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Fast Estimation of the Parameters of Alpha-Stable Impulsive Interference

by

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Abstract

We address the problem of estimation of the parameters of the recently proposed symmetric, alpha–stable model for impulsive interference. We propose new estimators based on asymptotic extreme value theory, order statistics, and fractional lower–order moments, which can be computed fast and are, therefore, suitable for the design of real–time signal processing algorithms. The performance of the new estimators is theoretically evaluated, verified via Monte–Carlo simulation, and compared to the performance of maximum likelihood estimators.

Key words: Impulsive Noise, Stable Distribution, Asymptotic Extreme Value Theory, Order Statistic, Fractional Lower–Order Moment.

1. Introduction

The signal processing and the communications literature has traditionally been dominated by Gaussianity assumptions for the data generation processes and the corresponding algorithms have been derived on the basis of the properties of Gaussian statistical signal models. The reason for this tradition is threefold: (i) The well known Central Limit Theorem [1] suggests that a Gaussian model is approximately valid provided that the data generation process includes contributions from a large number of sources, (ii) The Gaussian model has been extensively studied by probabilists and mathematicians and the design of algorithms on the basis of a Gaussianity assumption is a well understood procedure, and (iii) The resulting algorithms are usually of a simple, linear nature which can be implemented in real time without the need for advanced computer software or hardware. However, these advantages of Gaussian signal processing come at the expense of reduced performance of the resulting algorithms. In almost all cases of non–Gaussian environments, a serious degradation in the performance of Gaussian signal processing algorithms is observed. In the past, such degradation might be acceptable due to lack of sufficiently fast computer software and hardware to run more complicated, non–Gaussian signal processing algorithms in real time. With today's availability of inexpensive computer software and hardware, however, a loss in algorithmic performance, in exchange for simplicity and execution gains, is no longer tolerable. This fact has boosted the consideration of non–Gaussian models for statistical signal processing applications and the subsequent development of more complicated, yet more efficient, nonlinear algorithms [2].

One physical process which largely deviates from Gaussianity is the process that generates "impulsive" signal and/or noise bursts. These bursts occur in the form of short duration interferences, attaining large amplitudes with probability significantly higher than the probability predicted by Gaussian distributions. Many natural, as well as man–made, sources of impulsive intereferences exist, including lightning in the atmosphere, switching transients in power lines and car ignitions, and accidental hits in telephone lines [3, 4, 5, 6, 7]. In underwater signals, impulsive noise is quite common and may arise from ice cracking in the arctic region, the presence of submarines and other underwater objects, and reflections from the seabed [6, 8, 9, 10, 11]. Impulsive interference can be particularly annoying in the operation of communication receivers and in the performance of signal detectors. When subjected to impulsive interference, traditional communication devices, that have been built on Gaussianity assumptions, suffer degradation in their performance to unacceptably low levels. However, significant gains in performance can be obtained if the design of the communication devices is based on more appropriate statistical–physical models for the impulsive interference [12, 13, 14, 15].

Classical statistical–physical models for impulsive interference have been proposed by Middleton [16, 17, 18, 19, 20, 21, 22] and are based on the filtered–impulse mechanism. The model includes three different classes of interference, namely A, B, and C. Interference in class A is "coherent" in narrowband receivers, causing a negligible amount of transients. Interference in class B, however, is "impulsive," consisting of a large number of overlapping transients. Finally, interference in class C is the sum of the other two interferences. The Middleton model has been shown to describe real impulsive interferences with high fidelity; however, it is mathematically involved for signal processing applications. Very recently, an alternative to the Middleton model was proposed, which was based on the theory of symmetric, α–stable (SαS) distributions.

In particular, it was shown in [7] that, under very general assumptions, the first order distribution of impulsive interference does, indeed, follow a SαS law. The stable model was then tested with a variety of real data and was found in all cases examined to match the data with excellent fidelity [7]. The performance of optimum and suboptimum receivers in the presence of SαS impulsive interference was examined in [14], both theoretically and via Monte–Carlo simulation, and a method was presented for the real time implementation of the optimum nonlinearities. From this study, it was found that the corresponding optimum receivers perform in the presence of SαS impulsive interference quite well, while the performance of Gaussian and other suboptimum receivers is unacceptably low. It was also shown that a receiver designed on a Cauchy assumption for the first order distribution of the impulsive interference performed only slightly below the corresponding optimum receiver, provided that a reasonable estimate of the noise dispersion was available. These results clearly indicate a need for algorithms for real–time estimation of the parameters of a SαS model from measured observations.

