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SIGNAL EXTRAPOLATION IN WAVELET SUBSPACES*

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Abstract. The Papoulis-Gerchberg (PG) algorithm is well known for band-limited signal extrapolation. We consider the generalization of the PG algorithm to signals in the wavelet subspaces in this research. The uniqueness of the extrapolation for continuous-time signals is examined, and sufficient conditions on signals and wavelet bases for the generalized PG (GPG) algorithm to converge are given. We also propose a discrete GPG algorithm for discrete-time signal extrapolation, and investigate its convergence. Numerical examples are given to illustrate the performance of the discrete GPG algorithm.

Key words. extrapolation, Papoulis-Gerchberg algorithm, wavelets.

AMS(MOS) subject classifications. 41, 65D, 65F, 94.

1. Introduction. Band-limited signal interpolation (or sampling) and extrapolation have many applications in both mathematics and engineering, including radio astronomy, radar target detection, geophysical exploration, medical image processing and communication theory. The Shannon sampling theorem provides a signal interpolation formula from discrete samples of a band-limited signal if the sampling rate is above the Nyquist rate. In the 70's, Papoulis [12] and Gerchberg [8] developed an algorithm for extrapolating a band-limited signal outside a known interval. There have been many extensions and modifications of these two fundamental signal interpolation and extrapolation schemes [2], [3], [10], [13], [16], [17], [18], [19], [20], [22], [24], [25], [26], [27], [28]. However, all of them were derived from the Fourier transform viewpoint. Wavelet theory has been extensively studied for last several years [4], [5], [7]. It provides various attractive multiresolution bases for signal representation with a good time-frequency localization property. In particular, if the scaling function is chosen to be the sinc function, the corresponding wavelet subspaces are those formed by band-limited signals. By extending the Shannon sampling theorem for band-limited signals, Walter [21] derived a general sampling theorem applicable to signals in wavelet subspaces. In this research, we are interested in generalizing the Papoulis-Gerchberg (PG) algorithm from band-limited signals to signals in the wavelet subspaces.

Let us use the following two simple examples to illustrate the nature of the extrapolation problem. Consider first the Haar wavelet, where the scaling function $\phi_H(t) = \chi_{[0,1)}(t)$, which is 1 when $0 \leq t < 1$ and 0 otherwise. The wavelet subspaces

$$V_j = \left\{ f(t) : f(t) \text{ is constant in each interval } \left[\frac{k}{2^j}, \frac{(k+1)}{2^j} \right), k \in \mathbf{Z} \right\}, \text{ where } j \in \mathbf{Z},$$

consist of piecewise constant functions on intervals of length 2^{-j} . For a signal $f(t) \in V_j$, even though we know the values of $f(t)$ in interval $[k_0/2^j, k_1/2^j)$, where $k_0, k_1 \in \mathbf{Z}$ and $k_0 < k_1$, there is no unique way to extend $f(t)$ outside the interval. Second, consider the sinc wavelet as mentioned earlier where $\phi_s = \frac{\sin \pi t}{\pi t}$ and the wavelet subspaces V_j are

$$V_j = \{ f(t) : f(t) \text{ is } 2^j \pi \text{ band-limited} \}.$$

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For this case, any $f(t)$ in V_j is an entire function [1] so that it can be uniquely determined by its arbitrary piece. One such extrapolation procedure is in fact provided by the Papoulis-Gerchberg algorithm. An important focus of this research is to study the convergence of the generalized Papoulis-Gerchberg (GPG) algorithm and the uniqueness of the extrapolated signal. Several sufficient conditions on signals and wavelet bases for convergence and uniqueness will be given in §3 and 4. To implement extrapolation numerically, we propose a discrete GPG algorithm which extrapolates a scale-time limited sequence.

This paper is organized as follows. In §2, we briefly review the PG algorithm for band-limited signals and basic results of wavelet theory. We consider the extrapolation for continuous-time and discrete-time signals in §3 and 4, respectively, and establish a connection between the continuous and discrete cases. Some numerical examples are given in §5.

2. Preliminaries. We briefly review the PG algorithm for band-limited signal extrapolation and orthogonal wavelets in this section. The following notations will be used throughout this paper. The $L^2(\mathbf{R})$ denotes all real square integrable functions (or signals) defined on \mathbf{R} . For $D > 0$, the $L^2[-D, D]$ denotes all signals $f(t)$ defined on $[-D, D]$ satisfying

$$\int_{-D}^D |f(t)|^2 dt < \infty.$$

Let $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ denote the inner product and the norm on $L^2(\mathbf{R})$, i.e.

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(t)g(t)dt, \quad \text{where } f(t), g(t) \in L^2(\mathbf{R}),$$

and $\|f\|^2 = \langle f, f \rangle$. Similarly, we use $\langle \cdot, \cdot \rangle_D$ and $\|\cdot\|_D$ denote the inner product and the norm on $L^2[-D, D]$. For $f(t) \in L^2(\mathbf{R})$, we define

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-it\omega} dt,$$

to be the Fourier transform of $f(t)$.

2.1. The Papoulis-Gerchberg Algorithm. A signal $f(t)$ is said to be Ω band-limited if its Fourier spectrum $\hat{f}(\omega) = 0$ for $|\omega| > \Omega$. Let $f(t)$ be Ω band-limited and $f(t)$ be given for $|t| < T$ with $T > 0$. The question is to recover $f(t)$ for $|t| \geq T$. Define two projection operators P_T and P_Ω as follows:

$$P_T f(t) = \begin{cases} f(t), & |t| < T, \\ 0, & |t| \geq T, \end{cases}$$

and

$$P_\Omega \hat{f}(\omega) = \begin{cases} \hat{f}(\omega), & |\omega| < \Omega, \\ 0, & |\omega| \geq \Omega, \end{cases}$$

where P_T and P_Ω act on signals in the time and frequency domains, respectively. Let \mathcal{F} and \mathcal{F}^{-1} be the Fourier transform operator and its inverse, respectively, and I be the identity operator. Then, the PG algorithm is defined by the following iterative procedure:

Papoulis-Gerchberg (PG) Algorithm:

$$(2.1) \quad f^{(0)}(t) = P_T f(t).$$

For $l = 0, 1, 2, \dots$,

$$(2.2) \quad f^{(l+1)}(t) = P_T f(t) + (I - P_T)\mathcal{F}^{-1}P_\Omega\mathcal{F}f^{(l)}(t).$$

In [12], it was shown that $\|f^{(l)} - f\|$ converges to 0 as l goes to ∞ . For the generalization and discretization of the PG algorithm for band-limited signals, we refer to [3], [9], [16], [17], [18], [19], [20], [22], [24], [26], [27], [28].

2.2. Orthogonal Wavelets. We focus on real orthogonal wavelets in this paper, and refer to [4], [5], [7] for more detailed discussion. Let $\phi(t)$ be a scaling function such that, for a fixed arbitrary integer j ,

$$\{\phi_{j,k}(t)\}_{k \in \mathbf{Z}}, \quad \text{where} \quad \phi_{j,k}(t) = 2^{j/2} \phi(2^j t - k)$$

is an orthonormal basis of the wavelet subspace V_j , and $\{V_j\}_{j \in \mathbf{Z}}$ is a multiresolution approximation of $L^2(\mathbf{R})$, i.e. $V_j \subset V_{j+1}$ and $\overline{\bigcup_j V_j} = L^2(\mathbf{R})$. The wavelet function corresponding to $\phi(t)$ is denoted by $\psi(t)$ and $\psi_{j,k}(t) = 2^{j/2} \psi(2^j t - k)$. The associated quadrature mirror filters can be expressed as

$$(2.3) \quad H(\omega) = \sum_k h_k e^{-ik\omega}, \quad \text{and} \quad G(\omega) = \sum_k g_k e^{-ik\omega},$$

where $g_k = (-1)^k h_{1-k}$ and

$$\hat{\phi}(2\omega) = H(\omega)\hat{\phi}(\omega), \quad \text{and} \quad \hat{\psi}(2\omega) = G(\omega)\hat{\phi}(\omega).$$

