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Sums of Independent, Symmetric Stable Random Variables of Different Characteristic Exponents: Study of their Distribution and Application to Stochastic Transient Detection

by

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Abstract

We study the structure of the probability density function of random variables which are formed as the sum of two or more independent, symmetric stable random variables of different characteristic exponents. We present two asymptotic series expansions, valid for small and for large arguments, respectively. As an application of the theory, we develop a receiver which detects impulsive stochastic transients superimposed on Gaussian background noise and show that the new detector outperforms square– and ν^{th}–law detectors.

Key words: Stable distribution, impulsive stochastic transient, optimum detector, suboptimum detectors.

1. Introduction

Statistical processing of physical signals has traditionally relied heavily on the assumption of Gaussian models for the underlying random processes generating the data. However, this assumption is not always justifiable and is often made only for simplicity, mathematical tractability and reduction of the higher computational complexity usually associated with algorithms based on non–Gaussian models. With today's availability of inexpensive computer hardware of very high speed, a reduced algorithmic performance due to simplistic mathematical modeling is no longer acceptable. Thus, the classes of non–Gaussian random processes are becoming increasingly attractive to the signal processing community as models for signals occurring naturally [1, 2].

One physical process, which is not adequately described in terms of Gaussian models, is the process that generates "impulsive" noise bursts. Impulsive bursts occur in the form of short duration interferences, attaining large amplitudes with probability significantly higher than the probability predicted by a Gaussian pdf. Many natural, as well as man–made, sources of impulsive interference exist, including lightning in the atmosphere, switching transients in power lines and car ignitions, accidental hits in telephone lines, and ice cracking in the arctic region [3, 4]. On several occasions, the impulsive interference causes significant degradation of the performance of communication systems and needs to be filtered out [5]; on other occasions, however, the interference carries information and its detection is, actually, the first goal of the statistical signal processing [6, 7]. In both cases, optimal or close to optimal signal processors can be designed only if appropriate statistical models are defined for the impulsive interference.

Symmetric stable processes form a class of random models which present several similarities to the Gaussian processes, such as the stability property and a generalized form of the central limit theorem, and, in fact, contain the Gaussian processes as a subclass. However, several differences exist between the Gaussian and the non–Gaussian stable processes, as explained briefly in Section 2, which make the general stable processes very attractive statistical models for several physical phenomena involving impulsive noise [8, 2, 4]. For example, the Cauchy distribution, which is a stable distribution, was considered in [9] as a model for severe impulsive noise, while Stuck and Kleiner [10] experimentally observed that the noise over certain telephone lines was best described by almost Gaussian stable processes. Very recently, it was theoretically shown that, under general assumptions, the first order statistics of a broad class of impulsive noise can, indeed, be described via an analytically tractable and mathematically appealing model based on the theory of symmetric stable distributions [4].

In this paper, we examine the structure of the probability density function (pdf) of random variables, which consist of sums of two or more independent, symmetric stable random variables of different characteristic exponents, and present an application of the theory. In particular, the paper is organized as follows: Section 2 provides a brief review of the basic definitions and properties associated with the theory of symmetric stable pdfs and proceeds to study the pdf of the superposition of two independent, stable random variables of different characteristic exponents. Since no closed form expression is available for this pdf, we derive asymptotic series expansions, as well as polynomial fits, which allow the real time computation of this pdf at an arbitrary argument. In Section 3, we present an application of the theory of Section 2 in the detection of impulsive stochastic transients in a background of Gaussian noise. We show that the derived optimum receiver outperforms the square– and ν^{th}– law receivers previously presented in the literature [6]. Finally, Section 4 summarizes the key results and suggests possible future research topics.

2. Symmetric Stable Random Variables and their Sums

2.1 The class of SαS pdfs

The general class of symmetric α–stable (SαS) pdfs f_α of *characteristic exponent* α ($0 < \alpha \leq 2$) is obtained [11] via the inverse Fourier transform

$$f_\alpha(\gamma, \delta; x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(i\delta\omega - \gamma|\omega|^\alpha) e^{-i\omega x}\, d\omega. \qquad (2-1)$$

In this equation, δ ($-\infty < \delta < \infty$) is the *location parameter* and γ ($\gamma > 0$) is the *dispersion* of the SαS pdf. Closed form expressions for f_α are available only for the cases of $\alpha = 2$, corresponding to the Gaussian distribution with mean δ and variance 2γ, and of $\alpha = 1$, corresponding to the Cauchy distribution with dispersion γ and median δ:

$$f_2(\gamma, \delta; x) = \frac{1}{\sqrt{4\pi\gamma}} \exp[-\frac{(x-\delta)^2}{4\gamma}] \quad \text{(Gaussian)} \qquad (2\text{-}2)$$

$$f_1(\gamma, \delta; x) = \frac{1}{\pi} \frac{\gamma}{\gamma^2 + (x-\delta)^2} \quad \text{(Cauchy)}\,. \qquad (2\text{-}3)$$

No closed form expressions exist for SαS pdfs other than the Gaussian and the Cauchy; however, asymptotic expansions for all SαS pdfs are known [12, 13, 8], valid for either small (i.e., $|x - \delta| \to 0$) or large (i.e., $|x - \delta| \to \infty$) argument x; a method for real time computation of $f_\alpha(\gamma, \delta; \cdot)$ at arbitrary argument x was devised in [5].

The SαS pdfs present several similarities to the Gaussian pdf: They are smooth and bell–shaped, have the stability property, and naturally arise via a generalized form of the central limit theorem. However, they also differ from the Gaussian pdf in several significant ways. For example, the SαS pdfs have sharper maxima than the Gaussian pdf and algebraic (inverse power) tails in contrast to the exponential tails of the Gaussian pdf. As a result, the p^{th} order moments of the SαS pdfs are finite only for $0 < p < \alpha$. These properties of the SαS pdfs have allowed more accurate modeling of certain economical, physical, biological, and hydrological phenomena and may also indicate applications in statistical signal processing and communications [8].

Eventhough the SαS pdfs have been extensively studied in the literature [11, 13] and their applications are sought in the area of statistical signal processing and communications, little study has been made of sums of symmetric stable random variables of different characteristic exponents. However, such sums often arise in practical problems (see Section 3 for one such case) and, thus, there is a need for expressions and computation of their pdf in real time. We address and provide a solution to this problem here. For simplicity, we restrict the presentation to sums of two symmetric stable random variables of different exponents; however, the general case of sums of an arbitrary number of terms can be treated in a similar straightforward manner.

2.2 Sums of symmetric stable random variables

Definition and general properties

Let X_1 and X_2 be two independent, symmetric stable random variables of characteristic exponents α_1 and α_2 and dispersions γ_1 and γ_2, respectively[1], and let X be their sum[2] $X = X_1 + X_2$. Then, the pdf of X will be given via the inverse Fourier transform

$$f(\alpha_1, \alpha_2, \gamma_1, \gamma_2; x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-\gamma_1|\omega|^{\alpha_1} - \gamma_2|\omega|^{\alpha_2}) e^{-i\omega x}\, d\omega, \qquad (2-4)$$

where we have used the discussion in [14, p. 189] concerning sums of independent random variables and Eq.(2-1). No closed-form expression exists for the above integral, except for the special cases of $\alpha_1 = \alpha_2 = 2$ and $\alpha_1 = \alpha_2 = 1$. Two asymptotic series representations will, however, be derived in the following subsection, valid for small (i.e., $|x| \to 0$) and for large (i.e., $|x| \to \infty$) argument x, respectively. First, we present in Figs. 1a, 1b, and 1c plots of the sum of a Gaussian (i.e., $\alpha_1 = 2$) and a non-Gaussian S($\alpha_2 = 0.5$)S random variable. The dispersions of the Gaussian and the non-Gaussian random variables are 0.5, 1.5 and 0 and 1, 0 and 1.5, respectively. Therefore, Figs. 1b and 1c correspond to a purely Gaussian and a purely S($\alpha_2 = 0.5$)S random variable, respectively. Fig. 1d illustrates the pdf of the sum of

[1]Without loss of generality, we assume that the location parameters of both random variables are equal to zero. This assumption is similar to the usual zero mean assumption concerning Gaussian processes and does not affect our results.

[2]For simplicity, we will refer to X as the "sum random variable" and to its pdf as the "sum pdf."

a S($\alpha_1 = 1.5$)S and a S($\alpha_2 = 0.5$)S process, each of which has dispersion equal to one. Sums of Gaussian and impulsive processes arise in communications systems operating in impulsive noise environments and will, in fact, be considered as models for the stochastic transient detection problem of Section 3.

From the plots in Figs. 1, we observe that the sum variables maintain the general symmetric bell shape, but the sharpness of their peaks and the heaviness of their tails depend on the relative values of their characteristic exponents and their dispersions. Thus, a large variety of observed pdfs can be well approximated with appropriate pdfs of sum variables. When at least one of the variables is non–Gaussian stable, the resulting sum variable has finite p^{th} order moments only for $0 < p < \min\{\alpha_1, \alpha_2\}$, as can be seen from the asymptotic expansion of Eq.(2-6) below.

Asymptotic expansions

The following theorem gives asymptotic expansions for the sum pdf of Eq.(2-4):

Theorem *Let $f(\alpha_1, \alpha_2, \gamma_1, \gamma_2; \cdot)$ be the sum pdf of Eq.(2-4). Then, for all α_1, α_2, γ_1, and γ_2*

$$f(\alpha_1, \alpha_2, \gamma_1, \gamma_2; x) \sim \frac{1}{\pi} \sum_{k=0}^{\infty} [\frac{(-1)^k}{(2k)!} \sum_{l=0}^{\infty} \frac{(-1)^l \gamma_1^l}{l!} \frac{\Gamma(\frac{2k+\alpha_1 l+1}{\alpha_2})}{\alpha_2} \gamma_2^{-\frac{2k+\alpha_1 l+1}{\alpha_2}}] x^{2k}, \qquad (2-5)$$

as $|x| \to 0$, and

$$f(\alpha_1, \alpha_2, \gamma_1, \gamma_2; x) \sim -\frac{1}{\pi} \sum_{k=1}^{\infty} (-1)^k \sum_{l=0}^{k} \sin[\frac{\pi}{2}(\alpha_1 l + \alpha_2 (k-l))] \frac{\gamma_1^l \gamma_2^{k-l}}{l!(k-l)!} \frac{\Gamma[\alpha_1 l + \alpha_2 (k-l) + 1]}{|x|^{\alpha_1 l + \alpha_2 (k-l) + 1}} \qquad (2-6)$$

as $|x| \to \infty$.

Proof The proof is given in the Appendix at the end of the paper.

