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Connection Between Continuous-Time and Discrete-Time Signal Extrapolation Problems in Wavelet Subspaces

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Connection between Continuous-Time and Discrete-Time Signal Extrapolation Problems in Wavelet Subspaces*

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Abstract

Signal extrapolation in wavelet subspaces for both continuous-time and discrete-time signals was considered in our previous work, where an algorithm obtained from the generalization of the Papoulis-Gerchberg algorithm for band-limited signal extrapolation was proposed. The relationship between the continuous-time and discrete-time extrapolation problems is investigated in this paper. We provide sufficient conditions on signals and wavelet bases so that the discrete-time problem converges to the continuous-time problem when the sampling rate in a given interval goes to infinite.

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1 Introduction

Signal interpolation and extrapolation are two important problems in signal reconstruction. The interpolation problem is to reconstruct the continuous-time signal $f(t)$ based on its discrete samples $f(n\Delta t)$ while the extrapolation problem is to reconstruct a signal $f(t)$ based on some segment of $f(t)$, say, $t \in [-T, T]$. The Shannon sampling theorem is well known for band-limited signal interpolation, whereas the Papoulis-Gerchberg (PG) algorithm is popular in the context of band-limited signal extrapolation. The sampling theorem has been extended to signals in wavelet subspaces by Walter [15]. In our recent work [19], we derived a generalized PG algorithm which extrapolates signals in wavelet subspaces. One potential advantage of the proposed new scheme is that it provides many attractive multiresolution wavelet bases for signal modelling and, as a consequence, the difficulty arising in band-limited extrapolation, i.e. the ill-conditioning of the problem due to the smoothness and the lack of time localization of the Fourier basis, can be overcome with the choice of appropriate wavelet bases.

The discrete-time signal extrapolation problem is important in practical applications, since only discrete samples of a signal can be computed numerically. For discrete-time band-limited signal extrapolation, the discretized PG algorithm [11], [13], [21], [19] is well studied. It has been proved that the discretized PG algorithm converges to its continuous counterpart [13], [21] as the sampling rate goes to infinity. The convergence of the generalized PG algorithm in wavelet subspaces for both continuous-time and discrete-time signals and the uniqueness of the extrapolation were examined in [19]. In this paper, we investigate the relationship between the continuous-time and discrete-time signal extrapolation problems in wavelet subspaces. We provide sufficient conditions on signals and wavelet bases so that the discrete-time problem converges to the continuous one.

This paper is organized as follows. In §2, we briefly review the generalized PG algorithm for both conitinuous-time and discrete-time signal extrapolation. In §3, we state and prove a main theorem on the convergence of the discrete-time problem to the continuous-time one. Concluding remarks are given §4. The following notation will be used throughout the paper. The $L^2(\mathbf{R})$ denotes all real square integrable functions (or signals) defined on $\mathbf{R}$. For $D > 0$, the $L^2[-D, D]$ denotes all signals $f(t)$ defined on $[-D, D]$ satisfying

$$\int_{-D}^{D} |f(t)|^2 dt < \infty.$$

Let $<,>$ and $\|\cdot\|$ denote the inner product and the norm on $L^2(\mathbf{R})$, i.e.

$$< f, g >= \int_{-\infty}^{\infty} f(t)g(t)dt, \quad \text{where} \quad f(t), g(t) \in L^2(\mathbf{R}),$$

and $\|f\|^2 =< f, f >$. Similarly, we use $<,>_D$ and $\|\cdot\|_D$ denote the inner product and the norm on $L^2[-D, D]$. For $f(t) \in L^2(\mathbf{R})$, we define

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-it\omega}dt,$$

to be the Fourier transform of $f(t)$.

2 Review of Signal Extrapolation in Wavelet Subspaces

For completeness of presentation, we review basic results given in [19] in this section.

2.1 Orthogonal Wavelet Bases

We only consider real wavelets in this work, and refer to [2], [3], [4] for more detailed discussion. Let $\phi(t)$ be a scaling function such that, for a fixed arbitrary integer j,

$$\{\phi_{jk}(t)\}_{k\in\mathbf{Z}}, \quad \text{where} \quad \phi_{jk}(t) = 2^{j/2}\phi(2^j t - k),$$

is an orthonormal basis of the wavelet subspace V_j, and $\{V_j\}_{j\in\mathbf{Z}}$ is a multiresolution approximation of $L^2(\mathbf{R})$, i.e. $V_j \subset V_{j+1}$ and $\overline{\bigcup_j V_j} = L^2(\mathbf{R})$. The wavelet function corresponding to $\phi(t)$ is denoted by $\psi(t)$ and $\psi_{jk}(t) = 2^{j/2}\psi(2^j t - k)$. The associated quadrature mirror filters can be expressed as

$$H(\omega) = \sum_k h_k e^{-ik\omega}, \quad \text{and} \quad G(\omega) = \sum_k g_k e^{-ik\omega}, \tag{2.1}$$

where $g_k = (-1)^k h_{1-k}$ and

$$\hat{\phi}(2\omega) = H(\omega)\hat{\phi}(\omega), \quad \text{and} \quad \hat{\psi}(2\omega) = G(\omega)\hat{\phi}(\omega).$$

