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On Symmetric Stable Models for Impulsive Noise

by

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Abstract

First-order statistics of impulsive noise processes are developed from the filtered-impulse mechanism. Under appropriate assumptions on the spatial and temporal distributions of noise sources and the propagation conditions, we show that the instantaneous amplitude of the received noise obeys the symmetric stable distribution, which is a natural generalization of the Gaussian distribution and enjoys many of its familiar properties. In the case of narrowband reception, the joint distribution of the quadrature components of the received noise is isotropic stable. The noise phase is then shown to be uniformly distributed on $[0, 2\pi]$ and independent of the envelope, as in the Gaussian case. The distribution of the envelope, on the other hand, is a heavy-tailed generalization of the Rayleigh distribution. Compared with existing models, such as Middleton's statistical-physical canonical models, the symmetric stable model is much simpler and mathematically more appealing. Direct comparisons with experimental data show that this model fits closely a variety of non-Gaussian noise.

1 Introduction

It is a common practice in statistical communication theory to assume that communication channels are corrupted by additive Gaussian noise processes. In many situations, this assumption is reasonable and can be justified by the central limit theorem. It also greatly simplifies the implementation and analysis of the receiver structures. Unfortunately, there are also many communication channels encountered in practice that are highly non-Gaussian, many of which are the so-called impulsive channels. These impulsive channels tend to produce large-amplitude interferences or sharp noise spikes more frequently than one would expect from the Gaussian channels. For example, the main source of interference in extremely low frequency (ELF) communications is the atmospheric noise, where local thunderstorm activities may produce such sharp noise spikes [1]. Other examples of impulsive communication channels include underwater sonar communication channels, urban radio networks and certain telephone lines [2, 3, 4].

For impulsive channels the mathematically appealing Gaussian noise model is no longer appropriate for designing optimal signal processing algorithms, because the large dynamic ranges of impulsive noise generally result in significant performance degradation for systems optimized under the Gaussian assumption. A well-known example is the matched filter for coherent reception of deterministic signals in Gaussian white noise. If the channel noise is actually impulsive, serious degradation in performance occurs, such as increased false alarm rate or error probability [5]. On the other hand, a modest degree of nonlinear signal processing based on the actual noise statistics can lead to a much better receiver than the matched filter [6, 7]. Consequently, knowledge of the statistical properties of impulsive noise processes is essential for designing effective signal detection and estimation algorithms.

There have been considerable efforts, over the last forty years, in developing accurate statistical models for non-Gaussian impulsive noise. The models that have been proposed so far may be roughly categorized into two groups: empirical models and statistical-physical models. Empirical models are the results of attempting to fit the experimental data by familiar mathematical expressions without considering physical grounds of the noise process. Commonly used empirical models include the hyperbolic distribution, Hall's generalized "t" distribution and Gaussian mixture models [10, 11, 12]. Empirical models are usually simple and thus may lead to analytically tractable signal processing algorithms. However, they may not be motivated by the physical mechanism that generates the noise process. Hence, their parameters are often physically meaningless. In addition, applications of the empirical models are usually limited to particular situations.

Statistical-physical models, on the other hand, are derived from the underlying physical process giving rise to the noise, which takes into account the distribution of noise sources in space and time, and their propagation to the receiver [13, 14, 9]. Among them, some of the most general models are those developed by Middleton [3]. Middleton's models may be divided into three classes according to the bandwidth of the noise relative to that of the receiver. The Class A model is used to describe the first-order statistics of narrowband noises, i.e., those with spectra comparable to or narrower than the receiver passband, whereas the Class B noise has a bandwidth broader than the passband of the receiver. The Class C model is a linear mixture of the Class A and Class B models. Each class is parameterized by physically meaningful and measurable parameters. In addition, these models are canonical in that the mathematical forms do not change with the changing physical conditions, and are known to fit closely a variety of non-Gaussian noise encountered in practice.

Although Middleton's canonical statistical-physical models for impulsive noise provide a realistic description of the underlying physics of noise processes, they are mathematically involved and not easy to derive. This is particularly true for his Class B model. It has seven parameters, one of which is purely empirical and hence in no way related to the underlying physical situation. In addition, it has to be approximated by two distinct distribution functions. Consequently, it is difficult to accurately estimate the model parameters from a finite number of observations [15]. Furthermore, mathematical approximations are used in its derivation in order to get usable expressions for the characteristic and density functions. As it is pointed out in [16], these approximations are equivalent to changes in the assumed physics of the noise. Since these approximations and the corresponding changes in the basic assumptions about the noise physics can not be directly related, it is not obvious how the final mathematical formulas and the physical scenario are connected.

In this paper, we present an analytically tractable and mathematically appealing model for certain man-made and natural impulsive phenomena, such as automobile ignition noise, atmospheric noise and noise observed on telephone lines. Specifically, *symmetric stable* distributions are suggested as an appropriate model for the first-order statistics of a general class of impulsive noise. It is shown that the stable model can be derived from the basic filtered-impulse mechanism of the noise process under appropriate assumptions on the distribution of noise sources in space and time and their propagation conditions. These assumptions are simple, yet reasonably realistic. As a result, the stable model is much less complicated than Middleton's models and more accessible to engineers. In particular, it has only two model parameters, both of which are physically meaningful and can be determined by the underlying physics of the noise process. Excellent agreement between the stable model and experiment is demonstrated for several types of man-made and natural impulsive noise, as shown by a variety of examples in the paper.

The use of stable distributions as a basic statistical modeling tool can also be justified by the generalized central limit theorem, which states that the limit distribution of the sum of independent and identically distributed random variables must be stable [17]. Hence, if an observed signal or noise can be thought of as the result of a large number of independent and identically distributed effects, the generalized central limit theorem suggests that the stable model may be appropriate. Stable distributions also enjoy many familiar properties of the Gaussian distribution such as the stability property, and in fact include the Gaussian distribution as a special case. They have been found to provide useful statistical models in many disciplines, such as financial economics, physics and electrical engineering. For a tutorial on the properties of stable distributions, their applications, and other topics in stable signal processing, see [18] and the references therein.

The paper is organized as follows. Section 2 outlines the filtered-impulse mechanism for impulsive noise processes. The spatial and temporal distributions of noise sources and the propagation conditions are also discussed in this section. In Section 3 it is shown that the first-order distribution of the instantaneous noise amplitude is symmetric stable. The basic properties of symmetric stable distributions, including the stability property and the generalized central limit theorem, are discussed in details in Section 4. Narrowband reception is considered in Section 5. It is shown that the quadrature components of the received narrowband noise are jointly isotropic stable. Based on this result, the first-order statistics of the noise envelope and phase are also derived. Section 6 compares the present symmetric stable model for impulsive noise with existing models and experimental data, followed by concluding remarks in Section 7.

2 Filtered-Impulse Mechanism of Noise Processes

Most of the natural and man-made impulsive interferences may be considered as the results of a large number of spatially and temporally distributed sources which produce random noise pulses of short duration. A typical example of such noise sources is the lighting discharges of thunderstorm activity, which are the main cause of atmospheric noise. Other examples include ice cracking, automobile ignition and telephone switching transients. The noise received at any location is the superposition of these pulses, and hence depends on the spatial and temporal distributions of the individual sources and the propagation of their pulses to the receiver.

On the basis of these observations, the familiar filtered-impulse mechanism [19, 20, 13, 3] is used in the development of the present model for impulsive noise. By this mechanism, the impulsive noise is considered to consist of the linear superposition of independent, randomly occurring elementary pulses originated from sources distributed over both space and time according to the Poisson law. The waveforms of these elementary pulses are assumed to be identical, although their amplitudes, phases and even their durations may vary in a random way with prespecified distributions. The strength of received pulses is assumed to decrease with the distance from the sources to the point of observation in a known fashion. Simplifications of the source distributions and propagation processes are introduced to yield analytically tractable results.

2.1 Noise Processes

Let us assume, without loss of generality, that the origin of the spatial coordinate system is at the point of observation. The time axis is taken in the direction of past with its origin at the time of observation, i.e., t is the time length from the time of pulse occurrence to the time of observation.

Consider a region Ω in R^n, where R^n may be a line ($n = 1$), a plane ($n = 2$) or the entire space ($n = 3$). For simplicity, we assume that Ω is a cone with vertex at the point of observation. Inside this region, there is a collection of noise sources (e.g., lighting discharges, automobile ignition) which randomly generate transient pulses. It is assumed that all sources share a common random mechanism so that these elementary pulses have the same type of waveform, $aD(t;\underline{\theta})$, where the symbol $\underline{\theta}$ represents a collection of time-invariant random parameters that determine the scale, duration, etc., and a is a random amplitude. We shall further assume that only a countable number of such sources exist inside the region Ω, distributed at random positions $\mathbf{x}_1, \mathbf{x}_2, \cdots$. The source-to-receiver distance will be designated by the random variable r, i.e., $r_i = |\mathbf{x}_i|$. These sources emit pulses, $a_i D(t;\underline{\theta}_i), i = 1, 2, \cdots$, independently at random time $t_1, t_2, \cdots$, respectively. This implies that the random amplitudes $\{a_1, a_2, \cdots\}$ and the random parameters $\{\underline{\theta}_1, \underline{\theta}_2, \cdots\}$ are both i.i.d sequences with the prespecified probability densities $p_a(a)$ and $p_{\underline{\theta}}(\underline{\theta})$ respectively. The location $\mathbf{x}_i$ and emission time t_i of the ith source, and the random parameter $\underline{\theta}_i$ and amplitude a_i are assumed to be independent for all $i = 1, 2, \cdots$. And all of them are also independent of all other random variables. The distribution of random amplitude

a, $p_a(a)$, is assumed to be symmetric, i.e.,

$$p_a(a) = p_a(-a), \tag{1}$$

which in particular implies that the mean of the noise is zero.

When an elementary transient pulse $aD(t;\underline{\theta})$ passes through the medium and the receiver, it will be distorted and attenuated. Let the noise pulse emitted from a source located at $\mathbf{x}$ possess a waveform representable at the output of the receiver as a function of time, t, by $W(t;\mathbf{x},\underline{\theta})$. The relation between $W(t;\mathbf{x},\underline{\theta})$ and $aD(t;\underline{\theta})$ will be discussed shortly. The receiver is assumed to linearly superimpose the noise pulses so that the observed instantaneous amplitude of the noise at the output of the receiver and at the time of observation is

$$X = \sum_{i=1}^{N} W(t_i;\mathbf{x}_i,\underline{\theta}_i) \tag{2}$$

where N is the total number of elementary noise pulses arriving at the receiver at the time of observation. More will be said about N later.

The statistical properties of the received noise amplitude X depend on two major factors: the prevailing propagation conditions and the distribution of noise sources in both space and time. Each of them is examined next in details.

2.2 Propagation Conditions

The physics of propagation and reception controls the strengths and waveforms of the received transient noise pulses. It can be determined from a knowledge of the beam patterns of source and antenna, source location, impulse response of receiver, and etc., as Middleton has demonstrated in [21]. Although his expression for the received waveform $W(t; \mathbf{x}, \underline{\theta})$ is very general and able to describe major classes of noise processes, it needs to be simplified to yield mathematically tractable results. We shall assume that the effects of the transmission medium and receiver on the transient pulses may be separated into two multiplicative factors: filtering and attenuation. Without attenuation, the medium and receiver may be treated as a deterministic linear time-invariant filter. In this case, the received transient pulse is the convolution of the impulse response of the equivalent filter and the original pulse waveform $aD(t; \underline{\theta})$. The result is designated by $aE(t; \underline{\theta})$. Note that, unlike previous models such as those developed by Middleton [3], we place no restrictions on the bandwidth of the receiver. Our analysis applies to narrowband as well as wideband receptions.

In its most general form, the attenuation factor, $g(\mathbf{x})$, is a function of the source location relative to the receiver, and may be derived from the beam patterns of source and antenna and physical laws of electronic-magnetic transmission in the medium [21]. As a first-order approximation, we shall assume that the sources within the region of consideration have the same isotropic radiation patterns and the receiver has an omin-directional antenna. Then the attenuation factor is simply a decreasing function of the distance from the source to the receiver. A good approximation is that the attenuation factor varies inversely with a power of the distance [14, 3], i.e.,

$$g(\mathbf{x}) = c_1/r^p \tag{3}$$

where c_1 is a positive constant and $p > 0$ is the attenuation rate, and where $r = |\mathbf{x}|$ is the distance from the source to the receiver. Typically, $1/2 \leq p < 2$. For example, the waveguide-mode theory of long-distance propagation in the atmosphere shows that the attenuation factor is given by

$$g(\mathbf{x}) = c\exp(-dr)/r^p$$

where $p = 1/2$ [22]. Since the constant d is usually very small, the above function may be approximated by (3) with $p = 1/2$.