Computation in real time

A combination of truncated versions of the asymptotic series of the previous subsection can be used in the real time computation of sum pdfs. In particular, the asymptotic series of Eq.(2-5) provides a good approximation to a sum pdf for small argument, while the asymptotic series of Eq.(2-6) provides a good approximation to a sum pdf for large argument. There exists, however, an interval of values of the argument of a sum pdf for which neither series provides a reasonable approximation and, therefore, different mathematical expressions need to be considered. In [5], we faced the same problem when we considered the computation of a SαS pdf at an arbitrary argument. The solution we devised in [5] consisted of first establishing a cutoff argument beyond which a large argument asymptotic series of very few terms was sufficiently accurate and then computing the coefficients of a polynomial of small degree which interpolated the pdf for argument values smaller than the cutoff. This procedure resulted in mathematical expressions which were simple enough to be computable in real time and at the same time yielded a very small error. We follow the same procedure for the sum pdfs of this paper. For illustration purposes, we compare in Fig. 2 the sum pdf of a Gaussian random variable of zero mean and unit variance and a S($\alpha = 0.5$)S random variable of zero location parameter and unit dispersion, as computed via the Fourier inversion formula of Eq.(2-4) (continuous line) and the procedure just described (point line). In the latter case, we used the first two terms of the asymptotic series (2-6) and a fourth degree polynomial fit and we set the cutoff argument to the value 3. Thus, the net formula utilized in the computation of this sum pdf is

$$f(2, 0.5, 0.5, 1; x) = \begin{cases} -0.0046x^4 + 0.0377x^3 - 0.0929x^2 + 0.0107x + 0.1861, & \text{if } |x| < 3 \\ \frac{0.1995}{|x|^{1.5}} - \frac{0.1592}{|x|^2} + \frac{0.3740}{|x|^{3.5}}, & \text{if } |x| \geq 3. \end{cases}$$

Clearly, the above expression provides an excellent approximation to the true sum pdf and can be easily computed in real time.

3. Detection of Impulsive Transients

The detection of deterministic or stochastic transients in time series is often encountered in statistical signal processing [15]. For example, the purpose of communication systems and active radar and sonar is to detect deterministic transient signals with unknown parameters and to estimate their parameters. On the other hand, transient events occurring in such natural phenomena, as seismic, biological, speech, and underwater, are better described with stochastic models. In general, the detection of stochastic transients is a more difficult problem than the detection of deterministic transients and requires the construction of efficient statistical models for the transient and non–transient parts of the observed time series. When both these parts of the observed data are modeled as white Gaussian random processes of different variance and the duration of the transient is known, the optimum detector is a square–law device. If the transient duration is unknown, the square–law detector is only suboptimum. The optimum detector for this case becomes a bank of square–law devices, each matched to a different transient duration. It was shown, however, in [6], that a significant improvement in the performance of a fixed–law detector was possible by choosing the law of the detector to be $\nu > 2$. The assumption, was, however, maintained in [6] of Gaussian models for the transient signal and the background noise in the time series, eventhough it was stated that the transient occurs in the form of an impulsive burst. In this paper, we model the transient as an independent SαS process superimposed over a background of white Gaussian noise. We derive the optimum detector on the basis of the results of the previous section and compare its performance to that of square- and ν^{th}–law detectors. As measure of performance of the detectors, we compute the probability of detection for fixed (given) probability of false alarm.

3.1 Problem Formulation

We consider the following hypothesis testing problem:

$$
\begin{aligned}
H_0 : x &= n_2, \\
H_1 : x &= n_2 + n_\alpha,
\end{aligned}
$$

where n_2 and n_α are independent realizations of a Gaussian and a SαS random variable of variance σ^2 and dispersion γ, respectively. The receiver needs to make a decision on which hypothesis is true, i.e., whether the current random observation x is due to background noise only, or if it contains an impulsive component as well.

3.2 Optimum Detector

To decide between the two hypotheses H_0 and H_1, the optimum receiver computes the test statistic $\log[\frac{f(2,\alpha,\frac{\sigma^2}{2},\gamma;x)}{f_2(\frac{\sigma^2}{2},0;x)}]$, or, equivalently

$$
\Lambda = \log f(2,\alpha,\frac{\sigma^2}{2},\gamma;x) + \frac{x^2}{2\sigma^2}, \qquad (3-1)
$$

and compares it to a preset threshold η. The receiver decides that H_1 is true when $\Lambda > \eta$ and that H_0 is true when $\Lambda < \eta$. The value of the threshold η is chosen so as to satisfy a certain performance level. From Eq.(3-1) above, we observe that the optimum nonlinearity for the detection of stable transients over a background of Gaussian noise consists of the sum of a square–law characteristic and an additional nonlinearity arising from the non–Gaussianity of the transient. This fact should be contrasted with the square–law characteristic of the optimum detector for Gaussian transients over a background of Gaussian noise (see the following subsection).

3.3 Square– and Other ν^{th}–Law Detectors

When $\alpha = 2$ and $2\gamma \neq \sigma^2$, i.e., when we are detecting Gaussian stochastic transients over a background of Gaussian noise, the optimum receiver reduces to a simple square–law device. The receiver, in this case, computes the test statistic

$$
\Lambda_s(x) = |x|^2. \qquad (3-2)
$$

and decides that a Gaussian transient is present when $\Lambda_s > \eta$, and that no transient is present when $\Lambda_s < \eta$.

Detectors based on a ν^{th}–law test statistic

$$\Lambda_\nu(x) = |x|^\nu, \quad \nu \geq 2 \tag{3-3}$$

arise when detecting Gaussian transients of unknown duration in long time series. In particular, it was shown in [6], that ν^{th}–law detectors, eventhough suboptimum, still present an improvement in performance over square–law detectors, which may exceed 1 dB. We also consider, here, the magnitude detector, the test statistic of which is derived from Eq.(3-3) with $\nu = 1$, and compare its performance to the performance of the other detectors.

3.4 Performance Evaluation

The characteristic nonlinearities of the optimum detector of Eq.(3-1) and the power law detectors of Eq.(3-3) are plotted in Fig. 3. We see that both the square–law and the magnitude detectors have a characteristic which approximates that of the optimum detector everywhere except at small arguments. It is this small argument behavior of the optimum nonlinearity that provides the highest performance. We compare the performance of the optimum and the ν^{th}–law detectors when detecting zero median, S($\alpha = 0.5$)S transients over a background of zero mean Gaussian noise. The variance of the Gaussian process is taken equal to $\sigma^2 = 1$ and the dispersion of the S($\alpha = 0.5$)S process is $\gamma = 1$. The pdf under the hypothesis H_1 is, therefore, presented in Fig. 1a. As measure of detector performance, we use the probability of detection of the receivers for probability of false alarm fixed to $P_{fa} = 0.005$. Since the pdf of the test statistic of the detectors is not easy to derive, we compute the probabilities of false alarm and detection following a method based on the characteristic function of the test statistics [16, 17]. In particular, let the characteristic function of the test statistic $L \in \{\Lambda, \Lambda_\nu, \nu = 1, 2, 3, 4\}$ be

$$\phi_L(\omega|H_0) = \int_{-\infty}^{\infty} e^{i\omega L(x)} f_2(\frac{\sigma^2}{2}, 0; x)\, dx = 2\int_0^{\infty} e^{i\omega L(x)} f_2(\frac{\sigma^2}{2}, 0; x)\, dx \tag{3-4}$$

and

$$\phi_L(\omega|H_1) = \int_{-\infty}^{\infty} e^{i\omega L(x)} f(2, \alpha, \frac{\sigma^2}{2}, \gamma; x)\, dx = 2\int_0^{\infty} e^{i\omega L(x)} f(2, \alpha, \frac{\sigma^2}{2}, \gamma; x)\, dx \tag{3-5}$$

under the hypotheses H_0 and H_1, respectively. Then for $L \in \{\Lambda_\nu, \nu = 1, 2, 3, 4\}$, we have [18]

$$\Pr\{L > \eta|H_j\} = \frac{2}{\pi}\int_0^{\infty} \Im\{\phi_L(\omega|H_j)\} \cos(\omega t)\, \frac{d\omega}{\omega}, \quad j = 0, 1. \tag{3-6}$$

When $L = \Lambda$, we have [18]

$$\Pr\{L > \eta | H_j\} = \frac{1}{\pi} \int_0^\infty \Im\{\phi_L(\omega|H_j)\exp(-i\omega t)\} \frac{d\omega}{\omega}, \quad j = 0, 1. \qquad (3-7)$$

In Eqs.(3-6) and (3-7)[3] above, $\Im$ denotes the imaginary part and η is the threshold to which the test statistic is compared and, when $j = 0$ or $j = 1$, the probabilities of false alarm or of detection are obtained, respectively, as a function of the detector threshold η.

We numerically computed the right hand side of Eqs.(3-4) and (3-5) for $0 < \omega \leq 60$. For higher accuracy in the computations, we employed a rotation of the integration paths [6]. Then, the probability of false alarm was computed as a function of the detector threshold from Eqs.(3-6) and (3-7). For accuracy at high threshold values, the numerical integration procedure employed was Filon's integration formula [19, p. 890]. From this computation, we derived the detector threshold required to achieve a probability of false alarm equal to $P_{fa} = 0.005$. In the following table, we show these required thresholds for the detectors that we examine:

Threshold Required to Achieve $P_{fa} = 0.005$	
detector	threshold
Magnitude	2.8
Square–law	7.8
Cube–law	21.0
4^{th}–law	54.7
Optimum	2.2

For these thresholds, the resulting probability of detection was computed using Eqs.(3-5) and (3-6) and is shown in the following table:

Probability of Detection for $P_{fa} = 0.005$	
detector	P_d
Magnitude	0.24
Square–law	0.34
Cube–law	0.17
4^{th}–law	0.07
Optimum	0.46

[3]Eqs.(3-6) and (3-7) differ since the power-law test statistics assume only positive values, while the optimum test statistic assumes both positive and negative values [18].

From the above table, we see that ν^{th}–law detectors of $\nu > 2$ (and, in particular, cube– and 4^{th}–law detectors) do not present any improvement in the detection of impulsive transients over a background of Gaussian noise when compared to square–law detectors. To the contrary, they have a probability of detection smaller than the probability of detection of the square–law detector for the same probability of false alarm. This can become clear by recalling Eq.(3-1), where the optimum test statistic is shown to contain the square–law characteristic as part of its expression. The magnitude detector, on the other hand, performs closely to the square–law detector, yet below it.

