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ON THE LIMITING PROCESS OF SAMPLED SIGNAL EXTRAPOLATION USING WAVELETS*

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Abstract. The extrapolation of sampled signals from a given interval using a wavelet model with various sampling rates is examined in this research. We present sufficient conditions on signals and wavelet bases so that the discrete-time extrapolated signal converges to its continuous-time counterpart when the sampling rate goes to infinity. Thus, this work provides a practical procedure to implement continuous-time signal extrapolation with a discrete one via carefully choosing the sampling rate and the wavelet basis. A numerical example is given to illustrate our theoretical result.

Key words. signal extrapolation, Papoulis-Gerchberg algorithm, wavelets.

1. Introduction. The signal extrapolation problem is to recover a signal from a given piece of the signal. For a continuous-time signal $f(t)$, it is to reconstruct $f(t)$ from $f(t)$ in $t \in [-T, T]$ for certain $T > 0$. For a discrete-time signal $x[n]$, it can be formulated as recovering $x[n]$ with given $x[n]$, $n \in \mathcal{N}$, for a finite subset $\mathcal{N}$ of integers. The performance of the extrapolation algorithm heavily depends on whether the signal of interest can be modeled in a proper way. For example, by assuming the bandlimitedness of the signal, we have the band-limited signal extrapolation problem [1], [6], [7], [9], [14], [17], [20]. It has applications in spectral estimation [8], medical image processing [10], geosciences, etc. Let

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{-it\omega} dt.$$

be the Fourier transform of f, we call $f(t)$ Ω-bandlimited if $\hat{f}(\omega) = 0$ for $|\omega| > \Omega > 0$. In this case, since $f(t)$ is an entire function when t is extended to the complex plane, $f(t)$ is uniquely determined by $f(t)$, $t \in [-T, T]$, for any fixed $T > 0$. The Papoulis-Gerchberg (PG) iterative algorithm [6], [14] provides a procedure to achieve the band-limited signal extrapolation. However, the PG algorithm for continuous-time signal extrapolation is only of theoretical interest, since one has to process a discrete-time signal which is sampled from its continuous-time counterpart in practice. A discretized version of the PG algorithm was studied in [17], [18], [25]. It was proved that the extrapolated discrete-time sequence from the samples of a band-limited signal in $[-T, T]$ converges to the extrapolated continuous-time sequence as the sampling rate goes to infinity. Thus, we obtain a practical routine for continuous-time band-limited signal extrapolation by applying the discretized PG algorithm to its sampled discrete-time sequences.

Recently, wavelets were extensively discussed and widely used because of their attractive properties such as the time-frequency localization property. By using wavelet theory, Xia, Kuo and Zhang examined a scale-time limited signal model which contains both the continuous- and discrete-time signals in [23]. Besides, a generalized PG (GPG) algorithm was proposed for continuous-time signal extrapolation whereas a discretized GPG (DGPG) was proposed for discrete-time signal extrapolation. In this research, we present sufficient conditions on signals and wavelet bases so that the discrete-time extrapolated signal converges to its continuous-time counterpart when the sampling rate goes to infinity. Thus, it provides a practical procedure to implement continuous-time signal extrapolation with a discrete one via carefully choosing the sampling rate and the wavelet basis.

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This paper is organized as follows. We briefly review wavelet theory, the scale-time limited signal model, and the GPG and DGPG algorithms in §2. The main result of the work is given in §3. A numerical example is presented in §4 to illustrate the convergence behavior. Some concluding remarks are given in §5.

2. Scale-time limited signal extrapolation. In this section, we review basic properties of wavelets and the GPG and DGPG signal extrapolation algorithms. The notation and equalities introduced will be needed in the proofs given in §3.

2.1. Review of Wavelet Theory. We only consider real wavelets in this work, and refer to [2], [3], [5] for more detailed discussion. Let $\phi(t)$ be a scaling function such that, for a fixed arbitrary integer j,

$$\{\phi_{jk}(t)\}_{k\in\mathbf{Z}}, \quad \text{where} \quad \phi_{jk}(t) = 2^{j/2}\phi(2^j t - k),$$

is an orthonormal basis of the wavelet subspace V_j, and $\{V_j\}_{j\in\mathbf{Z}}$ is a multiresolution approximation of $L^2(\mathbf{R})$, i.e. $V_j \subset V_{j+1}$ and $\overline{\bigcup_j V_j} = L^2(\mathbf{R})$. The wavelet function corresponding to $\phi(t)$ is denoted by $\psi(t)$ and $\psi_{jk}(t) = 2^{j/2}\psi(2^j t - k)$. The associated quadrature mirror filters (QMF) can be expressed as

$$H(\omega) = \sum_k h_k e^{-ik\omega}, \quad \text{and} \quad G(\omega) = \sum_k g_k e^{-ik\omega}, \tag{2.1}$$

where $g_k = (-1)^k h_{1-k}$ and

$$\hat{\phi}(2\omega) = H(\omega)\hat{\phi}(\omega) = \prod_{j=0}^{\infty} H(\frac{\omega}{2^j}), \quad \text{and} \quad \hat{\psi}(2\omega) = G(\omega)\hat{\phi}(\omega). \tag{2.2}$$

Then, we have

$$f(t) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} b_{j,k}\psi_{jk}(t), \tag{2.3}$$

for any $f(t) \in L^2(\mathbf{R})$ and

$$f(t) = \sum_{k=-\infty}^{\infty} c_{J,k}\phi_{Jk}(t) = \sum_{j<J} \sum_{k=-\infty}^{\infty} b_{j,k}\psi_{jk}(t), \tag{2.4}$$

for any $f(t) \in V_J$, where $b_{j,k} = \langle f, \psi_{jk}\rangle$ and $c_{J,k} = \langle f, \phi_{Jk}\rangle$. Moreover, the coefficients $b_{j,k}$ with $j < J$ can be obtained from coefficients $c_{J,k}$ by the recursive formulas:

$$\begin{aligned} c_{j-1,k} &= \sqrt{2}\textstyle\sum_n h_{n-2k}c_{j,n}, \\ b_{j-1,k} &= \sqrt{2}\textstyle\sum_n g_{n-2k}c_{j,n}, \end{aligned} \tag{2.5}$$

for $j = J, J-1, J-2, \cdots$. On the other hand, we have the following synthesis formula to compute coefficients $c_{J,k}$ from $c_{J_0,k}$ and $b_{j,k}$ with $J_0 \le j < J$ via

$$c_{j+1,n} = \sqrt{2}\left(\sum_k h_{n-2k}c_{j,k} + \sum_k g_{n-2k}b_{j,k}\right), \tag{2.6}$$

for $j = J_0, J_0+1, \cdots, J-1$. By viewing $c_{J,n}$ as a sequence $x[n]$, we call (2.5) the discrete wavelet transform (DWT) with parameters J_0 and J or simply DWT of the sequence $x[n]$ and (2.6) the inverse discrete wavelet transform (IDWT) with parameters J_0 and J or simply IDWT of coefficients $c_{J_0,k}$ and $b_{j,k}$.

To construct a scaling function $\phi(t)$ and a wavelet $\psi(t)$ from their QMF $H(\omega)$, there are several ways. One of them [4], [19], [22] is the following. Let $\mathbf{D}$, $\mathbf{H}$ and $\mathbf{G}$ be matrices with entries d_{km}, h_{mn} and g_{mn}, respectively, and

$$d_{km} \triangleq \delta(2k-m), \; h_{mn} \triangleq \sqrt{2}h_{n-m}, \; h_{mn} \triangleq \sqrt{2}g_{n-m},$$

where $\delta(k) = 1$ when $k = 0$ and 0 when $k \neq 0$. Then

$$\phi(t) = \lim_{j\to\infty} \sum_k [(\mathbf{DH})^j]_{0k} 2^j \chi(2^j t - k), \tag{2.7}$$

and

$$\psi(t) = \lim_{j\to\infty} \sum_k [(\mathbf{DH})^{j-1}(\mathbf{DG})]_{0k} 2^j \chi(2^j t - k), \tag{2.8}$$

where $[\mathbf{A}]_{mn}$ denotes the (m, n)th entry A_{mn} of the matrix $\mathbf{A}$ and

$$\chi(t) \triangleq \begin{cases} 1, & t \in [-1/2, 1/2], \\ 0, & \text{otherwise.} \end{cases}$$

Actually (2.7) and (2.8) can be viewed as a consequence of (2.1) and (2.2). For more details, we refer to [5], [19].