Combining the filtering and attenuation factors, one finally obtains the waveform of the received pulse which is originated from a source located at $\mathbf{x}$:

$$W(t;\mathbf{x},\underline{\theta}) = aU(t;\mathbf{x},\underline{\theta}) \tag{4}$$

where

$$U(t;\mathbf{x},\underline{\theta}) = \frac{c_1}{r^p}E(t;\underline{\theta}), \qquad p > 0. \tag{5}$$

By (2), the observed instantaneous amplitude of noise at the time of observation can be rewritten as

$$X = \sum_{i=1}^{N} a_i U(t_i; \mathbf{x}_i, \underline{\theta}_i). \tag{6}$$

2.3 Spatial and Temporal Source Distributions

Since the numbers of sources in non-overlapping regions may be considered as independent random variables and since there are usually many sources, each with an extremely small chance of falling into any particular region, it is reasonable to postulate that the total number of sources in a particular region is Poisson distributed [23]. Likewise, it may be assumed that the total number of elementary pulses emitted by the sources within a time interval is also Poisson distributed. Based on these characteristics, we shall assume that the number, N, of noise pulses arriving at the receiver is a Poisson point process in both space and time, whose intensity function is denoted by $\rho(\mathbf{x}, t)$. This basic assumption is one of the essential features of our model as well as many of the previous models for noise-generating processes [19, 13, 14, 21].

The intensity function $\rho(\mathbf{x}, t)$ represents the approximate probability that a noise pulse originated from a unit area or volume will arrive at the receiver during a unit time interval, and thus may be considered as the spatial and temporal density of the noise sources. Since the noise sources need not to be homogeneously distributed in either space or time, the most general form of the intensity function $\rho(\mathbf{x}, t)$ depends on both the source location $\mathbf{x}$ and its emission time t, where $\mathbf{x} \in \Omega$ and $t \in R^+ = [0, \infty)$. However, for simplicity, we shall restrict our consideration to the most common case where the source distribution is time-invariant so that $\rho(\mathbf{x}, t) = \rho(\mathbf{x})$. This is in no way a serious limitation in our model, considering the fact that the observation period is usually very short.

In most applications $\rho(\mathbf{x})$ is a non-increasing function of the range $r = |\mathbf{x}|$. The number of sources that occur close to the receiver is usually larger than the number of sources that occur farther away. This is certainly the case, for example, for the tropical atmospheric noise where most lighting discharges occur locally, and relatively few discharges occur at great distances [14]. If the source distribution is symmetric about the point of observation, i.e., there is no preferred direction from which the pulses arrive, then it is reasonable to assume that $\rho(\mathbf{x})$ varies inversely with a certain power of the distance r [3, 14]:

$$\rho(\mathbf{x}, t) = \frac{\rho_0}{r^\mu} \tag{7}$$

where $\mu, \rho_0 > 0$ are constants.

3 First-order Statistics of the Instantaneous Noise Amplitude

To compare with the experimental studies and to design optimum and suboptimum receivers for impulsive communication channels, we need to know the first-order statistics of the instantaneous noise amplitude X given in (6). By taking advantage of the proposed model for the noise process, we are able to derive a simple result for the characteristic function of X. But before we actually derive the characteristic function, let us briefly review some of the basic facts about Poisson processes which will be needed shortly. First, if $N_{V,T}$ is the number of noise pulses originated from sources inside a spatial region V and emitted within a time interval T, then $N_{V,T}$ is a Poisson random variable with the parameter

$$\lambda_{V,T} = \int_V \int_T \rho(\mathbf{x}, t) d\mathbf{x} dt, \tag{8}$$

so that

$$P(N_{V,T} = k) = \frac{\lambda_{V,T}^k}{k!} \exp(-\lambda_{V,T}), \qquad k = 0, 1, 2, \cdots. \tag{9}$$

And its *factorial* moment-generating function is given by

$$\begin{aligned} \varphi_{V,T}(t) &= \mathbf{E}(t^{N_{V,T}}) \\ &= \exp(\lambda_{V,T}(t-1)). \end{aligned} \tag{10}$$

On the other hand, if it is known that the receiver receives exactly $N_{V,T}$ pulses that come from a spatial region V within a time interval T and if the actual source locations and their emission times are $(\mathbf{x}_i, t_i), i = 1, \cdots, N$, then the random pairs $(\mathbf{x}_i\ t_i), i = 1, \cdots, N$, are independent and identically distributed. And their common joint density function is given by

$$f(\mathbf{x}, t) = \frac{\rho(\mathbf{x}, t)}{\lambda_{V,T}}, \qquad \mathbf{x} \in V, t \in T \tag{11}$$

where the normalizing constant $\lambda_{V,T}$ is given by (8). In addition, the number of sources, N, is independent of the locations and emission times of all the sources. All of the above results are the consequences of the basic Poisson assumption [23].

To calculate the characteristic function, $\varphi(\omega)$, of the noise amplitude X, we first restrict our attention to those noise pulses emitted from sources inside the region $\Omega(R_1, R_2)$ and within the time interval $[0, T)$, where

$$\Omega(R_1, R_2) = \Omega \cap \{\mathbf{x} : R_1 < |\mathbf{x}| < R_2\}. \tag{12}$$

Let $N_{R1,R2,T}$ be the number of pulses emitted from the space-time region $\Omega(R_1, R_2) \times [0, T)$. Then by (8) and (10), $N_{R_1,R_2,T}$ is a Poisson variable with parameter

$$\lambda_{T,R_1,R_2} = \int_{\Omega(R_1,R_2)} \int_0^T \rho(\mathbf{x}, t) d\mathbf{x} dt \tag{13}$$

and

$$\mathbf{E}(t^{N_{T,R_1,R_2}}) = \exp(\lambda_{T,R_1,R_2}(t-1)). \tag{14}$$

The amplitude of the truncated noise is given by

$$X_{T,R_1,R_2} = \sum_{i=1}^{N_{T,R_1,R_2}} a_i U(t_i; \mathbf{x}_i, \underline{\theta}_i). \tag{15}$$

The observed noise amplitude X is understood to be the limit of X_{T,R_1,R_2} as $T, R_2 \to \infty$ and $R_1 \to 0$ in some suitable sense.

By our previous assumptions, $\{a_i, t_i, \mathbf{x}_i, \underline{\theta}_i\}_{i=1}^{\infty}$ is an i.i.d sequence. Hence, X_{T,R_1,R_2} is a sum of i.i.d random variables with a random number of terms, whose characteristic function can be calculated as follows:

$$\begin{aligned} \varphi_{T,R_1,R_2}(\omega) &= \mathbf{E}\{\exp(j\omega X_{T,R_1,R_2})\} \\ &= \mathbf{E}[\mathbf{E}(\exp(j\omega X_{T,R_1,R_2})) \mid N_{T,R_1,R_2}] \\ &= \mathbf{E}\{[\psi_{T,R_1,R_2}(\omega)]^{N_{T,R_1,R_2}}\} \end{aligned} \tag{16}$$

where

$$\psi_{T,R_1,R_2}(\omega) = \mathbf{E}\{\exp(j\omega a_1 U(t_1; \mathbf{x}_1, \underline{\theta}_1)) \mid T, \Omega(R_1, R_2)\}. \tag{17}$$

By (14),

$$\varphi_{T,R_1,R_2}(\omega) = \exp(\lambda_{T,R_1,R_2}(\psi_{T,R_1,R_2}(\omega) - 1)). \tag{18}$$

Recall that $a_1, \underline{\theta}_1$ and $(\mathbf{x}_1, t_1)$ are independent, with density functions $p_a(a), p_{\underline{\theta}}(\underline{\theta})$ and

$f_{T,R_1,R_2}(\mathbf{x},t)$ respectively, where by (11),

$$f_{T,R_1,R_2}(\mathbf{x},t) = \frac{\rho(\mathbf{x},t)}{\lambda_{T,R_1,R_2}}, \qquad \mathbf{x} \in \Omega(R_1,R_2), \quad t \in [0,T). \tag{19}$$

Therefore,

$$\psi_{T,R_1,R_2}(\omega) = \int_{-\infty}^{\infty} p_a(a)da \int_{\Theta} p_{\underline{\theta}}(\underline{\theta})d\underline{\theta} \int_0^T dt \int_{\Omega(R_1,R_2)} \frac{\rho(\mathbf{x},t)}{\lambda_{T,R_1,R_2}} \exp(j\omega a U(t;\mathbf{x},\underline{\theta}))d\mathbf{x}. \tag{20}$$

Combining (5), (7), (20) and (18), one can easily show that the logarithm of the characteristic function of X_{T,R_1,R_2} is

$$\log \varphi_{T,R_1,R_2}(\omega) = \rho_0 \int_{-\infty}^{\infty} p_a(a)da \int_{\Theta} p_{\underline{\theta}}(\underline{\theta})d\underline{\theta} \int_0^T dt \int_{\Omega(R_1,R_2)} [\exp(j\omega a c_1 r^{-p} E(t;\underline{\theta})) - 1] r^{-\mu} d\mathbf{x} \tag{21}$$

where $r = |\mathbf{x}|$. Noting that the distribution of the random amplitude a is symmetric (see (1)), one can further simplify the above equation to get

$$\log \varphi_{T,R_1,R_2}(\omega) = 2\rho_0 \int_0^{\infty} p_a(a)da \int_{\Theta} p_{\underline{\theta}}(\underline{\theta})d\underline{\theta} \int_0^T dt \int_{\Omega(R)} [\cos(\omega a c_1 r^{-p} E(t;\underline{\theta})) - 1] r^{-\mu} d\mathbf{x}. \tag{22}$$

Now, $\log \varphi_{T,R_1,R_2}(\omega)$ approaches the following limit

$$\log \varphi(\omega) = 2\rho_0 \int_0^{\infty} p_a(a)da \int_{\Theta} p_{\underline{\theta}}(\underline{\theta})d\underline{\theta} \int_0^{\infty} dt \int_{\Omega} [\cos(\omega a c_1 r^{-p} E(t;\underline{\theta})) - 1] r^{-\mu} d\mathbf{x} \tag{23}$$

as $T, R_2 \to \infty$ and $R_1 \to 0$, where $\varphi(\omega)$ is of course the desired characteristic function of the noise amplitude X.

To further simplify $\varphi(\omega)$, we rewrite the above integral involving the spatial coordinates $\mathbf{x}$ using the polar coordinate system. Since Ω is a cone with vertex at the origin, (23) can be written as follows:

$$\log\varphi(\omega) = 2c_2\rho_0 \int_0^\infty p_a(a)da \int_\Theta p_{\underline{\theta}}(\underline{\theta})d\underline{\theta} \int_0^\infty dt \int_0^\infty [\cos(\omega a c_1 r^{-p} E(t;\underline{\theta})) - 1] r^{n-1-\mu} dr \quad (24)$$

where c_2 is a constant determined by the shape of Ω and n is the dimension of the space. We then make a change of variable, replacing r^{-p} by r, to get

$$\log\varphi(\omega) = \frac{2}{p}c_2\rho_0 \int_0^\infty p_a(a)da \int_\Theta p_{\underline{\theta}}(\underline{\theta})d\underline{\theta} \int_0^\infty dt \int_0^\infty [\cos(\omega a c_1 r E(t;\underline{\theta})) - 1] r^{-\alpha-1} dr \quad (25)$$

where

$$\alpha = \frac{n-\mu}{p} \quad (26)$$

is an effective measure of an average source density with range. As we will see in the next section, α determines the degree of impulsiveness of the noise.

