n=1}^{MN-|k|} \mathbf{E}\{\ x(n)x^3(n+k)\ \} - 3\ r(0)\cdot r(k)$$

$$= \mathbf{E}\{\ x(n)x^3(n+k)\ \} - 3\ r(0)\cdot r(k) = c4(k)$$

Assume that $k \geq 0, k+v \geq 0$, then

$$\mathrm{Cov}\left(d_2^u(k), d_2^u(k+v)\right) = \mathbf{E}\left\{d_2^u(k)d_2^u(k+v)\right\} - \mathbf{E}\left\{d_2^u(k)\right\}\mathbf{E}\left\{d_2^u(k+v)\right\}$$

$$= \frac{1}{(MN-k)(MN-k-v)} \sum_{n=1}^{(MN-k)} \sum_{m=1}^{(MN-k-v)} \mathbf{E}\{x(n)x^3(n+k)x(m)x^3(m+k+v)\}$$

$$-\frac{3}{(MN-k)(MN)(MN-k-v)} \sum_{n=1}^{(MN-k)} \sum_{i=1}^{(MN)} \sum_{m=1}^{(MN-k-v)} \mathbf{E}\{x(n)x^3(n+k)x^2(i)x(m)x^3(m+k+v)\}$$

$$-\frac{3}{(MN-k-v)(MN)(MN-k)} \sum_{n=1}^{(MN-k-v)} \sum_{i=1}^{(MN)} \sum_{m=1}^{(MN-k)} \mathbf{E}\{x(n)x^3(n+k+v)x^2(i)x(m)x^3(m+k)\}$$

$$+\frac{9}{(MN)^2(MN-k)(MN-k-v)} \sum_{i=1}^{(MN)} \sum_{j=1}^{(MN)} \sum_{n=1}^{(MN-k)} \sum_{m=1}^{(MN-k-v)} \mathbf{E}\{x^2(i)x(n)x^3(n+k)x^2(j)x(m)$$

$$x^3(m+k+v)\} - c4(k)c4(k+v)$$

Table 3: Means and standard deviations (std) of accumulative singular values for correlations using one realization (Fig. 6)

Amplitudes	Number of singular value								
		1	2	3	4	5	6	7	8
[1(0) 1(0)]	mean	0.2214	0.4292	0.5827	0.7238	0.7898	0.8485	0.8754	0.9018
	std	0.0027	0.0043	0.0061	0.0072	0.0051	0.0042	0.0033	0.0027
[1(0) 0.7(-3)]	mean	0.2385	0.4480	0.5657	0.6685	0.7471	0.8168	0.8496	0.8818
	std	0.0049	0.0092	0.0092	0.0093	0.0072	0.0062	0.0050	0.0040
[1(0) 0.65(-3.7)]	mean	0.2429	0.4563	0.5636	0.6582	0.7399	0.8118	0.8450	0.8777
	std	0.0045	0.0086	0.0082	0.0079	0.0051	0.0045	0.0037	0.0032
[1(0) 0.625(-4)]	mean	0.2444	0.4593	0.5623	0.6538	0.7370	0.8096	0.8433	0.8763
	std	0.0043	0.0082	0.0086	0.0086	0.0060	0.0052	0.0043	0.0036
[1(0) 0.6(-4.4)]	mean	0.2481	0.4664	0.5640	0.6517	0.7345	0.8066	0.8408	0.8745
	std	0.0050	0.0096	0.0091	0.0084	0.0059	0.0054	0.0044	0.0037
[1(0) 0.575(-4.8)]	mean	0.2501	0.4705	0.5634	0.6486	0.7321	0.8044	0.8388	0.8726
	std	0.0063	0.0118	0.0103	0.0080	0.0055	0.0043	0.0036	0.0030
[1(0) 0.55(-5.2)]	mean	0.2531	0.4761	0.5667	0.6503	0.7324	0.8034	0.8382	0.8723
	std	0.0060	0.0113	0.0095	0.0080	0.0070	0.0063	0.0054	0.0047
[1(0) 0.525(-5.6)]	mean	0.2552	0.4803	0.5691	0.6513	0.7316	0.8008	0.8362	0.8709
	std	0.0048	0.0092	0.0073	0.0066	0.0060	0.0058	0.0045	0.0033

Table 4: Means and standard deviations (std) of accumulative singular values for correlations using independent realizations (Fig. 7)

Amplitudes	Number of singular value								
		1	2	3	4	5	6	7	8
[1(0) 1(0)]	mean	0.2235	0.4340	0.5890	0.7317	0.7946	0.8506	0.8772	0.9027
	std	0.0026	0.0058	0.0073	0.0090	0.0066	0.0052	0.0041	0.0032
[1(0) 0.7(-3)]	mean	0.2409	0.4524	0.5727	0.6776	0.7528	0.8196	0.8517	0.8826
	std	0.0040	0.0075	0.0090	0.0107	0.0066	0.0044	0.0035	0.0029
[1(0) 0.65(-3.7)]	mean	0.2461	0.4623	0.5721	0.6687	0.7462	0.8148	0.8478	0.8796
	std	0.0049	0.0093	0.0096	0.0104	0.0070	0.0053	0.0041	0.0032
[1(0) 0.625(-4)]	mean	0.2483	0.4667	0.5714	0.6641	0.7426	0.8117	0.8453	0.8775
	std	0.0040	0.0076	0.0087	0.0101	0.0065	0.0049	0.0039	0.0032
[1(0) 0.6(-4.4)]	mean	0.2509	0.4718	0.5712	0.6599	0.7398	0.8096	0.8435	0.8760
	std	0.0040	0.0077	0.0082	0.0089	0.0064	0.0058	0.0045	0.0035
[1(0) 0.575(-4.8)]	mean	0.2536	0.4770	0.5719	0.6576	0.7375	0.8071	0.8414	0.8744
	std	0.0044	0.0084	0.0093	0.0092	0.0057	0.0045	0.0036	0.0030
[1(0) 0.55(-5.2)]	mean	0.2564	0.4823	0.5727	0.6556	0.7357	0.8050	0.8398	0.8732
	std	0.0052	0.0100	0.0091	0.0078	0.0059	0.0055	0.0042	0.0032
[1(0) 0.525(-5.6)]	mean	0.2586	0.4866	0.5742	0.6551	0.7340	0.8021	0.8373	0.8712
	std	0.0043	0.0081	0.0067	0.0057	0.0048	0.0049	0.0039	0.0033

Table 7: Means and standard deviations (std) of accumulative singular values for correlations using one realization (Fig. 12)

Frequencies	Number of singular value								
		1	2	3	4	5	6	7	8
[0.1 0.2]	mean	0.2214	0.4292	0.5827	0.7238	0.7898	0.8485	0.8754	0.9018
	std	0.0027	0.0043	0.0061	0.0072	0.0051	0.0042	0.0033	0.0027
[0.1 0.14]	mean	0.2616	0.5122	0.6249	0.7233	0.7891	0.8463	0.8735	0.9001
	std	0.0041	0.0084	0.0101	0.0109	0.0078	0.0061	0.0051	0.0043
[0.1 0.13]	mean	0.3052	0.5808	0.6623	0.7290	0.7940	0.8462	0.8732	0.8999
	std	0.0040	0.0075	0.0088	0.0072	0.0057	0.0054	0.0046	0.0040
[0.1 0.1275]	mean	0.3153	0.5956	0.6695	0.7351	0.7985	0.8467	0.8735	0.8997
	std	0.0048	0.0089	0.0091	0.0070	0.0055	0.0055	0.0047	0.0039
[0.1 0.125]	mean	0.3260	0.6117	0.6801	0.7430	0.8039	0.8469	0.8737	0.9001
	std	0.0054	0.0097	0.0085	0.0074	0.0064	0.0061	0.0053	0.0045
[0.1 0.1225]	mean	0.3356	0.6257	0.6923	0.7529	0.8084	0.8461	0.8731	0.8996
	std	0.0046	0.0087	0.0064	0.0054	0.0055	0.0055	0.0047	0.0041
[0.1 0.12]	mean	0.3449	0.6397	0.7061	0.7663	0.8143	0.8472	0.8738	0.9000
	std	0.0049	0.0090	0.0068	0.0056	0.0056	0.0055	0.0047	0.0040
[0.1 0.1175]	mean	0.3515	0.6496	0.7164	0.7768	0.8183	0.8473	0.8737	0.8999
	std	0.0048	0.0089	0.0066	0.0052	0.0054	0.0054	0.0047	0.0041

Table 8: Means and standard deviations (std) of accumulative singular values for correlations using independent realizations (Fig. 13)

