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Infinite and Finite Observation Interval Whitening Filters for Certain Fractional Stable Noises

by

George A. Tsihrintzis and Chrysostomos L. Nikias

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Signal and Image Processing Institute
UNIVERSITY OF SOUTHERN CALIFORNIA
Department of Electrical Engineering-Systems
3740 McClintock Avenue, Room 400
Los Angeles, CA 90089-2564 U.S.A.

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Abstract

We extend the Mandelbrot-Van Ness models of fractional Gaussian noise to define linear and log fractional symmetric α-stable (SαS) noises. We present both "engineering" definitions, based on certain fractional integrals of white SαS noise, as well as rigorous mathematical definitions via appropriate linear operators acting on subsets of the Banach space L^α, $1 < \alpha < 2$. For these noises we find a canonical representation and proceed to derive whitening filters for linear and log fractional SαS noises for both infinite and finite observation intervals. The overall style of the report is tutorial complemented with a large number of references which review the relevant literature.

1. INTRODUCTION

Fractional noise, sometimes also referred to as $\frac{1}{f}$ noise, is a nonstationary, self-similar random process having a (generalized) power spectral density of the form:

$$S(f) \approx \frac{1}{|f|^{\beta}}, \quad 0 \leq \beta \leq 2 \quad (\beta \approx 1 \text{ usually}). \tag{1-1}$$

Such processes experimentally arise in electronics [1], seem appropriate to describe long range dependence observed in hydrological data [2, 3] and other geophysical and meteorological phenomena [4, 5], better fit the variation of certain economic quantities and indices [6, 7, 8], and provide appropriate models for certain noises arising in communications systems [9, 10], as well as in the description of textured images [11, 12]. A review of these observations of fractional processes can be found in [13] and an extensive literature search is presented in [14]. Sometimes, color names are attributed to the different spectra corresponding to the range of values of the exponent β in Eq.(1-1). For example, the noises with a power spectrum of the form of Eq.(1-1) and $\beta = 0, 1,$ and 2 are called white, pink, and brown noise, respectively, while noises of the form of Eq.(1-1) and $\beta > 2$ are called black noises [15]. Despite the large literature [14] devoted to analyzing fractional processes, this task remains unfinished although a number of different approaches have been proposed to it.

Early traces on the analysis of self-similar processes can be found in the works of Berger and Mandelbrot [9] and Mandelbrot [10, 16]. In [10], a model was built to explain the self-similarity properties of the clustering of errors in real communications systems and the concepts of "conditional stationarity" and "conditional power spectral density" were introduced. A process was defined to be conditionally stationary, if its observed segment exhibited stationarity, eventhough the entire process was nonstationary. This segment of the process was then embedded in an (infinite) stationary process and the conditional spectrum of the nonstationary process was then defined to be the (usual) spectrum of the embedding process. This procedure was developed upon the philosophy [10] that one needed a non-Wieneran theory to account for fractional noises and, in particular, operations meant to measure the Wiener-Khintchin spectrum might unvoluntarily measure something else, such as the conditional spectrum or any of the generalized spectra examined later in this report.

Mandelbrot [16] was subsequently able to obtain an entire family of $\frac{1}{f}$ (fractional) spectra, ranging from "white noise" with a $\frac{1}{f^0}$ spectrum to "direct current" with a $\frac{1}{f^2}$ spectrum. The development was based on the definition of appropriate renewal processes [16], in which the interevent intervals were independent and followed a distribution that assigned high probability to very small as well as very large sized intervals and low probability to medium sized intervals.[1] It was shown [16] that $\frac{1}{|f|^\beta}$ processes, with $0 \leq \beta \leq 1$, could be Gaussian and obtained from a Wiener theory; however, $\frac{1}{|f|^\beta}$ processes, with $1 < \beta < 2$, required the concept of conditional spectra and behaved "erratically" leading to time averages with asymptotically stable distributions of infinite variance.

Another attempt to model $\frac{1}{f}$ noise was made by Barnes and Allan [19], who proposed modeling the phase noise in oscillators as a fractional integral [20] of a Brownian motion (or, equivalently, of white noise). The corresponding $\frac{1}{f}$-type frequency noise would then be modeled by the increments of the phase noise process. The particular fractional integral that Barnes and Allan proposed was

$$\frac{1}{\Gamma(H+\frac{1}{2})}\int_0^t (t-s)^{H-\frac{1}{2}}\, dB(s), \quad t \geq 0, \tag{1-2}$$

where $\{B(t) : t \geq 0\}$ is an ordinary Brownian motion [21] and $\frac{1}{2} < H < 1$. Unfortunately, this process does not have stationary increments.

A refinement of the Barnes and Allan model, with stationary increments, is the fractional Brownian

[1]The symmetric α-stable distributions ($0 < \alpha \leq 2$) are briefly discussed in section 2; the tails of the corresponding distribution function $Pr(U \leq u)$ of which are algebraic ($Pr(U \leq u) \approx 1 - u^{-\alpha}$ for large u), are examples of such distributions. Mandelbrot [10] suggested the use of such ("Paretian") distributions for the interevent intervals, as a mathematically rigorous development and expansion of the original Berger-Mandelbrot model [9], and showed that, for given $0 < \alpha \leq 2$, processes with power spectrum of the form $f^{\alpha-2}$ could be obtained. Later [16], he also derived similar spectra by considering general distributions having finite moments of order p, where $0 < p \leq 2$ only. For certain mathematical details see also [17] and for further expansions upon these results see also [18].

motion (FBM) of Mandelbrot and Van Ness [22]. This process is defined as

$$B_H(t) = \frac{1}{\Gamma(H+\frac{1}{2})}\{\int_{-\infty}^{0}[|t-s|^{H-\frac{1}{2}} - |s|^{H-\frac{1}{2}}]\,dB(s) + \int_0^t |t-s|^{H-\frac{1}{2}}\,dB(s)\}, \quad t \in R^1, \quad \frac{1}{2} < H < 1, \tag{1-3}$$

and enjoys the property [22] of having, in a certain sense, [2] a stationary derivative, called *fractional* (as opposed to white) Gaussian noise with spectral density f^{1-2H}, $\frac{1}{2} < H < 1$, $0 \neq f \in R^1$. Equivalently, fractional Gaussian noise has a covariance function of the form $\mathcal{R}_{B'_H}(\tau) = V_H H(2H-1)|\tau|^{2H-2}$, $0 \neq \tau \in R^1$, where $V_H =$ var$\{B_H(1)\}$ [22]. In addition to being stationary, the increments of FBM are also self-similar with parameter H, i.e., for all $c > 0$, $t_0 \in R^1$:

$$\{B_H(t_0 + c\tau) - B_H(t_0) : \tau \in R^1\} \equiv \{c^H[B_H(t_0+\tau) - B_H(t_0)] : \tau \in R^1\}, \tag{1-4}$$

where $\equiv$ in here denotes equality of all finite-dimensional distributions. Mandelbrot and Van Ness [22] thus considered ("defined") the power spectral density of the nonstationary process $\{B_H(t) : t \in R^1\}$ to be the "usual" power spectral density of its formal stationary derivative.

