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by

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Abstract

An algorithm was proposed by Shensa for computing the coefficients of wavelet series transform (WST) or continuous wavelet transform (CWT). With the Shensa algorithm, we first perform filtering on a sampled discrete-time signal and then apply the Mallat's discrete wavelet transform (DWT) algorithm to the filtered sequence, where the prefiltering is used to reduce the approximation error between the computed and desired coefficients. In this research, we consider the design of optimal causal and noncausal FIR prefilters which reduce the approximation error as much as possible with a fixed filter length. Numerical experiments are provided to demonstrate the performance of the designed optimal prefilters.

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1 Introduction

Two types of wavelet transforms have been used to analyze continuous-time signals in signal representation, detection and processing [2], [5]. They are the continuous wavelet transform (CWT) and the wavelet series transform (WST). Let $\mathbf{Z}$, $\mathbf{R}$, $L^2(\mathbf{R})$ denote the sets of integers and real numbers and the space of all square-integrable functions, respectively. Consider a suitable function $\psi(t)$ whose dilations and translations

$$\{\psi_{jk}(t) \triangleq 2^{j/2}\psi(2^j t - k)\}_{j,k\in\mathbf{Z}}$$

form an orthonormal basis of $L^2(\mathbf{R})$. The function $\psi(t)$ is usually known as the mother wavelet, and the basis is called a wavelet basis. For $f(t) \in L^2(\mathbf{R})$, its CWT with respect to the mother wavelet $\psi(t)$ is defined as

$$CWT\{f(t); a, b\} = \int_{-\infty}^{\infty} f(t)\psi_{a,b}(t)dt\,, \tag{1}$$

where

$$\psi_{a,b}(t) = \frac{1}{\sqrt{|a|}}\psi(\frac{t-b}{a})\,,\ a, b \in \mathbf{R}\,,\ a \neq 0\,,$$

and where a and b are called the scale and time parameters, respectively. The WST of $f(t)$ is obtained by sampling its CWT on the scale-time plane (a, b) with the so-called "dyadic" grid, i.e.

$$WST\{f(t); j, k\} = CWT\{f(t); a = 2^{-j}, b = k2^{-j}\}\,, \quad j, k \in \mathbf{Z}\,.$$

The WST coefficients denoted by $b_{j,k}$ can be determined by

$$b_{j,k} \triangleq WST\{f(t); j, k\} = \int_{-\infty}^{\infty} f(t)\psi_{jk}(t)dt\,, \quad j, k \in \mathbf{Z}\,. \tag{2}$$

Given a continuous-time signal $f(t)$, we may compute its CWT or WST coefficients by performing numerical integration with (1) or (2), respectively. However, it is not efficient to compute the integrals if a large number of wavelet coefficients are to be determined. For this case, one usually obtain a discrete-time approximation $f(m\Delta t)$ of $f(t)$ through sampling and apply the discrete wavelet transform (DWT) to the sampled sequence. There exists a very efficient pyramid-structured algorithm proposed by Mallat to compute the DWT coefficients [1], [3]. The computed DWT coefficients are then used as approximations of the desired CWT or WST coefficients. To reduce the approximation error, Shensa [4], [6] proposed to perform a prefiltering process on the sampled signal $f(m\Delta t)$ to obtain a new sequence $\tilde{f}(m\Delta t)$ so that the computed DWT coefficients of the new sequence provides a better approximation of the desired CWT or WST coefficients. We derived formulas to characterize the error between the computed and desired wavelet coefficients for the Mallat and Shensa algorithms in [7]. In this research, we study the design of optimal causal and noncausal FIR prefilters used in the Shensa algorithm to minimize the error.