Then, we have

$$f(t) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} b_{j,k}\psi_{jk}(t), \tag{2.2}$$

for any $f(t) \in L^2(\mathbf{R})$ and

$$f(t) = \sum_{k=-\infty}^{\infty} c_{J,k}\phi_{Jk}(t) = \sum_{j<J} \sum_{k=-\infty}^{\infty} b_{j,k}\psi_{jk}(t), \tag{2.3}$$

for any $f(t) \in V_J$, where $b_{j,k} =< f, \psi_{jk} >$ and $c_{J,k} =< f, \phi_{Jk} >$. The $b_{j,k}$ in (2.2) are called the wavelet series transform (WST) coefficients of $f(t)$, and (2.2) provides the inverse wavelet series transform (IWST) of $b_{j,k}$. On one hand, the WST coefficients $b_{j,k}$ with $j < J$ can be obtained from coefficients $c_{J,k}$ by the recursive formulas:

$$\begin{aligned} c_{j-1,k} &= \sqrt{2}\textstyle\sum_n h_{n-2k} c_{j,n}, \\ b_{j-1,k} &= \sqrt{2}\textstyle\sum_n g_{n-2k} c_{j,n}, \end{aligned} \tag{2.4}$$

for $j = J, J-1, J-2, \cdots$. On the other hand, we have the following synthesis formula to compute coefficients $c_{J,k}$ from $c_{J_0,k}$ and $b_{j,k}$ with $J_0 \leq j < J$ via

$$c_{j+1,n} = \sqrt{2}\left(\sum_k h_{n-2k} c_{j,k} + \sum_k g_{n-2k} b_{j,k}\right), \tag{2.5}$$

for $j = J_0, J_0+1, \cdots, J-1$. By viewing $c_{J,n}$ as a sequence $x[n]$, we call (2.4) the discrete wavelet transform (DWT) with parameters J_0 and J or simply DWT of the sequence $x[n]$ and (2.5) the inverse discrete wavelet transform (IDWT) with parameters J_0 and J or simply IDWT of coefficients $c_{J_0,k}$ and $b_{j,k}$.

2.2 Signal Extrapolation

2.2.1 Continuous-time Case

Let $f(t) \in V_J$ for a fixed integer J. Given the value of $f(t)$ for $|t| < T$ $(T > 0)$, we are concerned with the determination of the value $f(t)$ for $|t| \geq T$. We propose the following generalized PG algorithm for extrapolation:

Generalized PG (GPG) Algorithm:

$$f^{(0)}(t) = P_T f(t). \tag{2.6}$$

For $l = 0, 1, 2, \cdots$,

$$g^{(l)}(t) = \sum_k < f^{(l)}, \phi_{Jk} > \phi_{Jk}(t), \tag{2.7}$$

$$f^{(l+1)}(t) = P_T f(t) + (I - P_T) g^{(l)}(t). \tag{2.8}$$

When the scaling function $\phi(t)$ is the sinc function, that is, $\phi(t) = \frac{\sin \pi t}{\pi t}$, the GPG algorithm (2.6) - (2.8) reduces to the PG algorithm with $\Omega = 2^J \pi$ [19].

2.2.2 Discrete-time Case

In practice, we only have discrete values of $f(t)$ in $[-T, T]$. Therefore, the GPG algorithm needs to be discretized for discrete-time signals. Recall that the DWT of a sequence $c_{J,n} = x[n]$ can be implemented via (2.4) for a certain integer J. The discrete sequence $c_{J,n}$ is said to be (J, K) *scale-time limited* for certain integers J and $K > 0$ if its DWT coefficients (with lowest resolution J_0) satisfies that coefficients $c_{J_0,k}$ and $b_{j,k}$ may take nonzero values only when $|k| \leq K$ and $J_0 \leq j < J$. When J and K are sufficiently large, the (J, K) scale-time limited sequence provides a practical discrete-time signal model.

Let $x[n]$ be a (J, K) scale-time limited sequence. The values of $x[n]$, $n \in \mathcal{N}$, are given, where the cardinality $|\mathcal{N}| = N$ is finite. The extrapolation problem is to recover $x[n]$ for $n \notin \mathcal{N}$. Let $P_{\mathcal{N}}$ and $P_{J,K}$ be the following operators:

$$P_{\mathcal{N}} y[n] = \begin{cases} y[n], & n \in \mathcal{N}, \\ 0, & n \notin \mathcal{N}, \end{cases}$$

and

$$P_{J,K} d_{j,k} = \begin{cases} d_{j,k}, & |k| \leq K \text{ and } J_0 \leq j < J, \\ 0, & \text{otherwise} \end{cases}$$

Let I be the identity operator and $\mathcal{D}_{J_0,J}$ and $\mathcal{D}_{J_0,J}^{-1}$ be the DWT and IDWT operators with parameters J_0 and J as defined in §2.1. The discrete GPG algorithm can be stated as follows.

The Discrete GPG (DGPG) Algorithm:

$$x^{(0)}[n] = P_{\mathcal{N}} x[n], \tag{2.9}$$

For $l = 0, 1, 2, \cdots$,

$$x^{(l+1)}[n] = P_{\mathcal{N}} x[n] + (I - P_{\mathcal{N}}) \mathcal{D}_{J_0,J}^{-1} P_{J,K} \mathcal{D}_{J_0,J} x^{(l)}[n]. \tag{2.10}$$

2.3 Uniqueness and Convergence Results

We summarize results on the uniqueness of extrapolation and the convergence of the GPG and DGPG algorithms below.

2.3.1 Continuous-time Case

Let us define

$$Q(s,t) \triangleq \sum_{k=-\infty}^{\infty} \phi(s-k)\phi(t-k), \quad (s,t) \in \mathbf{R}^2, \tag{2.11}$$

and

$$Q_J(s,t) \triangleq 2^J Q(2^J s, 2^J t) = \sum_{k=-\infty}^{\infty} \phi_{Jk}(s)\phi_{Jk}(t). \tag{2.12}$$

When the decay of $\phi(t)$ satisfies $|\phi(t)| \leq O(1+|t|^{0.5+\epsilon})^{-1}$ for some $\epsilon > 0$, $Q(s,t)$ in (2.11) is finite almost surely for $s,t \in \mathbf{R}$. For a kernal $K(s,t)$ satisfying $K(s,t) = K(t,s)$ and

$$\sum_{i=1}^{N}\sum_{j=1}^{N} a_i \bar{a}_j K(t_i, t_j) \geq 0, \tag{2.13}$$

where the bar denotes the complex conjugate, for any integer $N > 0$, any N points $t_i \in [-T,T]$ and any N numbers a_i, we say $K(s,t)$ to be *symmetric nonnegative definite* in $[-T,T]^2$. If the inequality in (2.13) is strictly great than 0 when there is at least one $a_i \neq 0$, we say that $K(s,t)$ is *symmetric positive definite* in $[-T,T]^2$. Obviously, $Q_J(s,t)$ is symmetric nonnegative definite in $[-T,T]^2$. We say that a signal $f(t)$ can be *uniquely* determined in a signal set S from its segment $f(t)$ defined on interval $[A,B]$, if any $f(t), g(t) \in S$ with $f(t) = g(t)$ for $t \in [A,B]$, implies $f(t) = g(t)$ for $t \in \mathbf{R}$.