4. Conclusions

SαS random processes form a class of statistical models which present several similarities to the well known Gaussian model, but at the same time differ from it in a number of ways. Because of their properties, SαS pdfs receive increasing attention from the signal processing community and are expected to find a number of applications in engineering and communications. In this paper, we examined the structure of the pdf of sums of independent symmetric stable processes of different characteristic exponents and presented series expansions which are computable in real time. As an application of the theory, we presented a processor designed to detect stochastic impulsive transients over a background of Gaussian noise and showed that this detector outperforms, in terms of probability of detection, the existing square– and ν^{th}–law detectors.

In the future, we intend to address the problem of detection of unknown signals in impulsive noise modeled as a symmetric stable random process of zero median and unknown characteristic exponent and dispersion. This problem is a generalization of the well studied problem of detection of an unknown signal in zero mean Gaussian noise of unknown variance, for which Student's t test has been shown to satisfy certain optimality requirements. Since it is known that Student's t test, as well as other nonparametric detectors, perform poorly in impulsive noise environments, it is expected that our approach will provide significant improvement, while at the same time maintaining an acceptable performance in Gaussian noise environments. This research is currently pursued and its results will be announced shortly.

Appendix: Proof of the Asymptotic Expansions Theorem

Since $f(\alpha_1, \alpha_2, \gamma_1, \gamma_2; x) = f(\alpha_1, \alpha_2, \gamma_1, \gamma_2; -x)$, we assume without loss of generality that $x > 0$. To prove the expansion in Eq.(2-5), we begin with Eq.(2-1), which we rewrite as:

$$f(\alpha_1, \alpha_2, \gamma_1, \gamma_2; x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-\gamma_1 |\omega|^{\alpha_1} - \gamma_2 |\omega|^{\alpha_2}) e^{-ix\omega} \, d\omega$$

$$= \frac{1}{\pi}\Re\int_0^\infty \exp(-\gamma_1|\omega|^{\alpha_1} - \gamma_2|\omega|^{\alpha_2})e^{-ix\omega}\,d\omega, \tag{1}$$

where $\Re$ denotes the real part. We consider the identity [12, 6]

$$e^z = \sum_{k=0}^{N}\frac{z^k}{k!} + \frac{z^{N+1}}{N!}\int_0^1 e^{zt}(1-t)^N\,dt, \tag{2}$$

where z can be complex, in general. We apply the relation in Eq.(2) twice, once for $z = -\gamma_1\omega^{\alpha_1}$ and once for $z = -ix\omega$, and obtain the double series representation

$$\begin{aligned} f(\alpha_1,\alpha_2,\gamma_1,\gamma_2;x) &= \frac{1}{\pi}\Re\sum_{k=0}^{N}\sum_{l=0}^{L}\frac{(-ix)^k}{k!}\frac{(-\gamma_1)^l}{l!}\int_0^\infty e^{-\gamma_2\omega^{\alpha_2}}\omega^{k+\alpha_1 l}\,d\omega + E_{N,L}(x) \\ &= \frac{1}{\pi}\Re\sum_{k=0}^{N}x^k\frac{(-i)^k}{k!}\sum_{l=0}^{L}\frac{(-1)^l\gamma_1^l}{l!}\frac{\Gamma(\frac{k+\alpha_1 l+1}{\alpha_2})}{\alpha_2}\gamma_2^{-\frac{k+\alpha_1 l+1}{\alpha_2}} + E_{N,L}(x) \end{aligned} \tag{3}$$

where, to get to Eq.(3), we used the result in [19, p. 255]. The error term in Eq.(3) is

$$\begin{aligned} E_{N,L}(x) &= \frac{1}{\pi}\Re\int_0^\infty [\frac{(-\gamma_1\omega^{\alpha_1})^{L+1}}{L!}\int_0^1 e^{-\gamma_1\omega_1^{\alpha}t_1}(1-t_1)^L\,dt_1] \\ &\quad [\frac{(-ix\omega)^{N+1}}{N!}\int_0^1 e^{-ix\omega t_2}(1-t_2)^N\,dt_2]e^{-\gamma_2\omega^{\alpha_2}}\,d\omega \\ &= x^{N+1}\frac{1}{\pi}\Re\int_0^\infty [\frac{(-\gamma_1\omega^{\alpha_1})^{L+1}}{L!}\int_0^1 e^{-\gamma_1\omega_1^{\alpha}t_1}(1-t_1)^L\,dt_1] \\ &\quad [\frac{(-i\omega)^{N+1}}{N!}\int_0^1 e^{-ix\omega t_2}(1-t_2)^N\,dt_2]e^{-\gamma_2\omega^{\alpha_2}}\,d\omega. \end{aligned} \tag{4}$$

From Eqs.(3) and (4), we clearly see that the ratio of the error term $E_{N,L}$ over the Nth term in the Eq.(3) goes to zero as x goes to zero, for any N, L, therefore

$$f(\alpha_1,\alpha_2,\gamma_1,\gamma_2;x) \sim \frac{1}{\pi}\Re\sum_{k=0}^{\infty}x^k\frac{(-i)^k}{k!}\sum_{l=0}^{\infty}\frac{(-1)^l\gamma_1^l}{l!}\frac{\Gamma(\frac{k+\alpha_1 l+1}{\alpha_2})}{\alpha_2}\gamma_2^{-\frac{k+\alpha_1 l+1}{\alpha_2}}. \tag{5}$$

Moreover, calculation of the real part of the double sum in the right hand side of Eq.(4) and reindexing of the terms yields Eq.(2-5) and proves the first part of the asymptotic expansions theorem.

The second part of the theorem, i.e. Eq.(2-6), can be proved in a similar manner. We begin again with Eq.(1) of this appendix and use the identity of Eq.(2) with $z = -\gamma_1\omega^{\alpha_1} - \gamma_2\omega^{\alpha_2}$ to obtain

$$\begin{aligned} f(\alpha_1,\alpha_2,\gamma_1,\gamma_2;x) &= \frac{1}{\pi}\Re\sum_{k=0}^{N}\frac{(-1)^k}{k!}\int_0^\infty(\gamma_1\omega^{\alpha_1}+\gamma_2\omega^{\alpha_2})^k e^{-ix\omega}\,d\omega + E_N(x) \\ &= \frac{1}{\pi}\Re\sum_{k=0}^{N}\frac{(-1)^k}{k!}\sum_{l=0}^{k}\frac{k!}{l!(k-l)!}\gamma_1^l\gamma_2^{k-l}\int_0^\infty \omega^{\alpha_1 l+\alpha_2(k-l)}e^{-ix\omega}\,d\omega, + E_N(x) \end{aligned} \tag{6}$$

where we have used the binomial expansion formula. We have

$$\begin{aligned} \int_0^\infty \omega^{\alpha_1 l+\alpha_2(k-l)}e^{-ix\omega}\,d\omega &= (-i)e^{-i\frac{\pi}{2}[\alpha_1 l+\alpha_2(k-l)]}\int_0^\infty e^{-xr}r^{\alpha_1 l+\alpha_2(k-l)}\,dr \\ &= (-i)e^{-i\frac{\pi}{2}[\alpha_1 l+\alpha_2(k-l)]}\frac{\Gamma[\alpha_1 l+\alpha_2(k-l)+1]}{x^{\alpha_1 l+\alpha_2(k-l)+1]}}, \end{aligned} \tag{7}$$

where we have used a rotation of the integration path in Eq.(7) and the result in [19, p. 255]. The error term E_N can be computed in the same manner as the error term $E_{N,L}$ was computed in Eq.(4) and shown again to form a ratio over the Nth term in Eq.(6), which has zero limit as $x \to \infty$. Thus, after computation of the real part of the terms, the asymptotic series of Eq.(2-6) is obtained.

Bibliography

[1] C. L. Nikias and A. Petropulu, *Higher-Order Spectra Analysis: A Nonlinear Signal Processing Framework*. Englewood Cliffs, NJ: Prentice-Hall, 1993.

[2] M. Shao and C. L. Nikias, "Detection and adaptive estimation of stable processes with fractional lower order moments," in *Proceedings of Sixth IEEE Workshop on Statistical Signal and Array Processing*, (Victoria, BC, Canada), pp. 94–97, October, 1992.

[3] E. J. Wegman, S. G. Schwartz, and J. B. Thomas, eds., *Topics in Non-Gaussian Signal Processing*. New York: Academic Press, 1989.

[4] M. Shao and C. L. Nikias, "On symmetric stable models for impulsive noise," *USC-SIPI Report # 231*, 1993.

[5] G. A. Tsihrintzis and C. L. Nikias, "Performance of optimum and suboptimum receivers in the presence of impulsive noise modeled as an α–stable process," *IEEE Trans. Comm.*, 1993. (submitted).

[6] J. A. Fawcett and B. H. Maranda, "The optimal power law for the detection of a Gaussian burst in a background of Gaussian noise," *IEEE Trans. Inform. Theory*, vol. IT-37, pp. 209–214, 1991.

[7] P. Zakarauskas, "Detection and localization of non-deterministic transients in time series and application to ice-cracking sound," *Digital Signal Processing*, vol. 3, pp. 36–45, 1993.

[8] M. Shao and C. L. Nikias, "Signal processing with fractional lower order moments: Stable processes and their applications," *Proc. IEEE*, July, 1993.

[9] S. S. Rappaport and L. Kurz, "An optimal nonlinear detector for digital data transmission through non-Gaussian channels," *IEEE Trans. Comm. Techn.*, vol. COM-14, pp. 266–274, 1966.

[10] B. W. Stuck and B. Kleiner, "A statistical analysis of telephone noise," *The Bell System Technical Journal*, vol. 53, pp. 1262–1320, 1974.

[11] P. Lévy, *Calcul des Probabilités*, vol. II. Paris: Gauthier-Villards, 1925. chapter 6.

[12] H. Bergstrom, "On some expansions of stable distribution functions," *Arkiv for Matematik*, vol. 2, pp. 375–378, 1952.

[13] V. Zolotarev, *One-Dimensional Stable Distributions*. Providence, RI: American Mathematical Society, 1986.

[14] A. Papoulis, *Probability, Random Variables, and Stochastic Processes*. Tokyo: McGraw-Hill, Intern. stud. ed., 1981.

[15] B. Friedlander and B. Porat, "Detection of transient signals by the Gabor representation," *IEEE Trans. Acoust., Speech, and Signal Process.*, vol. ASSP-37, pp. 169–180, 1989.

[16] J. S. Bird, "Calculating detection probabilities for systems employing noncoherent integration," *IEEE Trans. Aerospace Electr. Syst.*, vol. AES-18, pp. 401–409, 1982.