2.2. Signal Extrapolation Algorithms. The generalized PG (GPG) and the discrete GPG (DGPG) algorithms were proposed in [23] for continuous- and discrete-time signal extrapolation. They are summarized below.

To extrapolate a continuous-time signal, i.e. to recover $f(t)$ from $f(t)$, $t \in [-T, T]$, where $f(t) \in V_J$ for a fixed integer J, we can perform the following GPG algorithm.

Generalized PG (GPG) Algorithm:

$$f^{(0)}(t) = P_T f(t). \tag{2.9}$$

For $l = 0, 1, 2, \cdots$,

$$g^{(l)}(t) = \sum_k < f^{(l)}, \phi_{Jk} > \phi_{Jk}(t), \tag{2.10}$$

$$f^{(l+1)}(t) = P_T f(t) + (I - P_T) g^{(l)}(t), \tag{2.11}$$

where $P_T f(t) = f(t)$ for $|t| \leq T$ and 0 otherwise. When the scaling function $\phi(t)$ is the sinc function, that is, $\phi(t) = \frac{\sin \pi t}{\pi t}$, the GPG algorithm (2.9) - (2.11) reduces to the PG algorithm with $\Omega = 2^J \pi$.

Before stating the DGPG algorithm, we have to introduce some definitions. A sequence $c_{J,n}$ is said to be (J, K) *scale-time limited* for certain integers J and $K \geq 0$ if its DWT coefficients (with lowest resolution J_0) satisfies that coefficients $c_{J_0,k}$ and $b_{j,k}$ may take nonzero values only when $|k| \leq K$ and $J_0 \leq j < J$. Note that when J and K are sufficiently large, the (J, K) scale-time limited sequence provides a practical discrete-time signal model. Let $x[n]$ be a (J, K) scale-time limited sequence. The values of $x[n]$, $n \in \mathcal{N}$, are given, where the cardinality $|\mathcal{N}| = N$ is finite. The extrapolation problem is to recover $x[n]$ for $n \notin \mathcal{N}$.

Let $P_{\mathcal{N}}$ and $P_{J,K}$ be the following operators:

$$P_{\mathcal{N}} y[n] = \begin{cases} y[n], & n \in \mathcal{N}, \\ 0, & n \notin \mathcal{N}, \end{cases} \quad \text{and } P_{J,K} d_{j,k} = \begin{cases} d_{j,k}, & |k| \leq K \text{ and } J_0 \leq j < J, \\ 0, & \text{otherwise} \end{cases}$$

Let I be the identity operator and $\mathcal{D}_{J_0,J}$ and $\mathcal{D}_{J_0,J}^{-1}$ be the DWT and IDWT operators with parameters J_0 and J defined as above. Then, we can state the DGPG algorithm as follows.

The Discretized GPG (DGPG) Algorithm:

$$x^{(0)}[n] = P_{\mathcal{N}} x[n], \tag{2.12}$$

For $l = 0, 1, 2, \cdots$,

$$x^{(l+1)}[n] = P_{\mathcal{N}} x[n] + (I - P_{\mathcal{N}}) \mathcal{D}_{J_0,J}^{-1} P_{J,K} \mathcal{D}_{J_0,J} x^{(l)}[n]. \tag{2.13}$$

It is known that the PG algorithm and its discretization for band-limited signal models converge very slowly and are sensitive to noises [17] due to the nature of the Fourier transform. In contrast, it was shown in [11], [12], [23] that fast and robust DGPG algorithms can be obtained. For the convergence of the iterative GPG and DGPG algorithms, we refer to [11], [12], [23].

3. Convergence of extrapolated sampled sequences. In this section, we are interested in the convergence of the extrapolated sequence obtained from the DGPG algorithm as the sampling rate goes to infinity.

3.1. Problem formulation. We now look at the question raised in §1 more carefully. Consider continuous-time signals in the wavelet subspace V_J, where each $f(t) \in V_J$ has the form

$$f(t) = \sum_{k=-\infty}^{\infty} c_{J,k}\phi_{Jk}(t) = \sum_{k=-\infty}^{\infty} c_{J_0,k}\phi_{J_0k}(t) + \sum_{J_0 \le j < J} \sum_{k=-\infty}^{\infty} b_{j,k}\psi_{jk}(t).$$

In practice, $f(t)$ is usually small for large $|t|$ so that $c_{J_0,k}$ and $b_{j,k}$ are also small for large $|k|$. Thus, it is important to consider signals in the following subspace of V_J,

$$V_{J,K} \triangleq \left\{ f(t) : f(t) = \sum_{k=-K}^{K} c_{J_0,k}\phi_{J_0k}(t) + \sum_{J_0 \le j < J} \sum_{k=-K}^{K} b_{j,k}\psi_{jk}(t) \text{for some constants} c_{J_0,k}, b_{j,k} \right\}.$$

We call signals in $V_{J,K}$ as (J,K) scale-time limited. For any $f(t) \in V_{J,K}$, we have

$$f(t) = \sum_{k} c_{J,k}\phi_{Jk}(t) = \sum_{k=-K}^{K} c_{J_0,k}\phi_{J_0k}(t) + \sum_{J_0 \le j < J} \sum_{k=-K}^{K} b_{j,k}\psi_{jk}(t),$$

where

$$c_{J,k} =< f, \phi_{Jk} >, \quad c_{J_0,k} =< f, \phi_{J_0k} >, \quad b_{j,k} =< f, \psi_{jk} >,$$

and $c_{J,k}$, $k \in \mathbf{Z}$, is a (J,K) scale-time limited sequence.

We can use two examples to illustrate the scale-time limited signal model defined above. First, let $\psi(t)$ be the sinc function. Then, the scale-time limited model reduces to the well known band-limited signal model. Second, consider the cubic cardinal B-spline wavelet [2], where we approximate a function by the linear combination of a set of basis functions, which are second-order polynomials between the knots with continuous first-order derivate at the knots. Thus, for a function $f(t)$ with only continuous first-order derivative, it is better to represent the function with the cardinal B-spline wavelet basis rather than the conventional Fourier basis. The advantage of using wavelets is that it provides many bases with good time-frequency localization for signal modeling. The signal extrapolation results obtained in this research are generally applicable as long as the wavelet bases of consideration satisfy the sufficient conditions of the theorems.

Since $\phi(t)$ behaves like a lowpass filter, $c_{J,k}$ is close to the sampling sequence $x_J[k] \triangleq 2^{-J/2}f(k/2^J)$ [13], [16] for sufficiently large J. Therefore, the sampling sequence $x_J[k]$ is roughly (J,K) scale-time limited and we may use the discretized GPG algorithm for (J,K) scale-time limited sequences to recover the sequence $x_J[k]$ when $x_J[k]$ for $|k/2^J| \le T$ are known. More generally, even if J is not large enough so that $c_{J,k}$ cannot be well approximated by $2^{-J/2}f(k/2^J)$, we can oversample the continuous-time signal $f(t)$ as $x_{J_1}[k] \triangleq 2^{-J_1/2}f(k/2^{J_1})$ with appropriate scale parameter $J_1 \ge J$. When J_1 is large, $x_{J_1}[k] \approx c_{J_1,k} =< f, \phi_{J_1,k} >$ which is still (J,K) scale-time limited because $f(t) \in V_{J,K}$. Then, we may use the discretized GPG algorithm for (J,K) scale-time limited sequences to recover $x_{J_1}[k]$ with knowns $x_{J_1}[k]$, $|k/2^{J_1}| \le T$. The *question* is that, when the sampling rate in $[-T,T]$ goes to infinity (or J_1 goes to infinity), if the extrapolated sequence of $x_{J_1}[n]$ from the knowns $x_{J_1}[k]$, $|k/2^{J_1}| \le T$, converges to $f(t)$ in a certain sense.