Then, we have

$$(2.4) \quad f(t) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} b_{j,k} \psi_{j,k}(t),$$

for any $f(t) \in L^2(\mathbf{R})$ and

$$(2.5) \quad f(t) = \sum_{k=-\infty}^{\infty} c_{J,k} \phi_{J,k}(t) = \sum_{j < J} \sum_{k=-\infty}^{\infty} b_{j,k} \psi_{j,k}(t),$$

for any $f(t) \in V_J$, where $b_{j,k} = \langle f, \psi_{j,k} \rangle$ and $c_{J,k} = \langle f, \phi_{J,k} \rangle$. The $b_{j,k}$ in (2.4) are called the wavelet series transform (WST) coefficients of $f(t)$, and (2.4) provides the inverse wavelet series transform (IWST) of $b_{j,k}$. On one hand, the WST coefficients $b_{j,k}$ with $j < J$ can be obtained from coefficients $c_{J,k}$ by the recursive formulas:

$$(2.6) \quad \begin{aligned} c_{j-1,k} &= \sqrt{2} \sum_n h_{n-2k} c_{j,n}, \\ b_{j-1,k} &= \sqrt{2} \sum_n g_{n-2k} c_{j,n}, \end{aligned}$$

for $j = J, J-1, J-2, \dots$. On the other hand, we have the following synthesis formula to compute coefficients $c_{J,k}$ from $c_{J_0,k}$ and $b_{j,k}$ with $J_0 \leq j < J$ via

$$(2.7) \quad c_{j+1,n} = \sqrt{2} \left(\sum_k h_{n-2k} c_{j,k} + \sum_k g_{n-2k} b_{j,k} \right),$$

for $j = J_0, J_0+1, \dots, J-1$. By viewing $c_{J,n}$ as a sequence $x[n]$, we call (2.6) the discrete wavelet transform (DWT) of the sequence $x[n]$ and (2.7) the inverse discrete wavelet transform (IDWT) of coefficients $c_{J_0,k}$ and $b_{j,k}$. By using the orthonormality of the wavelet basis, one can prove that both DWT and IDWT preserve energy.

3. Extrapolation of continuous-time signals. We examine the extrapolation for continuous-time signals in wavelet subspaces in this section.

3.1. Generalized PG algorithm. Let $f(t) \in V_J$ for a fixed integer J . Given the value of $f(t)$ for $|t| < T$ ($T > 0$), we are concerned with the determination of the value $f(t)$ for $|t| \geq T$. We propose the following generalized PG (GPG) algorithm for extrapolation:

Generalized PG Algorithm:

$$(3.1) \quad f^{(0)}(t) = P_T f(t).$$

For $l = 0, 1, 2, \dots$,

$$(3.2) \quad g^{(l)}(t) = \sum_k \langle f^{(l)}, \phi_{Jk} \rangle \phi_{Jk}(t),$$

$$(3.3) \quad f^{(l+1)}(t) = P_T f(t) + (I - P_T)g^{(l)}(t),$$

Note that the above iterative procedure reduces to the standard PG algorithm if $\phi(t) = \frac{\sin \pi t}{\pi t}$ as shown below. For this case, $f(t) \in V_J$ implies that $f(t)$ is $2^J \pi$ band-limited.

$$\begin{aligned} g^{(l)}(t) &= \sum_k 2^J \frac{\sin \pi(2^J t - k)}{\pi(2^J t - k)} \int_{-\infty}^{\infty} f^{(l)}(s) \frac{\sin \pi(2^J s - k)}{\pi(2^J s - k)} ds \\ &= 2^J \int_{-\infty}^{\infty} f^{(l)}(s) \sum_k \frac{\sin \pi(2^J s - k)}{\pi(2^J s - k)} \frac{\sin \pi(2^J t - k)}{\pi(2^J t - k)} ds \\ &\stackrel{(1)}{=} \int_{-\infty}^{\infty} f^{(l)}(s) \frac{\sin \pi 2^J (s - t)}{\pi(s - t)} ds, \end{aligned}$$

where the identity in (1)

$$(3.4) \quad \sum_k \frac{\sin \pi(2^J s - k)}{\pi(2^J s - k)} \frac{\sin \pi(2^J t - k)}{\pi(2^J t - k)} = \frac{\sin \pi 2^J (s - t)}{\pi 2^J (s - t)}$$

is obtained by the Shannon sampling theorem. Therefore, we have

$$\hat{g}^{(l)}(\omega) = P_{2^J \pi} \hat{f}^{(l)}(\omega),$$

or, equivalently,

$$g^{(l)}(t) = \mathcal{F}^{-1} P_{2^J \pi} \mathcal{F} f^{(l)}(t).$$

By using (3.3), we obtain

$$f^{(l+1)}(t) = P_T f(t) + (I - P_T) \mathcal{F}^{-1} P_{2^J \pi} \mathcal{F} f^{(l)}(t),$$

which is exactly the same as (2.2) with $\Omega = 2^J \pi$.

The GPG algorithm has a property similar to that of the PG algorithm. That is, it reduces the error energy during the iteration process. To see this, we know from (3.3) that

$$f^{(l+1)}(t) - f(t) = (I - P_T)(g^{(l)} - f)(t),$$

for $f(t) \in V_J$. By using (3.2) and (2.5), we have

$$\begin{aligned} \|f^{(l+1)} - f\|^2 &\leq \|g^{(l)} - f\|^2 = \sum_k |\langle f^{(l)} - f, \phi_{Jk} \rangle|^2 \\ &\leq \|f^{(l)} - f\|^2. \end{aligned}$$

This proves that, for a scaling function $\phi(t)$ and $f(t) \in V_J$,

$$(3.5) \quad \|f^{(l+1)} - f\|^2 \leq \|f^{(l)} - f\|^2, \quad l = 0, 1, 2, \dots$$

3.2. Convergence and uniqueness results. We say that a signal $f(t)$ can be *uniquely* determined in a signal set S from its segment $f(t)$ defined on interval $[A, B]$, if any $f(t), g(t) \in S$ with $f(t) = g(t)$ for $t \in [A, B]$, implies $f(t) = g(t)$ for $t \in \mathbb{R}$. In this subsection, we will perform some theoretical study on the convergence of the GPG algorithm, the uniqueness of extrapolated signals and their relationship.

We first focus on the convergence issue. To do so, results from operator theory [14] are needed. Let us define

$$(3.6) \quad Q(s, t) \triangleq \sum_{k=-\infty}^{\infty} \phi(s-k)\phi(t-k), \quad (s, t) \in \mathbf{R}^2,$$

and

$$(3.7) \quad Q_J(s, t) \triangleq 2^J Q(2^J s, 2^J t) = \sum_{k=-\infty}^{\infty} \phi_{Jk}(s)\phi_{Jk}(t).$$

The definitions (3.6) and (3.7) also appeared in [21], where $Q_J(s, t)$ was called the reproducing kernel (RK) for the reproducing kernel Hilbert Space (RKHS) V_J . In particular, when $\phi(t)$ is equal to the sinc function, we know from (3.4) that

$$Q_J(s, t) = \frac{\sin 2^J \pi(s-t)}{\pi(s-t)},$$

which is analytic in \mathbf{R}^2 . When the decay of $\phi(t)$ satisfies $|\phi(t)| \leq O(1 + |t|^{1+\epsilon})^{-1}$ for some $\epsilon > 0$, $Q(s, t)$ in (3.6) is always finite for all real s, t . In what follows, we assume that $Q_J(s, t)$ is continuous in $[-T, T]^2$ and finite in \mathbf{R}^2 .