The work of Mandelbrot and Van Ness was later expanded upon by Flandrin [23, 24] who presented an alternative "definition" of the power spectral density of the nonstationary FBM of [22]. Namely, Flandrin based his approach on a time-frequency analysis of FBM, via the Wigner-Ville spectrum, as well as a time-scale analysis, via the Continuous and Discrete Wavelet Transform. In both cases, Flandrin showed that the self-similarity and the $\frac{1}{f}$ spectral behavior of FBM are compatible with its time-frequency and time-scale analysis.

A new representation for ("nearly") $\frac{1}{f}$ processes was obtained very recently by Wornell [25]. In there, it was shown that "nearly" $\frac{1}{f}$ processes, i.e., processes with spectral densities $\mathcal{S}(f)$ satisfying:

$$\frac{k_1}{|f|^\beta} \leq \mathcal{S}(f) \leq \frac{k_2}{|f|^\beta}, \quad 0 < k_1 \leq k_2 < \infty, \tag{1-5}$$

[2]Rigorously, this derivative is defined as a generalized Gaussian random process, i.e., as a distribution in the Schwartz sense.

could be constructed via a wavelet orthonormal expansion [26] of the process as

$$x(t) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} d_n^m \psi(2^m t - n), \tag{1-6}$$

where the coefficients d_n^m are Gaussian random variables uncorrelated along and across the scales m and have at each scale m a variance of $2^{-\beta m}$:

$$\mathcal{E}\{d_n^m d_{n'}^{m'}\} = 2^{-\beta m} \delta_{mm'} \delta_{nn'}. \tag{1-7}$$

The "spectrum" $\mathcal{S}$ was defined as

$$\mathcal{S}(f) = \sum_{m=-\infty}^{\infty} 2^{-\beta m} |\Psi(2^{-m} f)|^2, \tag{1-8}$$

and was found to equal:

$$\mathcal{S}(f) = (2^{\beta} - 1) \sum_{m=-\infty}^{\infty} 2^{-\beta m} |\Phi(2^{-m} f)|^2, \tag{1-9}$$

where Φ is the Fourier transform of the scaling function ϕ corresponding to the Fourier transform Ψ of the wavelet ψ [26]. This spectrum could, indeed [25], be bound between two $\frac{1}{f}$ processes, as in Eq.(1-5), thus the Discrete Wavelet Transform of FBM was shown to approximately act as a whitening transform (filter) for fractional Gaussian noise (see also [27]). This construction was employed by Wornell and Oppenheim in [28] to estimate parameters of fractal signals and to reconstruct fractal signals from noisy observations of their Discrete Wavelet Transform. The assumption was made in [28] of Gaussian signal and noise processes and the optimum estimate of the fractal signal was shown to be obtainable via an appropriate Wiener filtering procedure in the Discrete Wavelet Transform domain.

The problem of detection and estimation of signals embedded in fractional Gaussian noise has been extensively addressed by Barton and Poor [29], who examined the structure of the reproducing kernel Hilbert space [30] of FBM and derived exact whitening filters for FBM for both the cases of infinite and finite observation intervals. The work in [29] provides a rigorous framework for addressing more general signal detection and estimation issues associated with a fractional Gaussian noise environment, such as the same authors' suggestion of sequential detection in a similar environment.

Most of the literature has concentrated so far on Gaussian fractional processes, despite the fact that certain non-Gaussian fractional processes might be more appropriate in certain occasions [22, 5]. The probable cause for this seems to be the unavailability at the time of useful results in the theory of non-Gaussian processes. The need to consider non-Gaussian fractional processes has been recognized by at least Mandelbrot and Van Ness [22], Lawrence and Kottegoda [3], and Taqqu [31, 5] for the modeling of certain physical phenomena and for certain electrical engineering applications. A class of non-Gaussian processes that has been receiving increasing attention recently, for both its mathematical challenges and its applications in statistical signal processing, is the class of non-Gaussian symmetric α-stable processes [32]. These processes share many of the properties of Gaussian processes, containing the latter as a subclass, but they also differ from them in many and significant ways. Currently, many of the results and techniques developed for Gaussian signal processing are not readily applicable, or available, for non-Gaussian signal processing. It stands thus a challenge to the signal processing community to address classical signal processing problems, such as signal detection, estimation and reconstruction within a non-Gaussian α-stable framework.

In the subsequent sections, we define fractional stable noises and construct whitening filters for both infinite and finite observation intervals. In particular, in section 2 of this report, we briefly define non-Gaussian symmetric α-stable random variables and processes and present two of their most important characteristic properties. In section 3, a definition is given of two classes of fractional stable noises. A rigorous definition of white and fractional stable noises is given in terms of generalized stable random processes, i.e., linear operators acting on appropriate linear spaces. In section 4, we find a canonical representation of the stable processes of section 3. In section 5, we present whitening filters for the fractional stable noises of section 3 for both infinite and finite observation intervals. In section 6, we briefly discuss discrete-time counterparts to the continuous-time models of the form of fractional autoregressive integrated moving average models. Finally, in section 7, we summarize the key results in the report, draw conclusions, and suggest possible avenues for future research.

2. Review of the Theory of Stable Random Variables and Processes

A real random variable (r.v.) X has a symmetric α-stable (SαS) distribution if its characteristic function ϕ is of the form

$$\phi(\omega) = \mathcal{E}\{e^{i\omega X}\} = \exp(i\delta\omega - \gamma|\omega|^{\alpha}), \quad \omega \in R^1. \tag{2-1}$$

In this equation, α is the *characteristic exponent* ($0 < \alpha \leq 2$), γ is the *dispersion* ($\gamma > 0$), and δ is the *location parameter* ($-\infty < \delta < \infty$) of the stable distribution. Without loss of generality, we are going to assume that $\delta = 0$, similarly to the zero mean assumption for Gaussian distributions. The class of Gaussian r.v.'s constitute examples of SαS distributed r.v.'s with characteristic exponent $\alpha = 2$. The class of r.v.'s with characteristic exponent $\alpha = 1$ consists of exactly the class of Cauchy r.v.'s.

A collection of r.v.'s $\{Z(t) : t \in T\}$, where T is an arbitrary index set, is said to constitute a real SαS stochastic process if all real linear combinations $\sum_{j=1}^{n} \lambda_j Z(t_j)$, $\lambda_j \in R^1$, $n \geq 1$, are SαS r.v.'s of the same characteristic exponent α. A complex-valued r.v. $Z = Z' + iZ''$ is SαS (or rotationally invariant α-stable) if Z', Z'' are jointly SαS and have a radially symmetric distribution. This is equivalent to requiring that for any $z \in C^1$:

$$\mathcal{E}\{e^{i\Re(\bar{z}Z)}\} = \exp(-\gamma|z|^{\alpha}) \tag{2-2}$$

for some $\gamma > 0$. A complex-valued stochastic process $\{Z(t) : t \in T\}$ is SαS if all linear combinations $\sum_{j=1}^{n} \overline{z_j} Z(t_j)$, $z_j \in C^1$, $n \geq 1$, are complex-valued SαS r.v.'s. Note that the overbar denotes the complex conjugate.