In what follows, we consider $f(t) \in V_{J,K}$ where the scale and time parameters J and K are arbitrary but fixed. Without loss of generality, we assume that samples $f(k/2^{J_1})$ are known in the interval $[-T,T] = [-1,1]$ with $J_1 \ge J$. Since $f(t) \in V_{J,K} \subset V_{J_1,K}$,

$$f(t) = \sum_{k} c_{J_1,k}\phi_{J_1k}(t). \tag{3.1}$$

Let

$$\mathcal{N}_{J_1} \triangleq \{n : -2^{J_1} \le n \le 2^{J_1}\},$$

and

$$x_{J_1}[n] = 2^{-J_1/2} f(\frac{n}{2^{J_1}}),\ n \in \mathcal{N}_{J_1}.$$

The DGPG algorithm (2.12)-(2.13) can be rewritten in the current setting as:

$$x_{J_1}^{(0)}[n] = P_{\mathcal{N}_{J_1}} x_{J_1}[n], \tag{3.2}$$

and for $l = 0, 1, 2, \cdots$,

$$x_{J_1}^{(l+1)}[n] = P_{\mathcal{N}_{J_1}} x_{J_1}[n] + (I - P_{\mathcal{N}_{J_1}}) \mathcal{D}_{J_0,J_1}^{-1} P_{J,K} \mathcal{D}_{J_0,J_1} x_{J_1}^{(l)}[n]. \tag{3.3}$$

Therefore, from the samples $f(k/2^{J_1})$, $k \in \mathcal{N}_{J_1}$, we obtain a discrete-time signal $x_{J_1}^{(l)}[n]$, $n \in \mathbf{Z}$. With $x_{J_1}^{(l)}[n]$, we form a continuous-time signal via

$$f_{J_1,l}(t) = \sum_k x_{J_1}^{(l)}[k] \phi_{J_1 k}(t),\ t \in \mathbf{R}. \tag{3.4}$$

Our main result is on the convergence of $f_{J_1,l}(t)$ to $f(t)$. Before going to the convergence result, we need a preparation for (J, K) scale-time limited sequences.

3.2. Representation of (J, K) scale-time limited sequences. We introduce two operators H and G related to the quadrature mirror filters $H(\omega)$ and $G(\omega)$ in (2.1) as follows:

$$Hy[k] \triangleq \sqrt{2} \sum_n h_{n-2k} y[n], \quad \text{and} \quad Gy[k] \triangleq \sqrt{2} \sum_n g_{n-2k} y[n].$$

Let H^* and G^* be their duals, respectively, i.e.

$$H^* y[n] \triangleq \sqrt{2} \sum_k h_{n-2k} y[k], \quad \text{and} \quad G^* y[n] \triangleq \sqrt{2} \sum_k g_{n-2k} y[k].$$

Then, from (2.6), we have

$$x[n] = \left((H^*)^{J-J_0} c_{J_0,k} + (H^*)^{J-J_0-1} G^* b_{J_0,k} + \cdots + H^* G^* b_{J-2,k} + G^* b_{J-1,k}\right)[n].$$

We can rewrite the above equation as

$$x[n] = \mathbf{w}_n \mathbf{p}, \quad n \in \mathbf{Z}, \tag{3.5}$$

where $\mathbf{p}$ and $\mathbf{w}_n$ are, respectively, column and row vectors of length $(2K+1)(J - J_0 + 1)$ of the form

$$\begin{aligned} \mathbf{p} &= (\mathbf{c}_{J_0}, \mathbf{b}_{J_0}, \mathbf{b}_{J_0+1}, \cdots, \mathbf{b}_{J-1})^T, \\ \mathbf{w}_n &= \left((H^*)_n^{J-J_0}, \left((H^*)^{J-J_0-1} G^*\right)_n, \cdots, (H^* G^*)_n, G_n^*\right), \end{aligned}$$

and where

$$\begin{aligned} \mathbf{c}_{J_0} &= (c_{J_0,-K}, c_{J_0,-K+1}, \cdots, c_{J_0,K}), \\ \mathbf{b}_j &= (b_{j,-K}, b_{j,-K+1}, \cdots, b_{j,K}), \\ G_n^* &= \sqrt{2}\,(g_{-K-2n}, g_{-K+1-2n}, \cdots, g_{K-2n}), \\ \left((H^*)^j G^*\right)_n &= (\sqrt{2})^{j+1} \left([(\mathbf{DH})^j (\mathbf{DG})]_{n,-K}, [(\mathbf{DH})^j (\mathbf{DG})]_{n,-K+1}, \cdots, [(\mathbf{DH})^j (\mathbf{DG})]_{n,K}\right), \\ (H^*)_n^{J'} &= (\sqrt{2})^{J'} \left([(\mathbf{DH})^{J'}]_{n,-K}, [(\mathbf{DH})^{J'}]_{n,-K+1}, \cdots, [(\mathbf{DH})^{J'}]_{n,K}\right), \end{aligned}$$

for $1 \leq j \leq J - J_0 - 1$ and $J' = J - J_0$. Now, by letting

$$\mathcal{N} = \{m_1, m_2, \cdots, m_N : m_1 < m_2 < \cdots < m_N\},$$

we obtain the following linear system

$$\mathbf{x} = \mathbf{W}\mathbf{p}, \tag{3.6}$$

where

$$\mathbf{x} = (x[m_1], x[m_2], \cdots x[m_N])^T, \quad \text{and} \quad \mathbf{W} = \left(\mathbf{w}_{m_1}^T, \mathbf{w}_{m_2}^T, \cdots, \mathbf{w}_{m_N}^T\right)^T.$$

Let $\tilde{x}_{J_1}[n] \triangleq c_{J_1,n} =< f, \phi_{J_1k} >$, $n \in \mathbf{Z}$. As mentioned before, the discrete sequence $\tilde{x}_{J_1}[n]$ is (J, K) scale-time limited. Furthermore, because of $f(t) \in V_{J,K}$ and $J_1 \geq J$, similar to (3.5) for $x[n]$ we have

$$\tilde{x}_{J_1}[n] = \mathbf{w}_n(J_1)\mathbf{p}, \quad n \in \mathbf{Z}, \tag{3.7}$$

where

$$\mathbf{w}_n(J_1) = \left(\left((H^*)^{J_1-J_0}\right)_n, \left((H^*)^{J_1-J_0-1}G^*\right)_n, \cdots, \left((H^*)^{J_1-J}G^*\right)_n\right),$$

and $\mathbf{p}$, $((H^*)^j)_n$ and $((H^*)^jG^*)_n$ are the same as before. It is clear that $\mathbf{w}_n(J_1) = \mathbf{w}_n$ when $J_1 = J$. Let

$$\tilde{\mathbf{x}}_{J_1} = (\tilde{x}_{J_1}[-2^{J_1}], \tilde{x}_{J_1}[-2^{J_1}+1], \cdots, \tilde{x}_{J_1}[2^{J_1}])^T,$$

and

$$\mathbf{W}(J_1) = \left(\mathbf{w}_{-2^{J_1}}^T(J_1), \mathbf{w}_{-2^{J_1}+1}^T(J_1), \cdots, \mathbf{w}_{2^{J_1}}^T(J_1)\right)^T. \tag{3.8}$$

Then,

$$\tilde{\mathbf{x}}_{J_1} = \mathbf{W}(J_1)\mathbf{p}. \tag{3.9}$$

Let

$$\lambda_1(J_1) \geq \lambda_2(J_1) \geq \cdots \geq \lambda_{r_0}(J_1) \geq \cdots \geq \lambda_{2^{J_1+1}+1}(J_1) \geq 0 \tag{3.10}$$

be the all eigenvalues of the matrix $\mathbf{W}(J_1)(\mathbf{W}(J_1))^T$, where $r_0 \triangleq (2K+1)(J - J_0 + 1)$.

To prove the convergence we need the following two lemmas.