2 Mallat and Shensa Algorithms

Some basic results in wavelet theory [1], [3], [6], [7] are reviewed below. Any $f(t) \in L^2(\mathbf{R})$ can be approximated by its projection on a subspace $V_J \subset L^2(\mathbf{R})$, i.e.

$$f_J(t) = \sum_{k=-\infty}^{\infty} c_{J,k}\phi_{Jk}(t) = \sum_{j=-\infty}^{J-1} \sum_{k=-\infty}^{\infty} b_{j,k}\psi_{jk}(t)$$

where $J \geq 0$ is a given integer,

$$c_{J,k} \triangleq \int_{-\infty}^{\infty} f(t)\phi_{Jk}(t)dt.$$

and $b_{j,k}$, $j \leq J-1, k \in \mathbf{Z}$, are called the WST coefficients of $f(t)$. They can be computed from $c_{J,k}$ with the recursion:

$$\begin{array}{l} c_{j-1,k} = \frac{1}{2}\sum_n h_{n-2k}c_{j,n}\,, \\ b_{j-1,k} = \frac{1}{2}\sum_n g_{n-2k}c_{j,n}\,, \end{array} \qquad \text{for } j = J, J-1, \cdots, \tag{3}$$

where h_n and g_n are impulse responses of lowpass and highpass filters, respectively.

If the initial sequence $c_{J,k}$ in (3) is replaced by the sampled signal $x[k] = f(k/2^J)$, it leads to the Mallat algorithm and we denote the computed coefficients by $b_{j,k}^{(M)}$. The Shensa algorithm is a generalization of the Mallat algorithm, where a prefiltered version $x'[k]$ of the sampled signal $x[k]$, i.e.

$$x'[k] = \sum_m x[m]q[k-m] \tag{4}$$

is used as the initial sequence in (3). The computed coefficients are denoted by $b_{j,k}^{(S)}$. Note that if $q[n]$ is the unit impulse sequence $\delta[n]$, the Shensa algorithm is the same as the Mallat algorithm.

The errors $b_{j,k} - b_{j,k}^{(M)}$ and $b_{j,k} - b_{j,k}^{(S)}$ were analyzed in [7]. With slight modification, we can restate one of the main results in [7] as follows.

Theorem 1 *If $f(t)$ is $2^J\pi$ band-limited, then*

$$\sum_{j\leq J-1} \sum_k |b_{j,k} - 2^{-J/2}b_{j,k}^{(S)}|^2 = \frac{2^{J-1}}{\pi}\int_{-\pi}^{\pi} |\hat{f}(-2^J\omega)|^2|Q(\omega) - \hat{\phi}(\omega)|^2 d\omega\,. \tag{5}$$

In (5), $\hat{f}(\omega)$ denotes the Fourier integral of a continuous-time function $f(t) \in L^2(\mathbf{R})$ and $Q(\omega)$ is the frequency response of a prefilter related to the sequence $q[n]$ via $Q(\omega) = \sum_n q[n]e^{in\omega}$.

3 Design of Optimal Prefilters

It is clear from Theorem 1 that if $f(t)$ is $2^J\pi$ band-limited, to minimize the error resulted from the Shensa algorithm is equivalent to the minimization of the cost functional

$$C_{f,\phi} \triangleq \int_{-\pi}^{\pi} |\hat{f}(-2^J\omega)|^2 |Q(\omega) - \hat{\phi}(\omega)|^2 d\omega, \tag{6}$$

for a given $J \geq 0$. The design of the optimal FIR prefilter with respect to a given $f(t)$ is equivalent to the search of a sequence $q[n]$ of finite length which minimizes $C_{f,\phi}$ in (6). To do so, we first expand $C_{f,\phi}$ as a function of $q[n]$ as

$$\begin{aligned} C_{f,\phi} &= \int_{-\pi}^{\pi} \left| \hat{f}(-2^J\omega)|^2 | \sum_n q[n]e^{in\omega} - \hat{\phi}(\omega) \right|^2 d\omega \\ &= A\sum_n q^2[n] - \sum_{n<m} B_{mn} q[n]q[m] + \sum_n C_n q[n] + D\,, \end{aligned}$$