Some basic results are summarized below (see [14]). The following operator defined from $L^2[-T, T]$ to itself,

$$\mathcal{Q}_J g(t) = \int_{-T}^T g(s)Q_J(s, t)ds, \quad t \in [-T, T],$$

is completely continuous and symmetric. We use λ_k and $\Phi_k(t)$ to denote the eigenvalues and normalized eigenfunctions of operator \mathcal{Q}_J , and arrange $|\lambda_k|$ in a descending order, i.e.

$$(3.8) \quad \mathcal{Q}_J \Phi_k(t) = \lambda_k \Phi_k(t), \quad t \in [-T, T],$$

for $k = 0, 1, 2, \dots$, where $\int_{-T}^T |\Phi_k(t)|^2 dt = 1$ and $\infty > |\lambda_0| \geq |\lambda_1| \geq \dots$. Then, $\{\Phi_k(t)\}_k$ is an orthogonal basis of the range space $\mathcal{Q}_J(L^2[-T, T])$ of operator \mathcal{Q}_J so that we can write

$$\mathcal{Q}_J g(t) = \sum_{k=0}^{\infty} \lambda_k \langle g, \Phi_k \rangle_T \Phi_k(t), \quad t \in [-T, T], \quad \forall g(t) \in L^2[-T, T],$$

Moreover, if all eigenvalues λ_k are not zero, $\{\Phi_k(t)\}_k$ is an orthonormal basis of $L^2[-T, T]$ so that we have

$$g(t) = \sum_{k=0}^{\infty} \langle g, \Phi_k \rangle_T \Phi_k(t), \quad t \in [-T, T], \quad \forall g(t) \in L^2[-T, T].$$

Now, let

$$\mathbf{K} = \{k : \lambda_k \neq 0, \text{ and } k \in \{0, 1, 2, \dots\}\}.$$

For $k \in \mathbf{K}$, we can extend $\Phi_k(t)$ from $[-T, T]$ to \mathbf{R} via (3.8) as

$$(3.9) \quad \Phi_k(t) = \frac{1}{\lambda_k} \int_{-T}^T \Phi_k(s)Q_J(s, t)ds$$

$$(3.10) \quad = \frac{1}{\lambda_k} \sum_n \left(\int_{-T}^T \Phi_k(s)\phi_{Jn}(s)ds \right) \phi_{Jn}(t),$$

where $t \in \mathbb{R}$.

In the following, the symbol $\Phi_k(t)$ with $k \in \mathbf{K}$ is always used to mean the extended one as given in (3.9). Properties of the extended $\Phi_k(t)$, $k \in \mathbf{K}$, are described in the following lemma.

LEMMA 1. *The extended $\Phi_k(t)$ with $k \in \mathbf{K}$ as defined in (3.9) are in V_J and orthogonal in $L^2(\mathbb{R})$, and the associated λ_k are greater than zero.*

Proof. Due to (3.10), $\lambda_k \neq 0$ and the fact that

$$\sum_n \left| \int_{-T}^T \Phi_k(s) \phi_{Jn}(s) ds \right|^2 \leq \|\Phi_k\|_T^2 = 1 < \infty,$$

we conclude that $\Phi_k(t) \in V_J$. To check the orthogonality of $\Phi_k(t)$ in $L^2(\mathbb{R})$, we have

$$\begin{aligned} \int_{-\infty}^{\infty} \Phi_{k_1}(t) \Phi_{k_2}(t) dt &= \int_{-\infty}^{\infty} \frac{1}{\lambda_{k_1} \lambda_{k_2}} \int_{-T}^T \Phi_{k_1}(s_1) Q_J(s_1, t) ds_1 \int_{-T}^T \Phi_{k_2}(s_2) Q_J(s_2, t) ds_2 \\ &= \frac{1}{\lambda_{k_1} \lambda_{k_2}} \int_{-T}^T \int_{-T}^T \Phi_{k_1}(s_1) \Phi_{k_2}(s_2) \int_{-\infty}^{\infty} Q_J(s_1, t) Q_J(s_2, t) dt ds_1 ds_2, \end{aligned}$$

for $k_1, k_2 \in \mathbf{K}$. By using

$$\begin{aligned} \int_{-\infty}^{\infty} Q_J(s_1, t) Q_J(s_2, t) dt &= \int_{-\infty}^{\infty} \sum_{k_1} \phi_{Jk_1}(s_1) \phi_{Jk_1}(t) \sum_{k_2} \phi_{Jk_2}(s_2) \phi_{Jk_2}(t) dt \\ (3.11) \qquad \qquad \qquad &= \sum_k \phi_{Jk}(s_1) \phi_{Jk}(s_2) = Q_J(s_1, s_2), \end{aligned}$$

we can further simplify the above expression to be

$$\begin{aligned} \int_{-\infty}^{\infty} \Phi_{k_1}(t) \Phi_{k_2}(t) dt &= \frac{1}{\lambda_{k_1} \lambda_{k_2}} \int_{-T}^T \int_{-T}^T \Phi_{k_1}(s_1) \Phi_{k_2}(s_2) Q_J(s_1, s_2) ds_1 ds_2 \\ &\stackrel{(2)}{=} \frac{1}{\lambda_{k_1}} \int_{-T}^T \Phi_{k_1}(s_1) \Phi_{k_2}(s_1) ds_1 \\ (3.12) \qquad \qquad \qquad &= \begin{cases} \frac{1}{\lambda_{k_1}}, & k_1 = k_2 \\ 0, & k_1 \neq k_2, \end{cases} \end{aligned}$$

where the equality labeled with (2) is due to the fact that $\{\Phi_k(t)\}_k$ is an orthonormal basis of $L^2[-T, T]$. It is also obvious that $\lambda_k > 0$. \square

Consider the space

$$(3.13) \quad U_J = \left\{ g(t) : g(t) = \sum_{k \in \mathbf{K}} a_k \Phi_k(t) \in L^2(\mathbb{R}) \text{ for some constants } a_k, k \in \mathbf{K} \right\},$$

generated by $\Phi_k(t)$ with $k \in \mathbf{K}$. For any $g(t) = \sum_{k \in \mathbf{K}} a_k \Phi_k(t) \in U_J$, we know from (3.12) that

$$\|g\|^2 = \sum_{k \in \mathbf{K}} \frac{|a_k|^2}{\lambda_k} < \infty,$$

so that U_J is a subspace of $L^2(\mathbb{R})$. Furthermore, since $\Phi_k(t) \in V_J$ for $k \in \mathbf{K}$, $U_J \subset V_J$. Now, we are ready to state our first main result on the convergence of the GPG algorithm.

THEOREM 1. *Let $Q_J(s, t)$ be defined by (3.7) and continuous in $[-T, T]^2$ and U_J be defined by (3.13). If $f(t) \in U_J$, then $\|f^{(l)} - f\| \rightarrow 0$ as $l \rightarrow \infty$, where $f^{(l)}(t)$ is obtained from the GPG algorithm described by (3.1)-(3.3).*

Proof. We have the following representation for any $f(t) \in U_J$,

$$f(t) = \sum_{k \in \mathbf{K}} a_k \Phi_k(t), \quad \text{where} \quad \sum_{k \in \mathbf{K}} \frac{|a_k|^2}{\lambda_k} < \infty.$$

Given $k \in \mathbf{K}$ and initialization $f_k(t) = a_k \Phi_k(t)$, $f_k^{(0)}(t) = P_T f_k(t)$, consider the iteration:

$$\begin{cases} g_k^{(l)}(t) &= \sum_n \langle f_k^{(l)}, \phi_{J_n} \rangle \phi_{J_n}(t), \\ f_k^{(l+1)}(t) &= P_T f_k(t) + (I - P_T) g_k^{(l)}(t). \end{cases}$$

We want to prove

$$(3.14) \quad f_k(t) - f_k^{(l)}(t) = (I - P_T)(1 - \lambda_k)^l a_k \Phi_k(t), \quad \text{for } l = 0, 1, 2, \dots$$

The equality (3.14) is trivial for $l = 0$. By assuming (3.14) is true for l , i.e.,

$$\begin{aligned} f_k^{(l)}(t) &= f_k(t) - (I - P_T)(1 - \lambda_k)^l a_k \Phi_k(t) \\ &= a_k \Phi_k(t) - (I - P_T)(1 - \lambda_k)^l a_k \Phi_k(t) \\ &= (1 - (1 - \lambda_k)^l) a_k \Phi_k(t) + (1 - \lambda_k)^l a_k P_T \Phi_k(t), \end{aligned}$$

it can be shown that (3.14) holds for $l + 1$. To see this, let us examine

$$\begin{aligned} g_k^{(l)}(t) &= \sum_n \langle f_k^{(l)}, \phi_{J_n} \rangle \phi_{J_n}(t) = \int_{-\infty}^{\infty} f_k^{(l)}(s) Q_J(s, t) ds \\ &= a_k (1 - (1 - \lambda_k)^l) \int_{-\infty}^{\infty} \Phi_k(s) Q_J(s, t) ds + a_k (1 - \lambda_k)^l \int_{-T}^T \Phi_k(s) Q_J(s, t) ds \\ &= a_k (1 - (1 - \lambda_k)^l) \Phi_k(t) + a_k \lambda_k (1 - \lambda_k)^l \Phi_k(t) \\ &= a_k (1 - (1 - \lambda_k)^{l+1}) \Phi_k(t). \end{aligned}$$