Frequencies	Number of singular value								
		1	2	3	4	5	6	7	8
[0.1 0.2]	mean	0.2235	0.4340	0.5890	0.7317	0.7946	0.8506	0.8772	0.9027
	std	0.0026	0.0058	0.0073	0.0090	0.0066	0.0052	0.0041	0.0032
[0.1 0.14]	mean	0.2650	0.5193	0.6333	0.7327	0.7954	0.8501	0.8767	0.9021
	std	0.0036	0.0073	0.0085	0.0091	0.0068	0.0058	0.0047	0.0037
[0.1 0.13]	mean	0.3100	0.5902	0.6720	0.7374	0.7996	0.8503	0.8768	0.9023
	std	0.0047	0.0086	0.0088	0.0074	0.0050	0.0043	0.0036	0.0031
[0.1 0.1275]	mean	0.3209	0.6063	0.6799	0.7422	0.8030	0.8502	0.8767	0.9022
	std	0.0044	0.0081	0.0083	0.0067	0.0058	0.0055	0.0044	0.0037
[0.1 0.125]	mean	0.3312	0.6213	0.6881	0.7489	0.8078	0.8505	0.8768	0.9022
	std	0.0045	0.0084	0.0081	0.0068	0.0061	0.0059	0.0048	0.0039
[0.1 0.1225]	mean	0.3404	0.6347	0.6983	0.7566	0.8119	0.8498	0.8762	0.9017
	std	0.0050	0.0093	0.0071	0.0058	0.0054	0.0054	0.0046	0.0039
[0.1 0.12]	mean	0.3493	0.6479	0.7109	0.7684	0.8170	0.8500	0.8763	0.9018
	std	0.0053	0.0099	0.0071	0.0056	0.0058	0.0058	0.0047	0.0038
[0.1 0.1175]	mean	0.3573	0.6601	0.7230	0.7802	0.8214	0.8501	0.8762	0.9018
	std	0.0039	0.0072	0.0056	0.0056	0.0055	0.0055	0.0046	0.0039

Table 11: Means and standard deviations of estimated amplitudes using independent realizations for Example 2

True Amplitude	1.0	1.0	
Correlation (2 harmonics)			
LS	1.0435 (0.0093)	1.0506 (0.0132)	
TLS	1.0662 (0.0095)	1.0734 (0.0126)	
CTLS	1.0434 (0.0093)	1.0504 (0.0133)	
Correlations (3 harmonics)			
LS	1.0331 (0.0096)	1.0452 (0.0134)	0.5548 (0.0174)
TLS	1.0415 (0.0099)	1.0537 (0.0132)	0.5578 (0.0176)
CTLS	1.0351 (0.0092)	1.0438 (0.0133)	0.5533 (0.0173)
Cumulants			
LS	0.9865 (0.0319)	0.9899 (0.0283)	
TLS	0.9879 (0.0315)	0.9914 (0.0281)	
CTLS	0.9885 (0.0314)	0.9895 (0.0282)	

Table 12: Means and standard deviations of estimated amplitudes for Example 3

True Amplitude	1.0	1.0	
Correlation (2 harmonics)			
LS	1.0899 (0.0189)	1.0991 (0.0215)	
TLS	1.1800 (0.0197)	1.1896 (0.0217)	
CTLS	1.0909 (0.0190)	1.0997 (0.0215)	
Correlations (3 harmonics)			
LS	1.0664 (0.0191)	1.0851 (0.0217)	0.7996 (0.0234)
TLS	1.0975 (0.0183)	1.1175 (0.0216)	0.8201 (0.0240)
CTLS	1.0739 (0.0190)	1.0859 (0.0216)	0.7969 (0.0234)
Cumulants			
LS	0.9928 (0.0534)	0.9985 (0.0696)	
TLS	1.0024 (0.0475)	0.9979 (0.0631)	
CTLS	0.9887 (0.0542)	0.9887 (0.0697)	

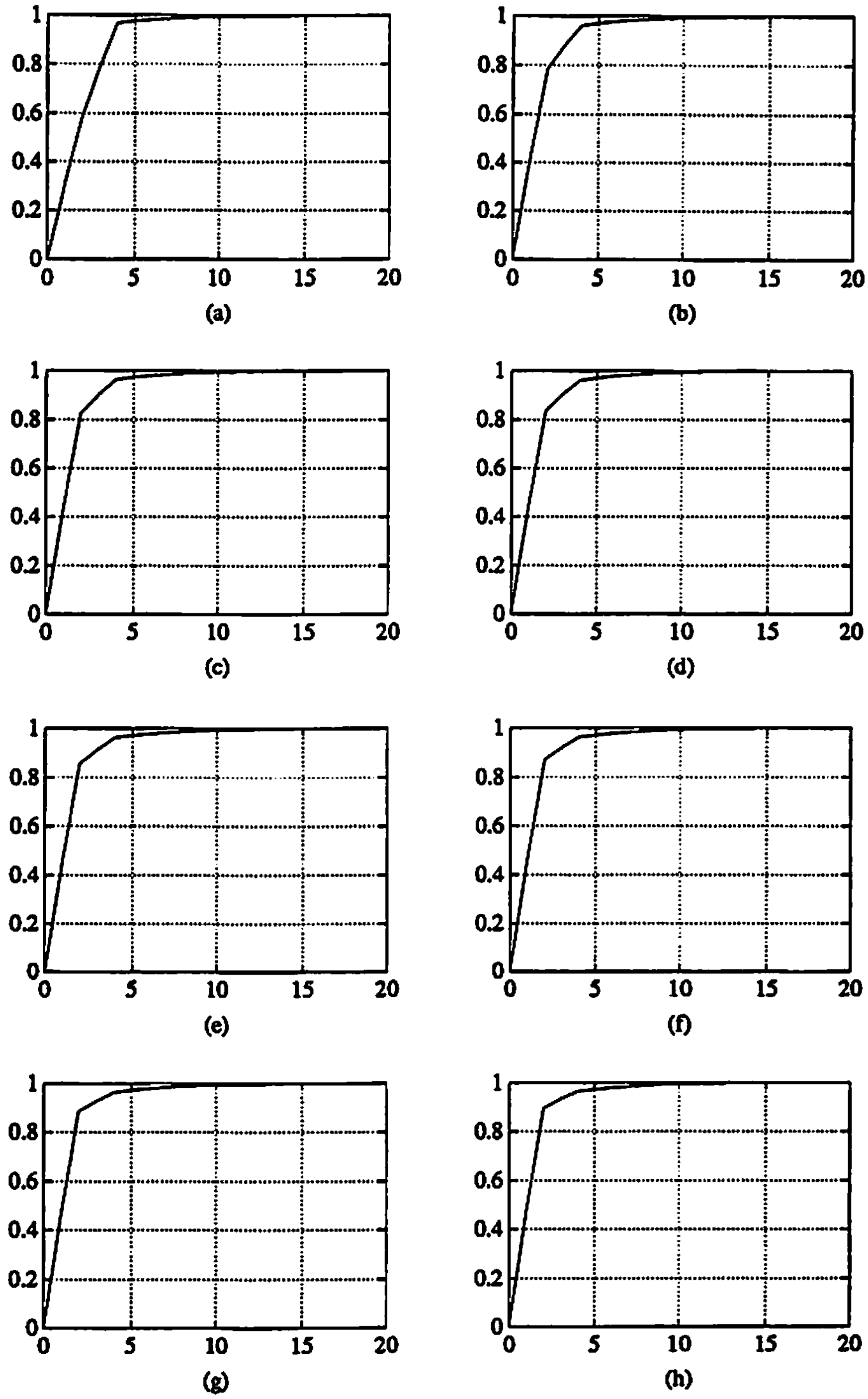

Figure 2: Accumulative singular values of cumulants using one realization when the fixed amplitude sinusoid at 0.1 has a local SNR of 5 dB and the amplitude of the sinusoid at 0.2 has a local SNR (dB) of : (a) 5 (b) 1.7 (c) 1 (d) 0.7 (e) 0.3 (f) 0 (g) -0.4 (h) -0.8