3.3. Two lemmas. Recall $\tilde{x}_{J_1}[n] = c_{J_1,n}$, which is (J, K) scale-time limited. Let us apply the DGPG algorithm to $\tilde{x}_{J_1}[n]$ with $n \in \mathcal{N}_{J_1}$ to reconstruct $\tilde{x}_{J_1}[n]$ for $n \in \mathbf{Z}$, i.e.

$$\tilde{x}_{J_1}^{(0)}[n] = P_{\mathcal{N}_{J_1}}\tilde{x}_{J_1}[n], \tag{3.11}$$

and for $l = 0, 1, 2, \cdots$,

$$\tilde{x}_{J_1}^{(l+1)}[n] = P_{\mathcal{N}_{J_1}}\tilde{x}_{J_1}[n] + (I - P_{\mathcal{N}_{J_1}})\mathcal{D}_{J_0,J_1}^{-1}P_{J,K}\mathcal{D}_{J_0,J_1}\tilde{x}_{J_1}^{(l)}[n]. \tag{3.12}$$

Then, for the error between the reconstruction $\tilde{x}_{J_1}^{(l)}[n]$ and the signal $\tilde{x}_{J_1}[n]$ we have the following lemma.

LEMMA 1. *If $\lambda_{r_0}(J_1) > 0$, then*

$$\sum_n \left|\tilde{x}_{J_1}^{(l)}[n] - \tilde{x}_{J_1}[n]\right|^2 \leq \|f\|^2 \sum_{i=1}^{r_0} \lambda_i(J_1)(1 - \lambda_i(J_1))^{2l}, \tag{3.13}$$

where $\lambda_i(J_1)$ is defined in (3.10).

Proof. Let $\mathbf{q}_i$ be the eigenvector of $\mathbf{W}(J_1)(\mathbf{W}(J_1))^T$ corresponding to $\lambda_i(J_1)$, i.e.

$$\mathbf{W}(J_1)(\mathbf{W}(J_1))^T\mathbf{q}_i = \lambda_i(J_1)\mathbf{q}_i, \; i = 1, 2, \cdots, 2^{J_1+1}+1, \tag{3.14}$$

where

$$\lambda_1(J_1) \geq \lambda_2(J_1) \geq \cdots \geq \lambda_{r_0}(J_1) > \lambda_{r_0+1}(J_1) = \cdots = \lambda_{2^{J_1+1}+1}(J_1) = 0. \tag{3.15}$$

so that $\mathbf{q}_i$ forms an orthonormal basis of $\mathbf{C}^{2^{J_1+1}+1}$, where $\mathbf{C}$ denotes the set of complex numbers. Let

$$(\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_{r_0}) \triangleq (\mathbf{W}(J_1))^T(\mathbf{q}_1, \mathbf{q}_2, \cdots, \mathbf{q}_{r_0}).$$

Since $\mathbf{q}_i$, $1 \leq i \leq r_0$, are linearly independent and the matrix $(\mathbf{W}(J_1))^T$ has rank r_0, we can choose $\mathbf{y}_i$ with $1 \leq i \leq r_0$ to be an orthonormal basis in $\mathbf{C}^{r_0}$. Therefore, there are r_0 constants a_i such that

$$\mathbf{p} = \sum_{i=1}^{r_0} a_i \mathbf{y}_i, \tag{3.16}$$

and

$$\sum_{i=1}^{r_0} |a_i|^2 = ||\mathbf{p}||^2 = \sum_{J_0 \leq j < J}\sum_{k=-K}^{K} |b_{j,k}|^2 + \sum_{k=-K}^{K} |c_{J_0,k}|^2 = ||f||^2. \tag{3.17}$$

It is clear from (3.14) that only $\mathbf{q}_i[n]$ with $n \in \mathcal{N}_{J_1}$ are known. For $1 \leq i \leq r_0$, we extend $\mathbf{q}_i[n]$ from $n \in \mathcal{N}_{J_1}$ to all integers via

$$\tilde{\mathbf{q}}_i[n] = \frac{1}{\lambda_i(J_1)} \mathbf{w}_n(J_1)(\mathbf{W}(J_1))^T\mathbf{q}_i, \qquad n \in \mathbf{Z}. \tag{3.18}$$

By (3.7), (3.16) and (3.18), we have

$$\tilde{x}_{J_1}[n] = \mathbf{w}_n(J_1)\mathbf{p} = \mathbf{w}_n(J_1)\sum_{i=1}^{r_0} a_i\mathbf{y}_i = \sum_{i=1}^{r_0} a_i \mathbf{w}_n(J_1)(\mathbf{W}(J_1))^T\mathbf{q}_i = \sum_{i=1}^{r_0} a_i\lambda_i(J_1)\tilde{\mathbf{q}}_i[n]. \tag{3.19}$$

We now prove that $\lambda_i(J_1) \leq 1$ for $1 \leq i \leq r_0$. Since

$$\begin{aligned}
||\tilde{\mathbf{q}}_i||^2 &= \sum_{n=-\infty}^{\infty} |\tilde{\mathbf{q}}_i[n]|^2 = \frac{1}{(\lambda_i(J_1))^2}\sum_{n=-\infty}^{\infty} |\mathbf{w}_n(J_1)(\mathbf{W}(J_1))^T\mathbf{q}_i|^2 \\
&= \frac{1}{(\lambda_i(J_1))^2}||\mathcal{D}_{J_0,J_1}^{-1}(\mathbf{W}(J_1))^T\mathbf{q}_i||^2 = \frac{1}{(\lambda_i(J_1))^2}||(\mathbf{W}(J_1))^T\mathbf{q}_i||^2 \\
&= \frac{1}{(\lambda_i(J_1))^2}||\mathcal{D}_{J_0,J_1}P_{\mathcal{N}_{J_1}}\tilde{\mathbf{q}}_i||^2 = \frac{1}{(\lambda_i(J_1))^2}||P_{\mathcal{N}_{J_1}}\tilde{\mathbf{q}}_i||^2 \\
&\leq \frac{1}{(\lambda_i(J_1))^2}||\tilde{\mathbf{q}}_i||^2,
\end{aligned}$$

where the property that both $\mathcal{D}_{J_0,J_1}$ and $\mathcal{D}_{J_0,J_1}^{-1}$ preserve the total energy is used, we conclude that $\lambda_i(J_1) \leq 1$ for $1 \leq i \leq r_0$.

Next, we use induction to prove

$$\tilde{x}_{J_1}[n] - \tilde{x}_{J_1}^{(l)}[n] = (I - P_{\mathcal{N}_{J_1}})\sum_{i=1}^{r_0} a_i\lambda_i(J_1)(1-\lambda_i(J_1))^l\tilde{\mathbf{q}}_i[n]. \tag{3.20}$$

When $l = 0$, (3.20) is trivial by (3.11) and (3.19). Assume that (3.20) holds for the lth iteration. Then,

$$\tilde{x}_{J_1}[n] - \tilde{x}_{J_1}^{(l+1)}[n]$$

$$
\begin{aligned}
&= (I-P_{\mathcal{N}_{J_1}})\mathcal{D}_{J_0,J_1}^{-1}P_{J,K}\mathcal{D}_{J_0,J_1}(\tilde{x}_{J_1}[n]-\tilde{x}_{J_1}^{(l)}[n]) \\
&\stackrel{1}{=} (I-P_{\mathcal{N}_{J_1}})\mathcal{D}_{J_0,J_1}^{-1}P_{J,K}\mathcal{D}_{J_0,J_1}(I-P_{\mathcal{N}_{J_1}})\sum_{i=1}^{r_0}a_i\lambda_i(J_1)(1-\lambda_i(J_1))^l\tilde{\mathrm{q}}_i[n]) \\
&= (I-P_{\mathcal{N}_{J_1}})\sum_{i=1}^{r_0}a_i\lambda_i(J_1)(1-\lambda_i(J_1))^l(\tilde{\mathrm{q}}_i[n]-\mathcal{D}_{J_0,J_1}^{-1}P_{J,K}\mathcal{D}_{J_0,J_1}P_{\mathcal{N}_{J_1}}\tilde{\mathrm{q}}_i[n]) \\
&\stackrel{2}{=} (I-P_{\mathcal{N}_{J_1}})\sum_{i=1}^{r_0}a_i\lambda_i(J_1)(1-\lambda_i(J_1))^l(\tilde{\mathrm{q}}_i[n]-\mathbf{w}_n(J_1)(\mathbf{W}(J_1))^T\mathrm{q}_i[n]) \\
&\stackrel{3}{=} (I-P_{\mathcal{N}_{J_1}})\sum_{i=1}^{r_0}a_i\lambda_i(J_1)(1-\lambda_i(J_1))^{l+1}\tilde{\mathrm{q}}_i[n],
\end{aligned}
$$

where step 1 is from the induction assumption, step 2 is from the definitions of $\mathbf{w}_n(J_1)$ and $\mathbf{W}(J_1)$ and step 3 is from (3.18). This proves (3.20) is true for all $l=0,1,2,\cdots$. From (3.18), for $1\le i\le r_0$,

$$
\|\tilde{\mathrm{q}}_i\|^2=\frac{1}{(\lambda_i(J_1))^2}(\mathrm{q}_i)^T\mathbf{W}(J_1)(\mathcal{D}_{J_0,J})^T\mathcal{D}_{J_0,J}(\mathbf{W}(J_1))^T\mathrm{q}_i.
$$