We shall assume that $0 < \alpha < 2$. Otherwise, the integrand in (25) is not absolutely integrable. Using the following elementary integral [24]

$$\int_0^\infty x^{\mu-1}\sin^2 ax dx = -\frac{\Gamma(\mu)\cos\frac{\mu\pi}{2}}{2^{\mu+1}a^\mu}, \qquad a > 0, \ -2 < \mu < 0$$

one can easily show that for $0 < \alpha < 2$,

$$\int_0^\infty \frac{1-\cos br}{r^{\alpha+1}} dr = \frac{\Gamma(1-\alpha)}{\alpha} |b|^\alpha \cos\frac{\pi}{2}\alpha = \frac{\pi}{2\alpha\Gamma(\alpha)\sin\frac{\pi}{2}\alpha} |b|^\alpha \tag{27}$$

where $\Gamma(\cdot)$ is the usual gamma function defined by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt. \tag{28}$$

Using the identity (27) and (25), one finally obtains the characteristic function of the instantaneous noise amplitude as follows:

$$\log\varphi(\omega) = -\gamma|\omega|^\alpha \tag{29}$$

where

$$\gamma = \frac{2c_1^\alpha c_2 \rho_0 \Gamma(1-\alpha)\cos\frac{\pi}{2}\alpha}{p\alpha} \int_0^\infty a^\alpha p_a(a) da \int_\Theta p_{\underline{\theta}}(\underline{\theta}) d\underline{\theta} \int_0^\infty dt |E(t;\underline{\theta})|^\alpha > 0. \tag{30}$$

A random variable whose characteristic function $\varphi(\omega)$ has the form defined by (29) is called *symmetric stable*. Hence, we have shown that the probability distribution of the instantaneous amplitude of impulsive noise is symmetric stable. Stable distributions are an important class of distributions in probability and statistics, whose properties are discussed in details in the next section.

4 Symmetric Stable Distributions

Symmetric stable distributions belong to a more general class of distributions called the *stable* distributions, which may be symmetric or asymmetric. As a natural generalization of the Gaussian distribution, stable distributions are increasingly important in both theory and applications [18]. In this paper, we will deal exclusively with symmetric stable distributions.

4.1 Characterizations of Symmetrical Stable Distributions

The symmetric stable distribution is defined by its characteristic function as follows:

$$\varphi(t) = \exp(jat - \gamma|t|^{\alpha}), \tag{31}$$

where the three parameters have to be confined to the domains

$$-\infty < a < \infty, \ \gamma \geq 0, \ 0 < \alpha \leq 2 \tag{32}$$

in order for (31) to be a valid characteristic function. It is easy to show that a probability distribution with its characteristic function given by (31) is indeed symmetric with respect to the location parameter a. Hence a is also the median of the distribution. The scale parameter γ is called the *dispersion.* If γ is limited to strictly positive values, the distribution will be non-degenerate. The most important parameter of a symmetric stable distribution is the *characteristic exponent*, α. It determines the shape of the distribution and plays a critical role in impulsive noise modeling, as we will see shortly. A symmetric stable distribution with characteristic exponent α is often called *symmetric α-stable*, or simply $S\alpha S$. Two important special cases of the $S\alpha S$ distributions are the Gaussian ($\alpha = 2$) and Cauchy ($\alpha = 1$) distributions.

Stable distributions inherit two most important characteristics of the Gaussian distribution, namely, the stability property and the central limit theorems [18]. They are responsible for much of the appeal of stable distributions as statistical models of uncertainty.

The stability property states that stable distributions are invariant under linear combinations. Specifically, if $X_1, \ldots, X_n$ are independent and symmetrical stable with the same characteristic exponent α, then for any constants $a_1, \ldots, a_n$, the linear combination $\sum_{i=1}^{n} a_i X_i$ is again $S\alpha S$. This can be readily shown by using the characteristic functions of stable distributions. More importantly, the converse is also true. Namely, stable distributions are the only family of distributions that are invariant under linear combinations. Intuitively, the stability property is very desirable in noise modeling, as it has been demonstrated in the Gaussian case.

The central limit theorem, on the other hand, often provides a theoretical justification for stable models, as it does for the Gaussian model. Recall that the ordinary central limit theorem says that the only possible limiting distributions for sums of independent, identically distributed random variables with finite variance are Gaussian. A more powerful version of the central limit theorem, called the *generalized central limit theorem*, states that the family of stable (symmetric or asymmetric) distributions contains *all* the limiting distributions of sums of i.i.d. random variables. In particular, if such variables have large or infinite variance, which is usually the case under an impulsive environment, their sum is approximately stable.

4.2 $S\alpha S$ Density and Distribution Functions

Without loss of generality, we shall assume that all $S\alpha S$ distributions are centered at the origin so that $a = 0$. In this case, a $S\alpha S$ distribution is determined by two parameters, $0 < \alpha \leq 2$ and $\gamma > 0$, through its characteristic function

$$\varphi_{\alpha,\gamma}(t) = \exp(-\gamma|t|^{\alpha}). \tag{33}$$

Its density and distribution functions are denoted by $f_{\alpha,\gamma}(x)$ and $F_{\alpha,\gamma}(x)$ respectively. If γ is equal to 1, the $S\alpha S$ distribution is called *standard.* Notice that by this definition, the standard Gaussian distribution has variance equal to 2, not 1.

Since the characteristic function $\varphi_{\alpha,\gamma}(t)$ in (33) is absolutely integrable, the corresponding probability density function $f_{\alpha,\gamma}(x)$ is continuous and can be found by taking the inverse Fourier transform of $\varphi_{\alpha,\gamma}(t)$, namely,

$$f_{\alpha,\gamma}(x) = \frac{1}{\pi}\int_0^{\infty} \cos(xt)\exp(-\gamma t^{\alpha})dt. \tag{34}$$

By integrating (34), one obtains the corresponding distribution function

$$F_{\alpha,\gamma}(x) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \frac{\sin xt}{t}\exp(-\gamma t^\alpha)dt. \tag{35}$$

Based on (34), one may investigate the analytical properties of $S\alpha S$ distributions. For example, all derivatives of the density function $f_{\alpha,\gamma}(x)$ exist and are given by

$$\frac{d^n}{dx^n}f_{\alpha,\gamma}(x) = \frac{1}{\pi}\int_0^\infty \left\{ \begin{array}{c} \pm\sin xt \\ \\ \pm\cos xt \end{array} \right\} t^n \exp(-\gamma t^\alpha)dt, \quad n \geq 0.$$

Using the basic definition of the gamma function in (28), one can easily show that

$$|\frac{d^n}{dx^n}f_{\alpha,\gamma}(x)| \leq \frac{1}{\pi\alpha\gamma^{(n+1)/\alpha}}\Gamma(\frac{n+1}{\alpha}), \tag{36}$$

i.e., $f_{\alpha,\gamma}(x)$ has bounded derivatives of arbitrary orders. Hence, $S\alpha S$ densities are very smooth and well-behaved functions.

Unfortunately, no closed-form expressions exist for general $S\alpha S$ densities, except for the Gaussian ($\alpha = 2$) and Cauchy ($\alpha = 1$) distributions. Instead, we derive power series expansions for the $S\alpha S$ density function $f_{\alpha,\gamma}(x)$. Since

$$\cos x = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}x^{2k}$$

one has

$$f_{\alpha,\gamma}(x) = \frac{1}{\pi}\int_0^\infty (\sum_{k=0}^\infty \frac{(-1)^k x^{2k}}{(2k)!} t^{2k}) \exp(-\gamma t^\alpha) dt.$$

To interchange the integral and summation, we need to have

$$\sum_{k=0}^\infty \frac{x^{2k}}{(2k)!}\int_0^\infty t^{2k}\exp(-\gamma t^\alpha)dt < \infty \tag{37}$$

so that Fubini's theorem may be used [25]. One can easily show from the properties of the gamma function that (37) is satisfied if $1 < \alpha \leq 2$. In this case, using the fact that

$$\int_0^\infty t^{2k}\exp(-\gamma t^\alpha)dt = \frac{1}{\alpha\gamma^{(2k+1)/\alpha}}\Gamma(\frac{2k+1}{\alpha})$$

one has

$$f_{\alpha,\gamma}(x) = \frac{1}{\pi\alpha\gamma^{1/\alpha}}\sum_{k=0}^\infty \frac{(-1)^k}{(2k)!}\Gamma(\frac{2k+1}{\alpha})(\frac{x}{\gamma^{1/\alpha}})^{2k}. \tag{38}$$

By expanding the function $\exp(-\gamma t^\alpha)$ and using contour integral, one may also show that [17] for $0 < \alpha < 1$ and $x \neq 0$,

$$f_{\alpha,\gamma}(x) = \frac{1}{\pi\gamma^{1/\alpha}}\sum_{k=1}^\infty \frac{(-1)^{k-1}}{k!}\Gamma(\alpha k + 1)\sin(\frac{k\alpha\pi}{2})(\frac{|x|}{\gamma^{1/\alpha}})^{-\alpha k - 1}. \tag{39}$$

In summary, the $S\alpha S$ density function with dispersion γ and zero location parameter is given by

$$f_{\alpha,\gamma}(x) = \begin{cases} \dfrac{1}{\pi\gamma^{1/\alpha}} \sum_{k=1}^{\infty} \dfrac{(-1)^{k-1}}{k!} \Gamma(\alpha k + 1) \sin(\dfrac{k\alpha\pi}{2}) (\dfrac{|x|}{\gamma^{1/\alpha}})^{-\alpha k - 1} & 0 < \alpha < 1 \\ \dfrac{\gamma}{\pi(x^2 + \gamma^2)} & \alpha = 1 \\ \dfrac{1}{\pi\alpha\gamma^{1/\alpha}} \sum_{k=0}^{\infty} \dfrac{(-1)^k}{(2k)!} \Gamma(\dfrac{2k+1}{\alpha}) (\dfrac{x}{\gamma^{1/\alpha}})^{2k} & 1 < \alpha < 2 \\ \dfrac{1}{2\sqrt{\gamma\pi}} \exp(-x^2/4\gamma) & \alpha = 2. \end{cases} \tag{40}$$

The graphs of the standard $S\alpha S$ density are shown in Fig. 1(a) for a few values of α, with $\alpha = 2$ corresponding to the Gaussian density and $\alpha = 1$ corresponding to the Cauchy density. Observe that $S\alpha S$ densities have many features of the Gaussian density function. For example, they are all smooth, unimodal, symmetric with respect to the median and bell-shaped. In addition, although not shown in the graphs, the $S\alpha S$ stable density may be shifted and scaled by the location parameter a and the dispersion γ, just as the Gaussian density by its mean and and variance. In fact when $\alpha = 2$, i.e., when the $S\alpha S$ density is Gaussian, the location parameter a is the mean and the dispersion γ is a half of the variance.

Although the behavior of a non-Gaussian $S\alpha S$ density is approximately Gaussian near the origin, it is visibly different at the tails from a similarly scaled and centered Gaussian density, as shown in Fig. 1(b). Specifically, the tails of non-Gaussian $S\alpha S$ densities decay at a lower rate than the Gaussian density tails. And the smaller α is, the lower the decay rate. Consequently, if a $S\alpha S$ random variable is observed, the smaller α is, the more likely