Non-Gaussian SαS distributions are characterized by two important properties [32]: (i) Unlike the Gaussian distributions for which pth order moments of all orders $0 < p < \infty$ are finite, non-Gaussian SαS distributions have finite pth order moments only for $0 < p < \alpha$ and (ii) Non-Gaussian SαS distributions have maxima more peaked than those of Gaussian distributions, and algebraic (inverse power), as opposed to exponential, tails. These two properties of SαS distributions may allow more accurate modeling of certain economical, physical, biological, and hydrological phenomena, and may find applications in electrical engineering [33]. An overview of the theory of SαS r.v.'s and processes, presented in tutorial style, can be found in [32]. In this report, we will concentrate on a certain class of SαS processes which exhibit very long range dependence. Such processes are reasonable models for many phenomena in nature and in today's technology [33] and are expected to be more accurate than existing ARMA and similar models.

A concept playing a key role in the theory of SαS (with $1 < \alpha \leq 2$) r.v.'s and processes is that of the *covariation.* The covariation of two complex-valued SαS r.v.'s Z_1, Z_2 is defined as the quantity

$$[Z_1, Z_2]_\alpha = \frac{\mathcal{E}\{Z_1 Z_2^{<p-1>}\}}{\mathcal{E}\{|Z_2|^p\}}\gamma_2, \quad 1 \leq p < \alpha, \quad 1 < \alpha \leq 2, \qquad (2-3)$$

where γ_2 is the dispersion in the characteristic function of the r.v. Z_2 and for any $z \in C^1$: $z^{<p>} \equiv |z|^{p-1}\overline{z}$, $\overline{z}$ being the complex conjugate of z. The above definition is mathematically equivalent [34] to the definition given in [35] and relates to a concept of orthogonality in a Banach space [36]. Since it can be shown [35] that there exists a constant $C(p, \alpha)$, [3] depending solely on p and α ($1 \leq p < \alpha$ and $1 < \alpha \leq 2$), such that for any SαS r.v. Z_2: $\gamma_2^{\frac{p}{\alpha}} = C(p, \alpha)\mathcal{E}\{|Z_2|^p\}$, we have

$$[Z_1, Z_2]_\alpha = C(p, \alpha)\mathcal{E}\{Z_1 Z_2^{<p-1>}\}\gamma_2^{\frac{\alpha}{p}}. \qquad (2-4)$$

By letting $Z_1 = Z_2$, we also observe that $[Z_2, Z_2]_\alpha = \gamma_2$, i.e., the covariation of a r.v. with itself is simply equal to the dispersion in the characteristic function of the r.v. The covariation function of a SαS random process $\{Z(t) : t \in T\}$ is in turn defined as the covariation of the r.v.'s $Z(t)$ and $Z(s)$ for $t, s \in T$. The concept of covariation is a generalization of the usual concept of covariance of Gaussian random variables and processes and reduces to it when $\alpha = 2$. However, several of the properties of the covariance fail to hold in the non-Gaussian SαS case of $\alpha < 2$ [32].

Of importance to this report is a representation theorem that holds for SαS processes [37, 38], similar to the canonical representation theorem for Gaussian processes [39]. We state, in here, the version of this theorem that will be usefull in subsequent sections and refer the reader to [37, 38] for further details and a proof.

[3] In particular, $C(p, \alpha) = [2\Gamma(1 - \frac{p}{\alpha})\Gamma(p)\sin\frac{\pi p}{2}]^{\frac{1}{p}}$, with $\Gamma(\cdot)$ the Gamma function.

Theorem 2.1 *(Canonical representation theorem for SαS processes). Let $\{Z(t) : t \in [0,1]\}$ be a continuous in probability SαS random process ($0 < \alpha \leq 2$). Then, there exists a SαS motion $\{Z_\alpha(t) : t \in [0,1]\}$, i.e., a SαS stochastic process with independent increments, satisfying*

$$\mathcal{E}\{e^{i\omega Z_\alpha(t)}\} = \exp(-t|\omega|^\alpha), \quad t \in [0,1], \quad \omega \in R^1, \tag{2-5}$$

and functions $\{f_t : t \in [0,1]\}$, with $f_t \in L^\alpha([0,1])$, such that

$$\{Z(t) : t \in [0,1]\} \equiv \{\int_0^1 f_t(s)\, dZ_\alpha(s) : t \in [0,1]\}, \tag{2-6}$$

where "≡" denotes equality of all finite-dimensional distributions. We say that the functions $\{f_t : t \in [0,1]\}$ represent *the process $\{Z(t) : t \in [0,1]\}$. Moreover, when $1 < \alpha \leq 2$, the covariation function of the process $\{Z(t) : t \in [0,1]\}$ is given by*

$$[Z(t), Z(s)]_\alpha = \int_0^1 f_t(u)[f_s(u)]^{<\alpha-1>}\, du, \quad t, s \in [0,1]. \tag{2-7}$$

Finally, we state an intermediate result that appears in [40, Theorem 3.1], of which will be making use in the subsequent sections of this report.

Theorem 2.2 *Let $\{Z_\alpha(t) : t \in R^1\}$ be a SαS motion and $\{\zeta_\alpha(\lambda) : \lambda \in R^1)\}$ the SαS process [4] defined by*

$$\zeta_\alpha(\lambda) = \int_{-\infty}^{\infty} \hat{I}_{[0,\lambda]}(t)\, dZ_\alpha(t), \tag{2-8}$$

where $\hat{I}_{[0,\lambda]}$ is the Fourier transform of the indicator function of the interval $[0, \lambda]$. Let $f \in L^\alpha$ with

$$\hat{f}(\lambda) = \int_{-\infty}^{\infty} e^{it\lambda} f(t)\, dt$$
$$\check{f}(\lambda) = \int_{-\infty}^{\infty} e^{-it\lambda} f(t)\, dt$$

the Fourier and inverse Fourier transform, respectively, of the function f. Then

$$\int_{-\infty}^{\infty} f(t)\, dZ_\alpha(t) = \int_{-\infty}^{\infty} \check{f}(\lambda)\, d\zeta_\alpha(\lambda). \tag{2-9}$$

[4] The process $\{\zeta_\alpha(\lambda) : \lambda \in R^1\}$ is an independent increments process only in the Gaussian case $\alpha = 2$.

3. Linear and Log Fractional SαS Noises

We begin this section by defining the linear and log fractional SαS processes. Let $\{Z_\alpha(t) : t \in R^1\}$ be a SαS motion, i.e., a SαS process with independent increments, satisfying

$$\mathcal{E}\{e^{i\omega Z_\alpha(t)}\} = \exp(-t|\omega|^\alpha), \quad t \in R^1, \quad \omega \in R^1. \tag{3-1}$$

The SαS motion $\{Z_\alpha(t) : t \in R^1\}$ can be regarded as the formal integral of white SαS noise:[5]

$$Z_\alpha(t) = \int_0^t W_\alpha(s)\,ds = \int_{-\infty}^{\infty} I_{[0,t]}(s)W_\alpha(s)\,ds, \quad t \in R^1, \tag{3-2}$$

where $\{W_a(t) : t \in R^1\}$ is a white SαS noise process and $I_{[0,t]}$ is the indicator function of the interval $[0, t]$.