[17] C. W. Helstrom and J. A. Ritcey, "Evaluating radar detection probabilities by steepest descent integration," *IEEE Trans. Aerospace Electr. Syst.*, vol. AES-20, pp. 624–633, 1984.

[18] A. H. Nuttall, "Alternate forms for numerical evaluation of cumulative probability distributions directly from characteristic functions," *Proc. IEEE*, vol. 58, pp. 1872–1873, 1970.

[19] M. Abramowitz and I. A. Stegun, eds., *Handbook of Mathematical Functions*. New York: Dover Publications, 1965.

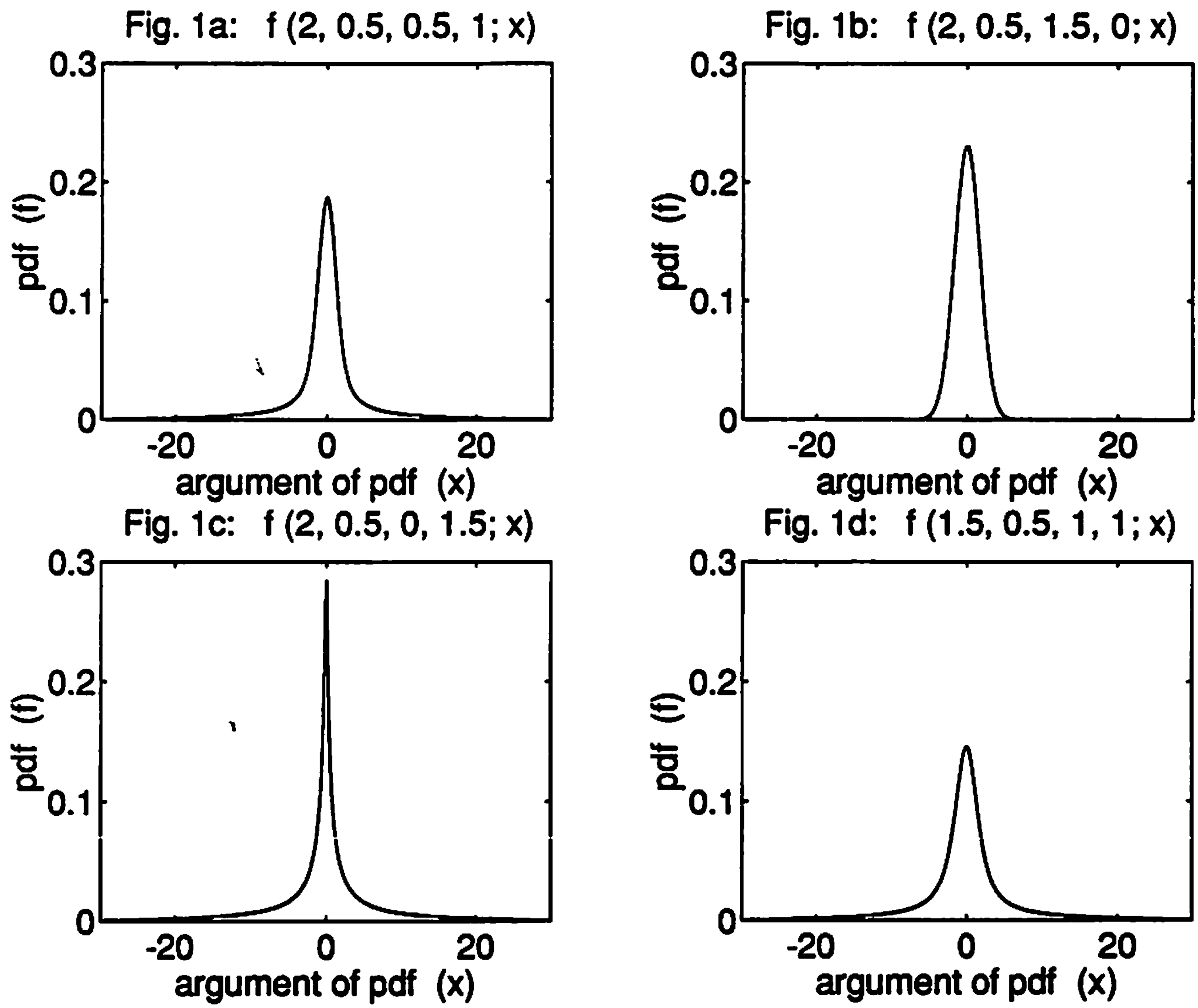
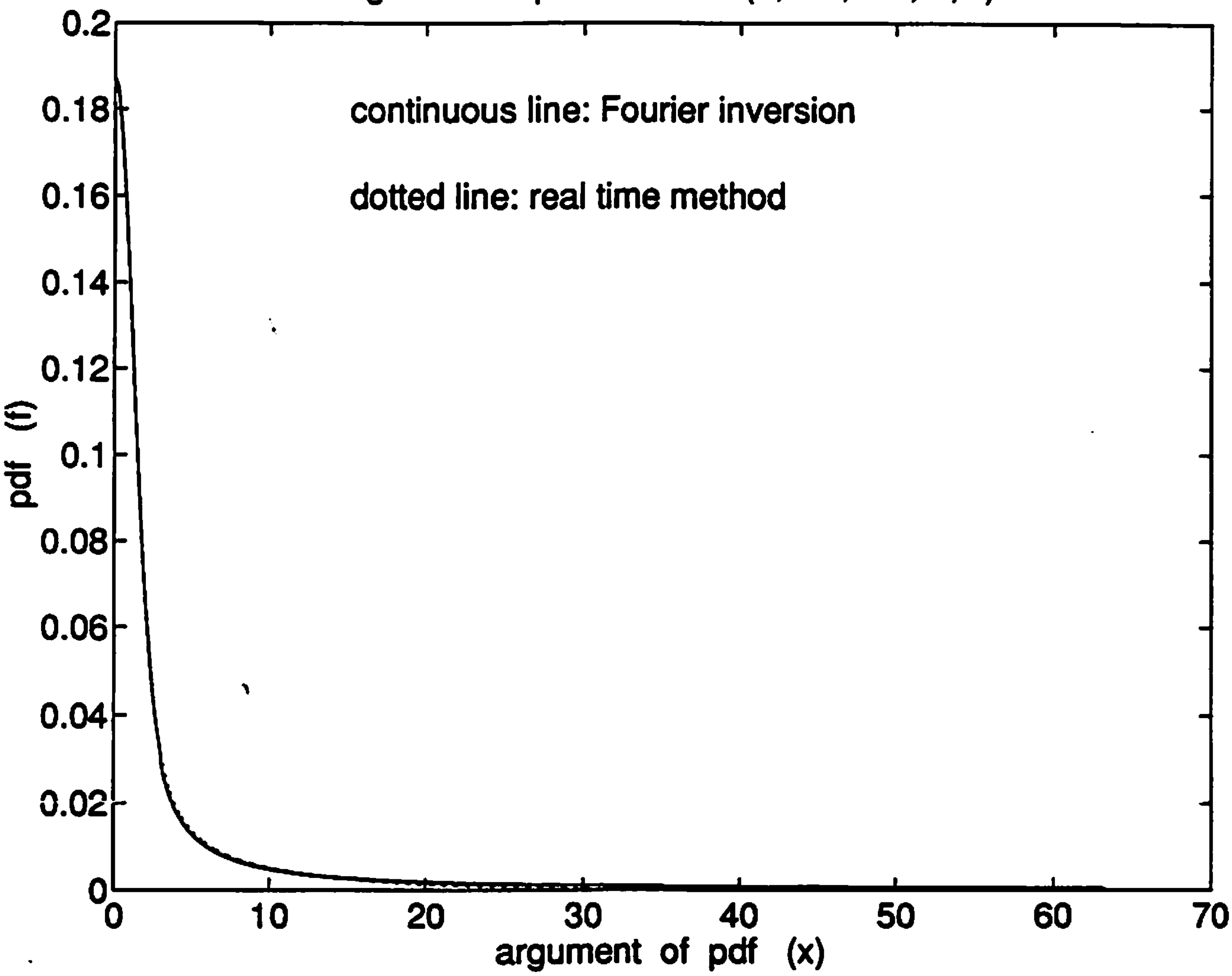


Fig. 1a: f (2, 0.5, 0.5, 1; x)
Fig. 1b: f (2, 0.5, 1.5, 0; x)
Fig. 1c: f (2, 0.5, 0, 1.5; x)
Fig. 1d: f (1.5, 0.5, 1, 1; x)
pdf (f)
argument of pdf (x)
0
0.1
0.2
0.3
-20
0
20

Fig. 2: Computation of f (2, 0.5, 0.5, 1; x)