The problem of estimation of the parameters of a SαS model has been addressed in the literature, mainly within the framework of Modern Statistics, and a number of approaches have been proposed to it. However, major difficulties are encountered when the classical estimation methods of Statistics are applied to this particular problem, mainly due to the lack of closed–form expressions for the general SαS pdf. Mandelbrot [23] and, in more detail, Fama [24] proposed a graphical procedure for estimating the characteristic exponent α of the stable distribution. Mandelbrot [25] also proposed approximating the stable distribution with a mixture of a uniform and a Pareto distribution and then applying the method of maximum likelihood in the estimation of the characteristic exponent α. DuMouchel [26] obtained approximate expressions for the maximum likelihood estimates of the characteristic exponent α and the dispersion γ of the SαS pdf under the assumption of zero location parameter ($\delta = 0$) and gave a table of the asymptotic standard deviations and correlations of the maximum likelihood estimates. In [27], DuMouchel considered the estimation of all the parameters of a SαS pdf, including the location parameter δ, and proved that the corresponding likelihood function has no maximum for arbitrary observations if the true characteristic exponent is allowed to range in the entire interval $0 < \alpha \leq 2$. However, restriction of the characteristic exponent α to the range $0 < \epsilon < \alpha \leq 2$, where ϵ can be arbitrarily small, provides (restricted) maximum likelihood estimates which are consistent and asymptotically normal, provided that the true characteristic exponent lies within the specified range of values. The actual estimation algorithm, even when the characteristic exponent can be restricted, is not readily available due to the lack of closed–form expressions for the SαS pdf. Zolotarev [28, 29] proposed a numerical method which begins with an integral form for the SαS pdf and iterative minimization. This approach was investigated via Monte–Carlo simulation by Brorsen and Yang [29] with fairly good results. However, this approach is extremely computation–intensive and no initialization or convergence analysis is available. As alternatives to the maximum likelihood method, the method of sample quantiles has been proposed by Fama and Roll [30] and later generalized by McCulloch [31]. Press [32], Paulson, Holcomb, and Leitch [33], and Koutrouvelis [34, 35] proposed estimation methods based on the empirical characteristic function of the data. It has been shown that, in terms of consistency, bias, and efficiency, Koutrouvelis's regression method is better than the other two. However, neither the sample quantile– nor the empirical characteristic function–based methods are suitable for real–time signal processing.

We propose alternative estimators for the parameters of SαS distributions. The proposed estimators are based on the asymptotic distributions of the extremes (maxima and minima) of collections of random variables, on order statistics, as well as on certain relations between fractional lower-order moments and the parameters of the distribution. These estimators are shown to maintain acceptable performance, while at the same time they are simple enough to be computable in real time. These two properties of the proposed estimators render them very useful for the design of algorithms for statistical signal processing applications.

More specifically, the paper is organized as follows: In Section 2, we briefly review the asymptotic theory of extreme order statistics with particular emphasis on the aspects of the theory that will be used in subsequent sections. This brief review has been considered necessary given the fact of lack of familiarity of the signal processing and the communication communities. A more detailed presentation of the theory from the signal processing point of view can be found in the relevant books (e.g., [36, 37]). In Section 3, we state the problem of estimation of the parameters of an independent, identically distributed (i.i.d.) sequence of stable random variables, which is the main concern of this paper, and propose estimators which can be computed very fast and are, therefore, suitable for real–time signal processing. We analyze the performance of these estimators both theoretically and via Monte–Carlo simulation, but postpone the proof of some of our results until Appendix A. We summarize the paper in Section 4, where we also draw consclusions and suggest possible avenues for future research. Finally, in Appendix B at the end of the paper, we present an analysis of the performance of maximum likelihood estimates of the parameters of a stable pdf and compare them to the performance of the estimates proposed in Section 3.

2. Asymptotic Extreme Value Theory

Extreme Value Theory (EVT) is the field of statistical analysis studying the distributions of extreme order statistics (maxima and minima) of collections of random variables. As such, it is very important in many engineering disciplines in which the laws of interest are governed by extremes. In the fields of communication theory and signal processing in particular, EVT has found application in the estimation via extrapolation of very small probabilities involved in the assessment of the performance of communication devices and signal processing algorithms [37]. In this paper, we present an application of EVT in the estimation of the characterstic exponent of the SαS model for impulsive interference. Because the field of EVT has remained relatively unpopularized in the communications and signal processing communities, we devoted this section of the paper to a brief review of the basic results of asymptotic EVT, summarized in Theorems 2.1, 2.2, and 2.3. A more complete presentation of the field and further applications can be found in the statistical literature [36, 37].