Thus, we have

$$\begin{aligned} f_k(t) - f_k^{(l+1)}(t) &= (I - P_T)[a_k \Phi_k(t) - a_k (1 - (1 - \lambda_k)^{l+1}) \Phi_k(t)] \\ &= (I - P_T) a_k (1 - \lambda_k)^{l+1} \Phi_k(t), \end{aligned}$$

and (3.14) is proved by induction. A direct consequence of (3.14) is

$$(3.15) \quad f(t) - f^{(l)}(t) = \sum_{k \in \mathbf{K}} [f_k(t) - f_k^{(l+1)}(t)] = (I - P_T) \sum_{k \in \mathbf{K}} (1 - \lambda_k)^l a_k \Phi_k(t).$$

In order to show $\|f^{(l)} - f\| \rightarrow 0$ as $l \rightarrow \infty$, we only have to prove $0 < \lambda_k \leq 1$ for $k \in \mathbf{K}$. The fact $\lambda_k > 0$ has been proved in Lemma 1. By using (3.12), we have

$$\frac{1}{\lambda_k} = \int_{-\infty}^{\infty} |\Phi_k(t)|^2 dt \geq \int_{-T}^T |\Phi_k(t)|^2 dt = 1,$$

so that $\lambda_k \leq 1$, which completes the proof of Theorem 1. \square

Based on Theorem 1, we have the following straightforward corollary on the uniqueness of extrapolation.

COROLLARY 1. *Let $Q_J(s, t)$ be defined by (3.7) and continuous in $[-T, T]^2$ and U_J be defined by (3.13). If $f(t) \in U_J$, then $f(t)$ is uniquely determined in U_J from the values of $f(t)$ with $t \in [-T, T]$.*

Although it may not be easy to check the condition $f(t) \in U_J$ practically, Theorem 1 does tell us that there exists a subspace U_J in V_J where the GPG algorithm converges. The observation is not only of theoretical interest but also provides an important step to the derivation of further results.

For a kernel $K(s, t)$ satisfying $K(s, t) = K(t, s)$ and

$$(3.16) \quad \sum_{i=1}^N \sum_{j=1}^N a_i \bar{a}_j K(t_i, t_j) \geq 0,$$

where the bar denotes the complex conjugate, for any integer $N > 0$, any N points $t_i \in [-T, T]$ and any N numbers a_i , we say $K(s, t)$ to be *symmetric nonnegative definite* in $[-T, T]^2$. If the inequality in (3.16) is strictly great than 0, we say that $K(s, t)$ is *symmetric positive definite* in $[-T, T]^2$. With a symmetric kernel $K(s, t)$ in $[-T, T]^2$, we can define an operator from $L^2[-T, T]$ to $L^2[-T, T]$ like \mathcal{Q}_J ,

$$\mathcal{K}g(t) = \int_{-T}^T K(s, t)g(s)ds, \quad t \in [-T, T]$$

Then, $K(s, t)$ is positive (or nonnegative) definite if and only if all eigenvalues of the operator \mathcal{K} are positive (or nonnegative). As an example, the kernel $Q_J(s, t)$ defined in (3.7) is symmetric nonnegative definite in $[-T, T]^2$ for any $\phi(t)$.

The following lemma is needed for the proof of Theorem 2.

LEMMA 2. For any $f(t) \in V_J$ and $k \in \mathbf{K}$,

$$(3.17) \quad \int_{-T}^T \Phi_k(s)f(s)ds = \lambda_k \int_{-\infty}^{\infty} \Phi_k(s)f(s)ds.$$

Proof. We first prove that (3.17) is true when $f(t) = \phi_{Jn}(t)$ for an arbitrary n .

$$\begin{aligned} \int_{-\infty}^{\infty} \Phi_k(s)\phi_{Jn}(s)ds &= \int_{-\infty}^{\infty} \frac{1}{\lambda_k} \int_{-T}^T \Phi_k(s')Q_J(s', s)ds'\phi_{Jn}(s)ds \\ &= \frac{1}{\lambda_k} \int_{-T}^T \Phi_k(s') \int_{-\infty}^{\infty} Q_J(s', s)\phi_{Jn}(s)ds ds' \\ &= \frac{1}{\lambda_k} \int_{-T}^T \Phi_k(s')\phi_{Jn}(s')ds'. \end{aligned}$$

For general $f(t) \in V_J$, since $f(t) = \sum_n a_n \phi_{Jn}(t)$ and $\sum_n |a_n|^2 < \infty$, (3.17) also holds. \square

The second convergence result is stated as follows.

THEOREM 2. If $Q_J(s, t)$ is continuous and positive definite in $[-T, T]^2$ and $f(t)$ is uniquely determined in V_J by $f(t)$, $t \in [-T, T]$, then $\|f^{(l)} - f\| \rightarrow 0$ when $l \rightarrow \infty$, where $f^{(l)}(t)$ is obtained from the GPG algorithm (3.1) -(3.3).

Proof. When $Q_J(s, t)$ is positive definite in $[-T, T]^2$, we have $\lambda_k > 0$ for $k = 0, 1, 2, \dots$, i.e. $\mathbf{K} = \{0, 1, 2, 3, \dots\}$. Therefore, by using the results summarized in the beginning of §3.2, $\{\Phi_k(t)\}_k$ is an orthonormal basis of $L^2[-T, T]$. For $f(t) \in V_J$, when $t \in [-T, T]$,

$$f(t) = \sum_{k=0}^{\infty} \langle f, \Phi_k \rangle_T \Phi_k(t).$$

By Lemma 2, when $t \in [-T, T]$,

$$f(t) = \sum_{k=0}^{\infty} \lambda_k \langle f, \Phi_k \rangle \Phi_k(t).$$

For $t \in \mathbf{R}$, let

$$\tilde{f}(t) = \sum_{k=0}^{\infty} \lambda_k \langle f, \Phi_k \rangle \Phi_k(t).$$

Then, $f(t) = \tilde{f}(t)$ for $t \in [-T, T]$ and

$$\tilde{f}(t) = \sum_{k=0}^{\infty} \langle f, \sqrt{\lambda_k} \Phi_k \rangle (\sqrt{\lambda_k} \Phi_k(t)).$$

By (3.12), $\{\sqrt{\lambda_k}\Phi_k(t)\}_k$ is orthonormal in $L^2(\mathbf{R})$. Therefore, $\|\tilde{f}\|^2 \leq \|f\|^2 < \infty$. This proves $\tilde{f}(t) \in U_J$. Since $U_J \subset V_J$, $\tilde{f}(t)$ is also in V_J . By assumption, we know that $f(t)$ is uniquely determined in V_J by $f(t)$, $t \in [-T, T]$ so that $f(t) = \tilde{f}(t)$ for all real t . Thus, $f(t) \in U_J$ and, by Theorem 1, Theorem 2 is proved. \square

Theorem 2 tells us that if $Q_J(s, t)$ is continuous and positive definite in $[-T, T]^2$, the uniqueness of extrapolation in V_J implies the convergence of the GPG algorithm. Instead of checking the uniqueness of extrapolation for various functions $f(t)$ of interest in V_J individually, the following theorem says that it is sufficient to check that for the scaling function $\phi(t)$ only.

THEOREM 3. *If $Q_J(s, t)$ is continuous and positive definite in $[-T, T]^2$ and the scaling function $\phi(t)$ is uniquely determined in V_J by any one of its segments $\phi(t)$, $t \in [-2^J T - k, 2^J T - k]$, $k \in \mathbf{Z}$, then $\|f^{(l)} - f\| \rightarrow 0$ as $l \rightarrow \infty$ where $f^{(l)}(t)$ is obtained from the GPG algorithm (3.1)-(3.3).*

Proof. Since $\phi(t)$ is uniquely determined in V_J by any one of the segments $\phi(t)$, $t \in [-2^J T - k, 2^J T - k]$, $k \in \mathbf{Z}$, $\phi_{Jk}(t)$ is also uniquely determined by $\phi_{Jk}(t)$, $t \in [-T, T]$, for any $k \in \mathbf{Z}$. Similar to the proof of Theorem 2, $\phi_{Jk}(t) \in U_J$ for all $k \in \mathbf{Z}$. This implies that $V_J \subset U_J$. Thus, $V_J = U_J$ by $U_J \subset V_J$. By Theorem 1, Theorem 3 is proved. \square

As a direct consequence of Theorem 3, we have

COROLLARY 2. *Under the same conditions as stated in Theorem 3, if $f(t) \in V_J$, then $f(t)$ is uniquely determined in V_J by its segment $f(t)$, $t \in [-T, T]$.*

From the definition of positive definite, if $Q_J(s, t)$ is continuous and positive definite in $[-T, T]^2$ then it is also continuous and positive definite in any $[A, B]^2$ with $-T \leq A < B \leq T$. The observation implies the following corollary.