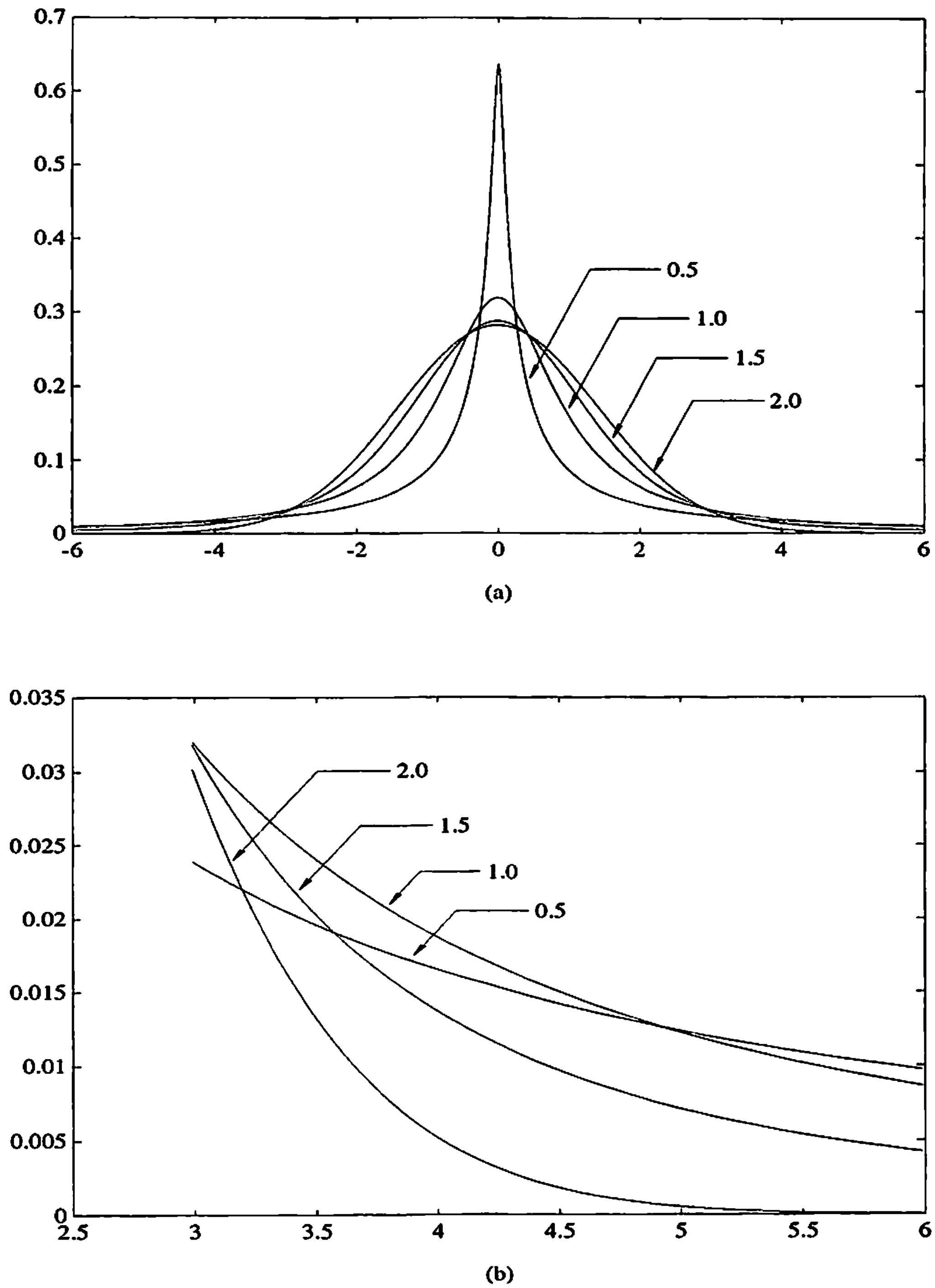

Figure 1: (a) Graphs of the standard $S\alpha S$ density function corresponding to the values $\alpha = 2.0, 1.5, 1.0$ and 0.5; (b) tails of the densities in (a).

it is to observe values of the random variable which are far from its central locations. In other words, the smaller α is, the more impulsive is a $S\alpha S$ random variable. The heavy-tailed character of non-Gaussian $S\alpha S$ distributions has important implications in modeling impulsive noise and is further discussed in the next section.

4.3 Tails and Moments

As shown in Fig. 1(b), the tail behavior of a $S\alpha S$ density is determined by its characteristic exponent α. The exact relation between α and the "thickness" of the tails can be made more explicit. Recall that the Gaussian density is an exponential function and hence its tails decay exponentially. The tails of non-Gaussian stable distributions, on the other hand, decay algebraically. For the Cauchy distribution ($\alpha = 1$), this is certainly true since its density in the standardized form is $1/\pi(x^2+1)$. For a general non-Gaussian $S\alpha S$ distribution with density $f_{\alpha,\gamma}(x)$ and $0 < \alpha < 2$, one can show, using an identity relating two stable densities with different characteristic exponents (see Equation (6.10) on pp. 549 in [17]), that

$$\lim_{|x|\to\infty} |x|^{\alpha+1} f_{\alpha,\gamma}(x) = C(\alpha,\gamma) \tag{41}$$

where $C(\alpha,\gamma)$ is a positive constant depending on α and γ. Hence, the decay rate of the tails of a $S\alpha S$ density is asymptotically on the order of $x^{-\alpha-1}$, much slower than that of the Gaussian density.

To determine $C(\alpha, \gamma)$, we note that $f_{\alpha,\gamma}$ is symmetric so that

$$\varphi_{\alpha,\gamma}(t) = 2\int_0^\infty f_{\alpha,\gamma}(x)\cos tx dx$$

and hence

$$1 - \varphi_{\alpha,\gamma}(t) = 2\int_0^\infty f_{\alpha,\gamma}(x)(1 - \cos tx)dx.$$

Assume $t > 0$. By a change of variable and dividing both sides by t^α, the above equation can be written as

$$(1 - \varphi_{\alpha,\gamma}(t))/t^\alpha = 2\int_0^\infty (u/t)^{\alpha+1} f_{\alpha,\gamma}(u/t)\frac{1 - \cos u}{u^{\alpha+1}}du.$$

Letting $t \to 0^+$ in the above equation and noting (41), (27), and the fact that

$$\lim_{t\to 0^+}(1 - \varphi_{\alpha,\gamma}(t))/t^\alpha = \lim_{t\to 0^+}(1 - \exp(-\gamma|t|^\alpha))/t^\alpha = \gamma$$

one finds that

$$C(\alpha, \gamma) = \gamma\frac{\alpha}{\pi}\Gamma(\alpha)\sin\frac{\pi\alpha}{2}, \qquad \text{for } 0 < \alpha < 2. \tag{42}$$

Note that by (42), $C(\alpha, \gamma) \to 0$ as $\alpha \to 2$. Consequently, the algebraic behavior of the tails of a $S\alpha S$ distribution becomes less obvious as α approaches to 2. In fact, one can show [26] that as α approaches to 2, the non-Gaussian $S\alpha S$ distribution approaches to the Gaussian distribution in a continuous fashion. Hence, there is no gap between the Gaussian distribution and the rest of the $S\alpha S$ family. By a proper choice of the characteristic exponent one may use $S\alpha S$ distributions to model a variety of different phenomena, from those which exhibit a Gaussian type of behavior to those which are highly impulsive.

An important consequence of (41) is the *non-existence* of the second-order moment of stable distributions, except for the limiting case $\alpha = 2$. Specifically, let X be a $S\alpha S$ random variable with zero location parameter and nonzero dispersion γ. If $0 < \alpha < 2$, then

$$\mathbf{E}|X|^p = \infty, \quad \text{if } \ p \geq \alpha \tag{43}$$

and

$$\mathbf{E}(|X|^p) = D(p,\alpha)\gamma^{\frac{p}{\alpha}} < \infty \qquad \text{if } \ 0 < p < \alpha \tag{44}$$

where

$$D(p,\alpha) = \frac{2^{p+1}\Gamma(\frac{p+1}{2})\Gamma(-p/\alpha)}{\alpha\sqrt{\pi}\Gamma(-p/2)} \tag{45}$$

depends only on α and p, not on X. For an elementary proof of (44), see [18].

Hence for $0 < \alpha \leq 1$, α-stable distributions have no finite first or higher-order moments; for $1 < \alpha < 2$, they have the first-order moment and all the moments of order p where $p < \alpha$; for $\alpha = 2$, all moments exist. In particular, *all non-Gaussian stable distributions have infinite variance.* The significance of infinite variance in signal processing is discussed in details in [18].

4.4 Amplitude Probability Distribution

A commonly measured statistical representation of impulsive noise is the amplitude probability distribution (APD), defined as the probability that the noise magnitude is above a threshold. Hence, if X is the instantaneous amplitude of impulsive noise, then its APD is given by

$$P(|X| > x) = 1 - F(x) + F(-x)$$

as a function of $x > 0$, where $F(x)$ is its distribution function. The APD can be easily measured in practice by counting the percentages of time for which the given threshold is exceeded by the noise magnitude during the period of observation.

In the case that X is $S\alpha S$ with dispersion γ, its APD can be calculated from (35) as

$$P(|X| > x) = 1 - \frac{2}{\pi}\int_0^\infty \frac{\sin tx}{t} \exp(-\gamma t^\alpha) dt. \tag{46}$$

By integrating (40), one can also represent the APD of $S\alpha S$ noise using power series as follows

$$P(|X| > x) = \begin{cases} \frac{2}{\pi\alpha}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k!k}\Gamma(\alpha k+1)\sin(\frac{k\alpha\pi}{2})(\frac{|x|}{\gamma^{1/\alpha}})^{-\alpha k} & 0 < \alpha < 1 \\ 1 - \frac{2}{\pi}\arctan(x/\gamma) & \alpha = 1 \\ 1 - \frac{2}{\pi\alpha}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}\Gamma(\frac{2k+1}{\alpha})(\frac{x}{\gamma^{1/\alpha}})^{2k+1} & 1 < \alpha < 2 \\ 1 - \mathrm{erf}(\frac{x}{2\sqrt{\gamma}}) & \alpha = 2 \end{cases} \tag{47}$$

where erf(x) is the standard error function defined by

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du.$$

By integrating (41), one can easily show that

$$\lim_{x \to \infty} x^\alpha P(|X| > x) = (2/\alpha) C(\alpha, \gamma). \tag{48}$$

Hence, the APD of $S\alpha S$ noise decays asymptotically on the order of $x^{-\alpha}$. As we will see in Section 6, this results are consistent with experimental observations.

Figs. 2 and 3 plot the APD of $S\alpha S$ noise for various values of α and γ. To fully represent the large range of the exceedence probability $P(|X| > x)$, the coordinate grid used in these two and subsequent figures employs a highly folded abscissa. Specifically, the axis for $P(|X| > x)$ is scaled according to $-\log(-\log P(|X| > x))$ and the axis for the exceedence level x is in decibel. As clearly shown in Fig. 3, $S\alpha S$ distributions have a Gaussian behavior when the amplitude is below a certain threshold.

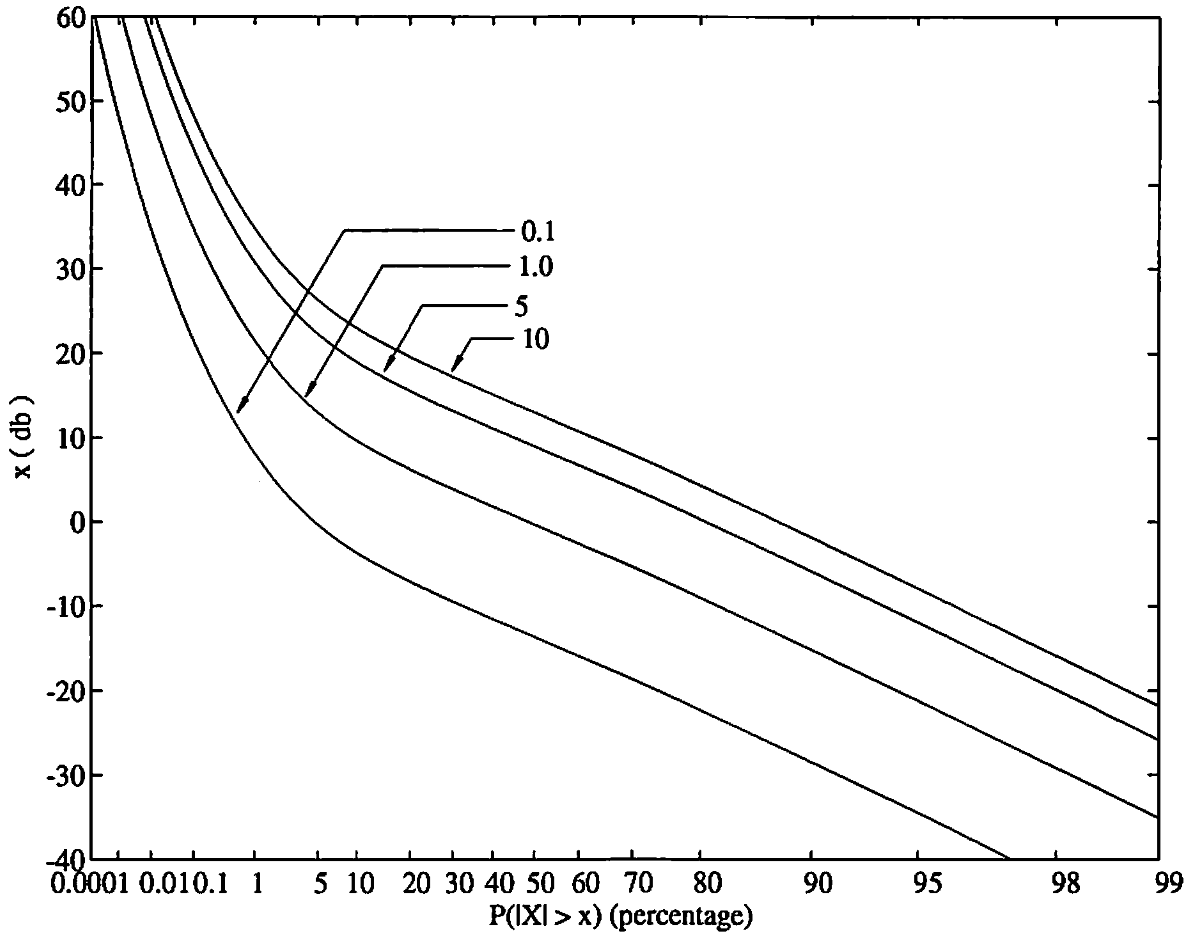

Figure 2: The APD of the instantaneous amplitude of $S\alpha S$ noise for $\alpha = 1.5$ and various values of γ.