Feasible asymptotic distributions of extreme order statistics

Let $X_1, X_2, \ldots, X_N$ be a collection of independent realizations of a random variable with pdf (*parent pdf*) $f(\cdot)$ and cumulative distribution function (cdf) $F(\cdot)$. Let X_M and X_m denote the maximum and the minimum in the sequence. We will refer to X_M and X_m as the *extreme order statistics* of the collection. The pdfs of X_M and X_m, respectively, are given by [37]

$$f_{M:N}(x) = NF^{N-1}(x)f(x) \tag{1}$$

$$f_{m:N}(x) = N[1-F(x)]^{N-1}f(x), \tag{2}$$

while their joint pdf $f_{M,m:N}(\cdot)$ is [37]:

$$f_{M,m:N}(x_M, x_m) = N(N-1)f(x_M)f(x_m)[F(x_M)-F(x_m)]^{N-2}. \tag{3}$$

From the above formulae, it is clear that the distributions (marginal or joint) of the maximum and the minimum of the random variable collection depend strongly on exact knowledge of the parent pdf $f(\cdot)$ and that this dependence becomes more significant at large sample sizes N. It is, thus, highly desirable to group the distributions of the extreme order statistics to a small number of asymptotic models which will be valid only at the limit of large sample sizes N, but will be very robust to errors in the estimation of the parent pdf. Indeed, such models have been found and can be summarized in the following theorem [36, 37]

Theorem 2.1 (Feasible asymptotic distributions for extreme order statistics) *If sequencies $\{a_N\}$, $\{b_N\}$, $\{c_N\}$, and $\{d_N\}$, exist, such that b_N, $d_N > 0$, for all N, and, as $N \to \infty$,*

$$b_N f_{M:N}(a_N + b_N x) \rightarrow f_M(x) \tag{4}$$

$$d_N f_{m:N}(c_N + d_N x) \quad \to \quad f_m(x), \tag{5}$$

then $f_M(\cdot)$ necessarily belongs to one of the following three families

$$\text{Frechet:} \quad f_{F,\beta}(x) = \begin{cases} x^{-\beta-1}\beta \exp(-x^{-\beta}) & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

$$\text{Weibull:} \quad f_{W,\beta}(x) = \begin{cases} 0 & \text{if } x \geq 0 \\ (-x)^{\beta-1}\beta \exp[-(-x)^{\beta}] & \text{otherwise} \end{cases} \tag{7}$$

$$\text{Gumbel:} \quad f_G(x) = e^{-x}\exp(-e^{-x}) \quad -\infty < x < \infty. \tag{8}$$

Similarly, $f_m(\cdot)$ necessarily belongs to one of the families

$$\text{Frechet:} \quad f_{F,\beta}(x) = \begin{cases} (-x)^{-\beta-1}\beta \exp[-(-x)^{-\beta}] & \text{if } x < 0 \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

$$\text{Weibull:} \quad f_{W,\beta}(x) = \begin{cases} & \text{if } x < 0 \\ x^{\beta-1}\beta \exp(-x^{\beta}) & \text{otherwise} \end{cases} \tag{10}$$

$$\text{Gumbel:} \quad f_G(x) = e^{x}\exp(-e^{x}) \quad -\infty < x < \infty. \tag{11}$$

Moreover, the two extreme order statistics are asymptotically independent, i.e., as $N \to \infty$,

$$b_N d_N f_{M,m:N}(a_N + b_N x_M, c_N + d_N x_m) \to f_M(x_M) f_m(x_m). \tag{12}$$

The family of asymptotic distributions to which are attracted the extreme order statistics of a collection of random variables can be determined from the tail behavior of the parent pdf [36, 37]. In particular, necessary and sufficient conditions for a parent pdf to be attracted to the Frechet distribution for maxima and minima are given by the following theorems [36, 37]:

Theorem 2.2 (Domain of attraction for asymptotic distributions of maxima) *A necessary and sufficient condition for the continuous parent pdf $f(x)$ (cdf $F(x)$) to belong to the domain* *of attraction for maxima of the Frechet distribution with parameter β is the following*

$$F(x) < 1, \text{ for all } x \tag{13}$$

and

$$\lim_{t\to\infty} \frac{1-F(tx)}{1-F(t)} = x^{-\beta}, \text{ for some } \beta > 0 \text{ and for all } x. \tag{14}$$