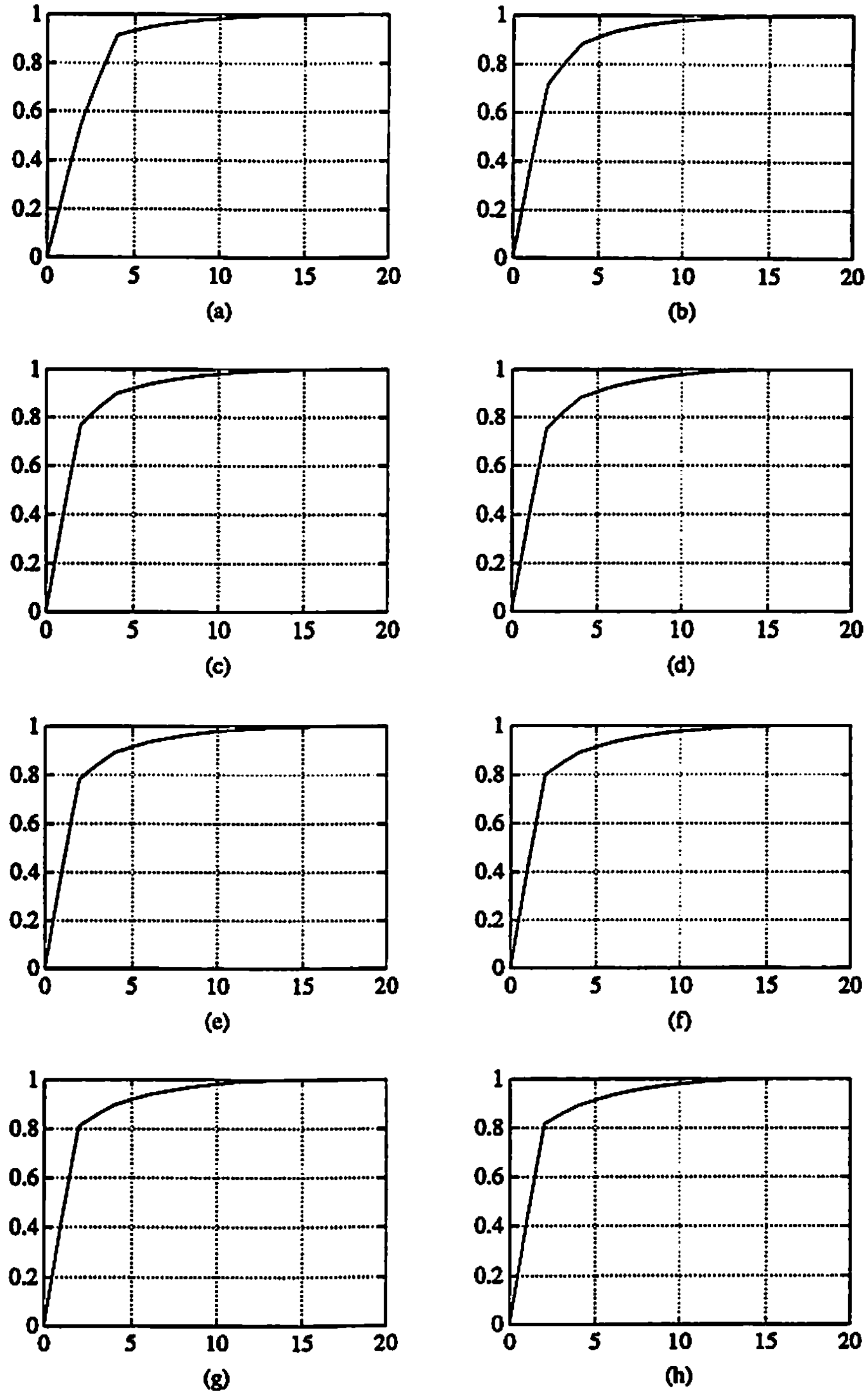

Figure 4: Accumulative singular values of cumulants using one realization when the fixed amplitude sinusoid at 0.1 has a local SNR of 0 dB and the amplitude of the sinusoid at 0.2 has a local SNR (dB) of : (a) 0 (b) -3 (c) -3.7 (d) -4 (e) -4.4 (f) -4.8 (g) -5.2 (h) -5.6

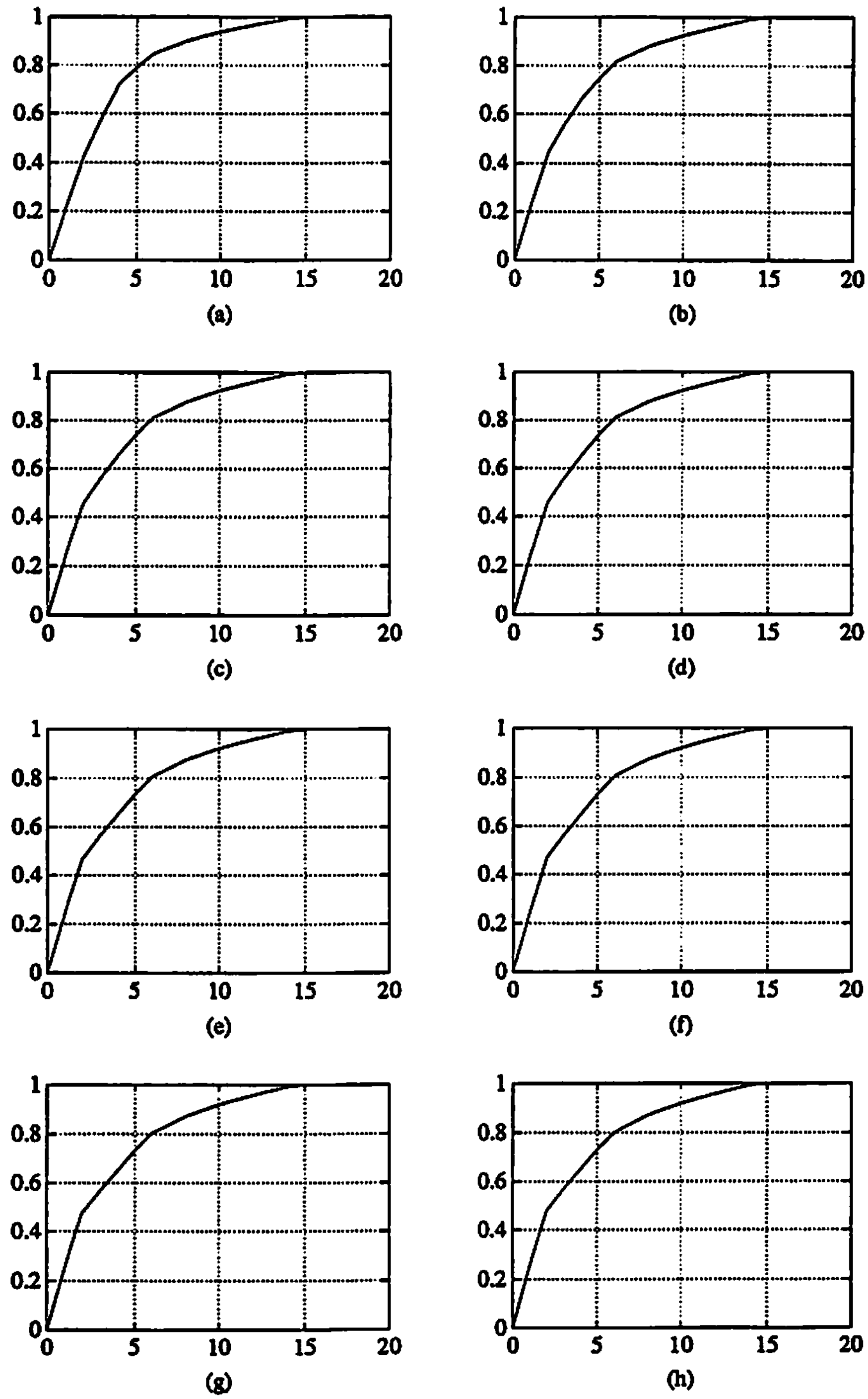

Figure 6: Accumulative singular values of correlations using one realization when the fixed amplitude sinusoid at 0.1 has a local SNR of 0 dB and the amplitude of the sinusoid at 0.2 has a local SNR (dB) of : (a) 0 (b) -3 (c) -3.7 (d) -4 (e) -4.4 (f) -4.8 (g) -5.2 (h) -5.6

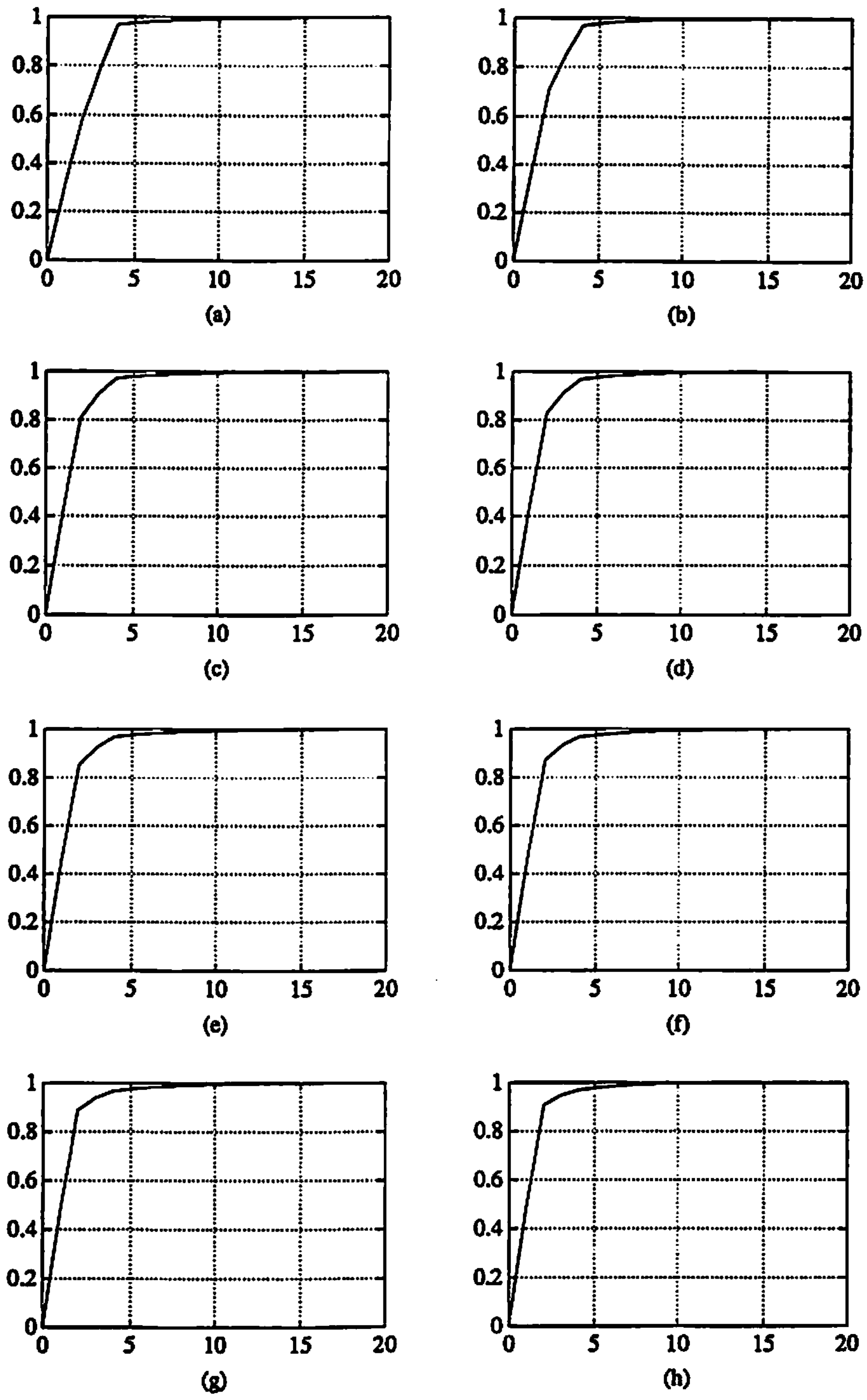

Figure 8: Accumulative singular values of cumulants using one realization when both amplitudes of sinusoids have local SNR's of 5 dB at $f_1 = 0.1$ and $f_2 =$: (a) 0.2 (b) 0.14 (c) 0.13 (d) 0.1275 (e) 0.125 (f) 0.1225 (g) 0.12 (h) 0.1175