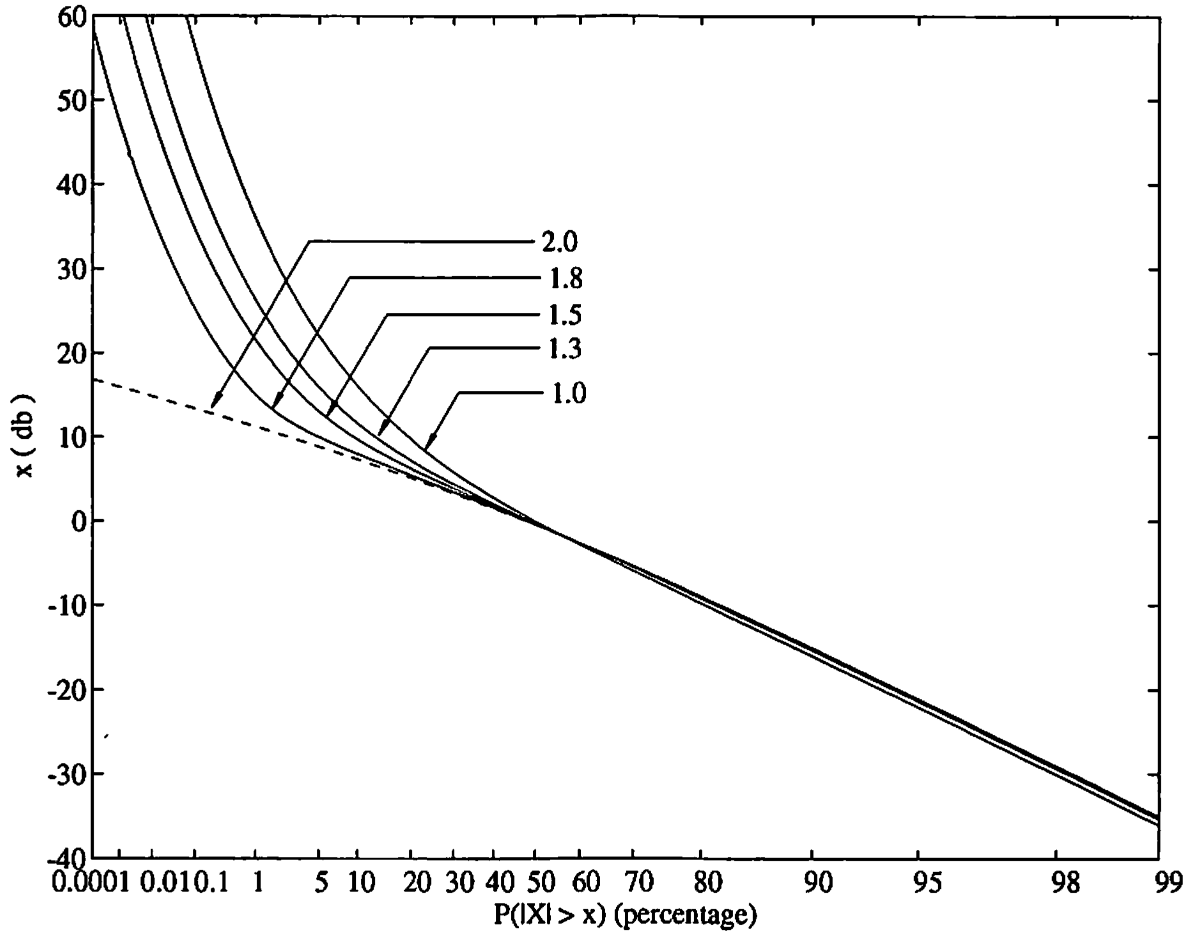

Figure 3: The APD of the instantaneous amplitude of $S\alpha S$ noise for $\gamma = 1$ and various values of α.

5 First-order Statistics of Envelope and Phase of Narrowband Noise

In the proceeding sections we have shown that the instantaneous noise amplitude is a $S\alpha S$ random variable and investigated the properties of $S\alpha S$ distributions. This result applies to both narrowband and wideband receptions. However, in most communication systems, the receiver is usually narrowband. In such cases, it is of interest to derive the first-order statistics of the envelope and phase of the received impulsive noise. Based on the general physical mechanism postulated in Section 2, we will show that the instantaneous phase is uniformly distributed in $[0, 2\pi]$, and the distribution of the noise envelope is a heavy-tailed generalization of the familiar Rayleigh distribution.

5.1 Joint Characteristic Function of the Quadrature Components

When the receiver is narrowband with the central frequency ω_0, the typical waveform, $E(t;\underline{\theta})$, of the noise pulse after reception (see Section 2.2) may be represented by its slow-varying envelope $V(t;\underline{\theta})$ and phase ϕ as follows

$$E(t;\underline{\theta}) = V(t;\underline{\theta})\cos(\omega_0 t + \phi). \tag{49}$$

Here the envelope again has a fixed waveform containing random parameters $\underline{\theta}$. The random phase ϕ of the pulse at the time of observation is assumed to be uniformly distributed in $[0, 2\pi]$ and independent of the envelope and all other random variables. Equivalently,

$$E(t;\underline{\theta}) = E_c(t;\underline{\theta}) \cos \omega_0 t - E_s(t;\underline{\theta}) \sin \omega_0 t \tag{50}$$

where the "in-phase" and "out-of-phase" quadrature components are defined by

$$E_c(t;\underline{\theta}) = V(t;\underline{\theta}) \cos \phi, \quad E_s(t;\underline{\theta}) = V(t;\underline{\theta}) \sin \phi. \tag{51}$$

By (6) and (5), the received narrowband noise at the time of observation is

$$X = X_c \cos \omega_0 t - X_s \sin \omega_0 t \tag{52}$$

where the quadrature components of the noise are given by

$$X_c = \sum_{i=1}^{N} a_i U_c(t_i; \mathbf{x}_i, \underline{\theta}_i, \phi_i), \quad X_s = \sum_{i=1}^{N} a_i U_c(t_i; \mathbf{x}_i, \underline{\theta}_i, \phi_i) \tag{53}$$

and where

$$U_c(t;\mathbf{x},\underline{\theta},\phi) = \frac{c_1}{r^p} E_c(t;\underline{\theta}), \quad U_c(t;\mathbf{x},\underline{\theta},\phi) = \frac{c_1}{r^p} E_s(t;\underline{\theta}). \tag{54}$$

To find the first-order statistics of the envelope and phase, we need the joint distribution of the quadrature components X_c and X_s of the noise process. In a similar way that

(25) was derived and taking into account the random phase, we can show that the joint characteristic function of X_c and X_s

$$\varphi(\omega_1, \omega_2) = \mathbf{E}(\exp(j(\omega_1 X_c + \omega_2 X_s)))$$

is given by

$$\log \varphi(\omega_1, \omega_2)) = \frac{1}{p\pi} c_2 \rho_0 \int_0^\infty p_a(a)da \int_\Theta p_{\underline{\theta}}(\underline{\theta})d\underline{\theta} \int_0^\infty dt \int_0^{2\pi} d\phi \int_0^\infty G_\alpha(a, \underline{\theta}, \phi, r)dr \quad (55)$$

where the integrand

$$G_\alpha(a, \underline{\theta}, \phi, r) = [\cos(ac_1 r(\omega_1 E_c(t; \underline{\theta}) + \omega_2 E_s(t; \underline{\theta}))) - 1]r^{-\alpha-1} \quad (56)$$

and α is again given by (26). Using (51), one can rewrite the integrand as follows

$$G_\alpha(a, \underline{\theta}, \phi, r) = [\cos(ac_1 r|\underline{\omega}|V(t; \underline{\theta}) \cos(\phi - \phi_\omega)) - 1]r^{-\alpha-1} \quad (57)$$

where

$$\underline{\omega} = (\omega_1, \omega_2), \quad |\underline{\omega}| = \sqrt{\omega_1^2 + \omega_2^2}, \quad \phi_{\underline{\omega}} = \arctan(\omega_2/\omega_1). \quad (58)$$

From (27), one finds that the joint characteristic function of the quadrature components of the noise is given by

$$\log \varphi(\omega_1, \omega_2)) = -\gamma|\underline{\omega}|^\alpha \quad (59)$$

where

$$\gamma = c_3 \int_0^\infty a^\alpha p_a(a)da \int_\Theta p_{\underline{\theta}}(\underline{\theta})d\underline{\theta} \int_0^\infty dt|V(t;\underline{\theta})|^\alpha \int_0^{2\pi} |\cos(\phi)|^\alpha d\phi > 0 \tag{60}$$

with

$$c_3 = \frac{c_1^\alpha c_2 \rho_0 \Gamma(1-\alpha)\cos\frac{\pi}{2}\alpha}{p\alpha\pi}.$$

A bivariate random vector whose characteristic function $\varphi(\omega_1, \omega_2)$ has the form defined by (59) is called *isotropic α-stable*, which is a special case of multidimensional $S\alpha S$ random variables.

5.2 Bivariate Isotropic Stable Distributions

Multivariate stable distributions are similarly characterized by the stability property and the generalized central limit theorem. They preserve the characteristic exponent $0 < \alpha \leq 2$ and include the multivariate Gaussian distributions ($\alpha = 2$) and the multivariate Cauchy distributions as special cases. In addition, they share many important properties of the multivariate Gaussian distributions. For a general discussion on the multidimensional stable distribution, see [18].

Multidimensional stable distributions are, however, much more difficult to describe than the univariate stable distributions mainly because they in general form a nonparametric family. An important exception is the class of multidimensional isotropic stable distributions, which has many properties that are similar to the univariate $S\alpha S$ distributions. Here, we investigate the two dimensional case, which is sufficient for our purposes.

A bivariate isotropic α-stable distribution is, in its most general form, defined by a characteristic function of the form

$$\varphi(t_1, t_2) = \exp(j(a_1 t_1 + a_2 t_2) - \gamma|\underline{t}|^\alpha) \tag{61}$$

where

$$-\infty < a_1, a_2 < \infty, \ \gamma > 0, \ 0 < \alpha \leq 2 \tag{62}$$

and $\underline{t} = (t_1, t_2)$, $|\underline{t}| = \sqrt{t_1^2 + t_2^2}$.

The parameters a_1, a_2 determine its central location. It is easy to check that a probability distribution defined by (61) is indeed isotropic with respect to the center (a_1, a_2). The parameters γ and α are the dispersion and the characteristic exponent, respectively. They play the same roles as in the univariate case. Two important special cases are the bivariate isotropic Gaussian ($\alpha = 2$) and Cauchy ($\alpha = 1$) distributions. Note that the two marginal distributions of the isotropic stable distribution defined by (61) are $S\alpha S$ with parameters (a_1, γ, α) and (a_1, γ, α) respectively.

As usual, we shall assume $(a_1, a_2) = (0, 0)$ so that all isotropic stable distributions are centered at the origin. In this case, a bivariate isotropic α-stable distribution is parameterized by the characteristic exponent α and dispersion γ as shown in its characteristic function

$$\varphi_{\alpha,\gamma}(t_1, t_2) = \exp(-\gamma|\underline{t}|^\alpha). \tag{63}$$

Denote its density and distribution functions by $f_{\alpha,\gamma}(x_1, x_2)$ and $F_{\alpha,\gamma}(x_1, x_2)$. If, furthermore, $\gamma = 1$, it is said to be the *standard* isotropic α-stable distribution.

The density function $f_{\alpha,\gamma}(x_1, x_2)$ is the inverse Fourier transform of the characteristic function in (63), i.e.,

$$f_{\alpha,\gamma}(x_1, x_2) = (2\pi)^{-2} \int_{t_1} \int_{t_2} \exp(-j(x_1 t_1 + x_2 t_2)) \exp(-\gamma |\underline{t}|^\alpha) dt_1 dt_2 \tag{64}$$

Let $\underline{x} = (x_1, x_2)$ and $|\underline{x}| = \sqrt{x_1^2 + x_2^2}$. Using the polar coordinate system for the integral, one can show that

$$f_{\alpha,\gamma}(x_1, x_2) = \chi_{\alpha,\gamma}(|\underline{x}|) \tag{65}$$

where

$$\chi_{\alpha,\gamma}(r) = \frac{1}{2\pi} \int_0^\infty s \exp(-\gamma s^\alpha) J_0(sr) ds, \quad r \geq 0. \tag{66}$$

Here,

$$J_n(x) = \frac{1}{\pi} \int_0^\pi \cos(x \sin\theta - n\theta) d\theta \tag{67}$$

is the n^{th} order Bessel function of the first kind.