Theorem 2.3 (Domain of attraction for asymptotic distributions of minima) *A necessary and sufficient condition for the continuous parent pdf $f(x)$ (cdf $F(x)$) to belong to the domain of attraction for minima of the Frechet distribution with parameter β is the following*

$$F(x) > 0, \text{ for all } x \tag{15}$$

and

$$\lim_{t\to-\infty} \frac{F(tx)}{F(t)} = x^{-\beta}, \text{ for some } \beta > 0 \text{ and for all } x. \tag{16}$$

3. Parameter Estimators for Stable Processes

3.1 The symmetric, α–stable probability density function

A symmetric, α–stable (SαS) pdf $f_\alpha(\gamma, \delta; \cdot)$ is best defined via the inverse Fourier transform integral [38, 39]

$$f_\alpha(\gamma, \delta; x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(i\delta\omega - \gamma|\omega|^\alpha) e^{-i\omega x} \, d\omega \tag{1}$$

and is completely characterized by the three parameters α (*characteristic exponent*, $0 < \alpha \leq 2$), γ (*dispersion*, $\gamma > 0$), and δ (*location parameter*, $-\infty < \delta < \infty$).

The characteristic exponent α relates directly to the heaviness of the tails of the SαS pdf. The smaller its value, the heavier the tails. The value $\alpha = 2$ corresponds to a Gaussian pdf, while the value $\alpha = 1$ corresponds to a Cauchy pdf. For these two pdfs, closed–form expressions exist [14], but for other values of the characteristic exponent, no closed–form expressions are known. All the SαS pdfs can be computed, however, at arbitrary argument with the real time method developed in [14]. The dispersion γ is a measure of the spread of the SαS pdf, in many ways similar to the variance of a Gaussian pdf and equal to half the variance of the pdf in the Gaussian case ($\alpha = 2$). Finally, the location parameter δ is the point of symmetry of the SαS pdf.

The non–Gaussian ($\alpha \neq 2$) SαS distributions maintain many similarities to the Gaussian distribution, but at the same time differ from it in some significant ways. For example, a non–Gaussian SαS pdf maintains the usual bell shape and, more importantly, non–Gaussian SαS random variables satisfy the stability property [38]. However, non–Gaussian SαS pdfs have much sharper peaks and much heavier tails than the Gaussian pdf. As a result, only their moments of order $p < \alpha$ are finite, in contrast with the Gaussian pdf which has finite moments of arbitrary order. These and other similarities and differences between Gaussian and non–Gaussian SαS pdfs and their implications on the design of signal processing algorithms are presented in detail in the tutorial paper [39] to which the interested reader is referred.

3.2 Proposed parameter estimators for the SαS pdf

Let $X_1, X_2, \ldots, X_N$ be observed independent realizations of a SαS random variable X of unknown characteristic exponent α, location parameter δ, and dispersion γ. We attempt to estimate the exact parameters of the SαS distribution of X from the observed realizations. The estimation procedure we propose has a hierarchical rather than a simultaneous structure. First an algorithm is proposed for the estimation of the characteristic exponent α, which does *not* involve knowledge or simultaneous estimation of the location parameter δ or the dispersion γ of the pdf. Then, an algorithm is proposed for the estimation of the location parameter δ of the pdf, which again does *not* require knowledge or simultaneous estimation of the characteristic exponent or the dispersion of the pdf. Finally, we propose an algorithm that utilizes either knowledge of the characteristic exponent and the location parameter or estimates thereof to obtain an estimate of the dispersion γ of the pdf. This hierarchical structure significantly reduces the computational complexity of the estimators and can, in fact, be implemented in real time for signal processing applications. The asymptotic properties of the estimates returned from the algorithm are summarized in the form of three theorems. The finite sample performance of the estimates is also assessed via a Monte–Carlo simulation evaluation.

a. Estimator of the characteristic exponent. For the estimation of the characteristic exponent α of the pdf, we propose the following algorithm. Consider a segmentation of the data into L nonoverlapping segments, each of length $K = N/L$:

$$\{X_1, X_2, \ldots, X_N\} = \{\mathbf{X}(1), \mathbf{X}(2), \ldots, \mathbf{X}(L)\}, \tag{2}$$

where $\mathbf{X}(l) = \{X_{(l-1)N/L+1}, X_{(l-1)N/L+2}, \ldots, X_{lN/L}\}$, $l = 1, 2, \ldots, L$. This segmentation is done arbitrarily for the time being and the reason for considering it will become apparent momentarily. Optimization of the segmentation is a topic of present and future research.