where

$$\begin{aligned} A &= \int_{-\pi}^{\pi} |\hat{f}(-2^J\omega)|^2 d\omega, & (7) \\ B_{mn} &= 2\int_{-\pi}^{\pi} |\hat{f}(-2^J\omega)|^2 \cos((n-m)\omega) d\omega, \ \text{ for } n \neq m, & (8) \\ C_n &= \int_{-\pi}^{\pi} |\hat{f}(-2^J\omega)|^2 \mathrm{Re}(\hat{\phi}(\omega)e^{-in\omega}) d\omega, & (9) \\ D &= \int_{-\pi}^{\pi} |\hat{f}(-2^J\omega)|^2 |\hat{\phi}(\omega)|^2 d\omega. \end{aligned}$$

To solve the above minimization problem, we set

$$\frac{\partial C_{f,\phi}}{\partial q[n]} = 0\,, \quad \forall n. \tag{10}$$

In particular, to design the optimal prefilter $q[n]$ with nonzero values in $N_1 \leq n \leq N_2 - 1$, we can express (10) to a linear system of equations

$$\mathbf{Tq} = \mathbf{C}, \tag{11}$$

where $\mathbf{T} = (t_{mn})_{N_1 \leq m,n < N_2}$ with

$$t_{mn} = \begin{cases} 2A, & n = m, \\ -B_{mn}, & n \neq m, \end{cases}$$

and

$$\begin{aligned}\mathbf{C} &= (-C_{N_1}, -C_{N_1+1}, \cdots, -C_{N_2-2}, -C_{N_2-1})^T, \\ \mathbf{q} &= (q[N_1], q[N_1+1], \cdots, q[N_2-2], q[N_2-1])^T,\end{aligned}$$

and where A, B_{mn} and C_n are defined by (7), (8) and (9), respectively.

Since $C_{f,\phi}$ depends on the signal $f(t)$, the optimal prefilter designed above is quite restricted. For some applications, we may want to consider a prefilter which performs well with respect to a class of functions rather than a particular one. It is therefore convenient to consider a signal independent cost functional

$$C_\phi = \int_{-\pi}^{\pi} F(\omega)|Q(\omega) - \hat{\phi}(\omega)|^2 d\omega, \tag{12}$$

where $F(\omega)$ is a nonnegative weighting function. Some a priori knowledge of signals such as the bandwidth and energy distribution are helpful in the determination of the nonnegative weighting function $F(\omega)$. A typical choice is the Gaussian-shaped function

$$F(\omega) = e^{-aw^2}, \tag{13}$$

where $a \geq 0$ is a parameter. Generally speaking, a should be larger (or smaller) for signals with narrower (or wider) bandwidths. Following a similar procedure, it can be shown that the optimal sequence $q[n]$ minimizing C_ϕ are obtainable by solving the linear system (11) except for the replacement of $|\hat{f}(-2^J\omega)|^2$ by the weighting function $F(\omega)$ in (7)-(9).

4 Determination of Optimal Interpolant

The purpose of prefiltering can be interpreted from another viewpoint. Let $\tilde{f}(t)$ be the interpolated signal of $x[n]$, i.e.

$$\tilde{f}(t) = \sum_n x[n]\chi(2^J t - n) = \sum_n f(\frac{n}{2^J})\chi(2^J t - n), \tag{14}$$

where $\chi(t)$ is a certain interpolant (or D/A converter) so that the computed coefficients $b_{j,k}^{(S)}$ satisfy

$$b_{j,k}^{(S)} = WST\{\tilde{f}(t); j, k\}, \quad j \leq J-1, k \in \mathbf{Z}.$$