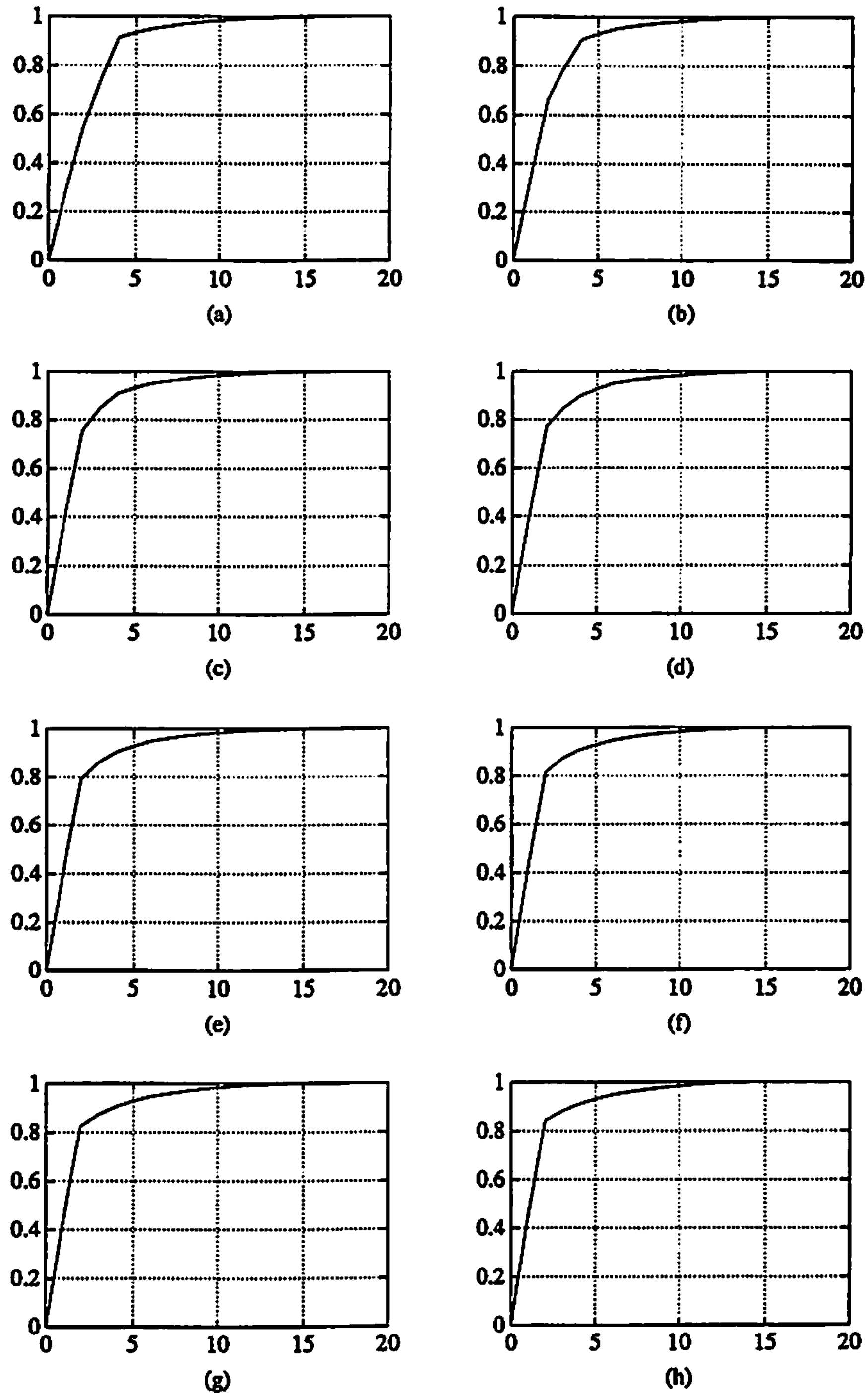

Figure 10: Accumulative singular values of cumulants using one realization when both amplitudes of sinusoids have local SNR's of 0 dB at $f_1 = 0.1$ and $f_2 =$: (a) 0.2 (b) 0.14 (c) 0.13 (d) 0.1275 (e) 0.125 (f) 0.1225 (g) 0.12 (h) 0.1175

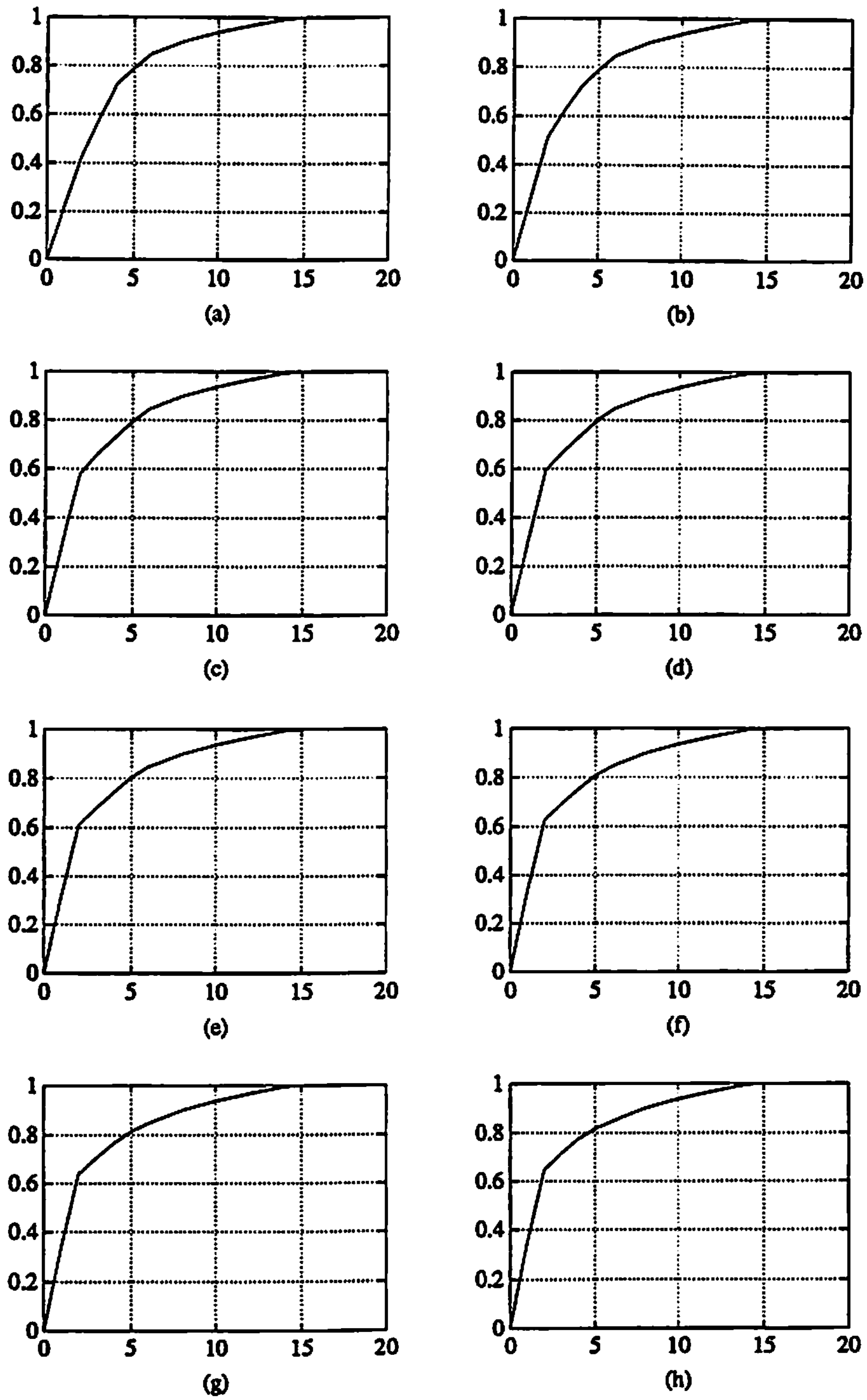

Figure 12: Accumulative singular values of correlations using one realization when both amplitudes of sinusoids have local SNR's of 0 dB at $f_1 = 0.1$ and $f_2 =$: (a) 0.2 (b) 0.14 (c) 0.13 (d) 0.1275 (e) 0.125 (f) 0.1225 (g) 0.12 (h) 0.1175