Since $(\mathcal{D}_{J_0,J})^T=(\mathcal{D}_{J_0,J})^{-1}$ and (3.14),

$$
\|\tilde{\mathrm{q}}_i\|^2=\frac{1}{\lambda_i(J_1)}.
$$

Therefore,

$$
\begin{aligned}
\left(\sum_n|\tilde{x}_{J_1}[n]-\tilde{x}_{J_1}^{(l)}[n]|^2\right)^{1/2} &\le \sum_{i=1}^{r_0}|a_i|\lambda_i(J_1)(1-\lambda_i(J_1))^l\|\tilde{\mathrm{q}}_i\| \\
&\le \left(\sum_{i=1}^{r_0}|a_i|^2\right)^{1/2}\left(\sum_{i=1}^{r_0}\lambda_i(J_1)(1-\lambda_i(J_1))^{2l}\right)^{1/2} \\
&\stackrel{4}{\le} \|f\|\left(\sum_{i=1}^{r_0}\lambda_i(J_1)(1-\lambda_i(J_1))^{2l}\right)^{1/2},
\end{aligned}
$$

where step 4 is (3.17). This proves Lemma 1. □

As mentioned in §3, the initial values in the algorithms (3.2)-(3.4) and (3.11)-(3.12) are close when J_1 is large. Thus we can expect that the reconstructions $x_{J_1}^{(l)}$ and $\tilde{x}_{J_1}^{(l)}$ are also close. We now estimate $\|x_{J_1}^{(l)}-\tilde{x}_{J_1}^{(l)}\|$ in detail in the following lemma.

Lemma 2. *If the scaling function $\phi(t)$ satisfy that $\hat{\phi}(\omega)\in L^1(\mathbf{R})$ is continuous at $\omega=0$ with $\hat{\phi}(0)=1$, then*

$$
\|x_{J_1}^{(l)}-\tilde{x}_{J_1}^{(l)}\|\le(l+1)(\Delta_{J_1})^{1/2},
$$

where

$$
\Delta_{J_1}\triangleq\|P_{\mathcal{N}_{J_1}}(x_{J_1}[n]-\tilde{x}_{J_1}[n])\|^2=\sum_{n\in\mathcal{N}_{J_1}}|x_{J_1}[n]-\tilde{x}_{J_1}[n]|^2\to0,\quad \textit{as } J_1\to\infty. \tag{3.21}
$$

Proof. By the definitions of $x_{J_1}[n]$ and $\tilde{x}_{J_1}[n]$,

$$
\|P_{\mathcal{N}_{J_1}}(x_{J_1}[n]-\tilde{x}_{J_1}[n])\|^2=\sum_{n=-2^{J_1}}^{2^{J_1}}|2^{-J_1/2}f(\frac{n}{2^{J_1}})-c_{J_1,n}|^2. \tag{3.22}
$$

We can rewrite the expression of the right-hand-side of (3.22) and obtain (see [24] for detailed derivation)

$$(3.23)\qquad \|P_{\mathcal{N}_{J_1}}(x_{J_1}[n]-\tilde{x}_{J_1}[n])\|^2=\sum_{k=-2^{J_1}}^{2^{J_1}}\frac{2^{-J_1}}{(2\pi)^2}\left|\int_{-\infty}^{\infty}\hat{f}(-\omega)(\hat{\phi}(\frac{\omega}{2^{J_1}})-1)e^{-ik\omega/2^{J_1}}d\omega\right|^2.$$

Let

$$(3.24)\qquad w(J_1)\triangleq\max_{\omega\in[-2^{J_1/2},2^{J_1/2}]}\left|\hat{\phi}(\frac{\omega}{2^{J_1}})-1\right|^2.$$

By the assumption of $\hat{\phi}(\omega)$ being continuous at $\omega=0$, we have that

$$(3.25)\qquad \lim_{J_1\to\infty}w(J_1)\to 0.$$

Let

$$(3.26)\qquad a(J_1)\triangleq\min\left\{2^{J_1/2},\,(w(J_1))^{-\alpha}\right\},$$

for certain constant α with $0<\alpha\le 1/2$. Then, $\lim_{J_1\to\infty}a(J_1)=\infty$. Since $\hat{\phi}(\omega)\in L^1(\mathbf{R})$, $\hat{\psi}(\omega)=\hat{\phi}(\frac{\omega}{2})G(\frac{\omega}{2})$ and $|G(\omega)|\le 1$, we have $\hat{\psi}(\omega)\in L^1(\mathbf{R})$. Therefore, $f(t)\in V_{J,K}$ implies that $f(t)\in L^1(\mathbf{R})$. That is, if we let

$$(3.27)\qquad \delta_{J_1}=\int_{-\infty}^{\infty}(I-P_{a(J_1)})|\hat{f}(\omega)|d\omega,$$

then,

$$(3.28)\qquad \delta_{J_1}\to 0,\qquad \text{when } J_1\to\infty.$$

By the orthonormality of the wavelet basis, we have

$$\sum_k|\hat{\phi}(\omega+2k\pi)|^2=1,\qquad \forall\omega\in\mathbf{R},$$

so that

$$(3.29)\qquad |\hat{\phi}(\omega)|\le 1,\qquad \forall\omega\in\mathbf{R}.$$

By using (3.24), (3.27) and (3.29), we can simplify the right-hand-side of (3.23) as,

$$\begin{aligned}
&\left|\int_{-\infty}^{\infty}\hat{f}(-\omega)(\hat{\phi}(\frac{\omega}{2^{J_1}})-1)e^{-ik\omega/2^{J_1}}d\omega\right|^2\\
\le\;& 2\left(\int_{-a(J_1)}^{a(J_1)}\left|\hat{f}(-\omega)\left(\hat{\phi}(\frac{\omega}{2^{J_1}})-1\right)\right|d\omega\right)^2+8\left(\int_{-\infty}^{\infty}(I-P_{a(J_1)})|\hat{f}(\omega)|d\omega\right)^2\\
\le\;& 4\pi\|f\|^2\int_{-a(J_1)}^{a(J_1)}\left|\hat{\phi}(\frac{\omega}{2^{J_1}})-1\right|^2d\omega+8\delta_{J_1}^2\\
\le\;& 8\pi\|f\|^2a(J_1)w(J_1)+8\delta_{J_1}^2\\
\le\;& 8\pi\|f\|^2(w(J_1))^{1-\alpha}+8\delta_{J_1}^2.
\end{aligned}$$

Therefore, by (3.23),

$$(3.30)\qquad \Delta_{J_1}=\|P_{\mathcal{N}_{J_1}}(x_{J_1}[n]-\tilde{x}_{J_1}[n])\|^2\le\frac{4+2^{-J_1+1}}{\pi}\|f\|^2(w(J_1))^{1-\alpha}+\frac{4+2^{-J_1+1}}{\pi^2}\delta_{J_1}^2.$$

Thus, by (3.25) and (3.28), we have proved (3.21). From (3.2)-(3.3) and (3.11)-(3.12), the difference $x_{J_1}^{(l)}[n]-\tilde{x}_{J_1}^{(l)}[n]$ is resulted from the difference $P_{\mathcal{N}_{J_1}}(x_{J_1}[n]-\tilde{x}_{J_1}[n])$. Furthermore, it is straightforward by induction to prove

$$\|x_{J_1}^{(l)}-\tilde{x}_{J_1}^{(l)}\|=\left(\sum_n |x_{J_1}^{(l)}[n]-\tilde{x}_{J_1}^{(l)}[n]|^2\right)^{1/2}\leq (l+1)(\Delta_{J_1})^{1/2}.$$

Thus, Lemma 2 is proved. □

3.4. Convergence of $f_{J_1,l}(t)$. For each real t, let $\mathrm{o}(t)$ denote the vector

$$\mathrm{o}(t)\triangleq(\phi_{J_0,-K}(t),\cdots,\phi_{J_0K}(t),\psi_{J_0,-K}(t),\cdots,\psi_{J_0K}(t),\psi_{J-1,-K}(t),\cdots,\psi_{J-1,K}(t)).$$

Then, for $f(t)\in V_{J,K}$,

$$f(t)=\mathrm{o}(t)\mathbf{p}, \tag{3.31}$$

where the components of the vector $\mathbf{p}$ are the wavelet transform coefficients of $f(t)$. We have the following the convergence result.