It is apparent from the following integrals [24]

$$\int_0^\infty z \exp(-az^2) J_0(bz) dz = \frac{1}{2a} \exp(-b^2/4a), \quad a > 0, b > 0$$

and

$$\int_0^\infty e^{-az} J_n(bz) z^{n+1} dz = \frac{a 2^{n+1} b^n \Gamma(n+\frac{3}{2})}{\sqrt{\pi}(a^2+b^2)^{n+3/2}}, \quad a>0, b>0, n>-1,$$

that

$$\chi_{2,\gamma}(r) = \frac{1}{4\pi\gamma}\exp(-r^2/4\gamma) \tag{68}$$

and

$$\chi_{1,\gamma}(r) = \frac{\gamma}{2\pi(r^2+\gamma^2)^{3/2}}. \tag{69}$$

When $\alpha \neq 1$ or 2, no closed-form expression is known for the integral in (66). But, by exploiting the basic properties of Bessel functions, one can find the following series expansion [27]

$$\chi_{\alpha,\gamma}(r) = \begin{cases} \dfrac{2^{\alpha k}}{\pi^2\gamma^{2/\alpha}} \sum_{k=1}^{\infty} \dfrac{(-1)^{k-1}}{k!} (\Gamma(\alpha k/2+1))^2 \sin(\dfrac{k\alpha\pi}{2})(r/\gamma^{1/\alpha})^{-\alpha k-2} & 0<\alpha<1 \\ \dfrac{1}{2^{2k+1}\pi\alpha\gamma^{2/\alpha}} \sum_{k=0}^{\infty} \dfrac{(-1)^k}{(k!)^2}\Gamma(\dfrac{2k+2}{\alpha})(r/\gamma^{1/\alpha})^{2k} & 1<\alpha<2. \end{cases} \tag{70}$$

In addition, $\chi_{\alpha,\gamma}(r)$ is also a heavy-tailed function. Specifically, one can show [27] that

$$\lim_{r\to\infty} r^{\alpha+2}\chi_{\alpha,\gamma}(r) = B(\alpha,\gamma) \tag{71}$$

where $B(\alpha,\gamma)$ is a positive constant. Hence, $\chi_{\alpha,\gamma}(r)$ has an algebraic tail of order $\alpha+2$.

5.3 First-order Statistics of Noise Envelope and Phase

From (52), the noise envelope A and phase Ψ are determined by the quadrature components X_c and X_s as follows

$$A = \sqrt{X_c^2 + X_s^2}, \qquad \Psi = \arctan(X_s/X_c). \tag{72}$$

Since X_c, X_s are jointly isotropic α-stable with density function $f_{\alpha,\gamma}(x_1, x_2)$, the joint density of the envelope and phase is given by

$$f(a, \psi) = a f_{\alpha,\gamma}(a\cos\psi, a\sin\psi), \quad a \geq 0,\ 0 \leq \psi \leq 2\pi. \tag{73}$$

From (65) and (66), it follows that

$$f(a, \psi) = a\chi_{\alpha,\gamma}(a) = \frac{a}{2\pi}\int_0^\infty s\exp(-\gamma s^\alpha)J_0(sa)ds. \tag{74}$$

Since the bivariate density $f(a, \psi)$ is independent of ψ, the random phase is *uniformly distributed* in $[0, 2\pi]$ and is independent of the envelope. This is a well-known fact for the Gaussian case.

The density of the envelope, on other hand, is given by

$$f(a) = a\int_0^\infty s\exp(-\gamma s^\alpha)J_0(as)ds, \quad a \geq 0. \tag{75}$$

By integrating (75) and noting the following identity of Bessel functions [28]

$$\int_0^x sJ_0(s)ds = xJ_1(x)$$

one obtains the envelope distribution function

$$F(a) = a\int_0^\infty \exp(-\gamma t^\alpha)J_1(at)dt, \quad a \geq 0. \tag{76}$$

From (68), (69) and (70) the envelope density is also given by the following series

$$f(a) = \begin{cases} \frac{2^{\alpha k+1}}{\pi\gamma^{1/\alpha}}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k!}(\Gamma(\alpha k/2+1))^2\sin(\frac{k\alpha\pi}{2})(\frac{a}{\gamma^{1/\alpha}})^{-\alpha k-1} & 0<\alpha<1 \\ \frac{a\gamma}{(a^2+\gamma^2)^{3/2}} & \alpha=1 \\ \frac{1}{2^{2k}\alpha\gamma^{1/\alpha}}\sum_{k=0}^{\infty}\frac{(-1)^k}{(k!)^2}\Gamma(\frac{2k+2}{\alpha})(\frac{a}{\gamma^{1/\alpha}})^{2k+1} & 1<\alpha<2 \\ \frac{a}{2\gamma}\exp(-a^2/4\gamma) & \alpha=2. \end{cases} \tag{77}$$

Note that when $\alpha = 2$, one obtains the familiar Rayleigh distribution for the envelope.

The APD of the envelope is given by

$$P(A > a) = 1 - a\int_0^\infty \exp(-\gamma t^\alpha)J_1(at)dt, \quad a \geq 0. \tag{78}$$

Or, by integrating (77), one obtains

$$f(a) = \begin{cases} \frac{2^{\alpha k+1}}{\pi\alpha}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k!k}(\Gamma(\alpha k/2+1))^2\sin(\frac{k\alpha\pi}{2})(\frac{a}{\gamma^{1/\alpha}})^{-\alpha k} & 0<\alpha<1 \\ \frac{\gamma}{(a^2+\gamma^2)^{1/2}} & \alpha=1 \\ 1-\frac{1}{2^{2k}\alpha}\sum_{k=0}^{\infty}\frac{(-1)^k}{(k!)^2(2k+2)}\Gamma(\frac{2k+2}{\alpha})(\frac{a}{\gamma^{1/\alpha}})^{2k+2} & 1<\alpha<2 \\ \exp(-a^2/4\gamma) & \alpha=2. \end{cases} \tag{79}$$

From (71), it follows that

$$\lim_{a\to\infty} a^{\alpha+1}f(a) = 2\pi B(\alpha,\gamma) > 0 \tag{80}$$

and hence

$$\lim_{a\to\infty} a^{\alpha}P(A>a) = (2\pi/\alpha)B(\alpha,\gamma) \tag{81}$$

i.e., the envelope distribution and density functions are again heavy-tailed. In particular, the APD of the envelope is asymptotically algebraic, i.e.,

$$P(A>a) = O(a^{-\alpha}) \quad \text{as } x\to\infty \tag{82}$$

Figs. 4 and 5 plot the APD of the $S\alpha S$ noise for various values of α and γ. Note that when $\alpha = 2$, i.e., when the envelope distribution is Rayleigh, one obtains a straight line with slope equal to $-\frac{1}{2}$. Fig. 5 again shows that at low amplitudes the $S\alpha S$ noise is basically Gaussian (Rayleigh).

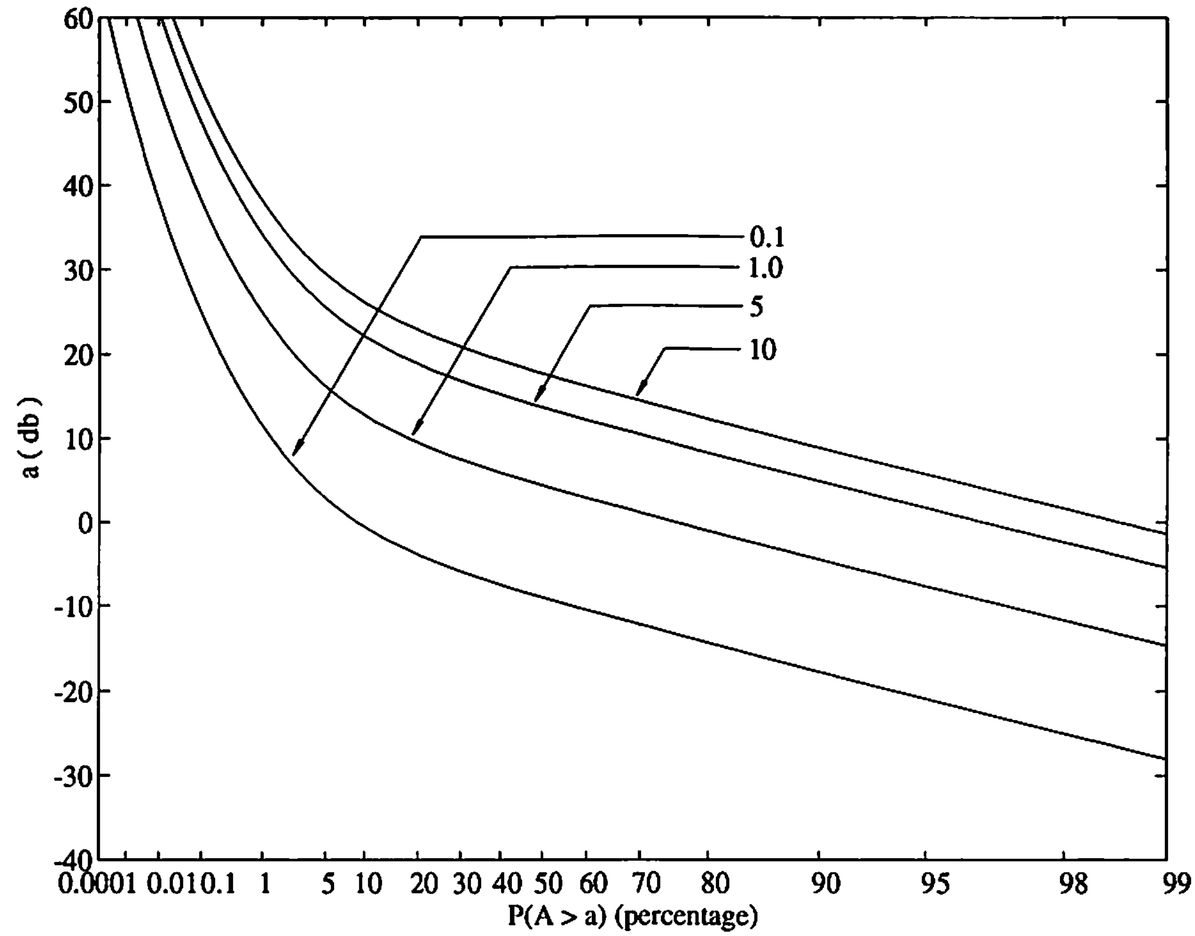

Figure 4: The APD of the envelope of narrowband $S\alpha S$ noise for $\alpha = 1.5$ and various values of γ.