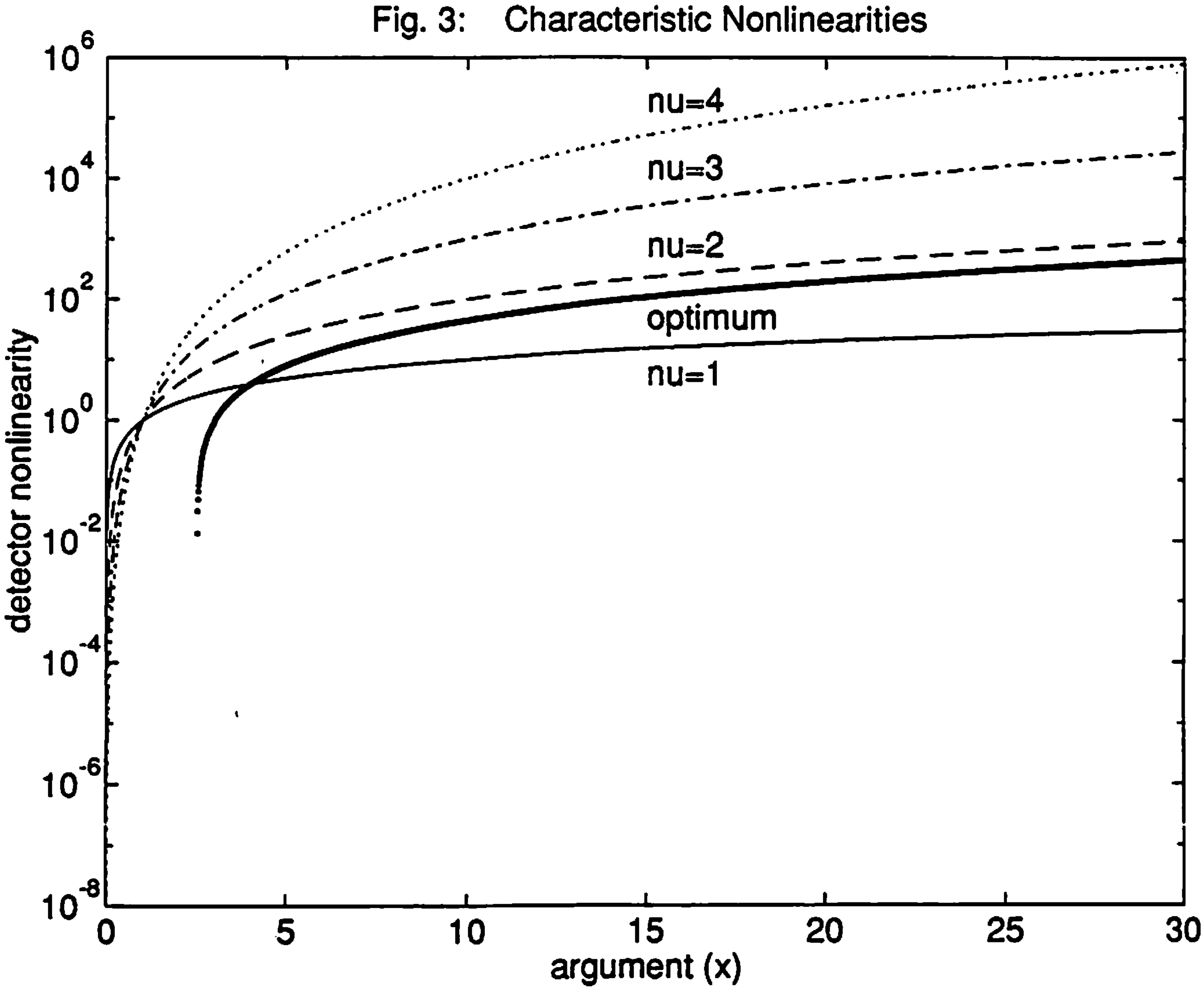


Fig. 3: Characteristic Nonlinearities
nu=4
nu=3
nu=2
optimum
nu=1
detector nonlinearity
argument (x)
10^6
10^4
10^2
10^0
10^-2
10^-4
10^-6
10^-8
0
5
10
15
20
25
30

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Sums of Independent, Symmetric Stable Random Variables of Different Characteristic Exponents: Study of their Distribution and Application to Stochastic Transient Detection

by

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Abstract

We study the structure of the probability density function of random variables which are formed as the sum of two or more independent, symmetric stable random variables of different characteristic exponents. We present two asymptotic series expansions, valid for small and for large arguments, respectively. As an application of the theory, we develop a receiver which detects impulsive stochastic transients superimposed on Gaussian background noise and show that the new detector outperforms square– and ν^{th}–law detectors.

Key words: Stable distribution, impulsive stochastic transient, optimum detector, suboptimum detectors.

1. Introduction

Statistical processing of physical signals has traditionally relied heavily on the assumption of Gaussian models for the underlying random processes generating the data. However, this assumption is not always justifiable and is often made only for simplicity, mathematical tractability and reduction of the higher computational complexity usually associated with algorithms based on non–Gaussian models. With today's availability of inexpensive computer hardware of very high speed, a reduced algorithmic performance due to simplistic mathematical modeling is no longer acceptable. Thus, the classes of non–Gaussian random processes are becoming increasingly attractive to the signal processing community as models for signals occurring naturally [1, 2].

One physical process, which is not adequately described in terms of Gaussian models, is the process that generates "impulsive" noise bursts. Impulsive bursts occur in the form of short duration interferences, attaining large amplitudes with probability significantly higher than the probability predicted by a Gaussian pdf. Many natural, as well as man–made, sources of impulsive interference exist, including lightning in the atmosphere, switching transients in power lines and car ignitions, accidental hits in telephone lines, and ice cracking in the arctic region [3, 4]. On several occasions, the impulsive interference causes significant degradation of the performance of communication systems and needs to be filtered out [5]; on other occasions, however, the interference carries information and its detection is, actually, the first goal of the statistical signal processing [6, 7]. In both cases, optimal or close to optimal signal processors can be designed only if appropriate statistical models are defined for the impulsive interference.

Symmetric stable processes form a class of random models which present several similarities to the Gaussian processes, such as the stability property and a generalized form of the central limit theorem, and, in fact, contain the Gaussian processes as a subclass. However, several differences exist between the Gaussian and the non–Gaussian stable processes, as explained briefly in Section 2, which make the general stable processes very attractive statistical models for several physical phenomena involving impulsive noise [8, 2, 4]. For example, the Cauchy distribution, which is a stable distribution, was considered in [9] as a model for severe impulsive noise, while Stuck and Kleiner [10] experimentally observed that the noise over certain telephone lines was best described by almost Gaussian stable processes. Very recently, it was theoretically shown that, under general assumptions, the first order statistics of a broad class of impulsive noise can, indeed, be described via an analytically tractable and mathematically appealing model based on the theory of symmetric stable distributions [4].

In this paper, we examine the structure of the probability density function (pdf) of random variables, which consist of sums of two or more independent, symmetric stable random variables of different characteristic exponents, and present an application of the theory. In particular, the paper is organized as follows: Section 2 provides a brief review of the basic definitions and properties associated with the theory of symmetric stable pdfs and proceeds to study the pdf of the superposition of two independent, stable random variables of different characteristic exponents. Since no closed form expression is available for this pdf, we derive asymptotic series expansions, as well as polynomial fits, which allow the real time computation of this pdf at an arbitrary argument. In Section 3, we present an application of the theory of Section 2 in the detection of impulsive stochastic transients in a background of Gaussian noise. We show that the derived optimum receiver outperforms the square– and ν^{th}– law receivers previously presented in the literature [6]. Finally, Section 4 summarizes the key results and suggests possible future research topics.

2. Symmetric Stable Random Variables and their Sums

2.1 The class of SαS pdfs

The general class of symmetric α–stable (SαS) pdfs f_α of *characteristic exponent* α ($0 < \alpha \leq 2$) is obtained [11] via the inverse Fourier transform

$$f_\alpha(\gamma, \delta; x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(i\delta\omega - \gamma|\omega|^\alpha) e^{-i\omega x}\, d\omega. \qquad (2-1)$$

In this equation, δ ($-\infty < \delta < \infty$) is the *location parameter* and γ ($\gamma > 0$) is the *dispersion* of the SαS pdf. Closed form expressions for f_α are available only for the cases of $\alpha = 2$, corresponding to the Gaussian distribution with mean δ and variance 2γ, and of $\alpha = 1$, corresponding to the Cauchy distribution with dispersion γ and median δ:

$$f_2(\gamma, \delta; x) = \frac{1}{\sqrt{4\pi\gamma}} \exp[-\frac{(x-\delta)^2}{4\gamma}] \quad \text{(Gaussian)} \qquad (2\text{-}2)$$

$$f_1(\gamma, \delta; x) = \frac{1}{\pi} \frac{\gamma}{\gamma^2 + (x-\delta)^2} \quad \text{(Cauchy)}\,. \qquad (2\text{-}3)$$

No closed form expressions exist for SαS pdfs other than the Gaussian and the Cauchy; however, asymptotic expansions for all SαS pdfs are known [12, 13, 8], valid for either small (i.e., $|x-\delta| \to 0$) or large (i.e., $|x-\delta| \to \infty$) argument x; a method for real time computation of $f_\alpha(\gamma, \delta; \cdot)$ at arbitrary argument x was devised in [5].

The SαS pdfs present several similarities to the Gaussian pdf: They are smooth and bell–shaped, have the stability property, and naturally arise via a generalized form of the central limit theorem. However, they also differ from the Gaussian pdf in several significant ways. For example, the SαS pdfs have sharper maxima than the Gaussian pdf and algebraic (inverse power) tails in contrast to the exponential tails of the Gaussian pdf. As a result, the p^{th} order moments of the SαS pdfs are finite only for $0 < p < \alpha$. These properties of the SαS pdfs have allowed more accurate modeling of certain economical, physical, biological, and hydrological phenomena and may also indicate applications in statistical signal processing and communications [8].

Eventhough the SαS pdfs have been extensively studied in the literature [11, 13] and their applications are sought in the area of statistical signal processing and communications, little study has been made of sums of symmetric stable random variables of different characteristic exponents. However, such sums often arise in practical problems (see Section 3 for one such case) and, thus, there is a need for expressions and computation of their pdf in real time. We address and provide a solution to this problem here. For simplicity, we restrict the presentation to sums of two symmetric stable random variables of different exponents; however, the general case of sums of an arbitrary number of terms can be treated in a similar straightforward manner.

2.2 Sums of symmetric stable random variables

Definition and general properties

Let X_1 and X_2 be two independent, symmetric stable random variables of characteristic exponents α_1 and α_2 and dispersions γ_1 and γ_2, respectively[1], and let X be their sum[2] $X = X_1 + X_2$. Then, the pdf of X will be given via the inverse Fourier transform

$$f(\alpha_1, \alpha_2, \gamma_1, \gamma_2; x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-\gamma_1|\omega|^{\alpha_1} - \gamma_2|\omega|^{\alpha_2}) e^{-i\omega x}\, d\omega, \qquad (2-4)$$

where we have used the discussion in [14, p. 189] concerning sums of independent random variables and Eq.(2-1). No closed-form expression exists for the above integral, except for the special cases of $\alpha_1 = \alpha_2 = 2$ and $\alpha_1 = \alpha_2 = 1$. Two asymptotic series representations will, however, be derived in the following subsection, valid for small (i.e., $|x| \to 0$) and for large (i.e., $|x| \to \infty$) argument x, respectively. First, we present in Figs. 1a, 1b, and 1c plots of the sum of a Gaussian (i.e., $\alpha_1 = 2$) and a non-Gaussian S($\alpha_2 = 0.5$)S random variable. The dispersions of the Gaussian and the non-Gaussian random variables are 0.5, 1.5 and 0 and 1, 0 and 1.5, respectively. Therefore, Figs. 1b and 1c correspond to a purely Gaussian and a purely S($\alpha_2 = 0.5$)S random variable, respectively. Fig. 1d illustrates the pdf of the sum of

[1] Without loss of generality, we assume that the location parameters of both random variables are equal to zero. This assumption is similar to the usual zero mean assumption concerning Gaussian processes and does not affect our results.

[2] For simplicity, we will refer to X as the "sum random variable" and to its pdf as the "sum pdf."

a S($\alpha_1 = 1.5$)S and a S($\alpha_2 = 0.5$)S process, each of which has dispersion equal to one. Sums of Gaussian and impulsive processes arise in communications systems operating in impulsive noise environments and will, in fact, be considered as models for the stochastic transient detection problem of Section 3.