COROLLARY 3. *Let $Q_J(s, t)$ be continuous and positive definite in $[-T, T]^2$. If $\phi(t)$ is uniquely determined in V_J by its any segment and $f(t) \in V_J$, then $\|f^{(l)} - f\| \rightarrow 0$ as $l \rightarrow \infty$ where $f^{(l)}(t)$ is obtained from (3.1)-(3.3) with $[-T, T]$ replaced by an arbitrarily fixed $[A, B]$ with $-T \leq A < B \leq T$, and also $f(t)$ is uniquely determined in V_J by its segment $f(t)$, $t \in [A, B]$.*

The proof is completely similar to that of Theorem 3 except replacing $[-T, T]$ by $[A, B]$. The details are therefore omitted.

It is easy to see that the Haar wavelet does not satisfy Theorem 3. For the sinc wavelet case where $\phi(t) = \phi_s(t) \triangleq \frac{\sin \pi t}{\pi t}$, we have the kernel $Q_J(s, t) = \frac{\sin 2^J \pi(s-t)}{\pi(s-t)}$ by (3.4). The corresponding operator Q_J is

$$(3.18) \quad Q_J g(t) = \int_{-T}^T g(s) \frac{\sin 2^J \pi(s-t)}{\pi(s-t)} ds, \quad t \in [-T, T], \quad g(s) \in L^2[-T, T].$$

We claim that $Q_J(s, t)$ is positive definite in $[-T, T]^2$ for any $T > 0$. It has been shown earlier that $Q_J(s, t)$ is nonnegative definite so that we only have to prove that all λ_k are non-zero. Suppose $\lambda_{k_0} = 0$ for some $k_0 \in \{0, 1, 2, \dots\}$. Then

$$(3.19) \quad \int_{-T}^T \Phi_{k_0}(s) \frac{\sin 2^J \pi(s-t)}{\pi(s-t)} ds = 0, \quad t \in [-T, T].$$

For all $t \in \mathbf{R}$, let

$$\tilde{\Phi}_{k_0}(t) = \int_{-T}^T \Phi_{k_0}(s) \frac{\sin 2^J \pi(s-t)}{\pi(s-t)} ds.$$

Then, $\tilde{\Phi}_{k_0}(t) = \Phi_{k_0}(t) = 0$ for $t \in [-T, T]$ and

$$\hat{\Phi}_{k_0}(\omega) = \begin{cases} P_T \widehat{\Phi}_{k_0}(\omega), & \omega \in [-2^J \pi, 2^J \pi], \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, $\tilde{\Phi}_{k_0}(t)$ is $2^J \pi$ band-limited so that $\tilde{\Phi}_{k_0}(z)$ is entire when t is extended to the complex variable z and $\tilde{\Phi}_{k_0}(z) = 0$ for all z . This implies $\tilde{\Phi}_{k_0}(t) = 0$ for all $t \in \mathbf{R}$ and $\hat{\Phi}_{k_0}(\omega) = 0$ for all $\omega \in \mathbf{R}$. Consequently, $P_T \widehat{\Phi}_{k_0}(\omega) = 0$ for $\omega \in [-2^J \pi, 2^J \pi]$. Since $P_T \widehat{\Phi}_{k_0}(\omega)$ is time-limited, similar to the above

argument $P_T \widehat{\Phi}_{k_0}(\omega) = 0$ for all $\omega \in \mathbb{R}$. Thus, $P_T \Phi_{k_0}(t) = 0$ for all $t \in \mathbb{R}$ or, equivalently, $\Phi_{k_0}(t) = 0$ for $t \in [-T, T]$. This contradicts that $\Phi_{k_0}(t)$ is an eigenfunction associated with λ_{k_0} . Therefore, we conclude $\lambda_{k_0} \neq 0$ and the claim is proved. Furthermore, since $\phi_s(t)$ is an entire function, $\phi_s(t)$ is determined by its any segment. It is clear that the sinc wavelet satisfies the conditions of Theorem 3 for any $T > 0$, and the conclusion in Theorem 3 matches the well known result of band-limited signal extrapolation.

4. Extrapolation of discrete-time signals. In this section, we consider the discrete version of the GPG algorithm in §3 for handling discrete-time signals. This is what we need practically. Recall that the DWT of a sequence $c_{j,n} = x[n]$ can be implemented via (2.6) for a certain integer J . The discrete sequence $c_{j,n}$ is said to be (J, K) *scale-time limited* for certain integers J and $K > 0$ if its DWT coefficients (with lowest resolution J_0) satisfies that coefficients $c_{j_0,k}$ and $b_{j,k}$ may take nonzero values only when $|k| \leq K$ and $J_0 \leq j < J$. When J and K are sufficiently large, the (J, K) scale-time limited sequence provides a practical discrete-time signal model.

4.1. Discrete GPG algorithm. Let $x[n]$ be a (J, K) scale-time limited sequence. The values of $x[n]$, $n \in \mathcal{N}$, are given, where the cardinality $|\mathcal{N}| = N$ is finite. The extrapolation problem is to recover $x[n]$ for $n \notin \mathcal{N}$. Let $P_{\mathcal{N}}$ and $P_{J,K}$ be the following operators:

$$P_{\mathcal{N}}y[n] = \begin{cases} y[n], & n \in \mathcal{N}, \\ 0, & n \notin \mathcal{N}, \end{cases}$$

and

$$P_{J,K}d_{j,k} = \begin{cases} d_{j,k}, & |k| \leq K \text{ and } J_0 \leq j < J, \\ 0, & \text{otherwise} \end{cases}$$

Let I be the identity operator and \mathcal{D} and \mathcal{D}^{-1} be the DWT and IDWT operators. The discrete GPG algorithm can be stated as follows.

The Discrete GPG (DGPG) Algorithm:

$$(4.1) \quad x^{(0)}[n] = P_{\mathcal{N}}x[n],$$

For $l = 0, 1, 2, \dots$,

$$(4.2) \quad x^{(l+1)}[n] = P_{\mathcal{N}}x[n] + (I - P_{\mathcal{N}})\mathcal{D}^{-1}P_{J,K}\mathcal{D}x^{(l)}[n],$$

Similar to (3.5) for the continuous-time GPG algorithm, it can be shown that the error energy of the discrete GPG algorithm is monotonically decreasing, i.e.

$$\sum_n |x^{(l+1)}[n] - x[n]|^2 \leq \sum_n |x^{(l)}[n] - x[n]|^2, \quad l = 0, 1, 2, \dots$$

4.2. Convergence and uniqueness results. To show the convergence of the discrete GPG algorithm (4.1)-(4.2), some tools are needed. We introduce two operators H and G related to the quadrature mirror filters $H(\omega)$ and $G(\omega)$ in (2.3) as follows:

$$Hy[k] \triangleq \sqrt{2} \sum_n h_{n-2k}y[n], \quad \text{and} \quad Gy[k] \triangleq \sqrt{2} \sum_n g_{n-2k}y[n].$$

Let H^* and G^* be their duals, respectively, i.e.

$$H^*y[n] \triangleq \sqrt{2} \sum_k h_{n-2k}y[k], \quad \text{and} \quad G^*y[n] \triangleq \sqrt{2} \sum_k g_{n-2k}y[k].$$

Then, from (2.7), we have

$$\begin{aligned} x[n] &= (H^*c_{J-1,k} + G^*b_{J-1,k})[n] \\ &= (H^*(H^*c_{J-2,k} + G^*b_{J-2,k}) + G^*b_{J-1,k})[n] \\ &= ((H^*)^{J-J_0}c_{J_0,k} + (H^*)^{J-J_0-1}G^*b_{J_0,k} + \dots + H^*G^*b_{J-2,k} + G^*b_{J-1,k})[n]. \end{aligned}$$