The first convergence result is stated as follows.

Proposition 1 *If $Q_J(s,t)$ is continuous and positive definite in $[-T,T]^2$ and $f(t)$ is uniquely determined in V_J by $f(t)$, $t \in [-T,T]$, then $\|f^{(l)} - f\| \to 0$ when $l \to \infty$, where $f^{(l)}(t)$ is obtained from the GPG algorithm* (2.6) -(2.8).

Proof. See the Proof of Theorem 1 in [19]. □

Proposition 1 tells us that if $Q_J(s,t)$ is continuous and positive definite in $[-T,T]^2$, the uniqueness of extrapolation in V_J implies the convergence of the GPG algorithm. Instead of checking the uniqueness of extrapolation for various functions $f(t)$ of interest in V_J individually, the following Proposition says that it is sufficient to check that for the scaling function $\phi(t)$ only.

Proposition 2 *If $Q_J(s,t)$ is continuous and positive definite in $[-T,T]^2$ and the scaling function $\phi(t)$ is uniquely determined in V_J by any one of its segments $\phi(t)$, $t \in [-2^J T - k, 2^J T - k]$, $k \in \mathbf{Z}$, then $\|f^{(l)} - f\| \to 0$ as $l \to \infty$ where $f^{(l)}(t)$ is obtained from the GPG algorithm* (2.6) -(2.8).

Proof. See the Proof of Theorem 2 in [19]. □

The conditions in Proposition 2 on a scaling function $\phi(t)$ are sufficient conditions for the uniqueness of the extrapolation and the convergence of the GPG algorithm. As a direct consequence of Proposition 2, we have the following corollary.

Corollary 1 *Under the same conditions as stated in Proposition 2, if $f(t) \in V_J$, then $f(t)$ is uniquely determined in V_J by its segment $f(t)$, $t \in [-T, T]$.*

2.3.2 Discrete-time Case

We introduce two operators H and G related to the quadrature mirror filters $H(\omega)$ and $G(\omega)$ in (2.1) as follows:

$$Hy[k] \triangleq \sqrt{2}\sum_n h_{n-2k}y[n], \quad \text{and} \quad Gy[k] \triangleq \sqrt{2}\sum_n g_{n-2k}y[n].$$

Let H^* and G^* be their duals, respectively, i.e.

$$H^*y[n] \triangleq \sqrt{2}\sum_k h_{n-2k}y[k], \quad \text{and} \quad G^*y[n] \triangleq \sqrt{2}\sum_k g_{n-2k}y[k].$$

Then, from (2.5), we have

$$x[n] = \left((H^*)^{J-J_0}c_{J_0,k} + (H^*)^{J-J_0-1}G^*b_{J_0,k} + \cdots + H^*G^*b_{J-2,k} + G^*b_{J-1,k}\right)[n].$$

We can rewrite the above equation as

$$x[n] = \mathbf{w}_n\mathbf{p}, \quad n \in \mathbf{Z}, \tag{2.14}$$

where $\mathbf{p}$ and $\mathbf{w}_n$ are, respectively, column and row vectors of length $(2K+1)(J-J_0+1)$ of the form

$$\begin{aligned}
\mathbf{p} &= (\mathbf{c}_{J_0}, \mathbf{b}_{J_0}, \mathbf{b}_{J_0+1}, \cdots, \mathbf{b}_{J-1})^T, \\
\mathbf{w}_n &= \left((H^*)_n^{J-J_0}, \left((H^*)^{J-J_0-1}G^*\right)_n, \cdots, (H^*G^*)_n, G_n^*\right),
\end{aligned}$$

and where

$$\begin{aligned}
\mathbf{c}_{J_0} &= (c_{J_0,-K}, c_{J_0,-K+1}, \cdots, c_{J_0,K}), \\
\mathbf{b}_j &= (b_{j,-K}, b_{j,-K+1}, \cdots, b_{j,K}), \\
G_n^* &= \sqrt{2}\,(g_{-K-2n}, g_{-K+1-2n}, \cdots, g_{K-2n}), \\
(H^*G^*)_n &= 2\left(\sum_{n_1} h_{n_1-2n}g_{-K-2n_1}, \sum_{n_1} h_{n_1-2n}g_{-K+1-2n_1}, \cdots, \sum_{n_1} h_{n_1-2n}g_{K-2n_1}\right), \\
\left((H^*)^jG^*\right)_n &= (\sqrt{2})^{j+1}\left(\sum_{n_1}\sum_{n_2}\cdots\sum_{n_j} h_{n_j-2n}h_{n_{j-1}-2n_j}\cdots h_{n_1-2n_2}g_{-K-2n_1},\right. \\
&\qquad \sum_{n_1}\sum_{n_2}\cdots\sum_{n_j} h_{n_j-2n}h_{n_{j-1}-2n_j}\cdots h_{n_1-2n_2}g_{-K+1-2n_1},
\end{aligned}$$

$$
\begin{aligned}
& \cdots, \\
& \sum_{n_1}\sum_{n_2}\cdots\sum_{n_j} h_{n_j-2n}h_{n_{j-1}-2n_j}\cdots h_{n_1-2n_2}g_{K-2n_1}\Bigg), \\
(H^*)_n^{J'} \;=\; & (\sqrt{2})^{J'}\Bigg(\sum_{n_1}\sum_{n_2}\cdots\sum_{n_{J'-1}} h_{n_{J'}-2n}h_{n_{J'-1}-2n_{J'}}\cdots h_{n_1-2n_2}h_{-K-2n_1}, \\
& \sum_{n_1}\sum_{n_2}\cdots\sum_{n_{J'-1}} h_{n_{J'}-2n}h_{n_{J'-1}-2n_{J'}}\cdots h_{n_1-2n_2}h_{-K+1-2n_1}, \\
& \cdots, \\
& \sum_{n_1}\sum_{n_2}\cdots\sum_{n_{J'-1}} h_{n_{J'}-2n}h_{n_{J'-1}-2n_{J'}}\cdots h_{n_1-2n_2}h_{K-2n_1}\Bigg),
\end{aligned}
$$