Let $\overline{X}_l$ and $\underline{X}_l$ be the maximum and the minimum of the data segment $\mathbf{X}(l)$. We then define

$$\overline{x}_l = \log \overline{X}_l \tag{3}$$

$$\underline{x}_l = -\log(-\underline{X}_l) \tag{4}$$

and the corresponding standard deviations

$$\overline{s} = \sqrt{\frac{1}{L-1}\sum_{l=1}^{L}(\overline{x}_l - \overline{x})^2}; \qquad \overline{x} = \frac{1}{L}\sum_{l=1}^{L}\overline{x}_l \tag{5}$$

$$\underline{s} = \sqrt{\frac{1}{L-1}\sum_{l=1}^{L}(\underline{x}_l - \underline{x})^2}; \qquad \underline{x} = \frac{1}{L}\sum_{l=1}^{L}\underline{x}_l. \tag{6}$$

With these definitions in mind, the estimate for the characteristic exponent α of the SαS pdf takes the form

$$\hat{\alpha} = \frac{\pi}{2\sqrt{6}}\left(\frac{1}{\bar{s}} + \frac{1}{\underline{s}}\right). \tag{7}$$

The justification for this choice of an estimator for the characteristic exponent of the SαS pdf becomes apparent in Appendix A. First, however, we present estimators for the location parameter and the dispersion of the SαS pdf.

b. Estimator of the location parameter. For the estimation of the location parameter δ of a SαS pdf, we propose the use of the sample median of the observations, i.e.

$$\hat{\delta} = \text{ median } \{X_1, X_2, \ldots, X_N\}, \tag{8}$$

where the sample median is defined as follows: If the sample consists of an odd number N of observations, the median is defined as the center order statistic. If the sample consists of an even number N of observations, the median is defined as the average of the two center statistics. The sample median forms the maximum likelihood estimate of the location parameter of a Laplace (double exponential) distribution and, therefore, enjoys all the properties of maximum likelihood estimators in that case. Its performance as an estimator for the location parameter δ of a SαS pdf can be expected to be very robust. In fact, we will show momentarily that this estimator performs very closely to the maximum likelihood estimator for the case of a SαS pdf.

c. Estimator of the dispersion. For the estimation of the dispersion γ of a SαS pdf, we propose the following estimator which is based on the theory of fractional lower order moments of the pdf:

$$\hat{\gamma} = \left[\frac{\frac{1}{N}\sum_{k=1}^{N} |X_k - \hat{\delta}|^p}{C(p, \hat{\alpha})}\right]^{\frac{\hat{\alpha}}{p}}, \tag{9}$$

where $C(p, \hat{\alpha})$ has been defined as

$$C(p, \hat{\alpha}) = \frac{1}{\cos(\frac{\pi}{2}p)} \frac{\Gamma(1 - p/\hat{\alpha})}{\Gamma(1 - p)} \tag{10}$$

and the choice of the order p $(0 < p < \frac{\hat{\alpha}}{2})$ of the fractional moment is arbitrary.

As we can see, the dispersion estimator requires knowledge of the characteristic exponent and the location parameter of the SαS pdf. Thus, the dispersion estimate must be computed *after* estimates for the characteristic exponent and the location parameter have been obtained.

3.3 Theoretical performance of the proposed estimators

Theorem 3.1 (Estimator of the characteristic exponent) *The estimator $\hat{\alpha}$ of the characteristic exponent α of a SαS distribution is consistent and asymptotically normal with mean equal to the true exponent α and variance $\frac{9}{2L\pi^4}(\mu_{4,G} - \frac{L-3}{L-1}\frac{\pi^4}{36\alpha^4})\alpha^6$, as $N \to \infty$ and $L \to \infty$ such that $N/L \to \infty$. In the expression for the asymptotic variance of the estimator, $\mu_{4,G}$ is on the order of $\frac{1}{\alpha^4}$; therefore, the asymptotic variance of the estimator is on the order of α^2.*

Proof The proof of the theorem is lengthy and, thus, is given in Appendix A to the paper.