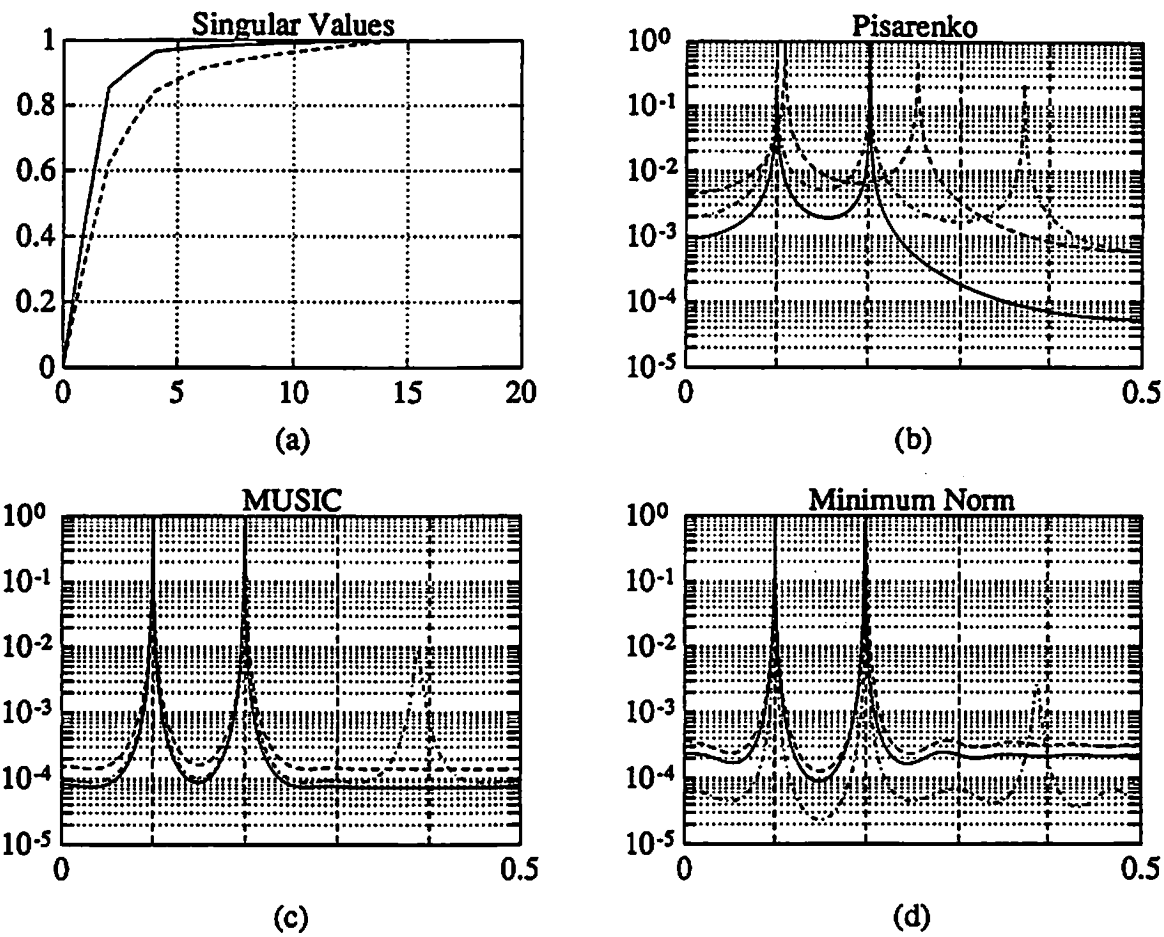

Figure 14: Results of estimated frequencies for Example 1: (a) mean of accumulative singular values (solid curve is for cumulants, dashed curve is for correlations); mean estimated spectra using (b) Pisarenko method, (c) MUSIC method, and (d) minimum norm method. In panels (b)-(d), the solid lines correspond to cumulant-based estimates, with p=2, and the dashed and dash-dotted lines correspond to correlation-based estimates with p=3, respectively.

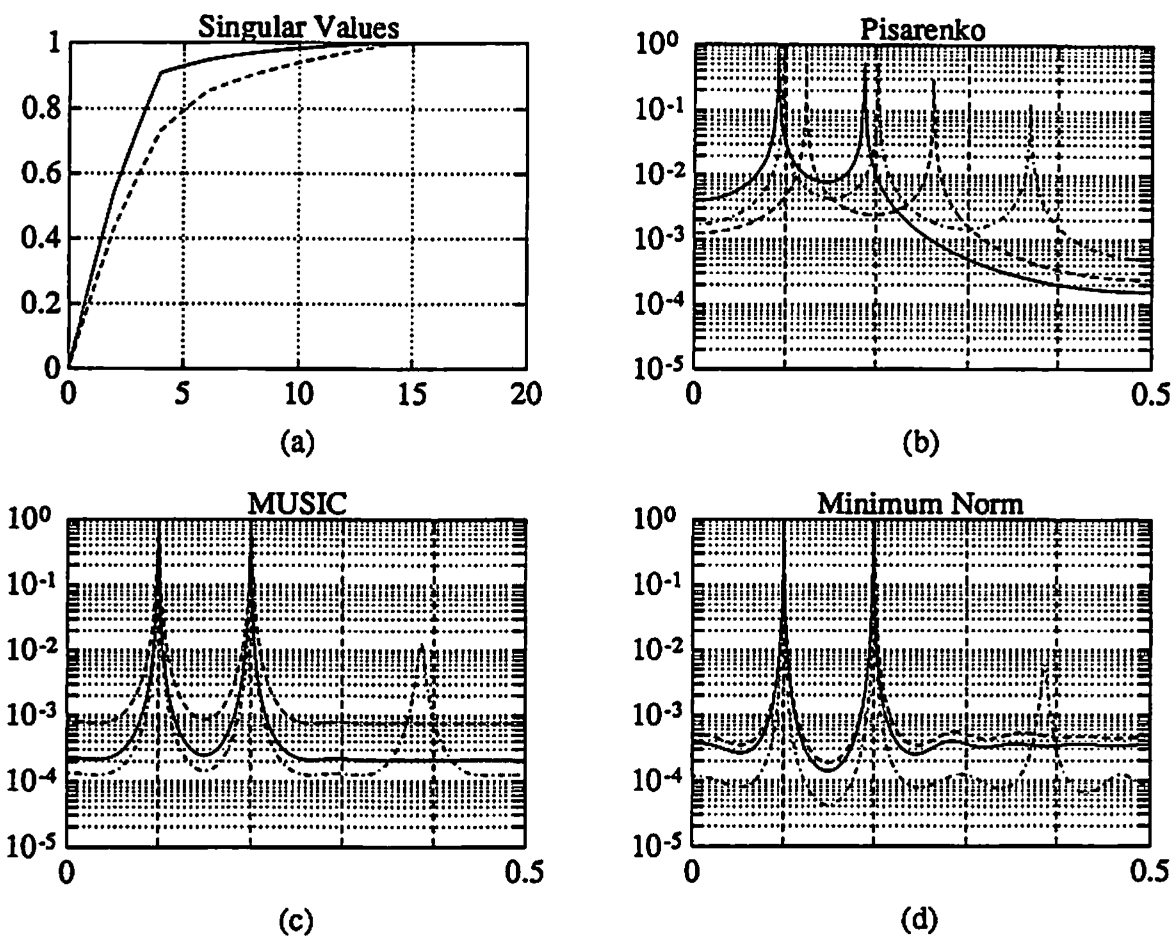

Figure 16: Results of estimated frequencies using independent realizations for Example 2: (a) mean of accumulative singular values (solid curve is for cumulants, dashed curve is for correlations); mean estimated spectra using (b) Pisarenko method, (c) MUSIC method, and (d) minimum norm method. In panels (b)-(d), the solid lines correspond to cumulant-based estimates, with p=2, and the dashed and dash-dotted lines correspond to correlation-based estimates with p=2 and p=3, respectively.