If $\tilde{f}(t)$ is close to $f(t)$, $b_{j,k}^{(S)}$ provides a good approximation of $b_{j,k}$. It can be proved [4], [7] that the impulse response $q[n]$ of the prefilter in (4) is related to the interpolant $\chi(t)$ in (14) via

$$q[n] = 2^{-J/2} \int \chi(t)\phi(t-n)dt. \tag{15}$$

By using (15), we can determine the optimal interpolant $\chi(t)$ from the optimal filter response $q[n]$. Let

$$q(s) = 2^{-J/2} \int \chi(t)\phi(t-s)dt, \quad s \in \mathbf{R},$$

and $\hat{q}(\omega)$ be the Fourier integral of $q(s)$. Since $q[n] = q(n)$ for all $n \in \mathbf{Z}$, $q(s)$ is a continuous-time signal which interpolates $q[n]$ at integer points. By transforming (15) to the Fourier domain, we have

$$\hat{q}(\omega) = 2^{-J/2}\hat{\chi}(\omega)\hat{\phi}(-\omega), \quad \omega \in \mathbf{R}. \tag{16}$$

Besides, we assume that

$$\hat{\phi}(\omega) \neq 0, \quad \text{for } |\omega| < \pi,$$

which is in fact satisfied by most wavelet bases and choose

$$q(s) = \sum_n q[n] \frac{\sin \pi(s-n)}{\pi(s-n)}. \tag{17}$$

Since there are only a finite number of nonzero values in $q[n]$, the $q(s)$ is π band-limited. Thus we can solve (16) for $\hat{\chi}(\omega)$, i.e.

$$\hat{\chi}(\omega) = \frac{2^{J/2}\hat{q}(-\omega)}{\hat{\phi}(-\omega)}\delta_{[-\pi,\pi]}(\omega) = \frac{2^{J/2}Q(-\omega)}{\hat{\phi}(-\omega)}\delta_{[-\pi,\pi]}(\omega). \tag{18}$$

where $\delta_{[a,b]}(\omega)$ is an indicator function whose value is 1 for $\omega \in [a,b]$ and 0 otherwise. Note that the $\chi(t)$ obtained from (18) is not the unique solution to (15) for a given $q[n]$.

5 Numerical Results

The test function $f(t)$ is a $2^J\pi$-limited signal with $J = 6$. It has the spectrum $\hat{f}(\omega) = e^{-(\omega/100)^2}$ for $|\omega| < 2^6\pi$. Both causal and non-causal prefilters are designed. In the non-causal case, we choose $-N_1 \leq N_2 - 1 \leq -N_1 + 1$ and $N = N_2 - N_1$ is the filter length. The performance of the signal-dependent optimal prefilter is shown in Figure 1, where we plot the error between $b_{j,k}$ and $b_{j,k}^{(S)}$ given by the left-hand-side expression of (5) as a function of the filter length N. The Haar, the Daubechies D_4 and D_8 wavelet bases are compared. Note that the Mallat algorithm corresponds to the case $N = 1$. We see a significant improvement of the Shensa algorithm even with a small value of N. Also, the non-causal prefilter performs better than the causal one when $N \geq 4$. For signal-independent optimal prefilter design, we consider two weighting functions of the form (13) with $a = 0$ and 0.1. For $a = 0$, the optimal prefilter $q[n]$ can be solved from

$$\min_{q[n]} \int_{-\pi}^{\pi} \left| \sum_{N_1 \le n < N_2} q[n] e^{in\omega} - \hat{\phi}(\omega) \right|^2 d\omega.$$

Thus the $q[n]$ is exactly the same as the Fourier series coefficient d_n, $N_1 \le n < N_2$, of $\hat{\phi}(\omega)$ in $[-\pi, \pi]$. In Table 1, we list the computed filter response sequences $q[n]$ of the optimal causal prefilters with $1 \le N \le 10$ and $a = 0, 0.1$ for the Daubechies D_4 wavelet basis. The performance of the signal-independent optimal prefilter is shown in Figure 2. We also plot in Figure 3 the interpolant $\chi(t)$ corresponding to the $q[n]$ with $N = 2$ and 4 in Table 1. We see that the interpolants $\chi(t)$ associated with the same filter length have similar waveforms.