Theorem 3.2 (Estimator of the location parameter) *The estimator $\hat{\delta}$ of the location parameter δ of a SαS distribution is consistent and asymptotically normal with mean equal to the true parameter δ and variance $(\frac{\pi\alpha\gamma^{1/\alpha}}{2\Gamma(1/\alpha)})^2\frac{1}{N}$, as $N \to \infty$.*

Proof From [40, p. 369], it follows that the sample median of the observations is asymptotically normal with mean equal to the true median (location parameter δ) and variance $(\frac{1}{2f_\alpha(\gamma,\delta;\delta)})^2\frac{1}{N}$. But, $f_\alpha(\gamma, \delta; x) = \gamma^{-1/\alpha} f_\alpha[1, 0; (x - \delta)\gamma^{-1/\alpha}]$, as can be seen from the defining Eq.(3-1). Moreover, $f_\alpha(1, 0; \delta) = \frac{1}{\pi\alpha}\Gamma(\frac{1}{\alpha})$ [41]. Combining the last two relations, we get

$$(\frac{1}{2f_\alpha(\gamma,\delta;\delta)})^2\frac{1}{N}=(\frac{\pi\alpha\gamma^{1/\alpha}}{2\Gamma(1/\alpha)})^2\frac{1}{N},$$

as the asymptotic variance of the estimator $\hat{\delta}$. **Q.E.D.**

Theorem 3.3 (Estimator of the dispersion) *The estimator $\hat{\gamma}$ of the dispersion of a $S\alpha S$ distribution is consistent and asymptotically normal with mean equal to the true dispersion γ and variance $\frac{1}{N}(m_{2p}-m_p^2)\{\frac{\alpha}{p}\cdot\frac{(\frac{m_p}{C(p,\alpha)})^{\frac{\alpha}{p}}}{m_p}\}^2$, as $N\to\infty$ and $L\to\infty$ with $K=N/L\to\infty$. With m_p and m_{2p}, we have denoted the pdf moments of fractional orders p and $2p$, respectively.*

Proof The result of Theorem 3.3 for the asymptotic performance of the dispersion estimator makes use of Theorems 3.1 and 3.2 and holds in the limit of N and $K=N/L$ sufficiently large to allow the assumption $\hat{\alpha}\approx\alpha$ and $\hat{\delta}\approx\delta$. As we verify in the Monte-Carlo simulations of the following section, these assumptions are realistic. We, then, have

$$\mathcal{E}\{\frac{1}{N}\sum_{k=1}^{N}|X_k-\hat{\delta}|^p\}=m_p,$$

while, asymptotically [42, p. 367]

$$\text{var}\,\{\frac{1}{N}\sum_{k=1}^{N}|X_k-\hat{\delta}|^p\}=\frac{1}{N}(m_{2p}-m_p^2).$$

In the above, m_p and m_{2p} denote the moments of a SαS pdf of fractional orders p and $2p$, respectively. According to the discussion in section 3, these moments exist, provided that $2p<\alpha$ as has been assumed.

We now have that

$$\sqrt{N}[\frac{1}{N}\sum_{k=1}^{N}|X_k-\hat{\delta}|^p-m_p]\to\mathcal{N}[0,(m_{2p}-m_p^2)],$$

since the random variable $\frac{1}{N}\sum_{k=1}^{N}|X_k-\hat{\delta}|^p$ is the sum of a large number of random variables of finite variance [1]. Thus, [42, p. 319]

$$\sqrt{N}[(\frac{\frac{1}{N}\sum_{k=1}^{N}|X_k-\hat{\delta}|^p}{C(p,\hat{\alpha})})^{\frac{\hat{\alpha}}{p}} - (\frac{m_p}{C(p,\hat{\alpha})})^{\frac{\hat{\alpha}}{p}}] \rightarrow \mathcal{N}[0,(m_{2p}-m_p^2)\{\frac{\hat{\alpha}}{p}\cdot\frac{(\frac{m_p}{C(p,\hat{\alpha})})^{\frac{\hat{\alpha}}{p}}}{m_p}\}^2],$$

or

$$\sqrt{N}[(\frac{\frac{1}{N}\sum_{k=1}^{N}|X_k-\hat{\delta}|^p}{C(p,\hat{\alpha})})^{\frac{\hat{\alpha}}{p}} - \gamma] \rightarrow \mathcal{N}[0,(m_{2p}-m_p^2)\{\frac{\hat{\alpha}}{p}\cdot\frac{(\frac{m_p}{C(p,\hat{\alpha})})^{\frac{\hat{\alpha}}{p}}}{m_p}\}^2].$$

The relation of the asymptotic variance of the proposed estimator $\hat{\gamma}$ to the true characteristic exponent α and the true dispersion γ becomes clearer if the relationship between the dispersion of a SαS pdf and its fractional, lower-order moments are considered [39]

$$\gamma = (\frac{m_p}{C(p,\alpha)})^{\frac{\alpha}{p}} = (\frac{m_{2p}}{C(2p,\alpha)})^{\frac{\alpha}{2p}},$$

where the function $C(\cdot,\cdot)$ has been defined in Eq.(3-10).