We can rewrite the above equation as

$$(4.3) \quad \mathbf{x}[n] = \mathbf{w}_n \mathbf{p}, \quad n \in \mathbb{Z},$$

where \mathbf{p} and \mathbf{w}_n are, respectively, column and row vectors of length $(2K+1)(J-J_0+1)$ of the form

$$\begin{aligned} \mathbf{p} &= (c_{J_0}, b_{J_0}, b_{J_0+1}, \dots, b_{J-1})^T, \\ \mathbf{w}_n &= ((H^*)^n_{J-J_0}, ((H^*)^{J-J_0-1} G^*)_n, \dots, (H^* G^*)_n, G^*_n), \end{aligned}$$

and where

$$\begin{aligned} c_{J_0} &= (c_{J_0, -K}, c_{J_0, -K+1}, \dots, c_{J_0, K}), \\ b_j &= (b_{j, -K}, b_{j, -K+1}, \dots, b_{j, K}), \\ G^*_n &= \sqrt{2} (g_{-K-2n}, g_{-K+1-2n}, \dots, g_{K-2n}), \\ (H^* G^*)_n &= 2 \left(\sum_{n_1} h_{n_1-2n} g_{-K-2n_1}, \sum_{n_1} h_{n_1-2n} g_{-K+1-2n_1}, \dots, \sum_{n_1} h_{n_1-2n} g_{K-2n_1} \right), \\ ((H^*)^j G^*)_n &= (\sqrt{2})^{j+1} \left(\sum_{n_1} \sum_{n_2} \dots \sum_{n_j} h_{n_j-2n} h_{n_{j-1}-2n_j} \dots h_{n_1-2n_2} g_{-K-2n_1}, \right. \\ &\quad \sum_{n_1} \sum_{n_2} \dots \sum_{n_j} h_{n_j-2n} h_{n_{j-1}-2n_j} \dots h_{n_1-2n_2} g_{-K+1-2n_1}, \\ &\quad \dots, \\ &\quad \left. \sum_{n_1} \sum_{n_2} \dots \sum_{n_j} h_{n_j-2n} h_{n_{j-1}-2n_j} \dots h_{n_1-2n_2} g_{K-2n_1} \right), \\ (H^*)^{J_1}_n &= (\sqrt{2})^{J_1} \left(\sum_{n_1} \sum_{n_2} \dots \sum_{n_{J_1-1}} h_{n_{J_1-2n} h_{n_{J_1-1}-2n_{J_1}} \dots h_{n_1-2n_2} h_{-K-2n_1}, \right. \\ &\quad \sum_{n_1} \sum_{n_2} \dots \sum_{n_{J_1-1}} h_{n_{J_1-2n} h_{n_{J_1-1}-2n_{J_1}} \dots h_{n_1-2n_2} h_{-K+1-2n_1}, \\ &\quad \dots, \\ &\quad \left. \sum_{n_1} \sum_{n_2} \dots \sum_{n_{J_1-1}} h_{n_{J_1-2n} h_{n_{J_1-1}-2n_{J_1}} \dots h_{n_1-2n_2} h_{K-2n_1} \right), \end{aligned}$$

for $1 \leq j \leq J-J_0-1$ and $J_1 = J-J_0$. Now, by letting

$$\mathcal{N} = \{m_1, m_2, \dots, m_N : m_1 < m_2 < \dots < m_N\},$$

we obtain the following linear system

$$(4.4) \quad \mathbf{x} = \mathbf{W} \mathbf{p},$$

where

$$(4.5) \quad \mathbf{x} = (x[m_1], x[m_2], \dots, x[m_N])^T, \quad \text{and} \quad \mathbf{W} = (\mathbf{w}_{m_1}^T, \mathbf{w}_{m_2}^T, \dots, \mathbf{w}_{m_N}^T)^T,$$

are known. If \mathbf{p} can be uniquely solved from (4.4), then $x[n]$ with $n \notin \mathcal{N}$ can be extrapolated from $x[n]$ with $n \in \mathcal{N}$. For \mathbf{p} , we have $(2K+1)(J-J_0+1) \triangleq r_0$ unknowns. Therefore, to uniquely determine $x[n]$, it is required that $N \geq r_0$ and that the rank of \mathbf{W} has to be r_0 . The above arguments prove the following theorem.

THEOREM 4. *Let $x[n]$ be a (J, K) scale-time limited sequence. Then, $x[n]$ can be uniquely determined from $x[n]$, $n \in \mathcal{N}$, if and only if the rank of \mathbf{W} is $r_0 = (2K+1)(J-J_0+1)$.*

To extrapolate $x[n]$ outside \mathcal{N} via the discrete GPG algorithm is equivalent to the solution of (4.4) for \mathbf{p} . There are two reasons to avoid solving (4.4) directly. One is that the direct computation of \mathbf{W} is expensive. The other is that, even though \mathbf{W} is known, to solve the linear system (4.4) is also expensive. We now go back to the convergence of the discrete GPG algorithm.

THEOREM 5. *Let $x[n]$ be a (J, K) scale-time limited sequence. If the rank of \mathbf{W} is $r_0 = (2K + 1)(J - J_0 + 1)$, then*

$$(4.6) \quad \sum_{n=-\infty}^{\infty} |x^{(l)}[n] - x[n]|^2 \rightarrow 0, \quad \text{as } l \rightarrow \infty.$$

On the other hand, if (4.6) is true for all (J, K) scale-time limited sequences, then the rank of the matrix \mathbf{W} is r_0 .

Proof. When the rank of \mathbf{W} is $r_0 = (2K + 1)(J - J_0 + 1)$, the DWT coefficients \mathbf{p} is uniquely determined from \mathbf{x} by solving the linear system (4.4). Also, $\mathbf{W}\mathbf{W}^T$ has rank r_0 and is nonnegative definite matrix. Let λ_i , $i = 1, 2, \dots, N$, be the eigenvalues of $\mathbf{W}\mathbf{W}^T$ and \mathbf{q}_i , $i = 1, 2, \dots, N$, be the corresponding eigenvectors, i.e.

$$(4.7) \quad \mathbf{W}\mathbf{W}^T \mathbf{q}_i = \lambda_i \mathbf{q}_i, \quad i = 1, 2, \dots, N,$$

where

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{r_0} > \lambda_{r_0+1} = \dots = \lambda_N = 0,$$

and \mathbf{q}_i forms an orthonormal basis of \mathbb{C}^N , and where \mathbb{C} denotes the set of complex numbers. Let

$$(\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_{r_0}) \triangleq \mathbf{W}^T(\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_{r_0}).$$

Since \mathbf{q}_i , $1 \leq i \leq r_0$, are linearly independent and the matrix \mathbf{W}^T has rank r_0 , \mathbf{y}_i with $1 \leq i \leq r_0$ forms a basis of \mathbb{C}^{r_0} . Therefore, there are r_0 constants a_i such that

$$\mathbf{p} = \sum_{i=1}^{r_0} a_i \mathbf{y}_i.$$

From (4.7), only $\mathbf{q}_i[n]$ with $n \in \mathcal{N}$ are given. For $1 \leq i \leq r_0$, we extend $\mathbf{q}_i[n]$ from $n \in \mathcal{N}$ to all integers via

$$(4.8) \quad \tilde{\mathbf{q}}_i[n] = \frac{1}{\lambda_i} \mathbf{w}_n \mathbf{W}^T \mathbf{q}_i, \quad n \in \mathbb{Z}.$$

By (4.3) and (4.8), we have

$$(4.9) \quad x[n] = \mathbf{w}_n \mathbf{p} = \mathbf{w}_n \sum_{i=1}^{r_0} a_i \mathbf{y}_i = \sum_{i=1}^{r_0} a_i \mathbf{w}_n \mathbf{W}^T \mathbf{q}_i = \sum_{i=1}^{r_0} a_i \lambda_i \tilde{\mathbf{q}}_i[n].$$

Note that the reason for proving (4.9) is similar to that for proving $f(t) \in U_J$ in the proof of Theorem 2 or 3.