From the plots in Figs. 1, we observe that the sum variables maintain the general symmetric bell shape, but the sharpness of their peaks and the heaviness of their tails depend on the relative values of their characteristic exponents and their dispersions. Thus, a large variety of observed pdfs can be well approximated with appropriate pdfs of sum variables. When at least one of the variables is non-Gaussian stable, the resulting sum variable has finite p^{th} order moments only for $0 < p < \min\{\alpha_1, \alpha_2\}$, as can be seen from the asymptotic expansion of Eq.(2-6) below.

Asymptotic expansions

The following theorem gives asymptotic expansions for the sum pdf of Eq.(2-4):

Theorem *Let $f(\alpha_1, \alpha_2, \gamma_1, \gamma_2; \cdot)$ be the sum pdf of Eq.(2-4). Then, for all α_1, α_2, γ_1, and γ_2*

$$f(\alpha_1, \alpha_2, \gamma_1, \gamma_2; x) \sim \frac{1}{\pi} \sum_{k=0}^{\infty} \Big[\frac{(-1)^k}{(2k)!} \sum_{l=0}^{\infty} \frac{(-1)^l \gamma_1^l}{l!} \frac{\Gamma(\frac{2k+\alpha_1 l+1}{\alpha_2})}{\alpha_2} \gamma_2^{-\frac{2k+\alpha_1 l+1}{\alpha_2}} \Big] x^{2k}, \qquad (2-5)$$

as $|x| \to 0$, and

$$f(\alpha_1, \alpha_2, \gamma_1, \gamma_2; x) \sim -\frac{1}{\pi} \sum_{k=1}^{\infty} (-1)^k \sum_{l=0}^{k} \sin\Big[\frac{\pi}{2}(\alpha_1 l + \alpha_2 (k-l))\Big] \frac{\gamma_1^l \gamma_2^{k-l}}{l!(k-l)!} \frac{\Gamma[\alpha_1 l + \alpha_2(k-l) + 1]}{|x|^{\alpha_1 l + \alpha_2 (k-l) + 1}} \qquad (2-6)$$

as $|x| \to \infty$.

Proof The proof is given in the Appendix at the end of the paper.

Computation in real time

A combination of truncated versions of the asymptotic series of the previous subsection can be used in the real time computation of sum pdfs. In particular, the asymptotic series of Eq.(2-5) provides a good approximation to a sum pdf for small argument, while the asymptotic series of Eq.(2-6) provides a good approximation to a sum pdf for large argument. There exists, however, an interval of values of the argument of a sum pdf for which neither series provides a reasonable approximation and, therefore, different mathematical expressions need to be considered. In [5], we faced the same problem when we considered the computation of a SαS pdf at an arbitrary argument. The solution we devised in [5] consisted of first establishing a cutoff argument beyond which a large argument asymptotic series of very few terms was sufficiently accurate and then computing the coefficients of a polynomial of small degree which interpolated the pdf for argument values smaller than the cutoff. This procedure resulted in mathematical expressions which were simple enough to be computable in real time and at the same time yielded a very small error. We follow the same procedure for the sum pdfs of this paper. For illustration purposes, we compare in Fig. 2 the sum pdf of a Gaussian random variable of zero mean and unit variance and a S($\alpha = 0.5$)S random variable of zero location parameter and unit dispersion, as computed via the Fourier inversion formula of Eq.(2-4) (continuous line) and the procedure just described (point line). In the latter case, we used the first two terms of the asymptotic series (2-6) and a fourth degree polynomial fit and we set the cutoff argument to the value 3. Thus, the net formula utilized in the computation of this sum pdf is

$$f(2, 0.5, 0.5, 1; x) = \begin{cases} -0.0046x^4 + 0.0377x^3 - 0.0929x^2 + 0.0107x + 0.1861, & \text{if } |x| < 3 \\ \frac{0.1995}{|x|^{1.5}} - \frac{0.1592}{|x|^2} + \frac{0.3740}{|x|^{3.5}}, & \text{if } |x| \geq 3. \end{cases}$$

Clearly, the above expression provides an excellent approximation to the true sum pdf and can be easily computed in real time.

3. Detection of Impulsive Transients

The detection of deterministic or stochastic transients in time series is often encountered in statistical signal processing [15]. For example, the purpose of communication systems and active radar and sonar is to detect deterministic transient signals with unknown parameters and to estimate their parameters. On the other hand, transient events occurring in such natural phenomena, as seismic, biological, speech, and underwater, are better described with stochastic models. In general, the detection of stochastic transients is a more difficult problem than the detection of deterministic transients and requires the construction of efficient statistical models for the transient and non–transient parts of the observed time series. When both these parts of the observed data are modeled as white Gaussian random processes of different variance and the duration of the transient is known, the optimum detector is a square–law device. If the transient duration is unknown, the square–law detector is only suboptimum. The optimum detector for this case becomes a bank of square–law devices, each matched to a different transient duration. It was shown, however, in [6], that a significant improvement in the performance of a fixed–law detector was possible by choosing the law of the detector to be $\nu > 2$. The assumption, was, however, maintained in [6] of Gaussian models for the transient signal and the background noise in the time series, eventhough it was stated that the transient occurs in the form of an impulsive burst. In this paper, we model the transient as an independent SαS process superimposed over a background of white Gaussian noise. We derive the optimum detector on the basis of the results of the previous section and compare its performance to that of square– and ν^{th}–law detectors. As measure of performance of the detectors, we compute the probability of detection for fixed (given) probability of false alarm.

3.1 Problem Formulation

We consider the following hypothesis testing problem:

$$\begin{aligned} H_0 : x &= n_2, \\ H_1 : x &= n_2 + n_\alpha, \end{aligned}$$

where n_2 and n_α are independent realizations of a Gaussian and a SαS random variable of variance σ^2 and dispersion γ, respectively. The receiver needs to make a decision on which hypothesis is true, i.e., whether the current random observation x is due to background noise only, or if it contains an impulsive component as well.

3.2 Optimum Detector

To decide between the two hypotheses H_0 and H_1, the optimum receiver computes the test statistic $\log[\frac{f(2,\alpha,\frac{\sigma^2}{2},\gamma;x)}{f_2(\frac{\sigma^2}{2},0;x)}]$, or, equivalently

$$\Lambda = \log f(2, \alpha, \frac{\sigma^2}{2}, \gamma; x) + \frac{x^2}{2\sigma^2}, \tag{3-1}$$

and compares it to a preset threshold η. The receiver decides that H_1 is true when $\Lambda > \eta$ and that H_0 is true when $\Lambda < \eta$. The value of the threshold η is chosen so as to satisfy a certain performance level. From Eq.(3-1) above, we observe that the optimum nonlinearity for the detection of stable transients over a background of Gaussian noise consists of the sum of a square–law characteristic and an additional nonlinearity arising from the non–Gaussianity of the transient. This fact should be contrasted with the square–law characteristic of the optimum detector for Gaussian transients over a background of Gaussian noise (see the following subsection).

3.3 Square– and Other ν^{th}–Law Detectors

When $\alpha = 2$ and $2\gamma \neq \sigma^2$, i.e., when we are detecting Gaussian stochastic transients over a background of Gaussian noise, the optimum receiver reduces to a simple square–law device. The receiver, in this case, computes the test statistic

$$\Lambda_s(x) = |x|^2. \tag{3-2}$$

and decides that a Gaussian transient is present when $\Lambda_s > \eta$, and that no transient is present when $\Lambda_s < \eta$.

Detectors based on a ν^{th}–law test statistic

$$\Lambda_\nu(x) = |x|^\nu, \quad \nu \geq 2 \tag{3-3}$$

arise when detecting Gaussian transients of unknown duration in long time series. In particular, it was shown in [6], that ν^{th}–law detectors, eventhough suboptimum, still present an improvement in performance over square–law detectors, which may exceed 1 dB. We also consider, here, the magnitude detector, the test statistic of which is derived from Eq.(3-3) with $\nu = 1$, and compare its performance to the performance of the other detectors.

3.4 Performance Evaluation

The characteristic nonlinearities of the optimum detector of Eq.(3-1) and the power law detectors of Eq.(3-3) are plotted in Fig. 3. We see that both the square–law and the magnitude detectors have a characteristic which approximates that of the optimum detector everywhere except at small arguments. It is this small argument behavior of the optimum nonlinearity that provides the highest performance. We compare the performance of the optimum and the ν^{th}–law detectors when detecting zero median, S($\alpha = 0.5$)S transients over a background of zero mean Gaussian noise. The variance of the Gaussian process is taken equal to $\sigma^2 = 1$ and the dispersion of the S($\alpha = 0.5$)S process is $\gamma = 1$. The pdf under the hypothesis H_1 is, therefore, presented in Fig. 1a. As measure of detector performance, we use the probability of detection of the receivers for probability of false alarm fixed to $P_{fa} = 0.005$. Since the pdf of the test statistic of the detectors is not easy to derive, we compute the probabilities of false alarm and detection following a method based on the characteristic function of the test statistics [16, 17]. In particular, let the characteristic function of the test statistic $L \in \{\Lambda, \Lambda_\nu, \nu = 1, 2, 3, 4\}$ be

$$\phi_L(\omega|H_0) = \int_{-\infty}^{\infty} e^{i\omega L(x)} f_2(\frac{\sigma^2}{2}, 0; x)\, dx = 2\int_0^{\infty} e^{i\omega L(x)} f_2(\frac{\sigma^2}{2}, 0; x)\, dx \tag{3-4}$$

and

$$\phi_L(\omega|H_1) = \int_{-\infty}^{\infty} e^{i\omega L(x)} f(2, \alpha, \frac{\sigma^2}{2}, \gamma; x)\, dx = 2\int_0^{\infty} e^{i\omega L(x)} f(2, \alpha, \frac{\sigma^2}{2}, \gamma; x)\, dx \tag{3-5}$$

under the hypotheses H_0 and H_1, respectively. Then for $L \in \{\Lambda_\nu, \nu = 1, 2, 3, 4\}$, we have [18]

$$\Pr\{L > \eta|H_j\} = \frac{2}{\pi}\int_0^{\infty} \Im\{\phi_L(\omega|H_j)\} \cos(\omega t)\, \frac{d\omega}{\omega}, \quad j = 0, 1. \tag{3-6}$$

When $L = \Lambda$, we have [18]

$$\Pr\{L > \eta | H_j\} = \frac{1}{\pi}\int_0^{\infty} \Im\{\phi_L(\omega|H_j)\exp(-i\omega t)\}\,\frac{d\omega}{\omega}, \quad j = 0, 1. \qquad (3-7)$$

In Eqs.(3-6) and (3-7)[3] above, $\Im$ denotes the imaginary part and η is the threshold to which the test statistic is compared and, when $j = 0$ or $j = 1$, the probabilities of false alarm or of detection are obtained, respectively, as a function of the detector threshold η.