We now give the definition [41]:

Definition 3.1 *Let $0 < H < 1$, $a, b \in R^1$, and $\{Z_\alpha(t) : t \in R^1\}$ be a SαS motion. For $0 < \alpha \leq 2$, $H \neq \frac{1}{\alpha}$, we define:*

$$B_{H,\alpha}(a, b; t) = \int_{-\infty}^{\infty} \{a[(t-s)_+^{H-\frac{1}{\alpha}} - (-s)_+^{H-\frac{1}{\alpha}}] + b[(t-s)_-^{H-\frac{1}{\alpha}} - (-s)_-^{H-\frac{1}{\alpha}}]\}\,dZ_\alpha(s), \quad t \in R^1 \tag{3-3}$$

with the convention $o^x = 0$ even for $x < 0$; and for $1 < \alpha < 2$, $H = \frac{1}{\alpha}$, we define:

$$B_{\frac{1}{\alpha},\alpha}(a; t) = a \int_{-\infty}^{\infty} \{\log|t-s| - \log|s|\} dZ_\alpha(s), \quad t \in R^1, \tag{3-4}$$

where in the above, we have adopted the notation $(x)_+ = \max(x, 0)$, $(x)_- = \max(-x, 0)$.

It was shown (originally in [41] for $1 < \alpha < 2$ and later in [42] for all $0 < \alpha < 2$ and arbitrary skewness parameter) that, each line in the parameter space $(a, b) \in R^2$ corresponded to a different in distribution

[5] We assume that the concept of white SαS noise is understood as a generalization of the usual white Gaussian noise with which electrical engineers are familiar. A rigorous definition of white and fractional SαS noises requires the definition of certain linear operators acting on appropriate linear spaces. Such a construction is possible and will be considered later on in this report.

SαS process in the non-Gaussian case $\alpha < 2$, while in the Gaussian case $\alpha = 2$, all different values of the parameters $(a, b) \in R^2$ correspond to multiples of the same Gaussian process, namely the fractional Brownian motion of Mandelbrot and Van Ness [22]. We call [41] $\{B_{H,\alpha}(a,b;t) : t \in R^1\}$, $H \neq \frac{1}{\alpha}$, a *linear fractional SαS motion* and $\{B_{\frac{1}{\alpha},\alpha}(a;t) : t \in R^1\}$ a *log fractional SαS motion.* It was also shown [41] that, both classes of processes have increments which are stationary and self-similar with parameter H, i.e., for all $c > 0$ and all $t_0 \in R^1$:

$$\{B_{H,\alpha}(a,b;t_0+c\tau) - B_{H,\alpha}(a,b;t_0) : \tau \in R^1\} \equiv \{c^H[B_{H,\alpha}(a,b;t_0+\tau) - B_{H,\alpha}(a,b;t_0)] : \tau \in R^1\}, 0 < H < 1, \tag{3-5}$$

where again "$\equiv$" denotes equality of all finite-dimensional distributions, and similarly for $\{B_{\frac{1}{\alpha},\alpha}(a;t) : t \in R^1)\}$. On the other hand, long range dependence in the increments of the processes $\{B_{H,\alpha}(t) : t \in R^1\}$ is observed only under the condition $\frac{1}{\alpha} < H < 1$, also requiring $\alpha > 1$ [5]. This fact also holds in the case of FBM, where long range dependence of the increments of FBM is observed only for $\frac{1}{2} < H < 1$ [9]. From now on, we will concentrate on linear and log fractional SαS processes, defined as in definition 3.1, with the stability exponent α restricted in the range $1 < \alpha < 2$ and self-similarity parameter $\frac{1}{\alpha} \leq H < 1$.

The stationarity of the increments of the processes $\{B_{H,\alpha}(a,b;t) : t \in R^1\}$ submits the idea that corresponding (stationary) fractional SαS noises $\{W_{H,\alpha}(a,b;t) : t \in R^1\}$ can be defined such that

$$B_{H,\alpha}(a,b;t) = \int_0^t W_{H,\alpha}(a,b;s)\,ds, \quad t \in R^1 \tag{3-6}$$

or, that

$$\{W_{H,\alpha}(a,b;t) : t \in R^1\} \equiv \frac{d}{dt}\{B_{H,\alpha}(a,b;t) : t \in R^1\}. \tag{3-7}$$

Unfortunately, it can be shown [41] that, the fractional SαS processes $\{B_{H,\alpha}(a,b;t) : t \in R^1\}$ are a.s. continuous, but a.s. non-differentiable. Thus, new approaches have to replace the formal relation in Eq.(3-7) above. The simplest of all is probably to define the new "smoothed" process

$$B_{H,\alpha;\delta}(a,b;t) = \frac{1}{\delta}\int_t^{t+\delta} B_{H,\alpha}(a,b;s)\,ds, \quad t \in R^1, \tag{3-8}$$

which can now be shown to have an a.s. continuous, a.s. non-differentiable stationary derivative

$$B'_{H,\alpha;\delta}(a,b;t) = \frac{1}{\delta}[B_{H,\alpha}(a,b;t+\delta) - B_{H,\alpha}(a,b;t)]. \qquad (3-9)$$

Provided that δ is small, the process $\{B_{H,\alpha}(a,b;t) : t \in R^1\}$ is almost non-distinguishable from the process $\{B_{H,\alpha;\delta}(a,b;t) : t \in R^1\}$, except for the high frequency effects to which the non-differentiability of the first process is due. We can then identify

$$\{W_{H,\alpha}(a,b;t) : t \in R^1\} \approx \frac{d}{dt}\{B_{H,\alpha;\delta}(a,b;t) : t \in R^1\}. \qquad (3-10)$$

Similar approaches can be followed for the definition of log fractional SαS noises.

Alternatively, let us also present an approach similar to that of Barnes and Allan [19], based on the concepts of fractional calculus [20].

(Linear fractional SαS noise). The linear fractional SαS noise processes $\{W_{H,\alpha}(a,b;t) : t \in R^1\}$ are fractional integrals of a white SαS noise process $\{W_\alpha(t) : t \in R^1\}$ of order $H - \frac{1}{\alpha}$, i.e.,

$$W_{H,\alpha}(a,b;t) = \frac{\Gamma(H - \frac{1}{\alpha} + 1)}{\Gamma(H - \frac{1}{\alpha})} \int_{-\infty}^{\infty} \{a(t-s)_+^{H-\frac{1}{\alpha}-1} + b(t-s)_-^{H-\frac{1}{\alpha}-1}\} W_\alpha(s)\, ds, \quad \frac{1}{\alpha} < H < 1, \quad t \in R^1 \qquad (3-11)$$

Similarly, for the log fractional SαS noises we have

(Log fractional SαS noise). The log fractional SαS noise processes $\{W_{\frac{1}{\alpha},\alpha}(a;t) : t \in R^1\}$ are integrals of a white SαS noise process of the form $\{W_\alpha(t) : t \in R^1\}$

$$W_{\frac{1}{\alpha},\alpha}(a;t) = \int_{-\infty}^{\infty} a|t-s|^{-1} W_\alpha(s)\, ds, \quad t \in R^1. \qquad (3-12)$$