THEOREM 1. *Let $f(t)\in V_{J,K}$ for certain integer $K\geq 0$. Let the scaling function $\phi(t)$ and the wavelet function $\psi(t)$ be continuous, and $\phi(t)$ satisfy that $\hat{\phi}(\omega)\in L^1(\mathbf{R})$ is continuous at $\omega=0$ with $\hat{\phi}(0)=1$. If there exit $t_1,t_2,\cdots,t_{r_0}\in[-1,1]$ so that the matrix $\mathrm{O}=(\mathrm{o}(t_1),\mathrm{o}(t_2),\cdots,\mathrm{o}(t_{r_0}))^T$ is non-singular, then*

$$\lim_{l\to\infty}\lim_{J_1\to\infty}\|f_{J_1,l}-f\|=0, \tag{3.32}$$

where o is defined as in (3.31) *and $f_{J_1,l}(t)$ is defined by* (3.4) *and* $r_0=(2K+1)(J-J_0+1)$.

Proof. By the constructions (2.7) and (2.8) of $\phi(t)$ and $\psi(t)$ from $H(\omega)$, we know that, when $n2^{-J_1}\approx t$ and J_1 is large enough, $\mathrm{o}(t)\approx 2^{J_1/2}\mathrm{w}_n(J_1)$. With the continuities of $\phi(t)$ and $\psi(t)$, when J_1 is large enough there is a submatrix $\mathbf{W}_1(J_1)$ with size $r_0\times r_0$ of $\mathbf{W}(J_1)$ so that the entries of $2^{J_1/2}\mathbf{W}_1(J_1)$ are close to the ones of O and $2^{J_1/2}[\mathbf{W}_1(J_1)]_{mn}\to[\mathrm{O}]_{mn}$ when $J_1\to\infty$, where $r_0=(2K+1)(J-J_0+1)$. Let $\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_{r_0}$ be the r_0 eigenvalues of the matrix $\mathrm{O}\mathrm{O}^T$. Then, from the conditions in Theorem 1, we have that $\lambda_{r_0}>0$. Let $\lambda_1'(J_1)\geq\lambda_2'(J_1)\geq\cdots\geq\lambda_{r_0}'(J_1)$ be the r_0 eigenvalues of $\mathbf{W}_1(J_1)(\mathbf{W}_1(J_1))^T$. Then, we have the following two consequences.

(a) $2^{J_1}\lambda_k'(J_1)\to\lambda_k$ when $J_1\to\infty$ and $1\leq k\leq r_0$,

(b) $\lambda_k(J_1)\geq\lambda_k'(J_1)$ for $1\leq k\leq r_0$,

where (a) is because of $2^{J_1/2}[\mathbf{W}_1(J_1)]_{mn}\to[\mathrm{O}]_{mn}$ when $J_1\to\infty$ and (b) is due to the fact that $\mathbf{W}_1(J_1)$ is a submatrix of $\mathbf{W}(J_1)$ (see [21]). Therefore, there is a $J_2>0$ so that when $J_1\geq J_2$

$$1\geq\lambda_1(J_1)\geq\lambda_2(J_1)\geq\cdots\geq\lambda_{r_0}(J_1)>\frac{1}{2}\lambda_{r_0}2^{-J_1}>0, \tag{3.33}$$

where the first inequality "$1\geq$" is from the proof of Lemma 1. In what follows, we always assume that $J_1\geq J_2$. With this assumption and the conditions in Theorem 1, the conditions in Lemmas 1-2 are satisfied. By (3.1), (3.4), (3.21), Lemmas 1 and 2, we have

$$\begin{aligned}\|f_{J_1,l}-f\| &= \left(\sum_n |x_{J_1}^{(l)}[n]-c_{J_1,n}|^2\right)^{1/2}\\ &\leq \left(\sum_n |x_{J_1}^{(l)}[n]-\tilde{x}_{J_1}^{(l)}[n]|^2\right)^{1/2}+\left(\sum_n |\tilde{x}_{J_1}^{(l)}[n]-\tilde{x}_{J_1}[n]|^2\right)^{1/2}\\ &\leq (l+1)(\Delta_{J_1})^{1/2}+\|f\|\left(\sum_{i=1}^{r_0}\lambda_i(J_1)(1-\lambda_i(J_1))^{2l}\right)^{1/2}.\end{aligned} \tag{3.34}$$

By (3.21) and the property (3.33) and Lemma 2, (3.32) is proved. This completes the proof of Theorem 1. □

Theorem 1 provides a sufficient condition for the convergence of the discrete-time extrapolations to the continuous-time one. Moreover, the algorithm (3.2)-(3.4) provides a practical extrapolation procedure for signals in $V_{J,K}$ by implementing the discretized GPG algorithm to oversampled data $f(n2^{-J_1})$. Numerical examples for this procedure are presented in §4. For the convergence rate of the reconstruction $f_{J_1,l}(t)$ via the sampling rate 2^{J_1} and the iteration steps l, we will derive a more detailed estimate below.

We first have a lemma to estimate Δ_{J_1} in (3.34).

Lemma 3. *If $f(t) \in V_{J,K}$ for certain integers $J, K \geq 0$, then*

$$\Delta_{J_1} \leq \frac{4+2^{-J_1+1}}{\pi}\left(\frac{\|f\|^2}{(c2^{J_1})^{1-\alpha}} + \frac{1}{\pi}\left(\int_{(c2^{J_1})^{\alpha}}^{\infty}(|\hat{f}(\omega)| + |\hat{f}(-\omega)|)d\omega\right)^2\right), \tag{3.35}$$

where Δ_{J_1} is defined by (3.21) and α is as in (3.26),

$$c = (\sum_{k=-\infty}^{\infty} |kh_k|)^{-2}, \tag{3.36}$$

h_k are the impulse response of the lowpass filter $H(\omega)$, and $r_0 = (2K+1)(J-J_0+1)$.

Proof. It is clear from (3.24), (3.26), (3.27) and (3.30) that to prove (3.35) we only need to estimate $\omega(J_1)$ in (3.24). For any $\omega \in \mathbf{R}$,

$$\begin{aligned} |\hat{\phi}(\omega) - 1| &= |\prod_{k=1}^{\infty} H(\frac{\omega}{2^k}) - 1| \\ &\leq |1 - H(\frac{\omega}{2})| + |H(\frac{\omega}{2})||1 - H(\frac{\omega}{2^2})| + |H(\frac{\omega}{2})H(\frac{\omega}{2^2})||1 - H(\frac{\omega}{2^3})| + \cdots. \end{aligned}$$

For orthogonal wavelets, $|H(\omega)| \leq 1$ for any real ω. Thus,

$$|\hat{\phi}(\omega) - 1| \leq \sum_{k=1}^{\infty} |1 - H(\frac{\omega}{2^k})|. \tag{3.37}$$

We now estimate $|1 - H(\omega)|$.

$$|1 - H(\omega)| \leq \max_{\omega} |H'(\omega)||\omega| \leq |\omega| \sum_{k=-\infty}^{\infty} |kh_k|.$$

Thus,

$$|\hat{\phi}(\omega) - 1| \leq \left(\sum_{k=1}^{\infty} \frac{|\omega|}{2^k}\right) \sum_{k=-\infty}^{\infty} |kh_k| = |\omega| \sum_{k=-\infty}^{\infty} |kh_k|.$$

This turns out $\omega(J_1) \leq c^{-1}2^{-J_1}$. The facts that $H(0) = 1$, $\alpha \leq 1/2$, and

$$\sum_{k=-\infty}^{\infty} |kh_k| > 1,$$

imply $a(J_1) = (c2^{J_1})^{\alpha}$. This concludes the proof. □

Combining (3.34) and (3.35), we have the following detailed error estimate of $\|f_{J_1,l} - f\|$.