for $1 \le j \le J - J_0 - 1$ and $J' = J - J_0$. Now, by letting

$$\mathcal{N} = \{m_1, m_2, \cdots, m_N : m_1 < m_2 < \cdots < m_N\},$$

we obtain the following linear system

$$\mathbf{x} = \mathbf{W}\mathbf{p}, \tag{2.15}$$

where

$$\mathbf{x} = (x[m_1], x[m_2], \cdots x[m_N])^T, \quad \text{and} \quad \mathbf{W} = \left(\mathbf{w}_{m_1}^T, \mathbf{w}_{m_2}^T, \cdots, \mathbf{w}_{m_N}^T\right)^T,$$

are known. If $\mathbf{p}$ can be uniquely solved from (2.15), then $x[n]$ with $n \notin \mathcal{N}$ can be extrapolated from $x[n]$ with $n \in \mathcal{N}$. For $\mathbf{p}$, we have $(2K+1)(J-J_0+1) \triangleq r_0$ unknowns. Therefore, to uniquely determine $x[n]$, it is required that $N \ge r_0$ and that the rank of $\mathbf{W}$ has to be r_0. The above arguments prove the following Proposition.

Proposition 3 *Let $x[n]$ be a (J, K) scale-time limited sequence. Then, $x[n]$ can be uniquely determined from $x[n]$, $n \in \mathcal{N}$, if and only if the rank of $\mathbf{W}$ is $r_0 = (2K+1)(J-J_0+1)$.*

To extrapolate $x[n]$ outside $\mathcal{N}$ via the discrete GPG algorithm is equivalent to the solution of (2.15) for $\mathbf{p}$. There are two reasons to avoid solving (2.15) directly. One is that the direct computation of $\mathbf{W}$ is expensive. The other is that, even though $\mathbf{W}$ is known, to solve the linear system (2.15) is also expensive. We now go back to the convergence of the discrete GPG algorithm.

Proposition 4 *Let $x[n]$ be a (J,K) scale-time limited sequence. If the rank of* $\mathbf{W}$ *is* $r_0 = (2K+1)(J-J_0+1)$, *then*

$$\sum_{n=-\infty}^{\infty} |x^{(l)}[n] - x[n]|^2 \to 0, \quad as\ l \to \infty. \tag{2.16}$$

On the other hand, if (2.16) *is true for all (J,K) scale-time limited sequences, then the rank of the matrix* $\mathbf{W}$ *is* r_0.

Proof. See the Proof of Theorem 4 in [19]. □

3 Connection between the Continuous- and Discrete-time Signal Extrapolation

In this section, we investigate the connection between continuous- and discrete-time signal extrapolation described in §2.

3.1 Importance of the Problem

Consider continuous-time signals in the wavelet subspace V_J, where each $f(t) \in V_J$ has the form

$$f(t) = \sum_{k=-\infty}^{\infty} c_{J,k}\phi_{Jk}(t) = \sum_{k=-\infty}^{\infty} c_{J_0,k}\phi_{J_0k}(t) + \sum_{J_0 \le j < J} \sum_{k=-\infty}^{\infty} b_{j,k}\psi_{jk}(t).$$

In practice, $f(t)$ is small for large $|t|$ so that $c_{J_0,k}$ and $b_{j,k}$ are also small for large $|k|$. Thus, it is important to consider signals in the following subspace of V_J,

$$V_{J,K} \triangleq \left\{ f(t) : f(t) = \sum_{k=-K}^{K} c_{J_0,k}\phi_{J_0k}(t) + \sum_{J_0 \le j < J} \sum_{k=-K}^{K} b_{j,k}\psi_{jk}(t) \text{for some constants} c_{J_0,k}, b_{j,k} \right\}.$$

We call signals in $V_{J,K}$ as (J,K) scale-time limited. For $f(t) \in V_{J,K}$, we have

$$f(t) = \sum_k c_{J,k}\phi_{Jk}(t) = \sum_{k=-K}^{K} c_{J_0,k}\phi_{J_0k}(t) + \sum_{J_0 \le j < J} \sum_{k=-K}^{K} b_{j,k}\psi_{jk}(t),$$

where

$$c_{J,k} = < f, \phi_{Jk} >, \quad c_{J_0,k} = < f, \phi_{J_0k} >, \quad b_{j,k} = < f, \psi_{jk} > .$$

Since $\phi(t)$ behaves like a lowpass filter, $c_{J,k}$ is close to $2^{-J/2}f(k/2^J)$ [6], [8], [17] for sufficiently large J. Therefore, we may replace $c_{J,k}$ or $x[k]$ with samples $2^{-J/2}f(k/2^J)$ and use the discrete-time GPG algorithm to provide an approximation for continuous-time signal

extrapolation. More generally, even if J is not large enough so that $c_{J,k}$ cannot be well approximated by $2^{-J/2}f(k/2^J)$, we can still use $2^{-J_1/2}f(k/2^{J_1})$ to approximate $c_{J_1,k} = x_{J_1}[n]$ with appropriate scale parameter $J_1 \geq J$. The question is that, when the sampling rate in $[-T,T]$ goes to infinity (or J_1 goes to infinity), if the extrapolated sequence $x_{J_1}[n]$ converges to $f(t)$ in a certain sense.