3.4 Monte–Carlo evaluation of the performance of the proposed estimators

The previous section analyzed the asymptotic performance of the estimators for the parameters (α, γ, δ) of a SαS distribution in the limit $N \rightarrow \infty$ and $L \rightarrow \infty$ such that $K = N/L \rightarrow \infty$. The finite data performance of the algorithm can only be evaluated via Monte–Carlo simulation. In this section, we report the results from an extensive Monte–Carlo simulation of the estimators of the previous section. For our simulations, we made use of an efficient method for generation of SαS deviates of arbitrary characteristic exponent α and dispersion γ [43, 14].

We ran 1,000 Monte–Carlo runs of the estimators that we propose in this paper. In particular, we examined the cases of $N = 5,000$ and 20,000, L = 20, 50, and 200, α = 0.1, 0.5, 1.0, and 1.5, and γ = 1.0 and 10.0 in all possible combinations. For the location parameter, we assumed $\delta = 1$. For the estimation of the dispersion γ, we used a fractional moment of order $p = \frac{\hat{\alpha}}{4}$, where $\hat{\alpha}$ was the estimated characteristic exponent. The arithmetic mean and the standard deviation (in brackets) over the 1,000 Monte–Carlo runs of the estimators $\hat{\alpha}$, $\hat{\delta}$, and $\hat{\gamma}$ are collectively presented in Tables 1, 2, and 3, respectively.

After careful study of the Monte–Carlo results, we reach the following conclusions:

1. The estimator $\hat{\alpha}$ of the characteristic exponent α becomes less accurate in terms of both bias and error variance as the true exponent α increases and gets closer to the Gaussian value $\alpha = 2$, but remains independent either of the true location parameter δ or the true dispersion γ. As the true exponent gets closer to the Gaussian value $\alpha = 2$, a small bias is introduced in the estimate $\hat{\alpha}$, which is explained by the fact that the convergence of the true extreme distributions to the asymptotic ones is slower for larger α [36, 37]. This difficulty can be compensated for by use of longer data sets for the estimation procedure. However, it is clear that a careful choice of the segment length N/L needs to be made on the basis of the facts that (i) the bias in the estimate $\hat{\alpha}$ is decreased if longer segments (higher N/L, smaller L) are used and (ii) the error variance in the estimate $\hat{\alpha}$ is decreased if more segments (higher L) are used. For fixed total data length N, these two requirements are competing and a compromise between them needs to be made.

2. The estimator $\hat{\delta}$ of the location parameter δ is very efficient. In fact, it is shown in Appendix B, both theoretically and via Monte–Carlo simulation, that the proposed estimator is almost as efficient as the maximum likelihood estimator for the case of known characteristic exponent α. However, its performance depends on both the true characteristic exponent α and the true disperion γ and, in particular, its error variance can become significant for small α and high γ.

3. The estimator $\hat{\gamma}$ of the true dispersion γ contains a relatively high error variance,

which is inherited into it from the fact that it is a moment–based estimator. Its error variance is higher at higher true dispersions γ. At high values of the true characteristic exponent α, a small bias may be observed which is due to the corresponding bias in the estimation of the characteristic exponent.

The above observations are absolutely compatible with the qualitative theoretical predictions from the asymptotic analysis of the previous section. At this point, it should be mentioned that the difficulties in estimating the parameters of SαS interference that may be present in certain cases are not limitations of the proposed estimators, but are rather inherent in the estimation problem at hand. For example, it is known that estimation of a true characteristic exponent $\alpha > 1.5$ is expected to be highly unreliable, as the corresponding Cramér–Rao bounds are high [44]. The Monte–Carlo simulation study in Appendix B verifies the efficiency of the proposed estimators when compared to the maximum likelihood estimators and the corresponding theoretical Cramér–Rao bounds.