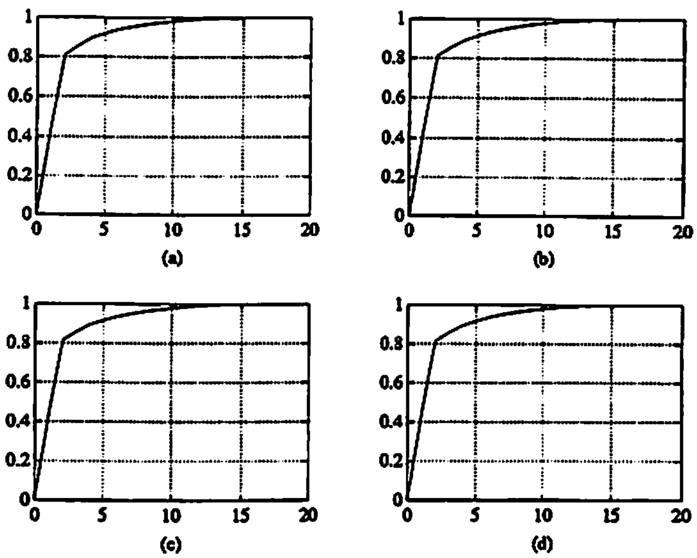

Figure 18: Accumulative singular values of cumulants using 64 x 64 (4096) one realization when the fixed amplitude sinusoid at 0.1 is unity and the amplitude of the sinusoid at 0.2 is : (a) 0.55* (b) 0.545 (c) 0.54 (d) 0.535

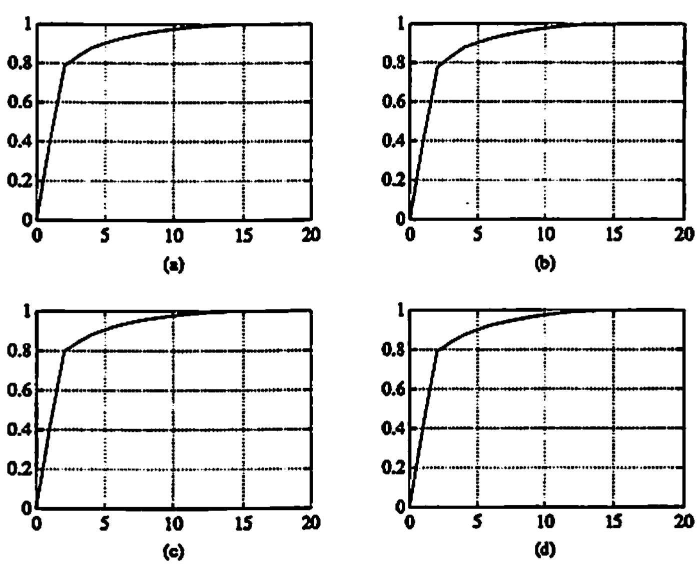

Figure 19: Accumulative singular values of cumulants using 50 x 64 (3200) one realization when the fixed amplitude sinusoid at 0.1 is unity and the amplitude of the sinusoid at 0.2 is : (a) 0.585 (b) 0.58* (c) 0.575 (d) 0.57

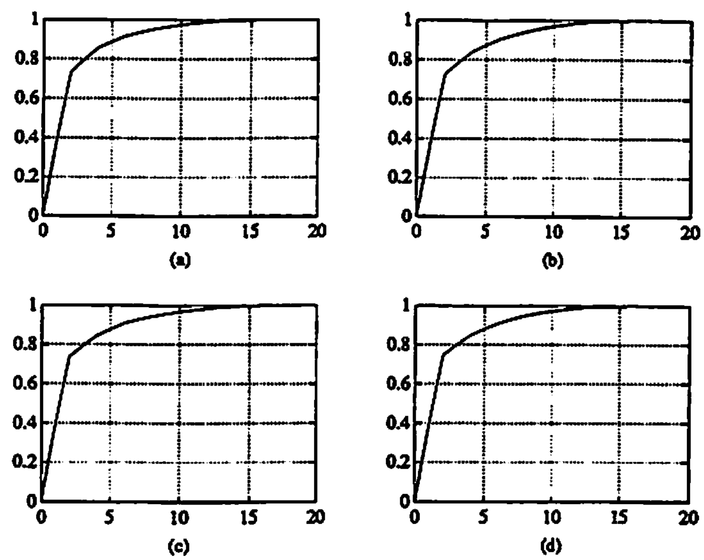

Figure 22: Accumulative singular values of cumulants using 25 × 64 (1600) one realization when the fixed amplitude sinusoid at 0.1 is unity and the amplitude of the sinusoid at 0.2 is : (a) 0.61* (b) 0.605 (c) 0.6 (d) 0.595

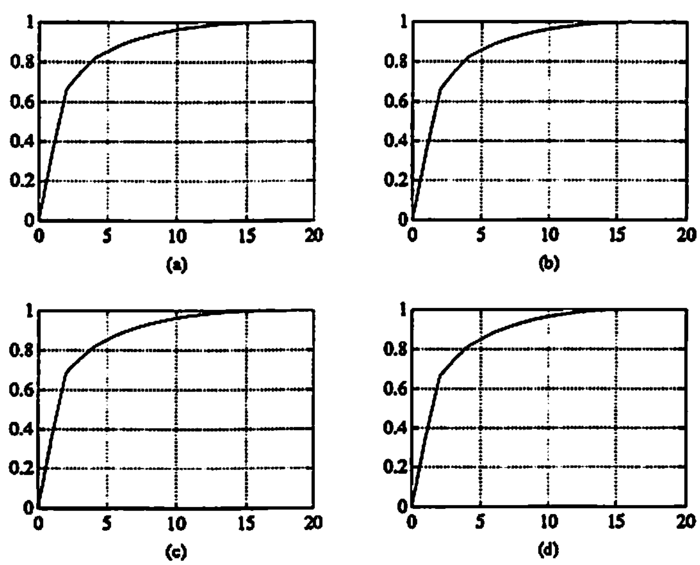

Figure 23: Accumulative singular values of cumulants using 16 × 64 (1024) one realization when the fixed amplitude sinusoid at 0.1 is unity and the amplitude of the sinusoid at 0.2 is : (a) 0.675 (b) 0.67* (c) 0.665 (d) 0.66

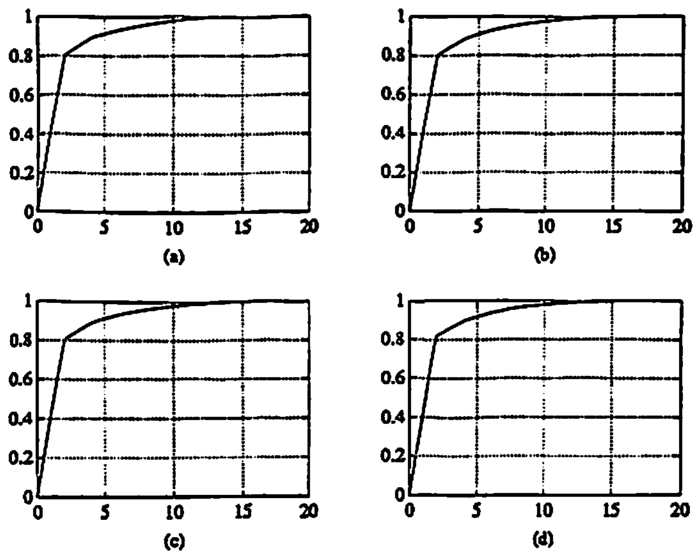

Figure 26: Accumulative singular values of cumulants using 64 x 64 (4096) one realization when the fixed amplitude sinusoid at 0.1 is unity and the amplitude of the sinusoid at 0.18 is : (a) 0.58 (b) 0.575 (c) 0.57* (d) 0.565

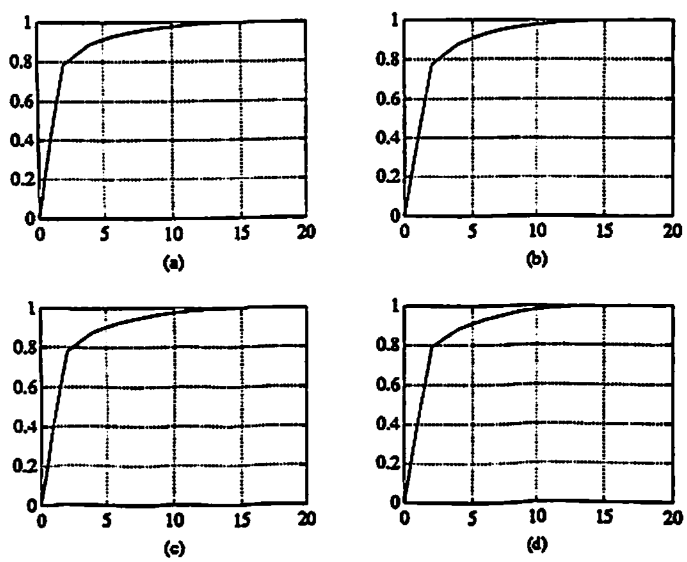

Figure 27: Accumulative singular values of cumulants using 50 x 64 (3200) one realization when the fixed amplitude sinusoid at 0.1 is unity and the amplitude of the sinusoid at 0.18 is : (a) 0.595 (b) 0.59 (c) 0.585* (d) 0.58

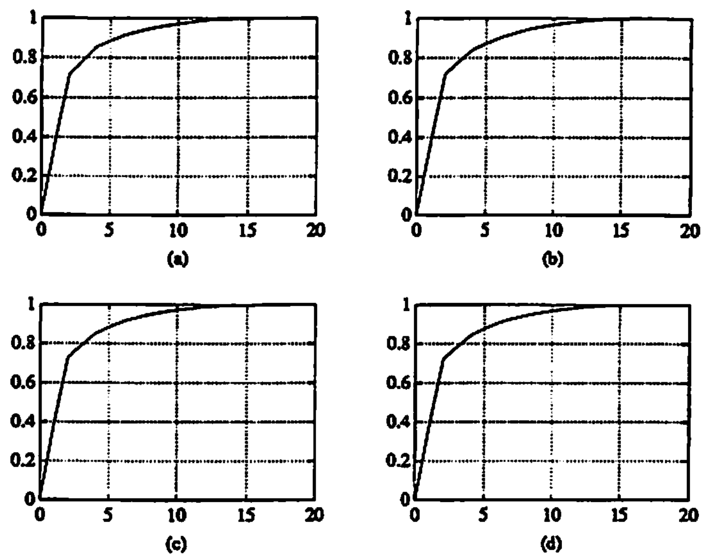

Figure 30: Accumulative singular values of cumulants using 25 x 64 (1600) one realization when the fixed amplitude sinusoid at 0.1 is unity and the amplitude of the sinusoid at 0.18 is : (a) 0.64 (b) 0.635* (c) 0.63 (d) 0.625