3.2 Main Result

In what follows, we consider $f(t) \in V_{J,K}$ where the scale and time parameters J and K are arbitrary but fixed. Without loss of generality, we assume that samples $f(k/2^{J_1})$ are known in the interval $[-T,T] = [-1,1]$ with $J_1 \geq J$. Since $f(t) \in V_{J,K} \subset V_{J_1,K}$,

$$f(t) = \sum_k c_{J_1,k}\phi_{J_1k}(t). \tag{3.1}$$

Let

$$\mathcal{N}_{J_1} \triangleq \{n : -2^{J_1} \leq n \leq 2^{J_1}\},$$

and

$$x_{J_1}[n] = 2^{-J_1/2}f(\frac{n}{2^{J_1}}),\ n \in \mathcal{N}_{J_1}.$$

The DGPG algorithm (2.9)-(2.10) can be rewritten in the current setting as:

$$x_{J_1}^{(0)}[n] = P_{\mathcal{N}_{J_1}}x_{J_1}[n], \tag{3.2}$$

and for $l = 0,1,2,\cdots$,

$$x_{J_1}^{(l+1)}[n] = P_{\mathcal{N}_{J_1}}x_{J_1}[n] + (I - P_{\mathcal{N}_{J_1}})\mathcal{D}_{J_0,J_1}^{-1}P_{J,K}\mathcal{D}_{J_0,J_1}x_{J_1}^{(l)}[n]. \tag{3.3}$$

Therefore, from the samples $f(k/2^{J_1})$, $k \in \mathcal{N}_{J_1}$, we obtain a discrete-time signal $x_{J_1}^{(l)}[n]$, $n \in \mathbf{Z}$. With $x_{J_1}^{(l)}[n]$, we form a continuous-time signal via

$$f_{J_1,l}(t) = \sum_k x_{J_1}^{(l)}[k]\phi_{J_1k}(t),\ t \in \mathbf{R}. \tag{3.4}$$

Our main result is on the convergence of $f_{J_1,l}(t)$ to $f(t)$,

Before stating the main convergence theorem, let us examine equations analogous to (2.14) and (2.15) in the current context. Let $\tilde{x}_{J_1}[n] = c_{J_1,n}$, $n \in \mathbf{Z}$. Then, similar to (2.14) for $x[n]$, we have

$$\tilde{x}_{J_1}[n] = \mathbf{w}_n(J_1)\mathbf{p}, \quad n \in \mathbf{Z}, \tag{3.5}$$

where

$$\mathbf{w}_n(J_1) = \left((H^*)_n^{J_1-J_0}, \left((H^*)^{J_1-J_0+1}G^*\right)_n, \cdots, \left((H^*)^{J_1-J}G^*\right)_n\right),$$

and $\mathbf{p}$, $((H^*)^j)_n$ and $((H^*)^jG^*)_n$ are the same as before. It is clear that $\mathbf{w}_n(J_1) = \mathbf{w}_n$ when $J_1 = J$. Let

$$\tilde{\mathbf{x}}_{J_1} = (\tilde{x}_{J_1}[-2^{J_1}], \tilde{x}_{J_1}[-2^{J_1}+1], \cdots, \tilde{x}_{J_1}[2^{J_1}])^T,$$

and

$$\mathbf{W}(J_1) = (\mathbf{w}_{-2^{J_1}}(J_1), \mathbf{w}_{-2^{J_1}+1}(J_1), \cdots, \mathbf{w}_{2^{J_1}}(J_1)). \tag{3.6}$$

Then,

$$\tilde{\mathbf{x}}_{J_1} = \mathbf{W}(J_1)\mathbf{p}. \tag{3.7}$$

Let $r_0 \triangleq (2K+1)(J-J_0+1)$ which is the same as in §2.3.2 and $r(J_1)$ be the rank of the matrix $\mathbf{W}(J_1)$. We see that the matrix $\mathbf{W}(J_1)$ only depends on the quadrature mirror filters H and G of a wavelet basis. We are now ready to state the convergence result.

Theorem 1 *Let $f(t) \in V_{J,K}$ for certain integer $K > 0$. If the scaling function $\phi(t)$ satisfies that $\hat{\phi}(\omega) \in L^1(\mathbf{R})$ is continuous at $\omega = 0$ with $\hat{\phi}(0) = 1$ and there is an integer J_1 with $J_1 \geq J$ such that the rank $r(J_1)$ of the matrix $\mathbf{W}(J_1)$ in (3.6) is $r_0 = (2K+1)(J-J_0+1)$, then*

$$\lim_{l\to\infty} \lim_{J_1\to\infty} \|f_{J_1,l} - f\| = 0, \tag{3.8}$$

where $f_{J_1,l}(t)$ is defined by (3.4).

Proof. See §3.3. □

It is clear that the condition in Proposition 4 also gurantees the convergence (3.8).

3.3 Proof of the Main Result

Let $\lambda_1(J_1), \lambda_2(J_1), \cdots, \lambda_{2^{J_1+1}+1}(J_1)$ be the eigenvalues of $\mathbf{W}(J_1)(\mathbf{W}(J_1))^T$ with

$$\lambda_1(J_1) \geq \lambda_2(J_1) \geq \cdots \geq \lambda_{2^{J_1+1}+1}(J_1) \geq 0.$$

For $J_2 \geq J_1$, $\mathbf{W}(J_1)$ is a submatrix of $\mathbf{W}(J_2)$ with the same number r_0 of columns. Therefore, we have (see [14])

$$\lambda_j(J_2) \geq \lambda_j(J_1), \quad 1 \leq j \leq 2^{J_1+1}+1. \tag{3.9}$$

In what follows, we assume that $r(J_1) = r_0$. This implies that the rank of $\mathbf{W}(J_2)(\mathbf{W}(J_2))^T$ is r_0 for all $J_2 \geq J_1$.

Recall $\tilde{x}_{J_1}[n] = c_{J_1,n}$. Let us apply the DGPG algorithm to $\tilde{x}_{J_1}[n]$ with $n \in \mathcal{N}_{J_1}$ to reconstruct $\tilde{x}_{J_1}[n]$ for $n \in \mathbf{Z}$, i.e.

$$\tilde{x}^{(0)}_{J_1}[n] = P_{\mathcal{N}_{J_1}} \tilde{x}_{J_1}[n], \tag{3.10}$$

and for $l = 0, 1, 2, \cdots$,

$$\tilde{x}^{(l+1)}_{J_1}[n] = P_{\mathcal{N}_{J_1}} \tilde{x}_{J_1}[n] + (I - P_{\mathcal{N}_{J_1}}) \mathcal{D}^{-1}_{J_0,J_1} P_{J,K} \mathcal{D}_{J_0,J_1} \tilde{x}^{(l)}_{J_1}[n]. \tag{3.11}$$

Then, we have the following lemma.