Corollary 1. *Under the same conditions with Theorem 1,*

$$\|f_{J_1,l} - f\| \leq C_0 \left(\frac{\|f\|^2}{(c2^{J_1})^{1-\alpha}} + \frac{1}{\pi}(\int_{(c2^{J_1})^{\alpha}}^{\infty}(|\hat{f}(\omega)| + |\hat{f}(-\omega)|)d\omega)^2\right)^{0.5} + \|f\| \left(\sum_{i=1}^{r_0} \lambda_i(J_1)(1-\lambda_i(J_1))^{2l}\right)^{0.5},$$

where

$$C_0 = \sqrt{\frac{4+2^{-J_1+1}}{\pi}},$$

α is arbitrarily fixed number with $0 < \alpha \leq 0.5$ and c is defined in (3.36).

We next want to estimate $\lambda_i(J_1)$. To do so, we define

$$Q(s,t) = \mathrm{o}(s)(\mathrm{o}(t))^T, \; s,t \in \mathbf{R}, \tag{3.38}$$

where $\mathrm{o}(t)$ is defined in (3.31). Then, there is a sequence of eigenvalues $\sigma_i \geq 0$ and eigenvectors $\Phi_i(t)$ of the operator Q so that (see [15])

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq 0, \tag{3.39}$$

and

$$(Q\Phi_i)(s) \triangleq \int_{-1}^{1} Q(s,t)\Phi_i(t)dt = \sigma_i \Phi_i(s), \; s \in [-1,1], \; i = 1,2,\cdots. \tag{3.40}$$

From the orthonormality of the wavelet basis, it can be proved that $\sigma_1 \leq 1$ similar to the proof of $\lambda_1(J_1) \leq 1$ in Lemma 1. From the assumption of the continuities of $\phi(t)$ and $\psi(t)$ and (2.7)-(2.8), $2^{J_1}\mathrm{w}_m(J_1)(\mathrm{w}_n(J_1))^T \approx Q(s,t)$ as long as $m2^{-J_1} \approx s$ and $n2^{-J_1} \approx t$ for $s,t \in [-1,1]$ when J_1 is large. Therefore, $2^{J_1}\lambda_1(J_1) \to \sigma_1$ as $J_1 \to \infty$. This implies that

$$\lambda_1(J_1) \leq O(2^{-J_1}). \tag{3.41}$$

We now have the following theorem for the error $||f_{J_1,l} - f||$.

THEOREM 2. *Let $f(t) \in V_{J,K}$ for certain integers $J, K \geq 0$. Let the scaling function $\phi(t)$ and the wavelet function $\psi(t)$ be continuous, and $\phi(t)$ satisfy that $\hat{\phi}(\omega) \in L^1(\mathbf{R})$ is continuous at $\omega = 0$ with $\hat{\phi}(0) = 1$ and $c > 0$ where c is as in Lemma 1. If there exit $t_1, t_2, \cdots, t_{r_0} \in [-1,1]$ so that the matrix $\mathbf{O} = (\mathrm{o}(t_1), \mathrm{o}(t_2), \cdots, \mathrm{o}(t_{r_0}))^T$ is non-singular and moreover*

$$|\hat{f}(\omega)| \leq O(\frac{1}{1+|\omega|^{1+\beta+\epsilon}}), \tag{3.42}$$

for certain constants $\epsilon > 0$ and $\beta \geq 0$, then

$$||f_{J_1,l} - f|| \leq O(l2^{-J_1\beta/(1+2\beta)}).$$

Proof. When the signal f satisfies (3.42), it can be proved that

$$\int_{(c2^{J_1})^\alpha}^{\infty} (|\hat{f}(\omega) + \hat{f}(-\omega)|)d\omega \leq O(2^{-J_1\alpha\beta}).$$

Based on (3.34), (3.35) and (3.41), we have

$$||f_{J_1,l} - f|| \leq O(l(2^{-J_1(1-\alpha)/2} + 2^{-J_1\alpha\beta}) + 2^{-J_1/2}).$$

By taking $\alpha = 1/(1+2\beta)$, Theorem 2 is proved. □

Since the sampling rate is $\Delta^{-1} = 2^{J_1}$, Theorem 2 tells us that the error $||f_{J_1,l} - f|| \leq O(h^{\beta/(1+2\beta)})$ where $h = \Delta$ and the number of iteration steps l is appropriately fixed. When $\beta \to \infty$, $||f_{J_1,l} - f|| \leq O(h^{0.5})$. In another words, when the order of the smoothness of the signal f goes to infinity, the error $||f_{J_1,l} - f||$ has the order $h^{0.5}$.

4. Numerical experiment. We use a numerical example to illustrate theoretical results derived in §3. The wavelet basis used is the Daubechies D_4, whose mother wavelet $\psi(t)$ and scaling function $\phi(t)$ are continuous. Besides, $\hat{\phi}(0) = 1$ and $\hat{\phi}(\omega)$ is continuous at $\omega = 0$. Consider a test signal $f(t) = \phi(t) + \psi(t)$ known in the interval $[-T, T] = [-1, 1]$. Since $f(t) \in V_{J,K}$ with $J = 1, K = 0$ and $J_0 = 0$, we have $r_0 = 2$. It is clear that $\mathrm{o}(t)$ in (3.31) is $\mathrm{o}(t) = (\phi(t), \psi(t))$ for a fixed t. and the matrix $\mathbf{O}$ in Theorem 1 for $t_1 \neq t_2 \in [-1, 1]$ is

$$\mathbf{O} = \begin{pmatrix} \phi(t_1) & \psi(t_1) \\ \phi(t_2) & \psi(t_2) \end{pmatrix}.$$

Let $t_1 = -0.5$ and $t_2 = 0.5$. Then,

$$\mathbf{O} = \begin{pmatrix} 0 & -0.246337 \\ 0.919341 & 1.26143 \end{pmatrix}.$$

The determinant $|\mathbf{O}| = 0.2265 \neq 0$. Since all conditions in Theorem 1 are satisfied, it holds for the space $V_{1,0}$.

The discrete-time sequence is obtained via sampling the continuous-time signal, i.e. $x_{J_1}[n] = f(n2^{-J_1})$. We perform the algorithm (3.2) - (3.3) with five sampling rates, i.e. 2^{-J_1} in $[-1, 1]$ with $J_1 = 1, 2, \cdots, 5$. The number of iterations is chosen to be $l = 6$. The reconstructed continuous-time signals $f_{J_1,l}(t)$ via (3.4) with $J_1 = 1, 2, \cdots, 5$ and $l = 6$ are shown in Figs. 1-5, where the solid lines indicate the original signal $f(t)$ while the dashed lines show the reconstructed signals $f_{J_1,l}(t)$. We calculate the mean square error between $f(t)$ and $f_{J_1,l}(t)$ for $-5 \leq t \leq 5$,

$$\mathrm{err}(J_1) = \frac{\sum_{n=-160}^{160} |f(\frac{n}{32}) - f_{J_1,l}(\frac{n}{32})|^2}{321}, \tag{4.1}$$

and plot them in Fig. 6.

To illustrate the convergence rate with respect to various sampling rates (i.e. $J_1 = 1, 2, \cdots, 5$), we list the ratios

$$r(J_1, l) \triangleq \frac{\|f_{J_1,l} - f\|}{2^{-J_1/2}} = \frac{\|f_{J_1,l} - f\|}{h^{0.5}}$$

with $l = 6$ in Table 1. We see that the error between the original and reconstructed signals decays faster with respect to a larger sampling rate.