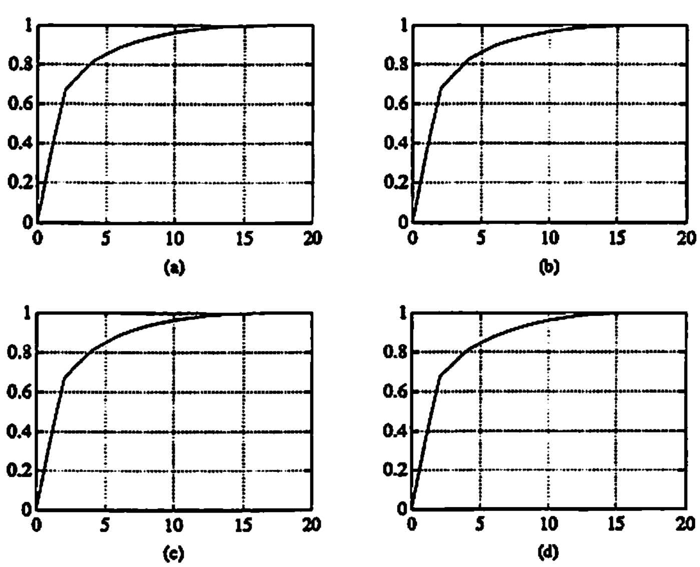

Figure 31: Accumulative singular values of cumulants using 16 x 64 (1024) one realization when the fixed amplitude sinusoid at 0.1 is unity and the amplitude of the sinusoid at 0.18 is : (a) 0.685 (b) 0.68* (c) 0.675 (d) 0.67

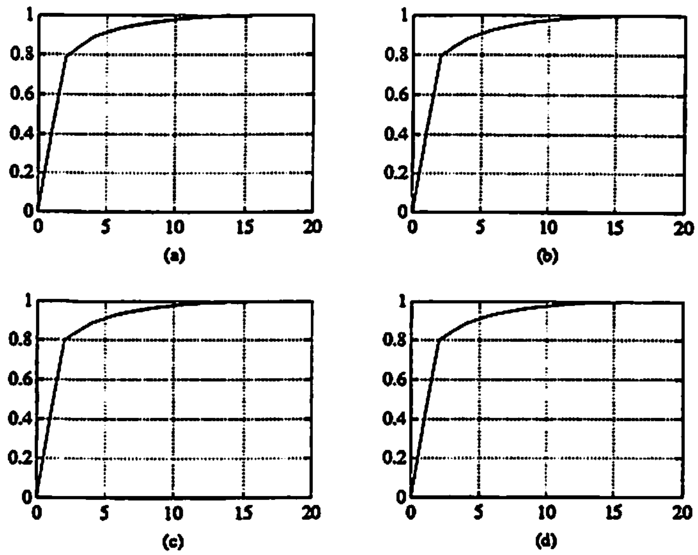

Figure 34: Accumulative singular values of cumulants using 64 x 64 (4096) one realization when the fixed amplitude sinusoid at 0.1 is unity and the amplitude of the sinusoid at 0.16 is : (a) 0.585* (b) 0.58 (c) 0.575 (d) 0.57

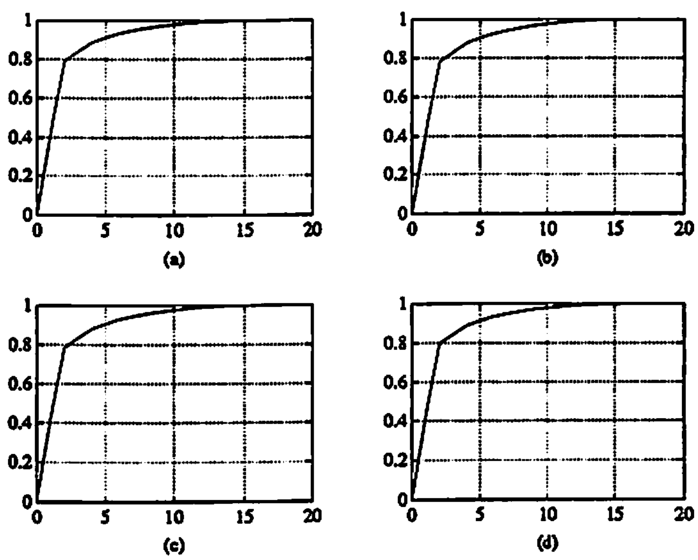

Figure 35: Accumulative singular values of cumulants using 50 x 64 (3200) one realization when the fixed amplitude sinusoid at 0.1 is unity and the amplitude of the sinusoid at 0.16 is : (a) 0.595 (b) 0.59* (c) 0.585 (d) 0.58

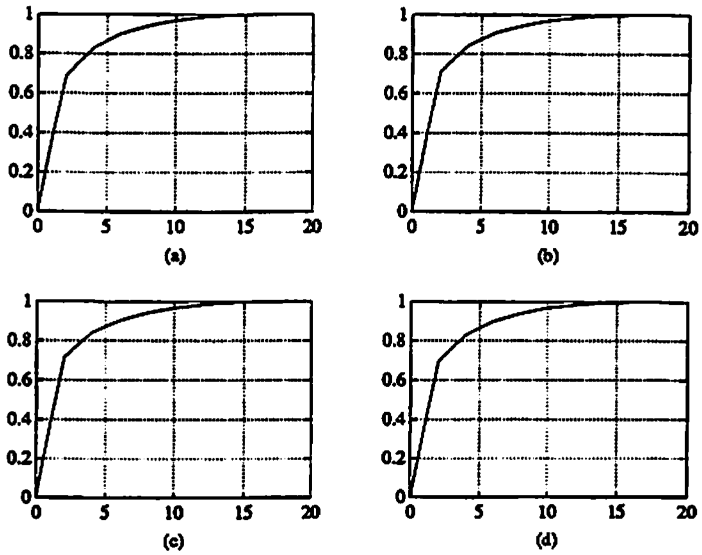

Figure 38: Accumulative singular values of cumulants using 25×64 (1600) one realization when the fixed amplitude sinusoid at 0.1 is unity and the amplitude of the sinusoid at 0.16 is : (a) 0.65 (b) 0.645 (c) 0.64* (d) 0.635

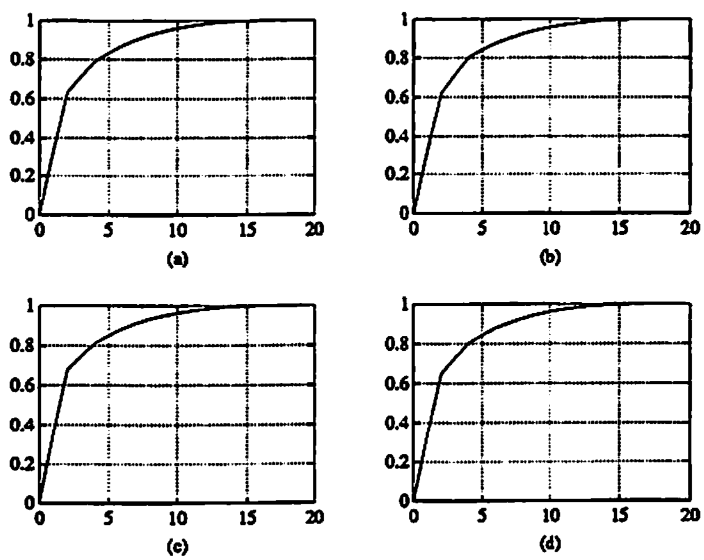

Figure 39: Accumulative singular values of cumulants using 16×64 (1024) one realization when the fixed amplitude sinusoid at 0.1 is unity and the amplitude of the sinusoid at 0.16 is : (a) 0.7 (b) 0.695* (c) 0.69 (d) 0.685

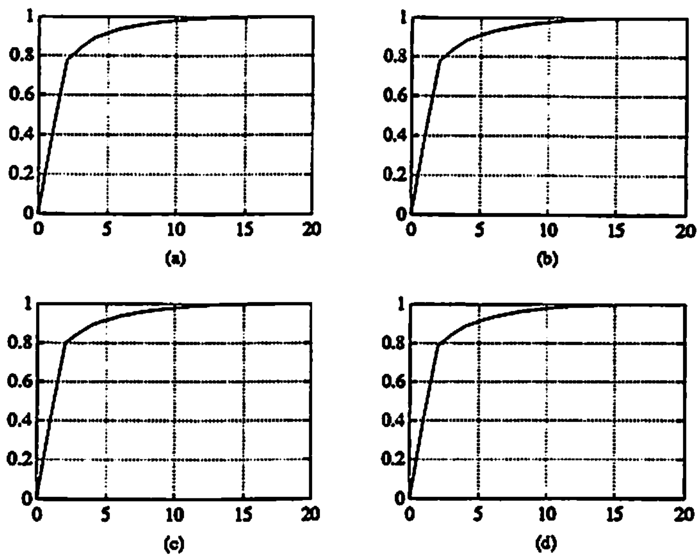

Figure 42: Accumulative singular values of cumulants using 64 x 64 (4096) one realization when the fixed amplitude sinusoid at 0.1 is unity and the amplitude of the sinusoid at 0.14 is : (a) 0.655 (b) 0.65 (c) 0.645* (d) 0.64