Lemma 1

$$\sum_n \left| \tilde{x}^{(l)}_{J_1}[n] - \tilde{x}_{J_1}[n] \right|^2 \leq \|f\|^2 r_0^2 (1 - \lambda_{r_0}(J_1))^{2l}. \tag{3.12}$$

Proof. Let $\mathbf{q}_i$ be the eigenvector of $\mathbf{W}(J_1)(\mathbf{W}(J_1))^T$ corresponding to $\lambda_i(J_1)$, i.e.

$$\mathbf{W}(J_1)(\mathbf{W}(J_1))^T \mathbf{q}_i = \lambda_i \mathbf{q}_i, \; i = 1, 2, \cdots, 2^{J_1+1} + 1, \tag{3.13}$$

where

$$\lambda_1(J_1) \geq \lambda_2(J_1) \geq \cdots \geq \lambda_{r_0}(J_1) > \lambda_{r_0+1}(J_1) = \cdots = \lambda_{2^{J_1+1}+1}(J_1) = 0. \tag{3.14}$$

Thus, $\mathbf{q}_i$ forms an orthonormal basis of $\mathbf{C}^{2^{J_1+1}+1}$, where $\mathbf{C}$ denotes the set of complex numbers. Let

$$(\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_{r_0}) \triangleq (\mathbf{W}(J_1))^T (\mathbf{q}_1, \mathbf{q}_2, \cdots, \mathbf{q}_{r_0}).$$

Since $\mathbf{q}_i$, $1 \leq i \leq r_0$, are linearly independent and the matrix $(\mathbf{W}(J_1))^T$ has rank r_0, we can choose $\mathbf{y}_i$ with $1 \leq i \leq r_0$ to be an orthonormal basis in $\mathbf{C}^{r_0}$. Therefore, there are r_0 constants a_i such that

$$\mathbf{p} = \sum_{i=1}^{r_0} a_i \mathbf{y}_i, \tag{3.15}$$

and

$$\sum_{i=1}^{r_0} |a_i|^2 = \|\mathbf{p}\|^2 = \sum_{J_0 \leq j < J} \sum_{k=-K}^{K} |b_{j,k}|^2 + \sum_{k=-K}^{K} |c_{J_0,k}|^2 = \|f\|^2. \tag{3.16}$$

From (3.13), only $\mathbf{q}_i[n]$ with $n \in \mathcal{N}_{J_1}$ are known. For $1 \leq i \leq r_0$, we extend $\mathbf{q}_i[n]$ from $n \in \mathcal{N}_{J_1}$ to all integers via

$$\tilde{\mathbf{q}}_i[n] = \frac{1}{\lambda_i(J_1)} \mathbf{w}_n(J_1)(\mathbf{W}(J_1))^T \mathbf{q}_i, \quad n \in \mathbf{Z}. \tag{3.17}$$

By (3.5), (3.15) and (3.17), we have

$$\tilde{x}_{J_1}[n] = \mathbf{w}_n(J_1)\mathbf{p} = \mathbf{w}_n(J_1)\sum_{i=1}^{r_0} a_i\mathbf{y}_i = \sum_{i=1}^{r_0} a_i\mathbf{w}_n(J_1)(\mathbf{W}(J_1))^T\mathbf{q}_i = \sum_{i=1}^{r_0} a_i\lambda_i(J_1)\tilde{\mathbf{q}}_i[n]. \tag{3.18}$$

We now prove that $\lambda_i(J_1) \leq 1$ for $1 \leq i \leq r_0$. Since

$$\begin{aligned}
\|\tilde{\mathbf{q}}_i\|^2 &= \sum_{n=-\infty}^{\infty} |\tilde{\mathbf{q}}_i[n]|^2 = \frac{1}{(\lambda_i(J_1))^2}\sum_{n=-\infty}^{\infty} |\mathbf{w}_n(J_1)(\mathbf{W}(J_1))^T\mathbf{q}_i|^2 \\
&= \frac{1}{(\lambda_i(J_1))^2}\|\mathcal{D}_{J_0,J_1}^{-1}(\mathbf{W}(J_1))^T\mathbf{q}_i\|^2 = \frac{1}{(\lambda_i(J_1))^2}\|(\mathbf{W}(J_1))^T\mathbf{q}_i\|^2 \\
&= \frac{1}{(\lambda_i(J_1))^2}\|\mathcal{D}_{J_0,J_1}P_{\mathcal{N}_{J_1}}\tilde{\mathbf{q}}_i\|^2 = \frac{1}{(\lambda_i(J_1))^2}\|P_{\mathcal{N}_{J_1}}\tilde{\mathbf{q}}_i\|^2 \\
&\leq \frac{1}{(\lambda_i(J_1))^2}\|\tilde{\mathbf{q}}_i\|^2,
\end{aligned}$$

where the property that both $\mathcal{D}_{J_0,J_1}$ and $\mathcal{D}_{J_0,J_1}^{-1}$ preserve the total energy is used, we conclude that $\lambda_i(J_1) \leq 1$ for $1 \leq i \leq r_0$.