6 Conclusion

In this article, we studied the optimal FIR prefilter design problem for the Shensa algorithm in computing the WST coefficients. It was shown numerically that the error for the computed WST coefficients is reduced significantly by using the desgined optimal prefilters than that obtained from the Mallat algorithm. The interpolant $\chi(t)$ corresponding to the optimal prefilter was also examined.

References

[1] I. Daubechies, "Orthonormal bases of compactly supported wavelets," *Comm. on Pure and Appl. Math.*, Vol. 41, pp. 909–996, 1988.

[2] F. Hlawatsch and G. Boudreaux-Bartels, "Linear and Quadratic time-frequency signal representations," *IEEE Signal Processing Magazine*, pp. 21–67, April 1992.

[3] S. Mallat, "A theory for multiresolution signal decomposition: the wavelet representation," *IEEE Trans. on Pattern Anal. and Mach. Intell.*, Vol. 11, pp. 674–693, 1989.

[4] O. Rioul and P. Duhamel, "Fast algorithms for discrete and continuous wavelet transform," *IEEE Trans. on Information Theory*, Vol. 38, pp. 569–586, Mar. 1992.

[5] O. Rioul and M. Vetterli, "Wavelets and signal processing," *IEEE Signal Processing Magazine*, pp. 14–38, Oct. 1991.

[6] M. J. Shensa, "Affine wavelets: Wedding the Atrous and Mallat algorithms," *IEEE Trans. on Signal Processing*, Oct. 1992. to appear.

[7] X.-G. Xia, C.-C. J. Kuo, and Z. Zhang, "On the accuracy of computed wavelet coefficients," tech. rep., Department of Electrical Engineering-Systems, University of Southern California, 1992. submitted to SIAM J. Numer. Anal.

Figure Captions

Table 1: Coefficients of optimal prefilters for D_4 basis with (a) $a = 0$ and (b) $a = 0.1$.

Figure 1: Errors for (a) causal and (b) noncausal signal dependent prefilters.

Figure 2: Errors for (a) causal and (b) noncausal signal independent prefilters.

Figure 3: The interpolants $\chi(t)$ correspond to the optimal prefilters $q[n]$ given in Table 1 with (a) $N = 2$ and $a = 0$, (b) $N = 4$ and $a = 0$, (c) $N = 2$ and $a = 0.1$, (d) $N = 4$ and $a = 0.1$.

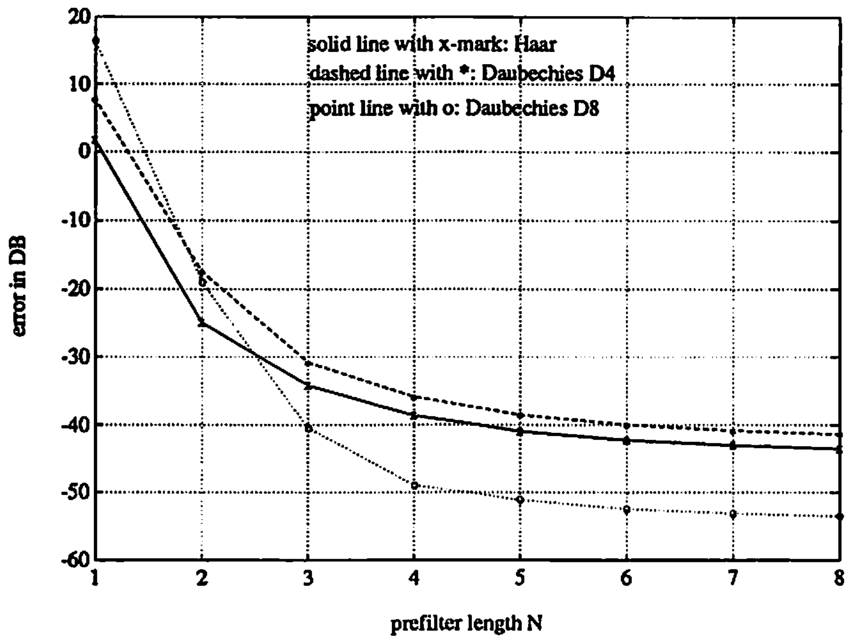
length N	$q[n]$, $0 \leq n \leq N-1$				
1	1.0000				
2	0.6466	0.5846			
3	0.6466	0.5846	-0.1975		
4	0.6466	0.5846	-0.1975	0.0980	
5	0.6466	0.5846	-0.1975	0.0980	-0.0675
6	0.6466	0.5846	-0.1975	0.0980	-0.0675
	0.0517				
7	0.6466	0.5846	-0.1975	0.0980	-0.0675
	0.0517	-0.0420			
8	0.6466	0.5846	-0.1975	0.0980	-0.0675
	0.0517	-0.0420	0.0353		
9	0.6466	0.5846	-0.1975	0.0980	-0.0675
	0.0517	-0.0420	0.0353	-0.0305	
10	0.6466	0.5846	-0.1975	0.0980	-0.0675
	0.0517	-0.0420	0.0353	-0.0305	0.0269