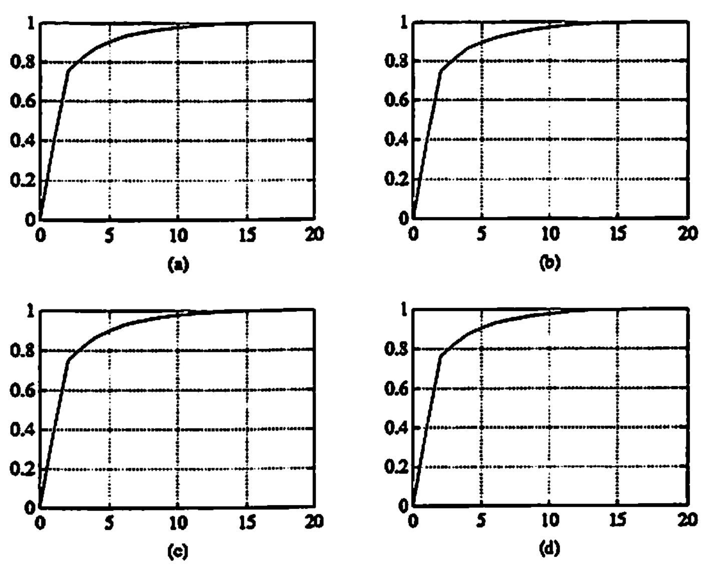

Figure 43: Accumulative singular values of cumulants using 50 x 64 (3200) one realization when the fixed amplitude sinusoid at 0.1 is unity and the amplitude of the sinusoid at 0.14 is : (a) 0.68 (b) 0.675* (c) 0.67 (d) 0.665

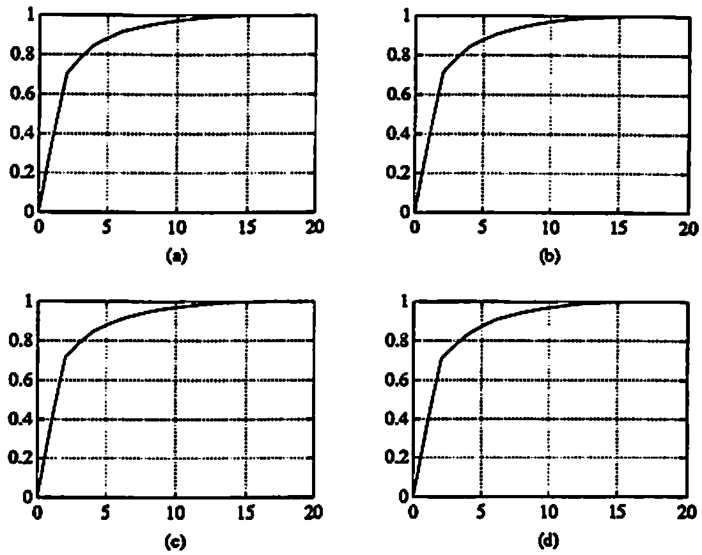

Figure 46: Accumulative singular values of cumulants using 25 x 64 (1600) one realization when the fixed amplitude sinusoid at 0.1 is unity and the amplitude of the sinusoid at 0.14 is : (a) 0.71 (b) 0.705* (c) 0.7 (d) 0.695

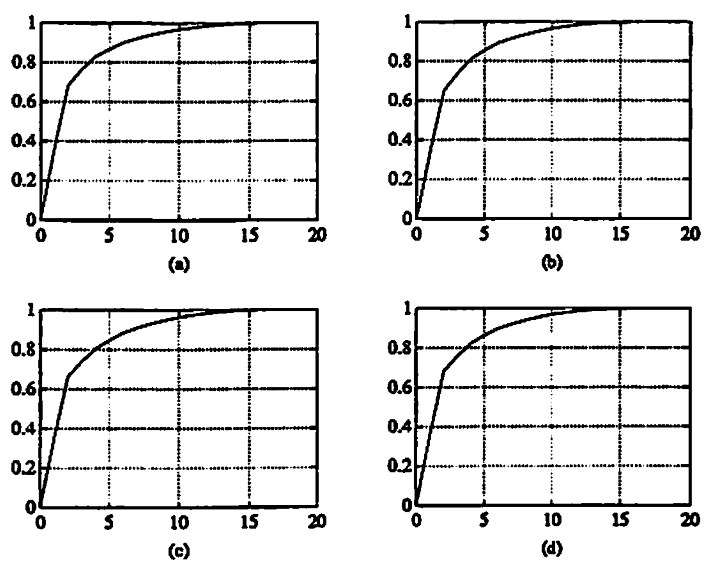

Figure 47: Accumulative singular values of cumulants using 16 x 64 (1024) one realization when the fixed amplitude sinusoid at 0.1 is unity and the amplitude of the sinusoid at 0.14 is : (a) 0.74* (b) 0.735 (c) 0.73 (d) 0.725

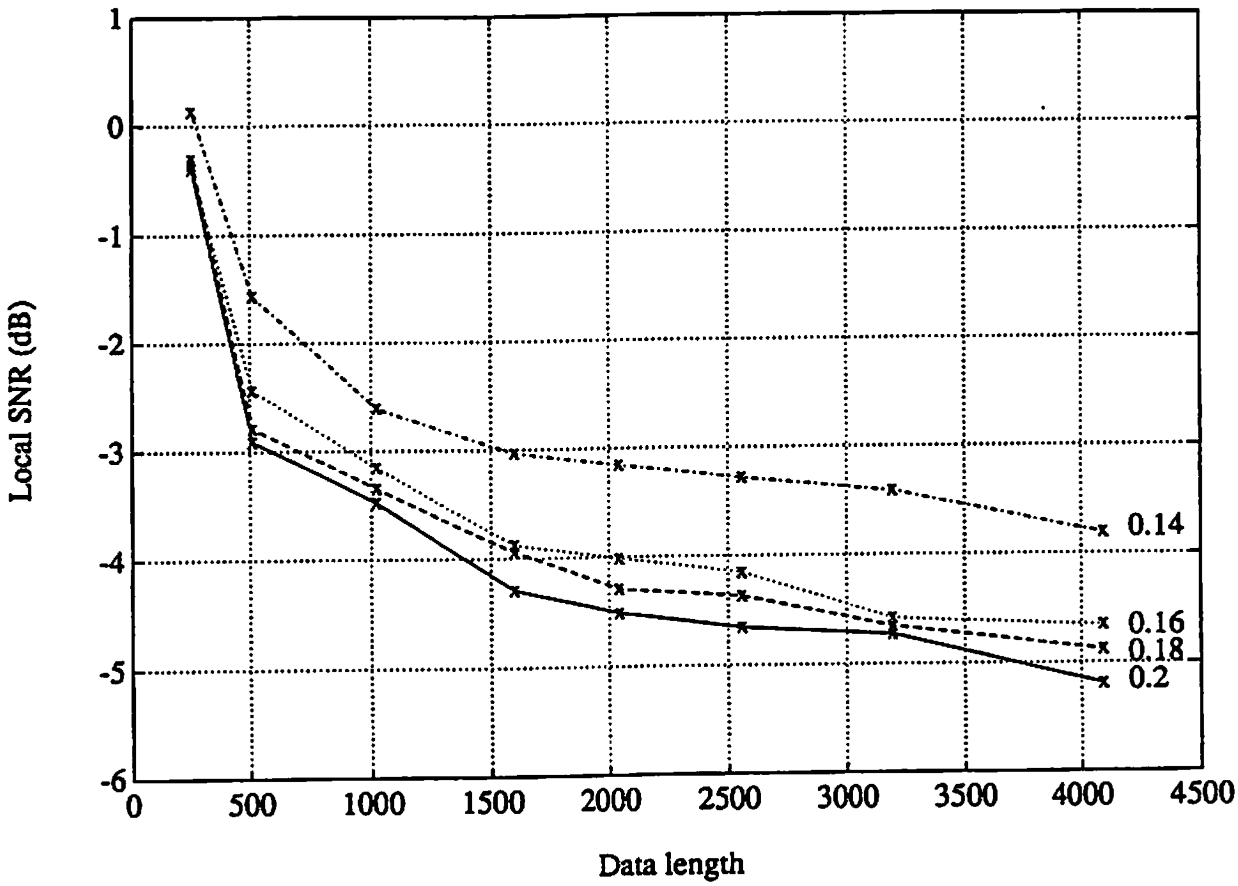

Figure 50: Minimum local SNR (dB) of the amplitude of the second sinusoid at f_2 when we can still determine the correct number of harmonics. The fixed amplitude at $f_1 = 0.1$ has local SNR of 0 dB. The crossed points denote the values that we got through simulations. The solid line is for $f_2 = 0.2$, and the dashed, dotted, and dash-dotted lines are for $f_2 = 0.18, 0.16$, and 0.14, respectively.