We can now derive the linear and log fractional SαS motions that we defined earlier as integrals of the corresponding fractional SαS noises. Indeed:

$$B_{H,\alpha}(a,b;t) = \int_0^t W_{H,\alpha}(a,b;\tau)\, d\tau, \quad t \in R^1, \qquad (3-13)$$

since from Eq.(3-11) we have:

$$\begin{aligned}
\int_0^t W_{H,\alpha}(a,b;\tau)\,d\tau &= \int_0^t d\tau \frac{\Gamma(H-\frac{1}{\alpha}+1)}{\Gamma(H-\frac{1}{\alpha})}\int_{-\infty}^{\infty} ds\{a(\tau-s)_+^{H-\frac{1}{\alpha}-1}+b(\tau-s)_-^{H-\frac{1}{\alpha}-1}\}W_\alpha(s) \\
&= \int_{-\infty}^{\infty} ds\, W_\alpha(s)\frac{\Gamma(H-\frac{1}{\alpha}+1)}{\Gamma(H-\frac{1}{\alpha})}\int_0^t d\tau\,\{a(\tau-s)_+^{H-\frac{1}{\alpha}-1}+b(\tau-s)_-^{H-\frac{1}{\alpha}-1}\} \\
&= \int_{-\infty}^{\infty} ds\, W_\alpha(s)\{a[(t-s]_+^{H-\frac{1}{\alpha}}-(-s)_+^{H-\frac{1}{\alpha}}]+b[(t-s)_-^{H-\frac{1}{\alpha}}-(-s)_-^{H-\frac{1}{\alpha}}]\} \\
&= B_{H,\alpha}(a,b;t). \qquad (3\text{-}14)
\end{aligned}$$

Similarly, for the log fractional SαS motion of Eq.(3-4) we have

$$B_{\frac{1}{\alpha},\alpha}(a;t) = \int_0^t W_{\frac{1}{\alpha},\alpha}(a;\tau)\,d\tau, \quad t \in R^1. \qquad (3-15)$$

The above "definitions" are only valid within an "engineering" framework. No stable process, such as W_α or $W_{H,\alpha}$, exists since it has infinite dispersion. However, these "definitions" are in the right direction since *generalized* stable random processes can be defined which behave similarly to the above loosely defined linear and log fractional stable noises. Indeed, we have:

Definition 3.2 *(White SαS noise). White SαS ($1 < \alpha \leq 2$) noise is defined as a linear operator*

$$W_\alpha : L^\alpha(R^1) \to L^\alpha(Z_\alpha) \qquad (3-16)$$

with

$$W_\alpha(f) \equiv \int_{-\infty}^{\infty} f(s)\,dZ_\alpha(s), \quad f \in L^\alpha(R^1). \qquad (3-17)$$

Definition 3.3 *(Linear and log fractional SαS noises). Linear ($1 < \alpha \leq 2$, $\frac{1}{\alpha} < H < 1$) and log ($1 < \alpha < 2$, $H = \frac{1}{\alpha}$) fractional SαS noise is defined as a linear operator*

$$W_{H,\alpha} : \Lambda^\alpha(R^1) \to L^\alpha(B_{H,\alpha}) \qquad (3-18)$$

with

$$W_{H,\alpha}(f) \equiv \int_{-\infty}^{\infty} (f * k_{H,\alpha})(s)\,dZ_\alpha(s), \quad f \in \Lambda^\alpha(R^1), \qquad (3-19)$$

where $\Lambda^\alpha(R^1) = \{f \in L^\alpha(R^1) : f * k_{H,\alpha} \in L^\alpha(R^1)\}$. *In this definition, "$*$" denotes the usual convolution operator and the convolution kernels* $k_{H,\alpha}$ *are defined as:*

$$k_{H,\alpha}(a,b;\tau) = \frac{\Gamma(H-\frac{1}{\alpha}+1)}{\Gamma(H-\frac{1}{\alpha})}[a(-\tau)_+^{H-\frac{1}{\alpha}-1} + b(-\tau)_-^{H-\frac{1}{\alpha}-1}], \quad 1<\alpha\le 2, \quad \frac{1}{\alpha}<H<1, \qquad (3-20)$$

and

$$k_{\frac{1}{\alpha},\alpha}(a;\tau) = a|\tau|^{-1}, \quad 1<\alpha<2. \qquad (3-21)$$

Definitions 3.2 and 3.3 above are compatible with the definitions of SαS and linear and log fractional SαS motions, respectively, and reproduce them as

$$Z_\alpha(t) = W_\alpha(I_{[0,t]}) \qquad (3\text{-}22)$$

$$B_{H,\alpha}(a,b;t) = W_{H,\alpha}(I_{[0,t]}) \qquad (3\text{-}23)$$

$$B_{\frac{1}{\alpha},\alpha}(a;t) = W_{\frac{1}{\alpha},\alpha}(I_{[0,t]}). \qquad (3\text{-}24)$$

Eq.(3-22) is readily verified as

$$\begin{aligned} W_\alpha(I_{[0,t]}) &= \int_{-\infty}^{\infty} I_{[0,t]}(s)\,dZ_\alpha(s) \\ &= \int_0^t dZ_\alpha(s) \\ &= Z_\alpha(t). \end{aligned}$$

To verify Eq.(3-23), we make use of theorem 2.2. We then have:

$$\begin{aligned} W_{H,\alpha}(I_{[0,t]}) &= \int_{-\infty}^{\infty} (I_{[0,t]} * k_{H,\alpha})(s)\,dZ_\alpha(s) \\ &= \int_{-\infty}^{\infty} \check{I}_{[0,t]}(\lambda)\check{k}_{H,\alpha}(a,b;\lambda)\,d\zeta_\alpha(\lambda) \\ &= \int_{-\infty}^{\infty} \frac{1-e^{-i\lambda t}}{i\lambda}\check{k}_{H,\alpha}(a,b;\lambda)\,d\zeta_\alpha(\lambda). \end{aligned} \qquad (3\text{-}25)$$

To compute $\check{k}_{H,\alpha}(a,b;\lambda)$, we make use of some results from the theory of fractional calculus [20, page 94] and find

$$\check{k}_{H,\alpha}(a,b;\lambda) = (-1)^{H-\frac{1}{\alpha}+1}\Gamma(H-\frac{1}{\alpha}+1)|\lambda|^{\frac{1}{\alpha}-H}[a\exp\{i\mathrm{sgn}(\lambda)(H-\frac{1}{\alpha})\frac{\pi}{2}\} + b\exp\{-i\mathrm{sgn}(\lambda)(H-\frac{1}{\alpha})\frac{\pi}{2}\}]. \qquad (3-26)$$

Substitution of Eq.(3-26) into Eq.(3-25) gives

$$W_{H,\alpha}(I_{[0,t]}) = \int_{-\infty}^{\infty} g_t(a,b;s)\, dZ_\alpha(s), \qquad (3-27)$$

where

$$g_t(a,b;s) = a[(t-s)_+^{H-\frac{1}{\alpha}} - (-s)_+^{H-\frac{1}{\alpha}}] + b[(t-s)_-^{H-\frac{1}{\alpha}} - (-s)_-^{H-\frac{1}{\alpha}}]$$

is the Fourier transform of $\check{I}_{[0,t]}(\lambda)\check{k}_{H,\alpha}(a,b;\lambda)$. Eq.(3-27) clearly verifies Eq.(3-23). To verify Eq.(3-24), we follow similar steps and consider, instead of the inverse Fourier transform in Eq.(3-26), the inverse Fourier transform

$$\check{k}_{\frac{1}{\alpha},\alpha}(a;\lambda) = i\pi \mathrm{sgn}(\lambda). \qquad (3-28)$$