(a)

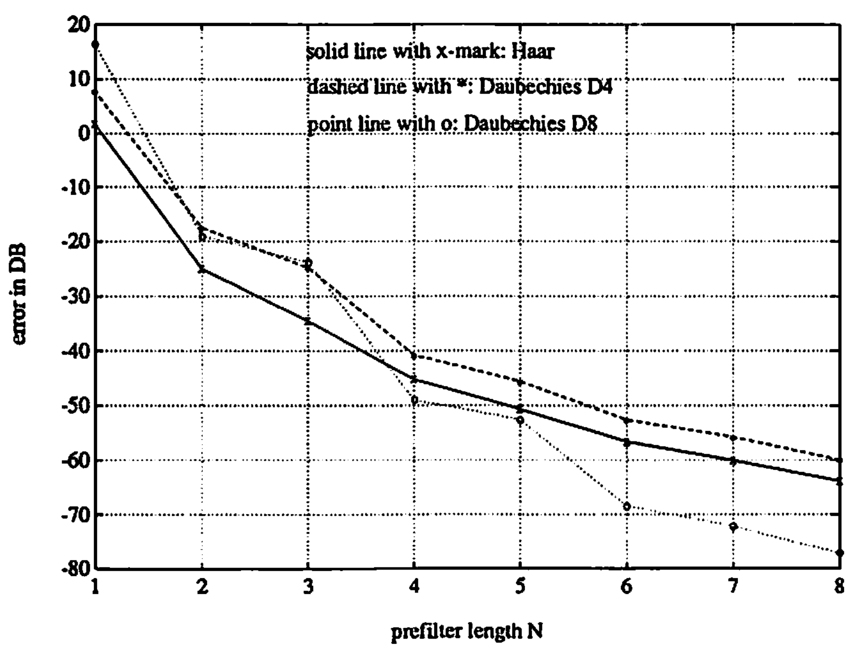
length N	$q[n]$, $0 \leq n \leq N-1$				
1	1.0000				
2	0.6292	0.5527			
3	0.6179	0.5884	-0.1805		
4	0.6154	0.5942	-0.1975	0.0854	
5	0.6145	0.5961	-0.2015	0.0968	-0.0573
6	0.6140	0.5969	-0.2029	0.0998	-0.0659
	0.0431				
7	0.6138	0.5973	-0.2036	0.1010	-0.0683
	0.0500	-0.0345			
8	0.6136	0.5975	-0.2039	0.1015	-0.0693
	0.0520	-0.0402	0.0287		
9	0.6135	0.5976	-0.2042	0.1018	-0.0698
	0.0528	-0.0419	0.0336	-0.0246	
10	0.6135	0.5977	-0.2043	0.1020	-0.0700
	0.0532	-0.0427	0.0351	-0.0288	0.0214

(b)

Table 1: Coefficients of optimal prefilters for D_4 basis with (a) $a = 0$ and (b) $a = 0.1$.