Next, we use induction to prove

$$\tilde{x}_{J_1}[n] - \tilde{x}_{J_1}^{(l)}[n] = (I - P_{\mathcal{N}_{J_1}})\sum_{i=1}^{r_0} a_i\lambda_i(J_1)(1-\lambda_i(J_1))^l\tilde{\mathbf{q}}_i[n]. \tag{3.19}$$

When $l = 0$, (3.19) is trivial by (3.10) and (3.18). Assume that (3.19) holds for the lth iteration. Then,

$$\begin{aligned}
&\tilde{x}_{J_1}[n] - \tilde{x}_{J_1}^{(l+1)}[n] \\
&= (I - P_{\mathcal{N}_{J_1}})\mathcal{D}_{J_0,J_1}^{-1}P_{J,K}\mathcal{D}_{J_0,J_1}(\tilde{x}_{J_1}[n] - \tilde{x}_{J_1}^{(l)}[n]) \\
&\overset{1}{=} (I - P_{\mathcal{N}_{J_1}})\mathcal{D}_{J_0,J_1}^{-1}P_{J,K}\mathcal{D}_{J_0,J_1}(I - P_{\mathcal{N}_{J_1}})\sum_{i=1}^{r_0} a_i\lambda_i(J_1)(1-\lambda_i(J_1))^l\tilde{\mathbf{q}}_i[n]) \\
&= (I - P_{\mathcal{N}_{J_1}})\sum_{i=1}^{r_0} a_i\lambda_i(J_1)(1-\lambda_i(J_1))^l(\tilde{\mathbf{q}}_i[n] - \mathcal{D}_{J_0,J_1}^{-1}P_{J,K}\mathcal{D}_{J_0,J_1}P_{\mathcal{N}_{J_1}}\tilde{\mathbf{q}}_i[n]) \\
&\overset{2}{=} (I - P_{\mathcal{N}_{J_1}})\sum_{i=1}^{r_0} a_i\lambda_i(J_1)(1-\lambda_i(J_1))^l(\tilde{\mathbf{q}}_i[n] - \mathbf{w}_n(J_1)(\mathbf{W}(J_1))^T\mathbf{q}_i[n]) \\
&\overset{3}{=} (I - P_{\mathcal{N}_{J_1}})\sum_{i=1}^{r_0} a_i\lambda_i(J_1)(1-\lambda_i(J_1))^{l+1}\tilde{\mathbf{q}}_i[n],
\end{aligned}$$

where step 1 is from the induction assumption, step 2 is from the definitions of $\mathbf{w}_n(J_1)$ and $\mathbf{W}(J_1)$ and step 3 is from (3.17). This proves (3.19) is true for all $l = 0, 1, 2, \cdots$. Therefore,

$$\sum_n |\tilde{x}_{J_1}[n] - \tilde{x}_{J_1}^{(l)}[n]|^2 \leq \sum_n |\sum_{i=1}^{r_0} a_i\lambda_i(J_1)(1-\lambda_i(J_1))^l\tilde{\mathbf{q}}_i[n]|^2$$

$$
\begin{aligned}
&\leq \sum_n \sum_{i=1}^{r_0} |a_i|^2 \sum_{i=1}^{r_0} (1-\lambda_i(J_1))^{2l} \sum_{i=1}^{r_0} |\lambda_i(J_1)\tilde{\mathbf{q}}_i[n]|^2 \\
&\overset{4}{\leq} \|f\|^2 r_0 (1-\lambda_{r_0}(J_1))^{2l} \sum_{i=1}^{r_0} \sum_n |\lambda_i(J_1)\tilde{\mathbf{q}}_i[n]|^2 \\
&= \|f\|^2 r_0 (1-\lambda_{r_0}(J_1))^{2l} \sum_{i=1}^{r_0} \sum_n |\mathbf{w}_n(J_1)(\mathbf{W}(J_1))^T \mathbf{q}_i|^2 \\
&\leq \|f\|^2 r_0 (1-\lambda_{r_0}(J_1))^{2l} \sum_{i=1}^{r_0} \|\mathcal{D}_{J_0,J_1}^{-1} (\mathbf{W}(J_1))^T \mathbf{q}_i\|^2 \\
&= \|f\|^2 r_0 (1-\lambda_{r_0}(J_1))^{2l} \sum_{i=1}^{r_0} \|(\mathbf{W}(J_1))^T \mathbf{q}_i\|^2 \\
&\leq \|f\|^2 r_0 (1-\lambda_{r_0}(J_1))^{2l} \sum_{i=1}^{r_0} \|\mathcal{D}_{J_0,J_1} P_{\mathcal{N}_{J_1}} \tilde{\mathbf{q}}_i\|^2 \\
&= \|f\|^2 r_0 (1-\lambda_{r_0}(J_1))^{2l} \sum_{i=1}^{r_0} \|P_{\mathcal{N}_{J_1}} \tilde{\mathbf{q}}_i\|^2 \\
&= \|f\|^2 r_0 (1-\lambda_{r_0}(J_1))^{2l} \sum_{i=1}^{r_0} \|\mathbf{q}_i\|^2 \\
&= \|f\|^2 r_0^2 (1-\lambda_{r_0}(J_1))^{2l},
\end{aligned}
$$

where step 4 is from (3.14), (3.16) and $0 \leq \lambda_i(J_1) \leq 1$. This proves Lemma 1. □

We next estimate $\|x_{J_1}^{(l)} - \tilde{x}_{J_1}^{(l)}\|$ in the following lemma.

Lemma 2

$$\|x_{J_1}^{(l)} - \tilde{x}_{J_1}^{(l)}\| \leq (l+1)(\Delta_{J_1})^{1/2},$$

where

$$\Delta_{J_1} \triangleq \|P_{\mathcal{N}_{J_1}}(x_{J_1}[n] - \tilde{x}_{J_1}[n])\|^2 = \sum_{n \in \mathcal{N}_{J_1}} |x_{J_1}[n] - \tilde{x}_{J_1}[n]|^2 \to 0, \quad \textit{as } J_1 \to \infty. \tag{3.20}$$

Proof: By the definitions of $x_{J_1}[n]$ and $\tilde{x}_{J_1}[n]$,

$$\|P_{\mathcal{N}_{J_1}}(x_{J_1}[n] - \tilde{x}_{J_1}[n])\|^2 = \sum_{n=-2^{J_1}}^{2^{J_1}} |2^{-J_1/2} f(\frac{n}{2^{J_1}}) - c_{J_1,n}|^2. \tag{3.21}$$