One comment that can be made here concerns the asymptotic behavior of the dependence structure of the fractional SαS noises that we defined above. We saw that fractional Gaussian noise has a covariance function of the form $\mathcal{R}_{B'_H}(\tau) =$ (constant) $|\tau|^{2H-2}$. SαS processes $\{Z(t) : t \in R^1\}$ do not have finite second moments, thus a new measure for the dependence structure has to be found for the fractional noises of this report. Levy and Taqqu [43] (for further details see also [44]) propose as such the quantity

$$r(\tau) = \mathcal{E}\{\exp(i[Z(\tau) + Z(0)])\} - \mathcal{E}\{\exp(iZ(\tau))\}\mathcal{E}\{\exp(iZ(0))\}, \qquad (3-29)$$

which in the Gaussian case is asymptotically (as $|\tau| \to \infty$) proportional to the covariance function, and show that, for the unit increment process of linear and log fractional SαS motions and as $\tau \to \infty$, $r(\tau)$ behaves as $r(\tau) =$ (constant) $|\tau|^{\alpha H - \alpha}$, if $1 < \alpha \leq 2$, $\frac{1}{\alpha} < H < 1$, and as $r(\tau) =$ (constant) $|\tau|^{1-\alpha}$, if $1 < \alpha < 2$, $H = \frac{1}{\alpha}$. In the following section, we examine the structure of the covariation function of the linear and log fractional SαS motions of this section and derive their corresponding canonical representations according to theorem 2.1.

4. Representation of Linear and Log Fractional SαS Motions

By examining the structure of the covariation function of the linear and log fractional SαS ($1 < \alpha < 2$) motions, we can derive corresponding canonical representations. We have the following theorems:

Theorem 4.1 *Let $\{B_{H,\alpha}(a,b;t) : t \in [0,1]\}$ be a linear fractional SαS ($1 < \alpha \leq 2$) process restricted to the compact interval $[0,1]$, with $B_{H,\alpha}(a,b;t)$ as in Eq.(3-3). Then:*

$$B_{H,\alpha}(a,b;t) = \int_0^1 f_t(u)\, dZ_\alpha(u), \tag{4-1}$$

where

$$f_t(u) = \frac{\Gamma(H - \frac{1}{\alpha} + 1)}{\Gamma[\frac{1}{2}(1 - \alpha + \alpha H)]} I_{[0,t]}(u) u^{-\frac{1}{2}(1-\alpha+\alpha H)}$$
$$\int_0^t \tau^{\frac{1}{2}(1-\alpha+\alpha H)} [a(\tau - u)_+^{-\frac{1}{2}(1+\alpha-\alpha H)} + b(\tau - u)_-^{-\frac{1}{2}(1+\alpha-\alpha H)}]\, d\tau. \tag{4-2}$$

Proof The derivation of Eq.(4-2) is algebraically complicated and requires the use of Eq.(15) in [45]. The result can be straightforwardly verified, however, by substitution of Eq.(4-2) into Eq.(4-1).

Similarly, for the log fractional SαS motions of Eq.(3-4), a canonical representation takes the form

Theorem 4.2 *Let $\{B_{\frac{1}{\alpha},\alpha}(a;t) : t \in [0,1]\}$ be a log fractional SαS ($1 < \alpha < 2$) process restricted to the compact interval $[0,1]$, with $B_{\frac{1}{\alpha},\alpha}(a;t)$ as in Eq.(3-4). Then:*

$$B_{\frac{1}{\alpha},\alpha}(a;t) = \int_0^1 f_t(u)\, dZ_\alpha(u), \tag{4-3}$$

where

$$f_t(u) = \frac{a}{\Gamma(1 - \frac{\alpha}{2})} I_{[0,t]}(u) u^{\frac{\alpha}{2}-1} \int_0^t \tau^{1-\frac{\alpha}{2}} |\tau - u|^{-\frac{1}{2}\alpha}\, d\tau. \tag{4-4}$$

5. Whitening Filters for Linear and Log Fractional SαS Noises

In this section, we derive infinite and finite observation interval whitening filters for the linear and log fractional SαS noises defined in section 3. We thus express the SαS motion $\{Z_\alpha(t) : t \in T\}$ in terms of an observed linear $\{B_{H,\alpha}(a,b;t) : t \in T\}$, $\frac{1}{\alpha} < H < 1$, or log $\{B_{\frac{1}{\alpha},\alpha}(a;t) : t \in T\}$ SαS motion, $1 < \alpha < 2$, where T is either the entire real line R^1 (infinite observation interval) or the compact interval $[0,1]$[6] thereof (finite observation interval). For the case when $T = R^1$, we have the following theorems:

[6] Appropriate translation and scaling make the formulation valid for any compact interval $T = [c_1, c_2]$ on the real line.

Theorem 5.1 *(Infinite observation interval for linear fractional SαS noise). Let $1 < \alpha \le 2$, $\{Z_\alpha(t) : t \in R^1\}$ be a SαS motion and $\{B_{H,\alpha}(a,b;t) : t \in R^1\}$, $\frac{1}{\alpha} < H < 1$, be the corresponding linear fractional SαS motion obtained via Eq.(3-3). The SαS motion $\{Z_\alpha(t) : t \in R^1\}$ can be expressed in terms of the linear fractional SαS motion $\{B_{H,\alpha}(a,b;t) : t \in R^1\}$ via the infinite interval whitening filtering procedure* [7]

$$\begin{aligned} Z_\alpha(t) &= \frac{1}{\Gamma(H-\frac{1}{\alpha}+1)\Gamma(\frac{1}{\alpha}-H+1)} \\ &\int_{-\infty}^{\infty} \{a^{-1}[(t-s)_+^{\frac{1}{\alpha}-H} - (-s)_+^{\frac{1}{\alpha}-H}] + b^{-1}[(t-s)_-^{\frac{1}{\alpha}-H} - (-s)_-^{\frac{1}{\alpha}-H}]\}\, dB_{H,\alpha}(a,b;s). \quad (5\text{-}1) \end{aligned}$$

In the above, we consider $a^{-1} = 0$ if $a = 0$ and similarly for b.

Proof To express $Z_\alpha(t)$ in terms of a linear fractional SαS motion, we need to find a function $g_t \in \Lambda^\alpha$, such that

$$W_{H,\alpha}(g_t) = Z_\alpha(t), \quad t \in R^1,$$

or, equivalently

$$g_t * k_{H,\alpha} = I_{[0,t]}, \quad t \in R^1,$$

or, according to theorem 2.2

$$\check{g}_t(\lambda) = \frac{\check{I}_{[0,t]}(\lambda)}{\check{k}_{H,\alpha}(\lambda)}.$$

Substitution of Eq.(3-26) into the last equation gives

$$\check{g}_t(\lambda) = (1 - e^{-it\lambda}) \frac{(-1)^{\frac{1}{\alpha}-H-1}}{\Gamma(H-\frac{1}{\alpha}+1)} \frac{|\lambda|^{H-\frac{1}{\alpha}}}{i\lambda} [a^{-1}\exp\{-i\mathrm{sgn}(\lambda)(\frac{1}{\alpha}-H)\frac{\pi}{2}\} + b^{-1}\exp\{i\mathrm{sgn}(\lambda)(\frac{1}{\alpha}-H)\frac{\pi}{2}\}],$$

and thus

$$g_t(s) = \frac{1}{\Gamma(H-\frac{1}{\alpha}+1)\Gamma(\frac{1}{\alpha}-H+1)} [a^{-1}\{(t-s)_+^{\frac{1}{\alpha}-H} - (-s)_+^{\frac{1}{\alpha}-H}\} + b^{-1}\{(t-s)_-^{\frac{1}{\alpha}-H} - (-s)_-^{\frac{1}{\alpha}-H}\}],$$

which proves theorem 5.1.