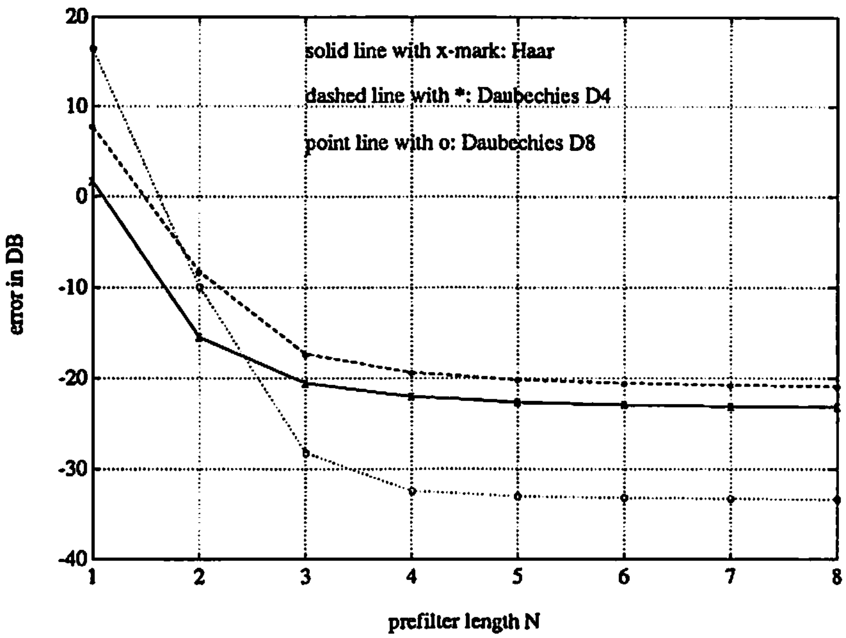


(a)

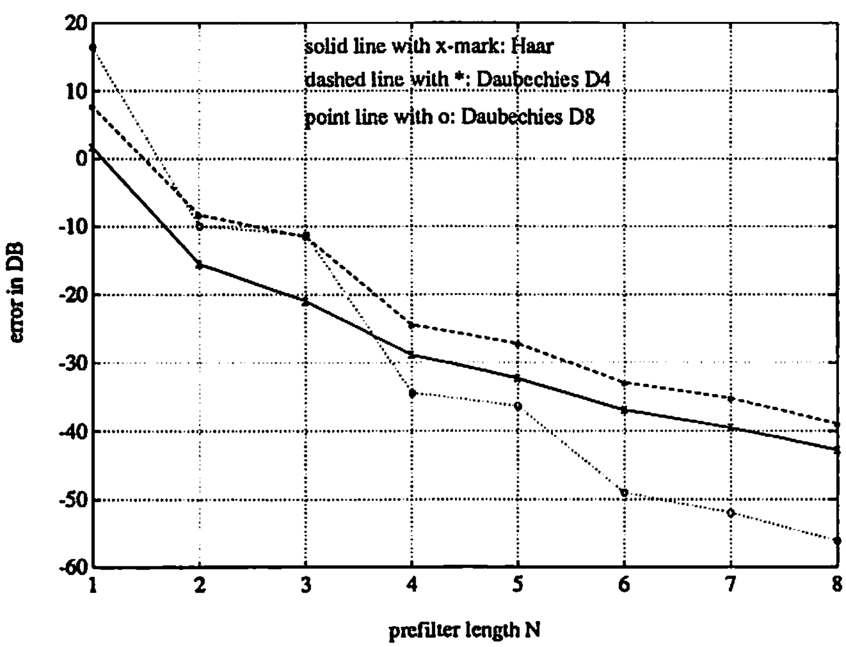


(b)

Figure 1: Errors for (a) causal and (b) noncausal signal dependent prefilters.