We numerically computed the right hand side of Eqs.(3-4) and (3-5) for $0 < \omega \leq 60$. For higher accuracy in the computations, we employed a rotation of the integration paths [6]. Then, the probability of false alarm was computed as a function of the detector threshold from Eqs.(3-6) and (3-7). For accuracy at high threshold values, the numerical integration procedure employed was Filon's integration formula [19, p. 890]. From this computation, we derived the detector threshold required to achieve a probability of false alarm equal to $P_{fa} = 0.005$. In the following table, we show these required thresholds for the detectors that we examine:

Threshold Required to Achieve $P_{fa} = 0.005$	
detector	threshold
Magnitude	2.8
Square–law	7.8
Cube–law	21.0
4^{th}–law	54.7
Optimum	2.2

For these thresholds, the resulting probability of detection was computed using Eqs.(3-5) and (3-6) and is shown in the following table:

Probability of Detection for $P_{fa} = 0.005$	
detector	P_d
Magnitude	0.24
Square–law	0.34
Cube–law	0.17
4^{th}–law	0.07
Optimum	0.46

[3] Eqs.(3-6) and (3-7) differ since the power–law test statistics assume only positive values, while the optimum test statistic assumes both positive and negative values [18].

From the above table, we see that ν^{th}–law detectors of $\nu > 2$ (and, in particular, cube– and 4^{th}–law detectors) do not present any improvement in the detection of impulsive transients over a background of Gaussian noise when compared to square–law detectors. To the contrary, they have a probability of detection smaller than the probability of detection of the square–law detector for the same probability of false alarm. This can become clear by recalling Eq.(3-1), where the optimum test statistic is shown to contain the square–law characteristic as part of its expression. The magnitude detector, on the other hand, performs closely to the square–law detector, yet below it.

4. Conclusions

SαS random processes form a class of statistical models which present several similarities to the well known Gaussian model, but at the same time differ from it in a number of ways. Because of their properties, SαS pdfs receive increasing attention from the signal processing community and are expected to find a number of applications in engineering and communications. In this paper, we examined the structure of the pdf of sums of independent symmetric stable processes of different characteristic exponents and presented series expansions which are computable in real time. As an application of the theory, we presented a processor designed to detect stochastic impulsive transients over a background of Gaussian noise and showed that this detector outperforms, in terms of probability of detection, the existing square– and ν^{th}–law detectors.

In the future, we intend to address the problem of detection of unknown signals in impulsive noise modeled as a symmetric stable random process of zero median and unknown characteristic exponent and dispersion. This problem is a generalization of the well studied problem of detection of an unknown signal in zero mean Gaussian noise of unknown variance, for which Student's t test has been shown to satisfy certain optimality requirements. Since it is known that Student's t test, as well as other nonparametric detectors, perform poorly in impulsive noise environments, it is expected that our approach will provide significant improvement, while at the same time maintaining an acceptable performance in Gaussian noise environments. This research is currently pursued and its results will be announced shortly.

Appendix: Proof of the Asymptotic Expansions Theorem

Since $f(\alpha_1, \alpha_2, \gamma_1, \gamma_2; x) = f(\alpha_1, \alpha_2, \gamma_1, \gamma_2; -x)$, we assume without loss of generality that $x > 0$. To prove the expansion in Eq.(2-5), we begin with Eq.(2-1), which we rewrite as:

$$f(\alpha_1, \alpha_2, \gamma_1, \gamma_2; x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-\gamma_1 |\omega|^{\alpha_1} - \gamma_2 |\omega|^{\alpha_2}) e^{-ix\omega} \, d\omega$$

$$= \frac{1}{\pi}\Re\int_0^\infty \exp(-\gamma_1|\omega|^{\alpha_1} - \gamma_2|\omega|^{\alpha_2})e^{-ix\omega}\,d\omega, \quad (1)$$

where $\Re$ denotes the real part. We consider the identity [12, 6]

$$e^z = \sum_{k=0}^{N}\frac{z^k}{k!} + \frac{z^{N+1}}{N!}\int_0^1 e^{zt}(1-t)^N\,dt, \quad (2)$$

where z can be complex, in general. We apply the relation in Eq.(2) twice, once for $z = -\gamma_1\omega^{\alpha_1}$ and once for $z = -ix\omega$, and obtain the double series representation

$$\begin{aligned} f(\alpha_1,\alpha_2,\gamma_1,\gamma_2;x) &= \frac{1}{\pi}\Re\sum_{k=0}^{N}\sum_{l=0}^{L}\frac{(-ix)^k}{k!}\frac{(-\gamma_1)^l}{l!}\int_0^\infty e^{-\gamma_2\omega^{\alpha_2}}\omega^{k+\alpha_1 l}\,d\omega + E_{N,L}(x) \\ &= \frac{1}{\pi}\Re\sum_{k=0}^{N}x^k\frac{(-i)^k}{k!}\sum_{l=0}^{L}\frac{(-1)^l\gamma_1^l}{l!}\frac{\Gamma(\frac{k+\alpha_1 l+1}{\alpha_2})}{\alpha_2}\gamma_2^{-\frac{k+\alpha_1 l+1}{\alpha_2}} + E_{N,L}(x) \end{aligned} \quad (3)$$

where, to get to Eq.(3), we used the result in [19, p. 255]. The error term in Eq.(3) is

$$\begin{aligned} E_{N,L}(x) &= \frac{1}{\pi}\Re\int_0^\infty [\frac{(-\gamma_1\omega^{\alpha_1})^{L+1}}{L!}\int_0^1 e^{-\gamma_1\omega_1^{\alpha}t_1}(1-t_1)^L\,dt_1] \\ &\quad [\frac{(-ix\omega)^{N+1}}{N!}\int_0^1 e^{-ix\omega t_2}(1-t_2)^N\,dt_2]e^{-\gamma_2\omega^{\alpha_2}}\,d\omega \\ &= x^{N+1}\frac{1}{\pi}\Re\int_0^\infty [\frac{(-\gamma_1\omega^{\alpha_1})^{L+1}}{L!}\int_0^1 e^{-\gamma_1\omega_1^{\alpha}t_1}(1-t_1)^L\,dt_1] \\ &\quad [\frac{(-i\omega)^{N+1}}{N!}\int_0^1 e^{-ix\omega t_2}(1-t_2)^N\,dt_2]e^{-\gamma_2\omega^{\alpha_2}}\,d\omega. \end{aligned} \quad (4)$$

From Eqs.(3) and (4), we clearly see that the ratio of the error term $E_{N,L}$ over the Nth term in the Eq.(3) goes to zero as x goes to zero, for any N, L, therefore

$$f(\alpha_1,\alpha_2,\gamma_1,\gamma_2;x) \sim \frac{1}{\pi}\Re\sum_{k=0}^{\infty}x^k\frac{(-i)^k}{k!}\sum_{l=0}^{\infty}\frac{(-1)^l\gamma_1^l}{l!}\frac{\Gamma(\frac{k+\alpha_1 l+1}{\alpha_2})}{\alpha_2}\gamma_2^{-\frac{k+\alpha_1 l+1}{\alpha_2}}. \quad (5)$$

Moreover, calculation of the real part of the double sum in the right hand side of Eq.(4) and reindexing of the terms yields Eq.(2-5) and proves the first part of the asymptotic expansions theorem.

The second part of the theorem, i.e. Eq.(2-6), can be proved in a similar manner. We begin again with Eq.(1) of this appendix and use the identity of Eq.(2) with $z = -\gamma_1\omega^{\alpha_1} - \gamma_2\omega^{\alpha_2}$ to obtain

$$\begin{aligned} f(\alpha_1,\alpha_2,\gamma_1,\gamma_2;x) &= \frac{1}{\pi}\Re\sum_{k=0}^{N}\frac{(-1)^k}{k!}\int_0^\infty(\gamma_1\omega^{\alpha_1}+\gamma_2\omega^{\alpha_2})^k e^{-ix\omega}\,d\omega + E_N(x) \\ &= \frac{1}{\pi}\Re\sum_{k=0}^{N}\frac{(-1)^k}{k!}\sum_{l=0}^{k}\frac{k!}{l!(k-l)!}\gamma_1^l\gamma_2^{k-l}\int_0^\infty \omega^{\alpha_1 l+\alpha_2(k-l)}e^{-ix\omega}\,d\omega, + E_N(x) \end{aligned} \quad (6)$$

where we have used the binomial expansion formula. We have

$$\begin{aligned} \int_0^\infty \omega^{\alpha_1 l+\alpha_2(k-l)}e^{-ix\omega}\,d\omega &= (-i)e^{-i\frac{\pi}{2}[\alpha_1 l+\alpha_2(k-l)]}\int_0^\infty e^{-xr}r^{\alpha_1 l+\alpha_2(k-l)}\,dr \\ &= (-i)e^{-i\frac{\pi}{2}[\alpha_1 l+\alpha_2(k-l)]}\frac{\Gamma[\alpha_1 l+\alpha_2(k-l)+1]}{x^{\alpha_1 l+\alpha_2(k-l)+1]}}, \end{aligned} \quad (7)$$

where we have used a rotation of the integration path in Eq.(7) and the result in [19, p. 255]. The error term E_N can be computed in the same manner as the error term $E_{N,L}$ was computed in Eq.(4) and shown again to form a ratio over the Nth term in Eq.(6), which has zero limit as $x \to \infty$. Thus, after computation of the real part of the terms, the asymptotic series of Eq.(2-6) is obtained.

Bibliography

[1] C. L. Nikias and A. Petropulu, *Higher-Order Spectra Analysis: A Nonlinear Signal Processing Framework*. Englewood Cliffs, NJ: Prentice-Hall, 1993.

[2] M. Shao and C. L. Nikias, "Detection and adaptive estimation of stable processes with fractional lower order moments," in *Proceedings of Sixth IEEE Workshop on Statistical Signal and Array Processing*, (Victoria, BC, Canada), pp. 94–97, October, 1992.

[3] E. J. Wegman, S. G. Schwartz, and J. B. Thomas, eds., *Topics in Non-Gaussian Signal Processing*. New York: Academic Press, 1989.

[4] M. Shao and C. L. Nikias, "On symmetric stable models for impulsive noise," *USC-SIPI Report # 231*, 1993.

[5] G. A. Tsihrintzis and C. L. Nikias, "Performance of optimum and suboptimum receivers in the presence of impulsive noise modeled as an α–stable process," *IEEE Trans. Comm.*, 1993. (submitted).