[7] Heuristically, the white SαS noise is expressed in terms of a linear fractional SαS noise as

$$W_\alpha(t) = \frac{1}{\Gamma(H-\frac{1}{\alpha}+1)\Gamma(\frac{1}{\alpha}-H)} \int_{-\infty}^{\infty} \{a^{-1}(t-s)_+^{\frac{1}{\alpha}-H-1} + b^{-1}(t-s)_-^{\frac{1}{\alpha}-H-1}\} W_{H,\alpha}(a,b;s)\, ds.$$

Theorem 5.2 *(Infinite observation interval for log fractional SαS noise). Let* $1 < \alpha < 2$, $\{Z_\alpha(t) : t \in R^1\}$ *be a SαS motion and* $\{B_{\frac{1}{\alpha},\alpha}(a;t) : t \in R^1\}$ *the corresponding log fractional SαS motion obtained via Eq.(3-4). The SαS motion* $\{Z_\alpha(t) : t \in R^1\}$ *can be expressed in terms of the log fractional SαS motion* $\{B_{\frac{1}{\alpha},\alpha}(a;t) : t \in R^1\}$ *via the infinite interval whitening filtering procedure* [8]

$$Z_\alpha(t) = -\frac{a^{-1}}{\pi^2}\int_{-\infty}^{\infty}\{\log|t-s| - \log|s|\}\, dB_{\frac{1}{\alpha},\alpha}(a;s). \qquad (5-2)$$

Proof We follow the same steps as in the proof of theorem 5.1 and seek a function $g_t \in \Lambda^\alpha$, such that

$$\check{g}_t(\lambda) = \frac{\check{I}_{[0,t]}(\lambda)}{\check{k}_{\frac{1}{\alpha},\alpha}(\lambda)},$$

which gives, if Eq.(3-28) is taken into account:

$$\check{g}_t(\lambda) = \frac{a^{-1}}{\pi}\frac{1-e^{-it\lambda}}{i\lambda}\text{isgn}\lambda.$$

Thus [46, p. 12]

$$g_t(s) = -\frac{a^{-1}}{\pi^2}(\log|t-s| - \log|s|),$$

which proves theorem 5.2.

Theorem 5.3 *(Finite observation interval for linear fractional SαS noise). Let* $\{B_{H,\alpha}(a,b;t) : t \in [0,1]\}$ *be a linear fractional SαS* ($1 < \alpha \le 2$) *process restricted to the compact interval* $[0,1]$, *with* $B_{H,\alpha}(a,b;t)$ *as in Eq.(3-3). There exists a process* $\{Z_\alpha(t) : t \in [0,1]\}$, *which is a SαS motion on the interval* $[0,1]$, *such that for* $t \in [0,1]$:

$$\begin{aligned} Z_\alpha(t) &= \frac{1}{\Gamma(H-\frac{1}{\alpha}+1)\Gamma[-\frac{1}{2}(1+\alpha-\alpha H)]}\int_0^t u^{\frac{1}{2}(1-\alpha+\alpha H)} \\ &\quad d_u\int_0^t \tau^{-\frac{1}{2}(1-\alpha+\alpha H)}[a^{-1}(u-\tau)_+^{-\frac{1}{2}(1-\alpha+\alpha H)} + b^{-1}(u-\tau)_-^{-\frac{1}{2}(1-\alpha+\alpha H)}]\, dB_{H,\alpha}(a,b;\tau), \quad (5\text{-}3) \end{aligned}$$

[8] Heuristically, the white SαS noise is expressed in terms of a log fractional SαS noise as

$$W_\alpha(t) = -\frac{a^{-1}}{\pi^2}\int_{-\infty}^{\infty}|t-s|^{-1}W_{\frac{1}{\alpha},\alpha}(s)\,ds.$$

where $a^{-1} = 0$ *if* $a = 0$ *and similarly for* b.

Proof We have:

$$\frac{1}{\Gamma[-\frac{1}{2}(1+\alpha-\alpha H)]}\int_0^t \tau^{-\frac{1}{2}(1-\alpha+\alpha H)}[a^{-1}(t-\tau)_+^{-\frac{1}{2}(1-\alpha+\alpha H)} + b^{-1}(t-\tau)_-^{-\frac{1}{2}(1-\alpha+\alpha H)}]\,dB_{H,\alpha}(a,b;\tau) =$$

$$= \frac{1}{\Gamma[-\frac{1}{2}(1+\alpha-\alpha H)]}\int_0^1 I_{[0,t]}(\tau)\tau^{-\frac{1}{2}(1-\alpha+\alpha H)}[a^{-1}(t-\tau)_+^{-\frac{1}{2}(1-\alpha+\alpha H)} + b^{-1}(t-\tau)_-^{-\frac{1}{2}(1-\alpha+\alpha H)}]$$
$$dB_{H,\alpha}(a,b;\tau) =$$

$$= \frac{1}{\Gamma[-\frac{1}{2}(1+\alpha-\alpha H)]}\int_0^1 \{\int_0^t \tau^{-\frac{1}{2}(1-\alpha+\alpha H)}[a^{-1}(t-\tau)_+^{-\frac{1}{2}(1-\alpha+\alpha H)} + b^{-1}(t-\tau)_-^{-\frac{1}{2}(1-\alpha+\alpha H)}]\,d_\tau f_\tau(u)\}$$
$$dZ_\alpha(u).$$

But, it is straightforward to show that

$$\frac{1}{\Gamma[-\frac{1}{2}(1+\alpha-\alpha H)]}\int_0^t \tau^{-\frac{1}{2}(1-\alpha+\alpha H)}[a^{-1}(t-\tau)_+^{-\frac{1}{2}(1-\alpha+\alpha H)} + b^{-1}(t-\tau)_-^{-\frac{1}{2}(1-\alpha+\alpha H)}]\,d_\tau f_\tau(u)$$
$$= \Gamma(H - \frac{1}{\alpha} + 1)I_{[0,t]}(u)u^{-\frac{1}{2}(1-\alpha+\alpha H)}.$$

This last relation, clearly verifies theorem 5.3.

Theorem 5.4 *(Finite observation interval for log fractional SαS noise). Let* $\{B_{\frac{1}{\alpha},\alpha}(a;t) : t \in [0,1]\}$ *be a log fractional SαS* $(1 < \alpha < 2)$ *process restricted to the compact interval* $[0,1]$, *with* $B_{\frac{1}{\alpha},\alpha}(a;t)$ *as in Eq.(3-4). There exists a process* $\{Z_\alpha(t) : t \in [0,1]\}$, *which is a SαS motion on the interval* $[0,1]$, *such that for* $t \in [0,1]$

$$Z_\alpha(t) = \frac{a^{-1}}{\Gamma(-\frac{\alpha}{2})}\int_0^t u^{1-\frac{\alpha}{2}}\,d_u \int_0^t \tau^{\frac{\alpha}{2}-1}|u-\tau|^{\frac{\alpha}{2}-1}\,dB_{\frac{1}{\alpha},\alpha}(a;\tau). \qquad (5-4)$$

Proof The proof is obtained by following steps similar to those followed in the proof of theorem 5.3.