We now prove that $\lambda_i \leq 1$ for $1 \leq i \leq r_0$. For $1 \leq i \leq r_0$,

$$\begin{aligned} \|\tilde{\mathbf{q}}_i\|^2 &= \sum_{n=-\infty}^{\infty} |\tilde{\mathbf{q}}_i[n]|^2 = \frac{1}{\lambda_i^2} \sum_{n=-\infty}^{\infty} |\mathbf{w}_n \mathbf{W}^T \mathbf{q}_i|^2 = \frac{1}{\lambda_i^2} \|\mathcal{D}^{-1} \mathbf{W}^T \mathbf{q}_i\|^2 \\ &= \frac{1}{\lambda_i^2} \|\mathbf{W}^T \mathbf{q}_i\|^2 = \frac{1}{\lambda_i^2} \|\mathcal{D} P_{\mathcal{N}} \tilde{\mathbf{q}}_i\|^2 = \frac{1}{\lambda_i^2} \|P_{\mathcal{N}} \tilde{\mathbf{q}}_i\|^2 \\ &\leq \frac{1}{\lambda_i^2} \|\tilde{\mathbf{q}}_i\|^2, \end{aligned}$$

where we use the properties that both \mathcal{D} and \mathcal{D}^{-1} preserve the total energy and that \mathbf{W}^T behaves like $\mathcal{D}P_{\mathcal{N}}$ when operating on (J, K) scale-time limited sequences. Thus, $\lambda_i \leq 1$ for $1 \leq i \leq r_0$. Similar to the proof of the error formula (3.15), we have

$$(4.10) \quad x[n] - x^{(l)}[n] = (I - P_{\mathcal{N}}) \sum_{i=1}^{r_0} a_i \lambda_i (1 - \lambda_i)^l \mathbf{q}_i[n].$$

By $0 < \lambda_i \leq 1, 1 \leq i \leq r_0$, and similar to the proof of the standard PG algorithm,

$$\sum_{n=-\infty}^{\infty} |x[n] - x^{(l)}[n]|^2 \rightarrow 0, \quad \text{as } l \rightarrow \infty.$$

This proves the first part of Theorem 5. If (4.6) is true for all (J, K) scale-time limited sequences $x[n]$, then $x[n]$ is uniquely determined by $x[n], n \in \mathcal{N}$. By Theorem 4, the second part of Theorem 5 is also proved. \square

Suppose that \mathbf{W} has rank r_0 and $\lambda_1, \lambda_2, \dots, \lambda_N$ are eigenvalues of the matrix $\mathbf{W}\mathbf{W}^T$ arranged in a descending order, i.e.

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{r_0} > \lambda_{r_0+1} = \dots = \lambda_N = 0.$$

Based on (4.10), we obtain an error estimate

$$|x^{(l)}[n] - x[n]| \leq O(\lambda_{r_0} (1 - \lambda_{r_0})^l),$$

for sufficiently large l . Even when the rank of \mathbf{W} is r_0 , the conventional discrete PG algorithm for band-limited signals [18], [20] usually converges slowly. This is because that the condition number of the corresponding $\mathbf{W}\mathbf{W}^T$ is usually quite large due to the smoothness of the sinc function. In contrast, the condition number of $\mathbf{W}\mathbf{W}^T$ by using suitable wavelet bases is smaller so that a faster convergence rate can be achieved (see numerical experiments in §5).

4.3. Connection between continuous-time and discrete-time extrapolation. We examined continuous-time signals in the wavelet subspace V_J in §3, where each $f(t) \in V_J$ has the form

$$f(t) = \sum_{k=-\infty}^{\infty} c_{J,k} \phi_{Jk}(t) = \sum_{k=-\infty}^{\infty} c_{J_0,k} \phi_{J_0k}(t) + \sum_{J_0 \leq j < J} \sum_{k=-\infty}^{\infty} b_{j,k} \psi_{jk}(t).$$

In practice, $f(t)$ is small for large $|t|$ so that $c_{J_0,k}$ and $b_{j,k}$ are also small for large $|k|$. Thus, it is important to consider signals in the following subspace of V_J ,

$$V_{J,K} \triangleq \left\{ f(t) : f(t) = \sum_{k=-K}^K c_{J_0,k} \phi_{J_0k}(t) + \sum_{J_0 \leq j < J} \sum_{k=-K}^K b_{j,k} \psi_{jk}(t) \text{ for some constants } c_{J_0,k}, b_{j,k} \right\}.$$

We call signals in $V_{J,K}$ as (J, K) scale-time limited. This explains the motivation of the definition of scale-time limited sequences. For $f(t) \in V_{J,K}$, we have

$$f(t) = \sum_k c_{J,k} \phi_{Jk}(t) = \sum_{k=-K}^K c_{J_0,k} \phi_{J_0k}(t) + \sum_{J_0 \leq j < J} \sum_{k=-K}^K b_{j,k} \psi_{jk}(t),$$

where

$$c_{J,k} = \langle f, \phi_{Jk} \rangle, \quad c_{J_0,k} = \langle f, \phi_{J_0k} \rangle, \quad b_{j,k} = \langle f, \psi_{jk} \rangle.$$

Since $\phi(t)$ behaves like a lowpass filter, $c_{J,k}$ is close to $f(k/2^J)$ [11], [15], [23] for sufficiently large J . Therefore, we can replace $c_{J,k}$ or $x[k]$ with samples $f(k/2^J)$ and use the discrete-time GPG algorithm to provide a good approximation for continuous-time signal extrapolation.

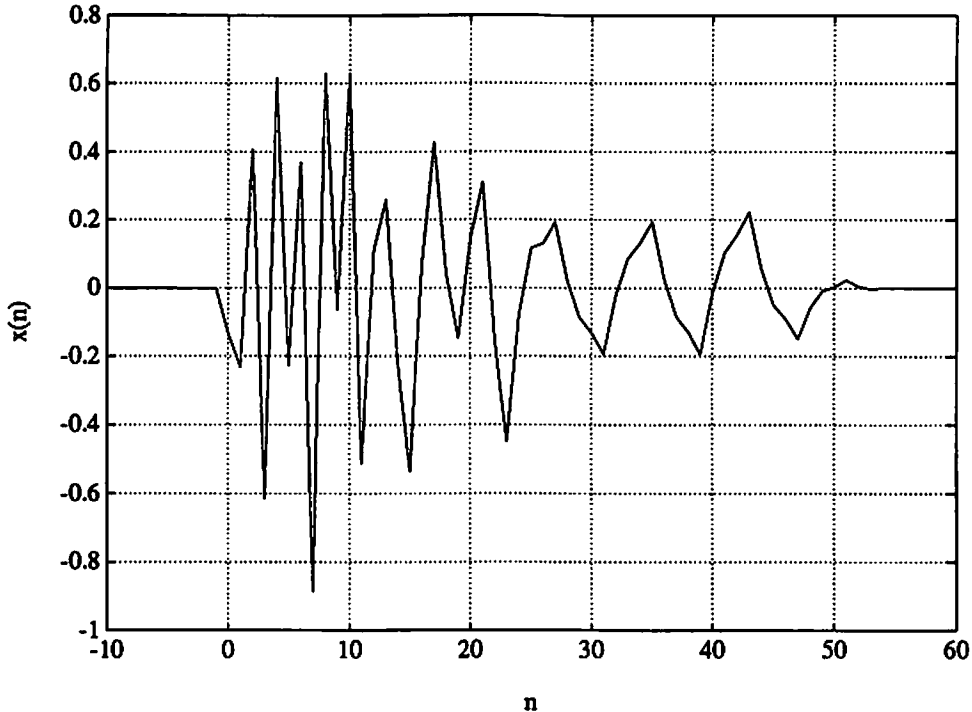


FIG. 1. The test signal $x[n]$.

5. Numerical Examples. In this section, numerical examples are used to illustrate the theory in §4. The wavelet basis used is the Daubechies D_4 [6]. The test signal $x[n]$ as given in Fig. 1 is a $(J, K) = (4, 2)$ scale-time limited signal with $J_0 = 1$, whose DWT coefficients are $c_{1,k} = 0$, $b_{j,k} = 1.0$, $1 \leq j \leq 3$, $1 \leq k \leq 5$. Since the low resolution coefficients $c_{1,k}$ are 0 for this case, $x[n]$ is in fact generated by the 15 wavelet coefficients $b_{j,k}$. Thus, $x[n]$ has $r_0 = 15$ degrees of freedom. We consider four test cases: the values $x[n]$ are given for $n \in \mathcal{N}$ with $\mathcal{N} = \{n : 1 \leq n \leq 15\}$, $\{n : 1 \leq n \leq 30\}$, $\{n : 1 \leq n \leq 42\}$ and $\{n : 1 \leq n \leq 45\}$.