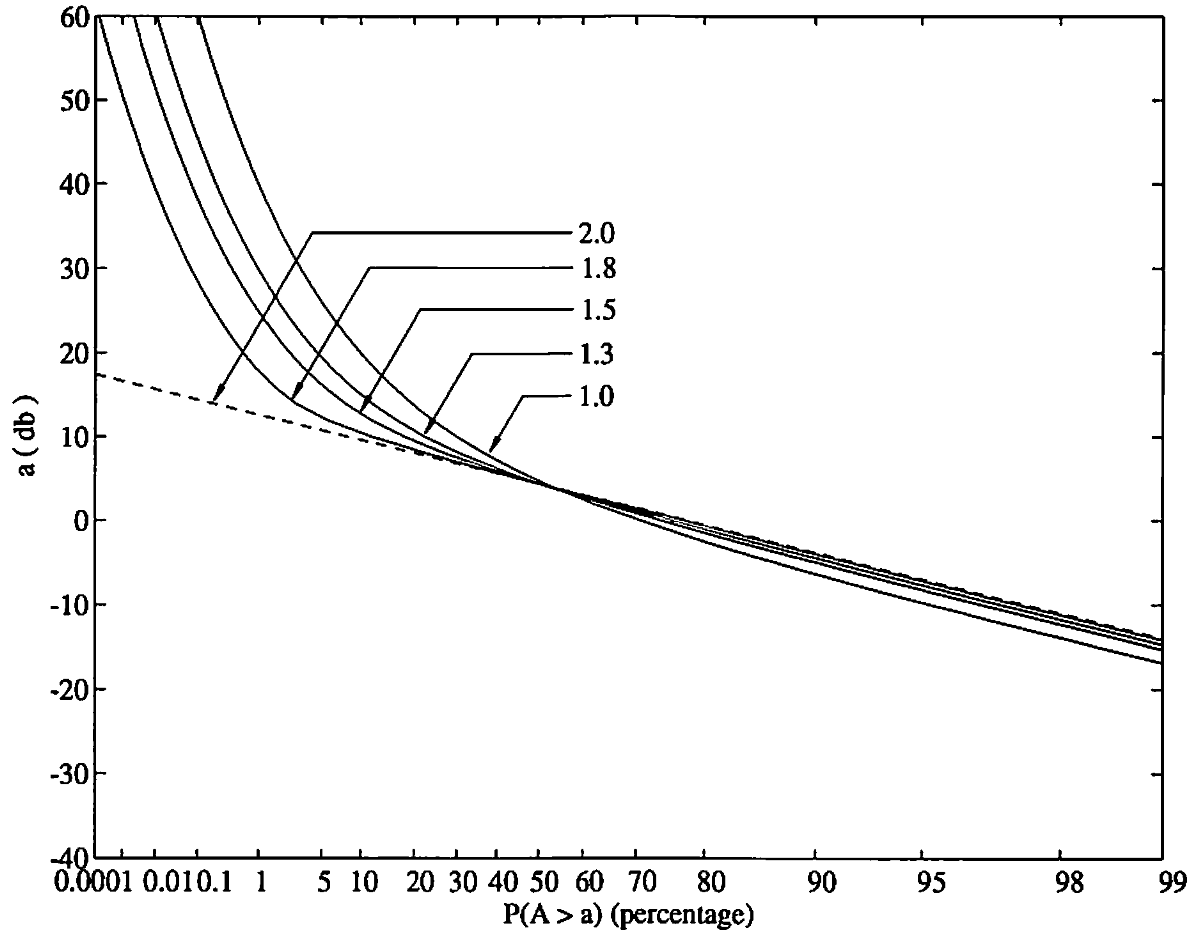

Figure 5: The APD of the envelope of narrowband $S\alpha S$ noise for $\gamma = 1$ and various values of α.

As a consequence of the algebraic character of its distribution, the envelope has no moments of order larger than α. All moments of order less than α are finite and given by

$$\mathbf{E}(A^p) = \frac{2^{p+1}\Gamma(-p/\alpha)\Gamma((2+p)/2)}{\alpha\Gamma(-p/2)}\gamma^{p/\alpha}, \quad \text{if } 0 < p < \alpha. \tag{83}$$

For a proof of (83), see [27].

6 Comparisons with Existing Models and Experimental Data

There exist mainly two types of models for impulsive noise: empirical models and statistical-physical models. Although they are usually very simple, empirical models are designed only to fit the measured statistics of the noise. They do not consider its underlying physics, and their uses are limited to particular situations. Statistical-physical models, on the other hand, are well motivated physically and specified by physically meaningful parameters. However, they often have the disadvantage of being lack of closed forms for the resulting probability distributions except in certain limiting situations.

In this section, we compare the $S\alpha S$ model with some of the typical empirical and statistical-physical models that have been proposed to date. Direct comparisons with experimental data are also made for various types of man-made and natural impulsive noise.

6.1 Middleton's Statistical-Physical Models

Statistical-physical models are typically based on the same filtered-impulse mechanism used to derive the $S\alpha S$ model earlier in this paper, with different assumptions for the noise source distribution and propagation condition [13, 9, 14, 3]. Among them, some of the most general are the Class A and Class B models suggested by Middleton [3]. In developing these models, considerable attention is given to the detailed structures of the basic waveforms of the emissions and the spatial-temporal distribution of the noise sources. In fact, under the assumption that reception is narrowband, a generally applicable, although complicated, expression is derived for the basic waveforms and used in deriving Class A and B models. As a result, these two models are canonical, i.e., they are invariant of the particular physical conditions as long as the received noise is narrowband. Unfortunately, they are also very complicated and difficult to derive. Furthermore, since several approximations have to be made in their derivations in order to get usable results, it is not obvious how the final mathematical formulas for the probability distributions of the Class A and Class B models and the underlying physics of the noise are connected [16].

The $S\alpha S$ model, on the other hand, is derived from simplified assumptions on the source density and propagation process. The main simplifications are that the beam patterns of the receiver antenna and noise sources are nondirectional and that the noise sources are istropically distributed in space. However, the inverse-power characteristic of the propagation law and source density is preserved. No further approximation is used in the derivation. And the derivation is much simpler than that of Middleton's models. It is interesting to note that the Class A model may be derived exactly if one uses similar simplications of the source distribution and propagation condition [16].

Because of the nature of its assumption on the received noise waveforms, the $S\alpha S$ model is necessarily constrained to Class B noise, which includes atmospheric noise, automotive ignition noise, and other non-intelligent noise. In fact, the $S\alpha S$ model and the Class B model are closely related. To see this, one notes that the characteristic function of the instantaneous amplitude of a Class B interference in the absence of Gaussian background noise[1] can be approximated by two distinct functions which are suitably joined at some appropriate threshold z_{0B} [29]. Specifically, when the noise amplitude is less than z_{0B}, the density function is approximately determined by the following characteristic function

$$\varphi_{B-I}(t) = \exp(-b_{1\alpha} A_B \hat{a}^{\alpha} |t|^{\alpha}). \tag{84}$$

When the noise amplitude is larger than z_{0B}, the approximate characteristic function is

$$\varphi_{B-II}(t) = e^{-A_B} \exp(A_B e^{-b_{2\alpha}\hat{a}^2|t|^2/2}). \tag{85}$$

All the parameter in (84) and (85) are non-negative and physically meaningful. In particular, the parameter α is exactly the same one defined by (26).

An immediate observation is that as the threshold z_{0B} approaches to the infinity, the $S\alpha S$ model and the Class B model coincide. Consequently, the $S\alpha S$ model may be viewed as a limiting case of Middleton's Class B model. One can show that the tails of the distribution function determined by (85) is much lighter than the algebraic tails of the $S\alpha S$ distribution. While this ensures that the Class B model has finite variance, it is often the algebraic behavior of the tails of the $S\alpha S$ distribution that is of interest in practice, as we will see later. Also, the full Class B model is very complicated and can

[1]Gaussian background noise is neglected in the development of the $S\alpha S$ model since its contribution is usually insignificant in the presence of impulsive noise.

not be easily used in designing signal processing algorithms. This further suggests that the $S\alpha S$ model is more desirable than the full Class B model. One should also note that the envelope distributions of $S\alpha S$ noise and Class B noise are generally different. While the characteristic function of the envelope of Class B noise can be approximated by two functions similar to those in (84) and (85) [3], that of the envelope of $S\alpha S$ noise is a heavy-tailed generalization of the Rayleigh distribution.

6.2 Empirical Models and Experimental Data

Many of the empirical models are based on the experimental observation that most of the impulsive noise, such as atmospheric noise, ice-cracking noise and automotive ignition noise, are approximately Gaussian at low amplitudes and impulsive at high amplitudes [8, 9, 30]. For example, the Gaussian behavior of atmospheric noise at low amplitudes is the result of many distant lightning discharges, whereas its impulsive behavior at high amplitudes is caused by strong spikes from nearby thunderstorms. A typical empirical model then approximates the probability distribution of the noise envelope by a Rayleigh distribution at low levels and a heavy-tailed distribution at high levels. In many cases, it has been observed that the heavy-tailed distribution can be assumed to follow some algebraic law x^{-n}, where n is typically between 1 and 3 [6, 10, 31, 6].

The behavior of the $S\alpha S$ model coincides with these empirical observations, i.e., $S\alpha S$ distributions exhibit Gaussian behavior at low amplitudes and decay algebraically at the tails. But unlike the empirical models, the $S\alpha S$ model provides physical insight into the noise generation process and is not limited to particular situations. It is certainly possible that other probability distributions could be formulated exhibiting these behaviors, but the $S\alpha S$ model is preferred for several reasons. First, stable distributions share many convenient properties with the Gaussian distribution, such as the stability property, as shown earlier. In addition, they are very flexible as a modeling tool in that the characteristic exponent α allows us to represent noise with a continuous range of impulsiveness. A small value of α implies that the noise is highly impulsive, while a value of α close to 2 indicates a Gaussian type of behavior. Lastly, it agrees very well with the measured data of a variety of man-made and natural noise, as demonstrated below.

Our first example is atmospheric noise, which is the predominant noise source at VLF and ELF. Fig. 6 compares the $S\alpha S$ model with experiment for typical ELF noise. The measured points for moderate-level Malta ELF noise in the bandwidth from 5 to 320 Hz are taken from [1]. Since the ratio of bandwidth to center frequency is not small at ELF, the APD of wideband $S\alpha S$ noise given by (47) is used. The characteristic exponent α and the dispersion γ are selected to best fit the data. Fig. 7 is analogous to Fig. 6, and compares the $S\alpha S$ model with experiment for typical VLF noise. The experimental APD is replotted from [32] and the theoretic APD is calculated from (79) be selecting best values of α and γ. The two figures show that the two-parameter representations of the APD by $S\alpha S$ distributions provide an excellent fit to the measurements of atmospheric noise.

Similar conclusions are also applicable to other man-made and natural impulsive noise, as shown in Figs. 8 and 9. Fig. 8 is an example of primarily urban automotive ignition noise, whereas Fig. 9 shows the APD for fluorescent lights in a mine shop [32]. In both cases, the agreement between theory and experiment is very good.

Another example of $S\alpha S$ noise is the impulsive noise observed on telephone lines. These impulsive interferences are caused by several sources including lighting, switching transients, and accidental hits during maintenance work. A detailed empirical study shows that noise on several telephone lines can be adequately modeled by $S\alpha S$ distributions with characteristic exponents close to 2 [4].

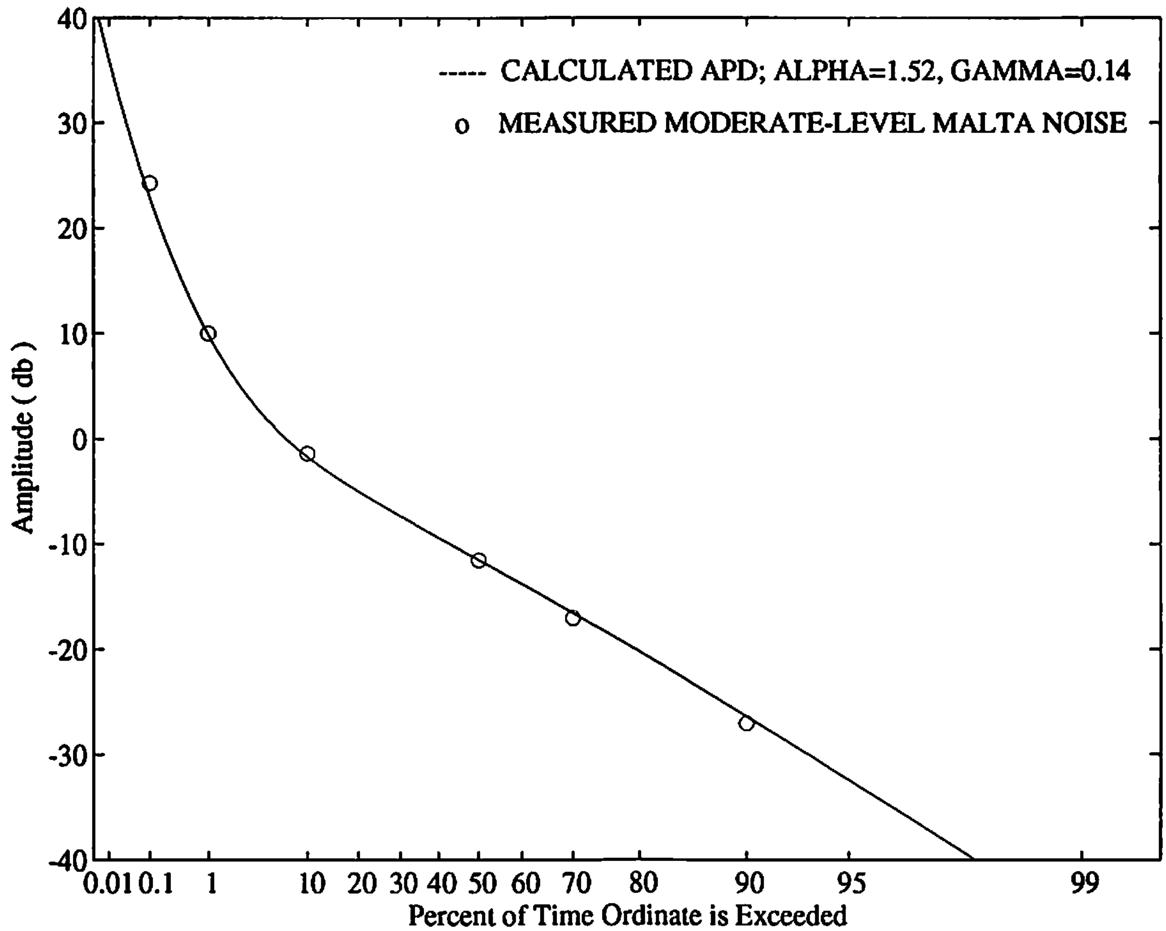

Figure 6: Comparison of a measured APD of ELF atmospheric noise with the $S\alpha S$ model (experimental data taken from [1]).