TABLE 1
$r(J_1, l)$ for $l = 6$

J_1	1	2	3	4	5
$r(J_1, l)$	2.5448	1.9688	1.4555	1.0750	0.8592

5. Conclusion. In this research, we investigated a numerical implementation of continuous-time signal extrapolation with a wavelet model. The procedure involves the application of the discretized GPG algorithm to a set of data sampled from the given segment of the continuous-time signal. We proved that under certain conditions the discrete-time extrapolated signal converges to its continuous-time counterpart as the sampling rate in the given interval goes to infinity. Numerical examples were given to illustrate the convergence behavior.

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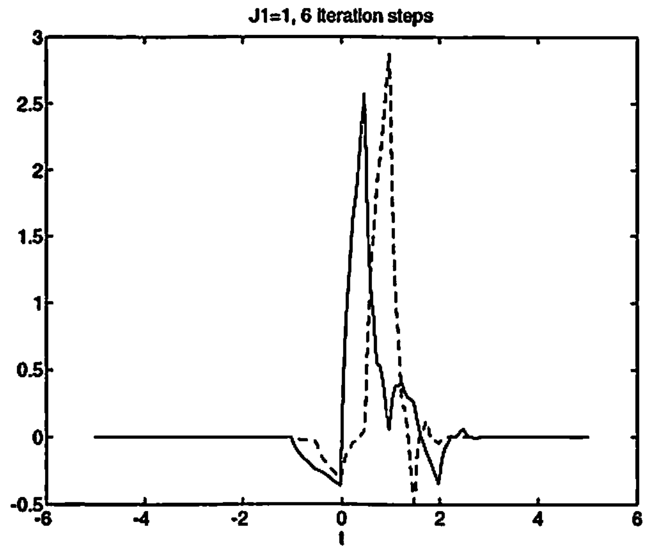

FIG. 1. *Solid line:* $f(t)$; *dashed line:* $f_{1,6}(t)$

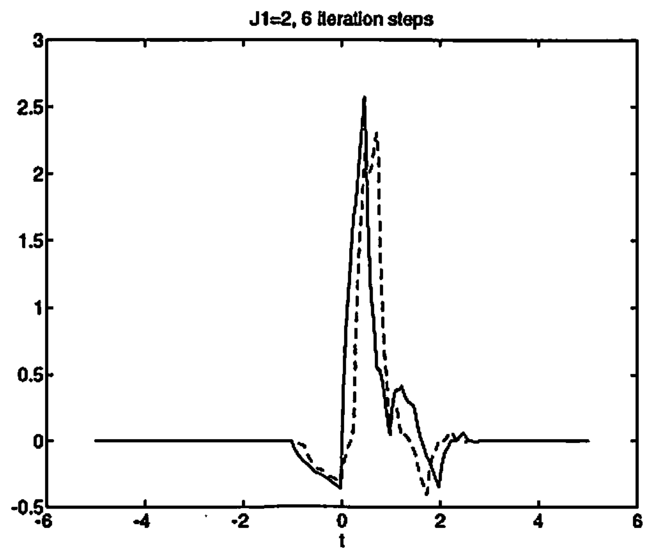

FIG. 2. *Solid line:* $f(t)$; *dashed line:* $f_{2,6}(t)$

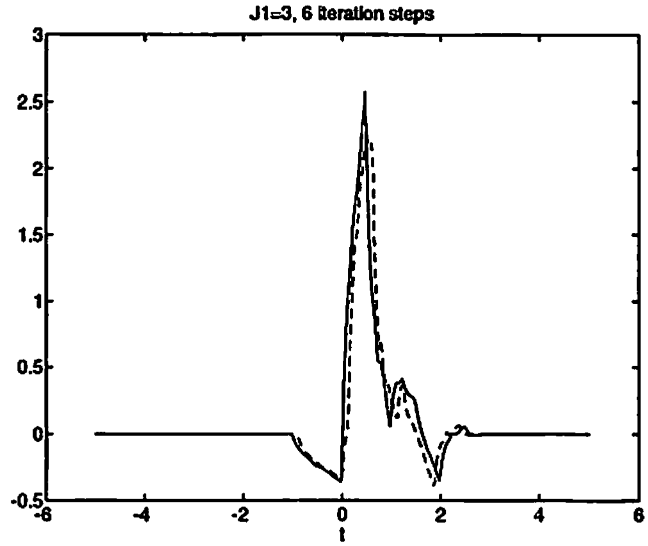

FIG. 3. *Solid line: $f(t)$; dashed line: $f_{3,6}(t)$*

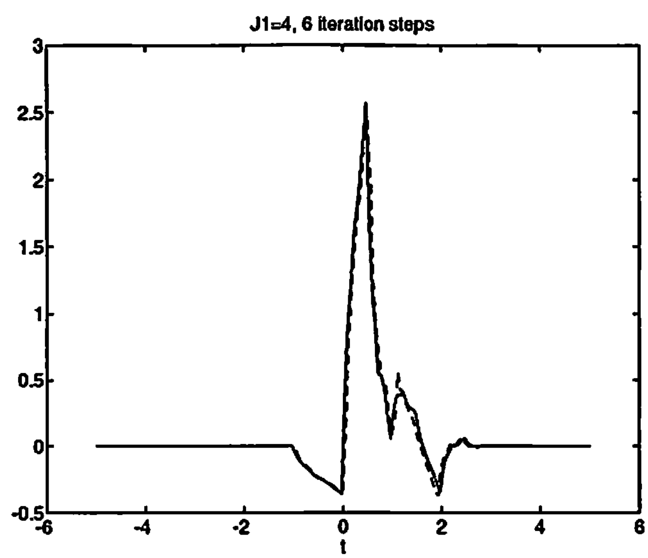

FIG. 4. *Solid line: $f(t)$; dashed line: $f_{4,6}(t)$*

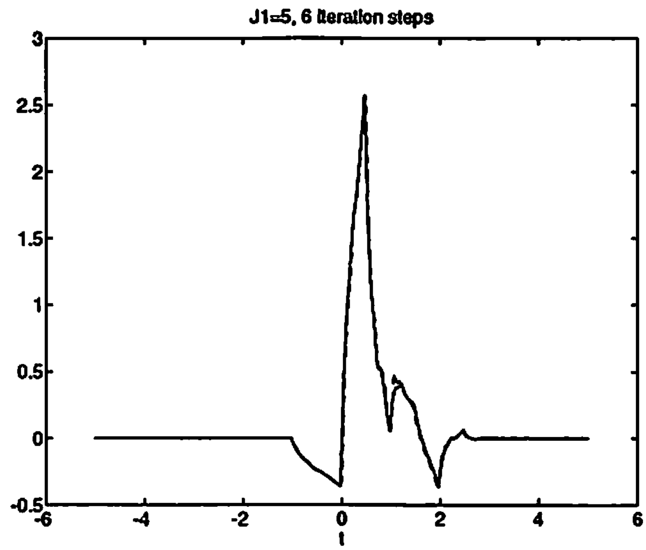

FIG. 5. *Solid line: $f(t)$; dashed line: $f_{5,6}(t)$*

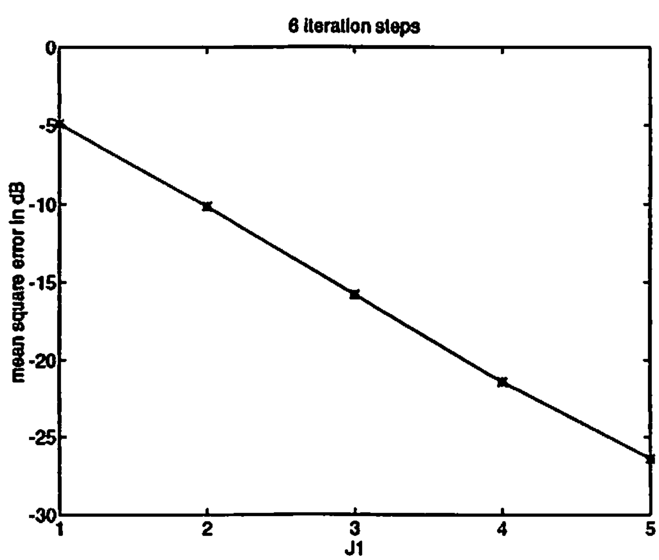

FIG. 6. *The mean square error $err(J_1)$ as a function of J_1.*

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