(a)



(b)

Figure 2: Errors for (a) causal and (b) noncausal signal independent prefilters.

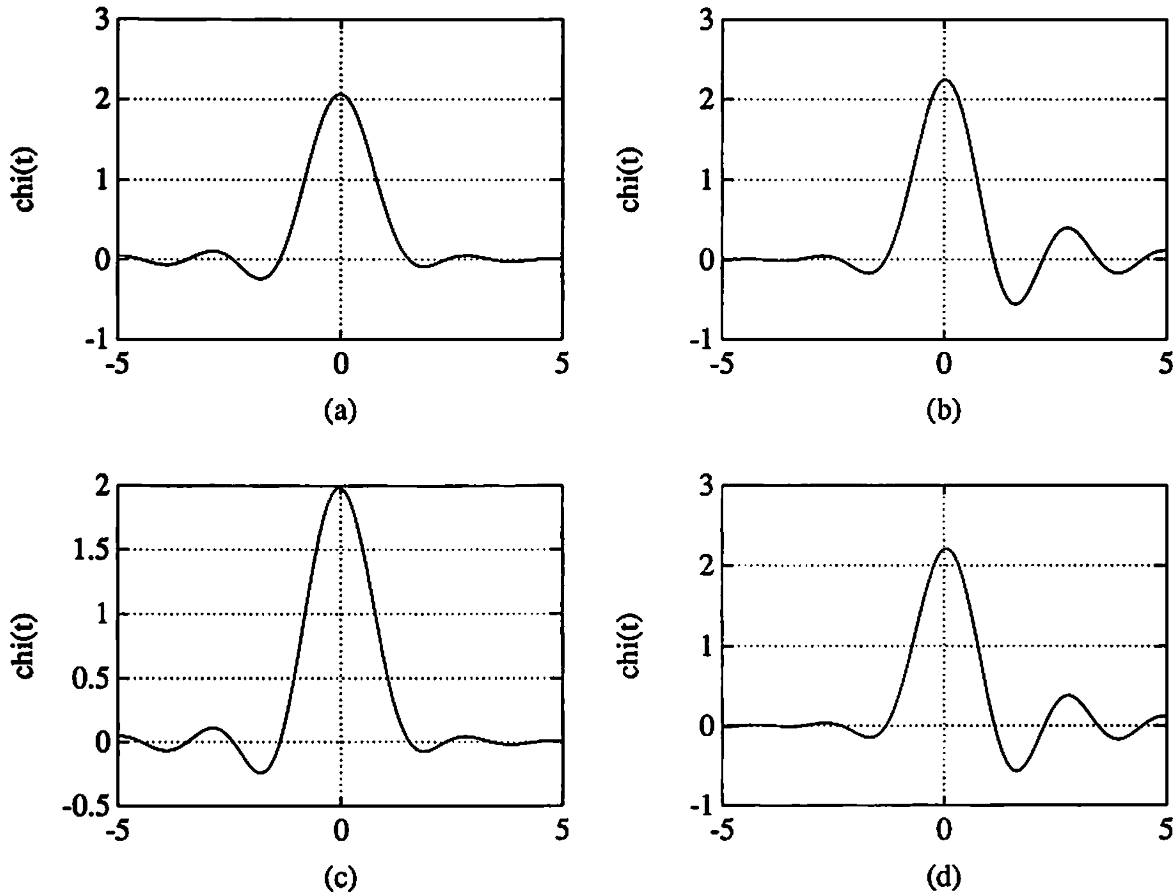


Figure 3: The interpolants $\chi(t)$ correspond to the optimal prefilters $q[n]$ given in Table 1 with (a) $N = 2$ and $a = 0$, (b) $N = 4$ and $a = 0$, (c) $N = 2$ and $a = 0.1$, (d) $N = 4$ and $a = 0.1$.