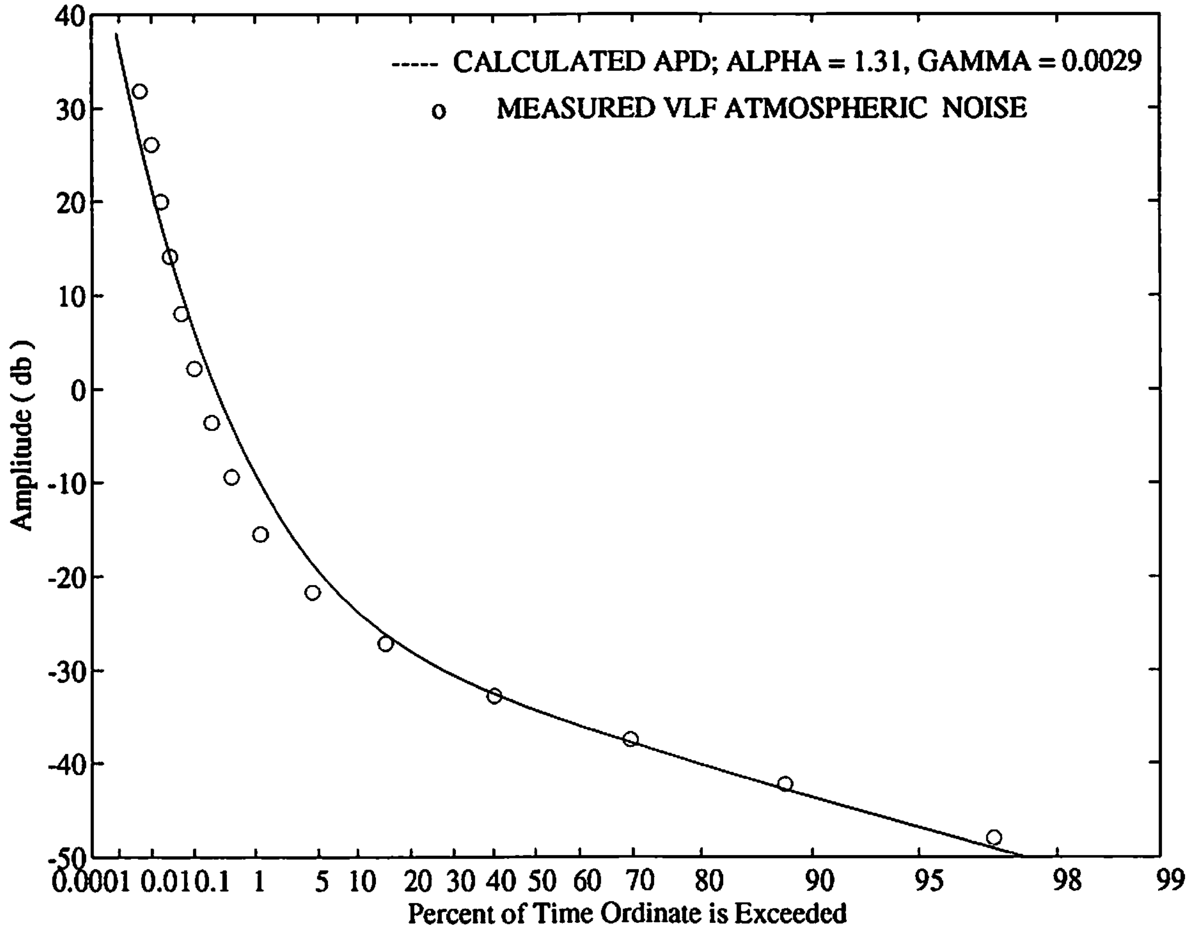

Figure 7: Comparison of a measured envelope APD of VLF atmospheric noise with the $S\alpha S$ model (experimental data taken from [32]).

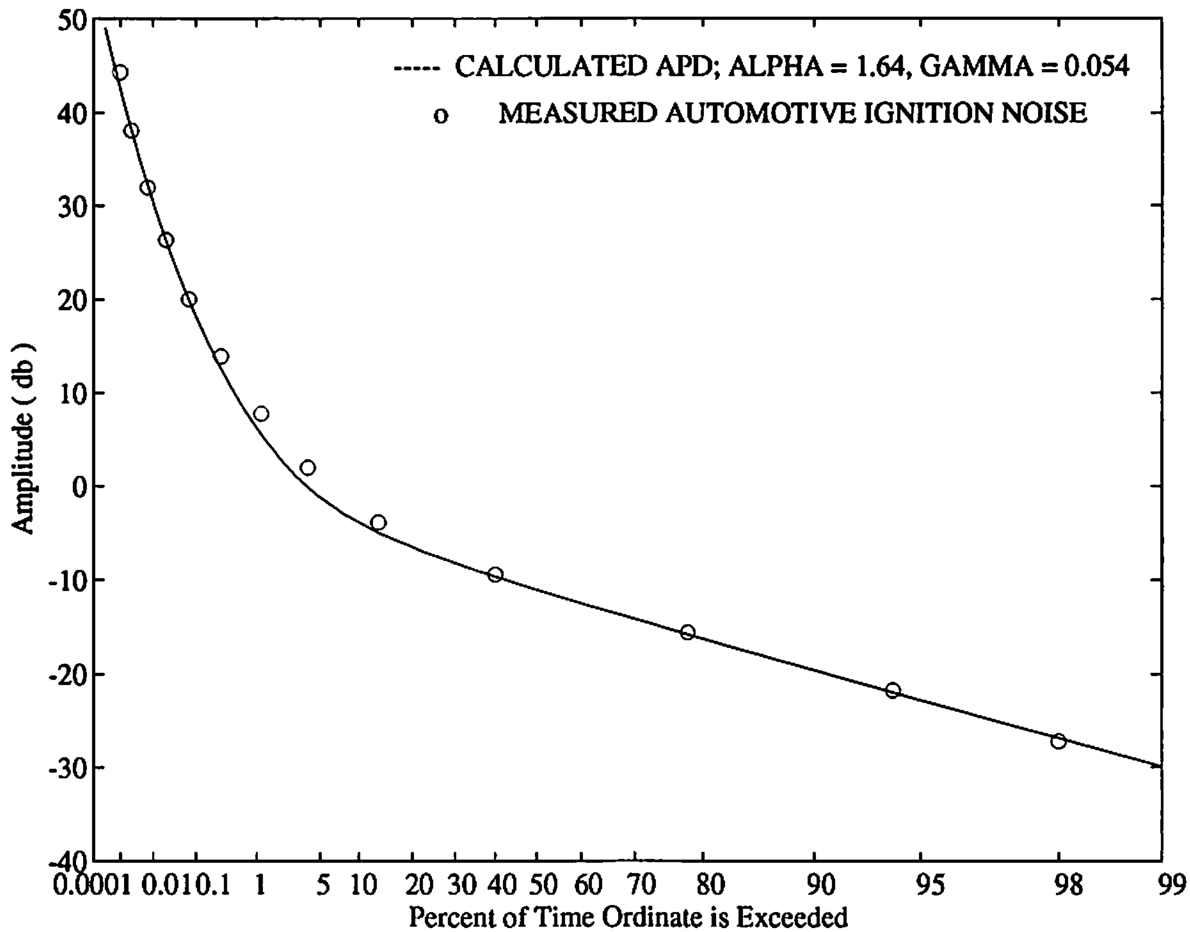

Figure 8: Comparison of a measured envelope APD of automotive ignition noise with the $S\alpha S$ model (experimental data taken from [32]).

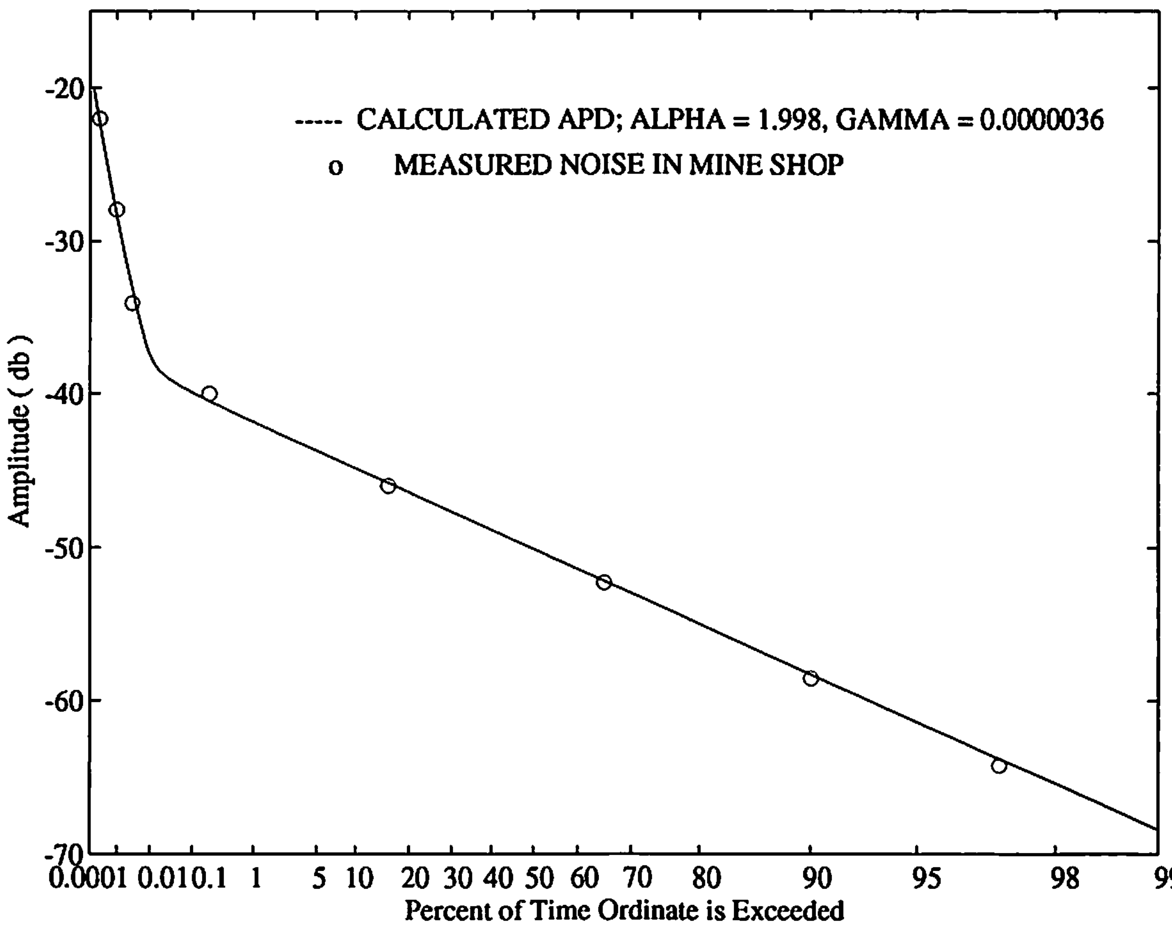

Figure 9: Comparison of a measured envelope APD of fluorescent lights in a mine shop office with the $S\alpha S$ model (experimental data taken from [32])

7 Conclusion

The symmetric stable model has been developed for impulsive noise from the familiar filtered-impulse mechanism of the noise generation process. It is shown to be a direct generalization of the Gaussian model and share many of its convenient and appealing properties, such as the stability property. The stable model is also very flexible and able to describe a wide variety of non-Gaussian phenomena, from those which only slightly deviate from the Gaussian to those which are severely impulsive. In addition, it is found to be consistent with both experimental data and prior theoretical results.

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