6. Fractional ARIMA(p,d,q) Models

In this section, we present difference models (see also [47] for a treatment of the Gaussian case) for linear fractional SαS noises which may be more appropriate for time series modeling and analysis than the continuous-time models of the previous sections. To this end, we assume that discrete-time random processes are indexed by a subscript variable t taking on integer values ($t = \ldots, -1, 0, 1, \ldots$) and define the backward shift operator B as

$$Bx_t = x_{t-1}. \tag{6-1}$$

Let d be a real number, so far arbitrary, and define the linear fractional backward difference operator Δ^d via the binomial expansion

$$\Delta^d = (1-B)^d = \sum_{k=0}^{\infty} \frac{d(d-1)\ldots(d-k+1)}{k!}(-B)^k = \sum_{k=0}^{\infty} \frac{d!}{k!(d-k)!}(-B)^k. \tag{6-2}$$

We can then have the following definitions:

White discrete-time SαS noise

White discrete-time SαS noise is defined as a collection $\{\epsilon_t\}_{t=-\infty}^{t=\infty}$ of independent, identically distributed random variables, each of which follows a SαS distribution of dispersion γ.

The following definition formally defines an ARIMA$(0, d, 0)$ process $\{B_t\}_{t=-\infty}^{t=\infty}$.

Linear fractional discrete-time SαS noise

Linear fractional discrete-time SαS noise is defined as a collection $\{B_t\}_{t=-\infty}^{t=\infty}$ of random variables each of which satisfies[9]

$$\Delta^d B_t = \epsilon_t, \quad t \in Z^1, \tag{6-3}$$

with $\{\epsilon_t\}_{t=-\infty}^{t=\infty}$ a white SαS noise.

The following theorem holds:

[9] In this definition, we consider fractional noises arising via a causal filtering operation on white noise. This is done only for simplicity and more general, non-causal fractional noises can be defined as a weighted sum of a causal and an anticausal term as was done in the previous sections of this report for the continuous-time case.

Theorem 6.1 *Let $\{B_t\}_{t=-\infty}^{t=\infty}$ be an ARIMA(0, d, 0) process as in Eq.(6-3). Then: (1) When $d < \frac{\alpha-1}{\alpha}$, $\{B_t\}_{t=-\infty}^{t=\infty}$ is stationary and admits the infinite moving average representation*

$$B_t = \sum_{k=0}^{\infty} \psi_k \epsilon_{t-k}, \qquad (6-4)$$

where

$$\psi_k = \frac{d(d+1)\ldots(d+k-1)}{k!} = \frac{(d+k-1)!}{k!(d-k)!}. \qquad (6-5)$$

As $k \to \infty$, $\psi_k \approx \frac{k^{d-1}}{(d-1)!}$. (2) When $d > \frac{1-\alpha}{\alpha}$, $\{B_t\}_{t=-\infty}^{t=\infty}$ is invertible and admits the infinite autoregressive representation

$$\sum_{k=0}^{\infty} \pi_k B_{t-k} = \epsilon_t, \qquad (6-6)$$

where

$$\pi_k = \frac{-d(-d+1)\ldots(-d+k-1)}{k!} = \frac{(-d+k-1)!}{k!}. \qquad (6-7)$$

As $k \to \infty$, $\pi_k \approx \frac{k^{-d-1}}{(-d-1)!}$.

Proof (1) For the Z-transforms $\hat{B}$ and $\hat{\epsilon}$ of the random sequences $\{B_t\}$ and $\{\epsilon_t\}$, respectively, we have $\hat{B} = (1-z^{-1})^{-d}\hat{\epsilon}$. Using the binomial expansion $(1-z^{-1})^{-d} = \sum_{k=0}^{\infty} \psi_k z^{-k}$, valid for $|z| \geq 1$, we get the moving average representation of Eq.(6-4), where $\psi_k \approx \frac{k^{d-1}}{(d-1)!}$ for $k \to \infty$. Since the random variables e_t are independent SαS with finite variance γ, for each t the series will converge in the α-mean to a SαS random variable B_t of finite dispersion $\gamma \sum_{k=0}^{\infty} |\psi_k|^{\alpha}$ [35] if and only if $\alpha(d-1) < -1$, i.e., if and only if $d < \frac{\alpha-1}{\alpha}$. (2) We need to follow steps similar to those in (1), only considering $-d$ in the place of d.

Note We can consider $d = H - \frac{1}{\alpha}$. Then, for the process $\{B_t\}$ to be stationary and invertible, we need to have $0 < \frac{2}{\alpha} - 1 < H < 1$, a condition compatible with the definitions of the continuous-time linear fractional SαS processes of the previous sections.

The ARIMA$(0, d, 0)$ model of Eq.(6-3) can be generalized to an ARIMA(p, d, q) model as

$$\phi(B)\Delta^d x_t = \theta(B)\epsilon_t, \qquad (6-8)$$

where

$$\phi(z) = 1 - \phi_1 z^{-1} - \ldots - \phi_p z^{-p}$$
$$\theta(z) = 1 - \theta_1 z^{-1} - \ldots \theta_q z^{-q}.$$

are polynomials and $\{\epsilon_t\}$ is white SαS noise. In this case, the dependence structure of the process $\{x_t\}$ is affected hyperbolically by the parameter d and exponentially by the parameters p, q. Thus, short term dependence is controlled by the parameters p, q, while long term dependence is modeled with the parameter d. We have the following theorem, the proof of which is easy to obtain:

Theorem 6.2 *Let $\{x_t\}_{t=-\infty}^{t=\infty}$ be an ARIMA(p, d, q) process as in Eq.(6-8). Then: (1) The process $\{x_t\}$ is stationary if $d < \frac{\alpha-1}{\alpha}$ and the roots of the polynomial equation $\phi(z) = 0$ lie inside the unit circle. (2) The process $\{x_t\}$ is invertible if $d > \frac{1-\alpha}{\alpha}$ and the roots of the equation $\theta(z) = 0$ lie inside the unit circle.*

7. Conclusions

We have addressed the problem of removing long range dependencies in data modeled as fractional stable processes. The particular fractional stable processes that we concentrated on were linear and log fractional stable motions. For these classes of processes we defined in a mathematically rigorous manner the corresponding fractional stable noise processes and also gave an "engineering" definition that provides deeper insight into their structure. We proceeded to derive whitening filters for both the cases of an infinite and a finite (compact) observation interval. We saw that for the case of an infinite observation interval, Fourier transform techniques almost similar to those employed in the theory of Gaussian processes, are applicable. However, for the case of a finite observation interval, the Wiener-Hopf theory is no longer applicable. The covariation function of a stable process is not the entire counterpart of the covariance function of Gaussian processes. In particular, two processes with zero covariation are not always independent, eventhough this is a necessary condition. Thus whitening of a stable process requires use of the canonical representation theorem for stable processes and an attempt to invert the stochastic integral equation implied by the canonical representation, a task that is non-trivial and for the accomplishment of which no general theory is available. In here, we performed this task for the two classes of fractional stable processes. It seems that in this process, it may generally be useful to employ results from the corresponding framework of Gaussian signal processing to gain insight.

In the future, it is worth to examine other classes of fractional stable processes, such as the harmonizable and the sub-Gaussian fractional stable processes. It also seems promising to examine the structure of optimum receivers for communications systems operating in similar fractional stable noise environments. Such research is currently underway and the results will be reported in the near future.

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