[6] J. A. Fawcett and B. H. Maranda, "The optimal power law for the detection of a Gaussian burst in a background of Gaussian noise," *IEEE Trans. Inform. Theory*, vol. IT-37, pp. 209–214, 1991.

[7] P. Zakarauskas, "Detection and localization of non-deterministic transients in time series and application to ice-cracking sound," *Digital Signal Processing*, vol. 3, pp. 36–45, 1993.

[8] M. Shao and C. L. Nikias, "Signal processing with fractional lower order moments: Stable processes and their applications," *Proc. IEEE*, July, 1993.

[9] S. S. Rappaport and L. Kurz, "An optimal nonlinear detector for digital data transmission through non-Gaussian channels," *IEEE Trans. Comm. Techn.*, vol. COM-14, pp. 266–274, 1966.

[10] B. W. Stuck and B. Kleiner, "A statistical analysis of telephone noise," *The Bell System Technical Journal*, vol. 53, pp. 1262–1320, 1974.

[11] P. Lévy, *Calcul des Probabilités*, vol. II. Paris: Gauthier-Villards, 1925. chapter 6.

[12] H. Bergstrom, "On some expansions of stable distribution functions," *Arkiv for Matematik*, vol. 2, pp. 375–378, 1952.

[13] V. Zolotarev, *One-Dimensional Stable Distributions*. Providence, RI: American Mathematical Society, 1986.

[14] A. Papoulis, *Probability, Random Variables, and Stochastic Processes*. Tokyo: McGraw-Hill, Intern. stud. ed., 1981.

[15] B. Friedlander and B. Porat, "Detection of transient signals by the Gabor representation," *IEEE Trans. Acoust., Speech, and Signal Process.*, vol. ASSP-37, pp. 169–180, 1989.

[16] J. S. Bird, "Calculating detection probabilities for systems employing noncoherent integration," *IEEE Trans. Aerospace Electr. Syst.*, vol. AES-18, pp. 401–409, 1982.

[17] C. W. Helstrom and J. A. Ritcey, "Evaluating radar detection probabilities by steepest descent integration," *IEEE Trans. Aerospace Electr. Syst.*, vol. AES-20, pp. 624–633, 1984.

[18] A. H. Nuttall, "Alternate forms for numerical evaluation of cumulative probability distributions directly from characteristic functions," *Proc. IEEE*, vol. 58, pp. 1872–1873, 1970.

[19] M. Abramowitz and I. A. Stegun, eds., *Handbook of Mathematical Functions*. New York: Dover Publications, 1965.

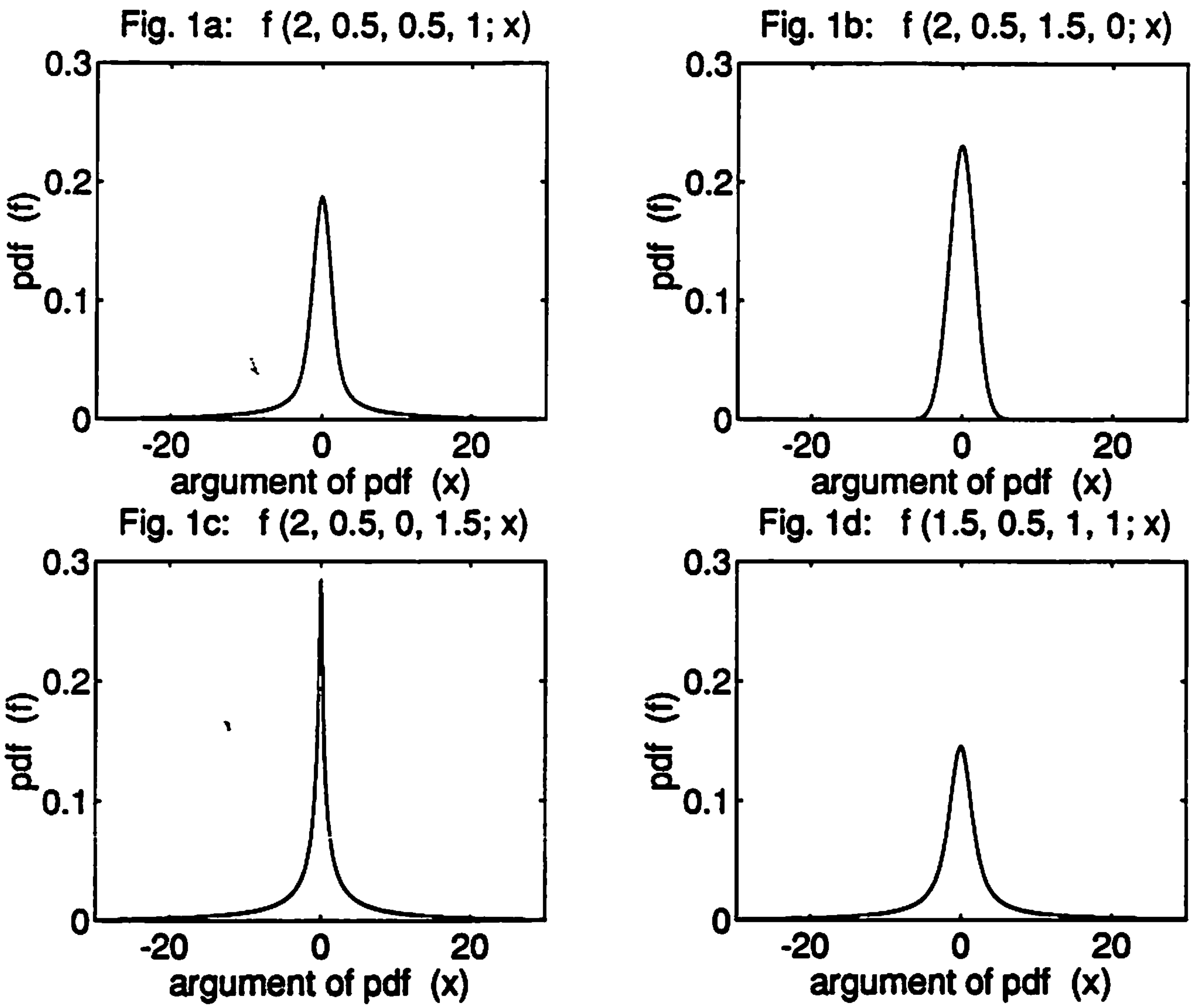

Fig. 1a: f (2, 0.5, 0.5, 1; x)
Fig. 1b: f (2, 0.5, 1.5, 0; x)
Fig. 1c: f (2, 0.5, 0, 1.5; x)
Fig. 1d: f (1.5, 0.5, 1, 1; x)
pdf (f)
0.3
0.2
0.1
0
-20
20
argument of pdf (x)

Fig. 2: Computation of f (2, 0.5, 0.5, 1; x)

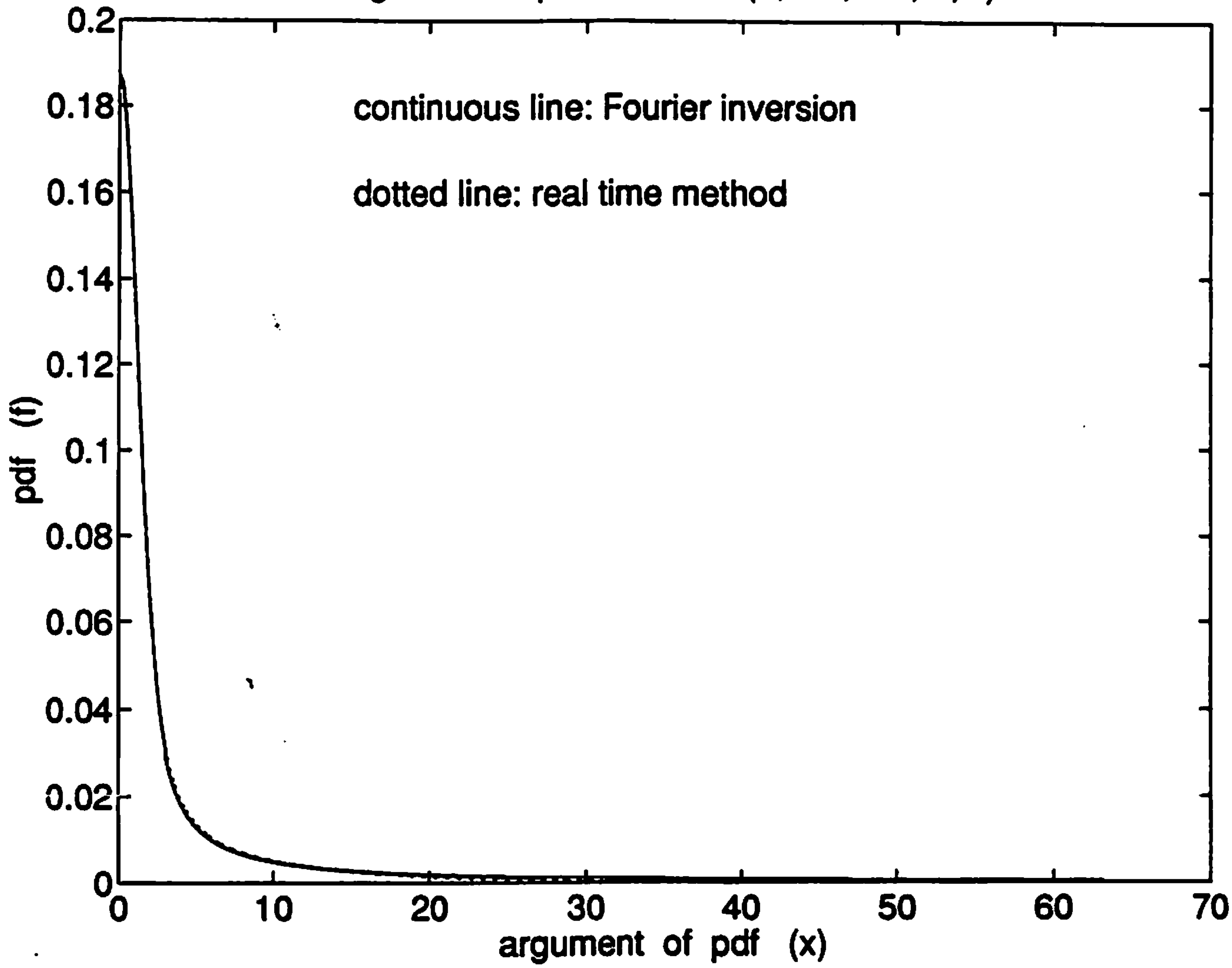

Fig. 3: Characteristic Nonlinearities