It is clear that the standard PG algorithm does not apply appropriately since $x[n]$ is not band-limited. We use the discrete GPG algorithm (4.1)-(4.2) to compute $x^{(l)}[n]$ iteratively. Let e_l denote the 2-norm of the extrapolation error at l th iteration, i.e.

$$e_l \triangleq \left(\sum_n |x^{(l)}[n] - x[n]|^2 \right)^{1/2}$$

The convergence history of e_l as a function of the number of iteration is plotted in Fig. 2. One can clearly see that the errors remain about the same for $N = 15, 30$ while the errors decrease as iteration proceeds for $N = 42, 45$. In other words, the discrete GPG algorithm converges to the correct solution only when $N = 42$ and 45 . This convergence behavior can be explained by examining Table 1, where we show the ranks and the first 15 eigenvalues of $\mathbf{W}\mathbf{W}^T$ for all four cases, where \mathbf{W} is defined by (4.5). Note that the ranks of \mathbf{W} are equal to the degree of freedom of $x[n]$ only when $N = 42$ and 45 . For $N = 15$ (or $N = 30$), there are 4 (or 1) zero eigenvalues among the largest 15 eigenvalues.

To determine $x[n]$ outside \mathcal{N} is equivalent to the determination of wavelet coefficients $b_{j,k}$, $1 \leq j \leq 3$ and $1 \leq k \leq 5$, which are all equal to 1 by design. Let $b_{j,k}^{(l)}$ be the corresponding DWT coefficients of $x^{(l)}[n]$. The convergence of $x^{(l)}[n]$ to $x[n]$ is equivalent to the convergence of $b_{j,k}^{(l)}$ to $b_{j,k}$. We observe from Tables 2 and 3 that all $b_{j,k}^{(l)}$ converge to the true values 1 of $b_{j,k}$ for $N = 42, 45$, and the convergence

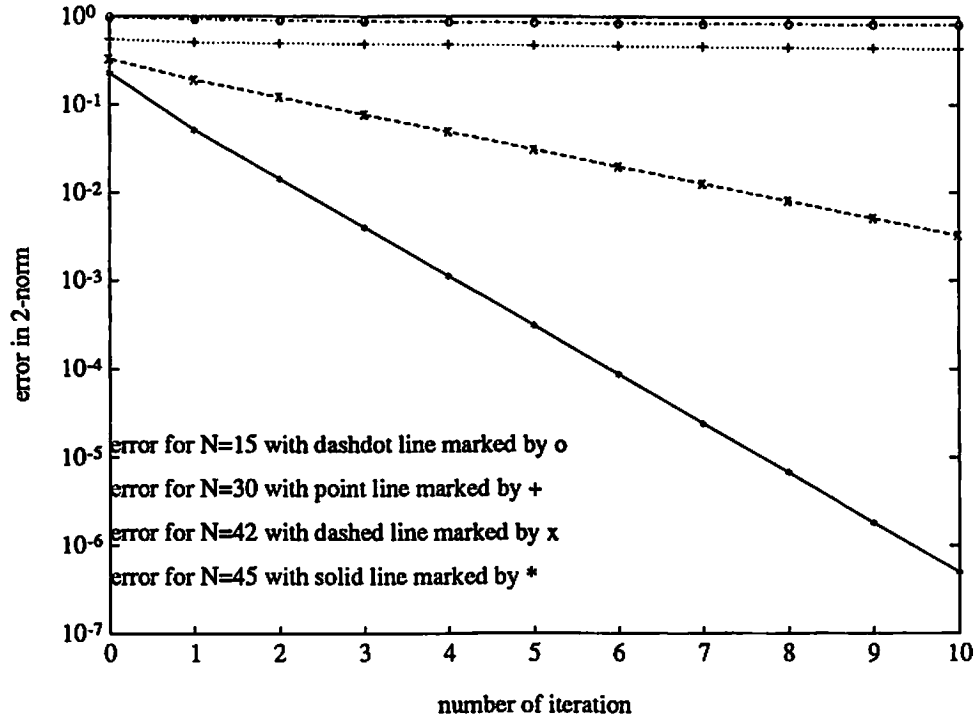


FIG. 2. Convergence history of the 2-norm of extrapolation errors.

TABLE 1

	first 15 eigenvalues of WW^T					rank of W
$N = 15$	1.00000	1.00000	1.00000	1.00000	1.00000	11
	1.00000	1.00000	0.97964	0.96402	0.11057	
	0.00065	0.00000	0.00000	0.00000	0.00000	
$N = 30$	1.00000	1.00000	1.00000	1.00000	1.00000	14
	1.00000	1.00000	1.00000	1.00000	1.00000	
	1.00000	0.97843	0.79403	0.40207	0.00000	
$N = 42$	1.00000	1.00000	1.00000	1.00000	1.00000	15
	1.00000	1.00000	1.00000	1.00000	1.00000	
	1.00000	1.00000	0.99793	0.97843	0.36378	
$N = 45$	1.00000	1.00000	1.00000	1.00000	1.00000	15
	1.00000	1.00000	1.00000	1.00000	1.00000	
	1.00000	1.00000	1.00000	0.97843	0.72182	

TABLE 2

	$b_{j,k}^{(5)}, 1 \leq j \leq 3, 1 \leq k \leq 5$				
$N = 15$	1.08760	0.59008	0.00000	0.00000	0.00000
	1.00013	1.00000	0.99501	0.47866	0.00000
	1.00019	1.00000	1.00000	1.00000	1.00000
$N = 30$	1.00000	1.00000	0.92549	0.15589	0.00000
	1.00000	1.00000	1.00000	1.00000	1.00000
	1.00000	1.00000	1.00000	1.00000	1.00000
$N = 42$	1.00000	1.00000	1.00000	0.99550	0.92973
	1.00000	1.00000	1.00000	1.00000	1.00000
	1.00000	1.00000	1.00000	1.00000	1.00000
$N = 45$	1.00000	1.00000	1.00000	1.00000	0.99954
	1.00000	1.00000	1.00000	1.00000	1.00000
	1.00000	1.00000	1.00000	1.00000	1.00000

TABLE 3

	$b_{j,k}^{(10)}, 1 \leq j \leq 3, 1 \leq k \leq 5$				
$N = 15$	1.06050	0.85387	0.00000	0.00000	0.00000
	1.00009	1.00000	0.99722	0.64568	0.00000
	1.00013	1.00000	1.00000	1.00000	1.00000
$N = 30$	1.00000	1.00000	0.93937	0.31247	0.00000
	1.00000	1.00000	1.00000	1.00000	1.00000
	1.00000	1.00000	1.00000	1.00000	1.00000
$N = 42$	1.00000	1.00000	1.00000	0.99953	0.99268
	1.00000	1.00000	1.00000	1.00000	1.00000
	1.00000	1.00000	1.00000	1.00000	1.00000
$N = 45$	1.00000	1.00000	1.00000	1.00000	1.00000
	1.00000	1.00000	1.00000	1.00000	1.00000
	1.00000	1.00000	1.00000	1.00000	1.00000

rate is fast. In contrast, there are 4 (or 1) $b_{j,k}^{(l)}$'s which do not converge to 1 for $N = 15$ (or $N = 30$) due to the zero eigenvalues existing in the matrix WW^T .

6. Conclusion. In this paper, we proposed a new iterative algorithm which extends the well known Papoulis-Gerchberg (PG) algorithm for band-limited signal extrapolation to signals in the wavelet subspaces. The convergence of the generalized PG (GPG) algorithm and the uniqueness of extrapolated signals for both continuous-time and discrete-time cases are investigated. Several conditions on signals and wavelet bases to achieve convergence and uniqueness are described.

There are many topics worth further investigation. One interesting problem is signal extrapolation with noisy data. For band-limited signal extrapolation, the relationship between the PG algorithm and its discrete version has been studied in [18], [20], [26], where the discrete-time model converges to the continuous-time one when the sampling rate goes to infinity. However, for extrapolation in general wavelet subspaces discussed in this work, the precise connection between continuous-time and discrete-time models is still not well understood. Furthermore, since various bases can be provided by wavelet theory, it is believed that the proposed GPG algorithm should have many potential applications in signal recovery.

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