We can rewrite the expression of the right-hand-side of (3.21) and obtain (see [18] for detailed derivation)

$$\|P_{\mathcal{N}_{J_1}}(x_{J_1}[n] - \tilde{x}_{J_1}[n])\|^2 = \sum_{n=-2^{J_1}}^{2^{J_1}} \frac{2^{-J_1}}{(2\pi)^2} \left| \int_{-\infty}^{\infty} \hat{f}(-\omega)(\hat{\phi}(\frac{\omega}{2^{J_1}}) - 1) e^{-ik\omega/2^{J_1}} d\omega \right|^2. \tag{3.22}$$

Let

$$w(J_1) \triangleq \max_{\omega \in [-2^{J_1/2}, 2^{J_1/2}]} \left| \hat{\phi}(\frac{\omega}{2^{J_1}}) - 1 \right|^2 .$$

By the assumption of $\hat{\phi}(\omega)$ being continuous at $\omega = 0$, we have that

$$\lim_{J_1 \to \infty} w(J_1) \to 0. \tag{3.23}$$

Let

$$a(J_1) \triangleq \min \left\{ 2^{J_1/2}, (w(J_1))^{-1/2} \right\}.$$

Then, $\lim_{J_1 \to \infty} a(J_1) = \infty$. Since $\hat{\phi}(\omega) \in L^1(\mathbf{R})$, $\hat{\psi}(\omega) = \hat{\phi}(\frac{\omega}{2}) G(\frac{\omega}{2})$ and $|G(\omega)| \leq 1$, we have $\hat{\psi}(\omega) \in L^1(\mathbf{R})$. Therefore, $f(t) \in V_{J,K}$ implies that $f(t) \in L^1(\mathbf{R})$. That is, if we let

$$\delta_{J_1} = \int_{-\infty}^{\infty} (I - P_{a(J_1)}) |\hat{f}(\omega)| d\omega, \tag{3.24}$$

then,

$$\delta_{J_1} \to 0, \qquad \text{when } J_1 \to \infty. \tag{3.25}$$

By the orthonormality of the wavelet basis, we have

$$\sum_k |\hat{\phi}(\omega + 2k\pi)|^2 = 1, \quad \forall \omega \in \mathbf{R},$$

so that

$$|\hat{\phi}(\omega)| \leq 1, \quad \forall \omega \in \mathbf{R}. \tag{3.26}$$

By using (3.26), we can simplify the right-hand-side of (3.22) as,

$$\begin{aligned}
&\left| \int_{-\infty}^{\infty} \hat{f}(-\omega)(\hat{\phi}(\frac{\omega}{2^{J_1}}) - 1) e^{-ik\omega/2^{J_1}} d\omega \right|^2 \\
&\leq \; 2 \left(\int_{-a(J_1)}^{a(J_1)} \left| \hat{f}(-\omega) \left(\hat{\phi}(\frac{\omega}{2^{J_1}}) - 1 \right) \right| d\omega \right)^2 + 8 \left(\int_{-\infty}^{\infty} (I - P_{a(J_1)}) |\hat{f}(\omega)| d\omega \right)^2 \\
&\leq \; 4\pi \|f\|^2 \int_{-a(J_1)}^{a(J_1)} \left| \hat{\phi}(\frac{\omega}{2^{J_1}}) - 1 \right|^2 d\omega + 8\delta_{J_1}^2 \\
&\leq \; 8\pi \|f\|^2 a(J_1) w(J_1) + 8\delta_{J_1}^2 \\
&\leq \; 8\pi \|f\|^2 (w(J_1))^{1/2} + 8\delta_{J_1}^2 .
\end{aligned}$$

Therefore, by (3.22),

$$\Delta_{J_1} = \|P_{N_{J_1}}(x_{J_1}[n] - \tilde{x}_{J_1}[n])\|^2 \leq \frac{4 + 2^{-J_1}}{\pi} \|f\|^2 (w(J_1))^{1/2} + \frac{4 + 2^{-J_1}}{\pi^2} \delta_{J_1}^2 .$$

Thus, by (3.23) and (3.25), we have proved (3.20). From (3.2)-(3.3) and (3.10)-(3.11), the difference $x_{J_1}^{(l)}[n] - \tilde{x}_{J_1}^{(l)}[n]$ is resulted from the difference $P_{\mathcal{N}_{J_1}}(x_{J_1}[n] - \tilde{x}_{J_1}[n])$. Furthermore, it is straightforward by induction to prove

$$\|x_{J_1}^{(l)} - \tilde{x}_{J_1}^{(l)}\| = \left(\sum_n |x_{J_1}^{(l)}[n] - \tilde{x}_{J_1}^{(l)}[n]|^2\right)^{1/2} \leq (l+1)(\Delta_{J_1})^{1/2}.$$

Thus, Lemma 2 is proved. □

We are now ready to prove the main result (3.8). By (3.1), (3.4), (3.20), Lemmas 1 and 2, we have

$$\begin{aligned}
\|f_{J_1,l} - f\| &= \left(\sum_n |x_{J_1}^{(l)}[n] - c_{J_1,n}|^2\right)^{1/2} \\
&\leq \left(\sum_n |x_{J_1}^{(l)}[n] - \tilde{x}_{J_1}^{(l)}[n]|^2\right)^{1/2} + \left(\sum_n |\tilde{x}_{J_1}^{(l)}[n] - \tilde{x}_{J_1}[n]|^2\right)^{1/2} \\
&\leq (l+1)(\Delta_{J_1})^{1/2} + r_0\|f\|(1 - \lambda_{r_0}(J_1))^l.
\end{aligned}$$

Since the rank $r(J_1) = r_0$, $\lambda_{r_0}(J_1) > 0$. By the property (3.9) and Lemma 2, (3.8) is proved. This completes the proof of Theorem 1.

4 Conclusion

In this paper, we proved that, under certain conditions, the discrete-time signal extrapolation problem converges to its continuous-time counterpart as the sampling rate in the known interval goes to infinity. This work solves one of the open problems stated in [19]. The convergence result obtained in this paper also provides a practical scheme for continuous-time signal extrapolation in a certain wavelet subspace when only discrete samples in an interval are given.

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