Table 1: Performance of Estimator $\hat{\alpha}$

		α							
		0.1		0.5		1.0		1.5	
		γ		γ		γ		γ	
N	L	1	10	1	10	1	10	1	10
5,000	20	0.1074	0.1075	0.5340	0.5452	1.0755	1.0858	1.6492	1.6568
		(0.0171)	(0.0175)	(0.0818)	(0.0888)	(0.1771)	(0.1707)	(0.2804)	(0.2787)
	50	0.1031	0.1030	0.5167	0.5116	1.0351	1.0386	1.6107	1.6140
		(0.0107)	(0.0109)	(0.0538)	(0.0532)	(0.1072)	(0.1066)	(0.1725)	(0.1760)
	200	0.0997	0.0995	0.4943	0.4970	0.9962	1.0106	1.6275	1.6805
		(0.0051)	(0.0051)	(0.0253)	(0.0256)	(0.0561)	(0.0538)	(0.0997)	(0.1052)
20,000	20	0.1072	0.1082	0.5367	0.5397	1.0787	1.0779	1.6103	1.6400
		(0.0166)	(0.0167)	(0.0863)	(0.0838)	(0.1683)	(0.1753)	(0.2583)	(0.2650)
	50	0.1033	0.1028	0.5147	0.5151	1.0289	1.0349	1.5677	1.5600
		(0.0109)	(0.0105)	(0.0524)	(0.0537)	(0.1095)	(0.1078)	(0.1633)	(0.1641)
	200	0.1003	0.1004	0.5008	0.5037	1.0088	1.0108	1.5740	1.5811
		(0.0052)	(0.0053)	(0.0267)	(0.0265)	(0.0550)	(0.0556)	(0.0854)	(0.0882)

Table 2: Performance of Estimator $\hat{\delta}$								
	α							
	0.1		0.5		1.0		1.5	
	γ		γ		γ		γ	
N	1	10	1	10	1	10	1	10
5,000	1.0000 (6.94×10^{-7})	58.6915 ($8.4604 \times 10^{+3}$)	0.9994 (0.0110)	0.9979 (1.1096)	0.9998 (0.0213)	1.0106 (0.2129)	0.9990 (0.0237)	1.0025 (0.1138)
20,000	1.0000 (1.18×10^{-7})	15.8975 ($1.11 \times 10^{+3}$)	1.0001 (0.0055)	0.9924 (0.5620)	1.0000 (0.0111)	1.0002 (0.1135)	0.9999 (0.0117)	0.9984 (0.0559)

Table 3: Performance of Estimator $\hat{\gamma}$

		α							
		0.1		0.5		1.0		1.5	
		γ		γ		γ		γ	
N	L	1	10	1	10	1	10	1	10
5,000	20	1.1153	15.9351	1.1027	16.8175	1.1191	16.2872	1.1545	18.1620
		(0.2275)	(12.9101)	(0.2147)	(13.8951)	(0.2518)	(12.8706)	(0.2865)	(16.2923)
	50	1.0421	11.8845	1.0468	11.5582	1.0480	12.2093	1.0966	14.0672
		(0.1209)	(4.8725)	(0.1201)	(4.4013)	(0.1200)	(5.5068)	(0.1429)	(6.3089)
	200	0.9975	9.9770	0.9885	9.9264	0.9969	10.5135	1.1030	15.4333
		(0.0493)	(1.6772)	(0.0472)	(1.6452)	(0.0540)	(1.8323)	(0.0819)	(3.3972)
20,000	20	1.1134	15.9542	1.1173	16.0716	1.1220	16.4911	1.1179	17.0753
		(0.2334)	(11.3094)	(0.2417)	(13.7210)	(0.2307)	(18.0993)	(0.2432)	(14.1963)
	50	1.0485	11.7752	1.0421	11.8994	1.0431	12.0774	1.0614	12.3398
		(0.1286)	(4.6033)	(0.1220)	(4.7826)	(0.1261)	(4.6853)	(0.1313)	(5.0753)
	200	1.0058	10.3116	1.0033	10.4156	1.0116	10.5573	1.0577	12.2613
		(0.0549)	(1.8329)	(0.0557)	(1.8470)	(0.0574)	(1.9775)	(0.0627)	(2.4282)

4. Summary, Conclusions, and Possible Future Research

In this paper, we have examined the problem of estimation of the parameters of SαS interference from a set of i.i.d. observations of its realizations. We have developed estimators which can be computed in small real time and yet maintain high efficiency. We analyzed the performance of the estimators both theoretically and via Monte–Carlo simulation and we found them to be appropriate for the design of algorithms for statistical signal processing applications.

Future topics in the same area of research, that need to be addressed, include the detection of multiple signals with unknown parameters embedded in i.i.d. impulsive noise using array sensors and their generalization to the case of linearly dependent impulsive noise. Of relevance are also algorithms for parameter estimation from data corrupted by impulsive interference. Possible applications of this research can be found in the detection of low-probability-of-intercept (spread spectrum) communication signals and the identification of incompletely specified communication channels. These and related topics are currently under investigation, the results of which are expected to be announced very shortly.

APPENDICES

1. Proof of Theorem 3.1

The proof is presented in three steps. Steps 1 and 2 clarify the rationale behind this choice for the estimator $\hat{\alpha}$, while step 3 wraps up steps 1 and 2 into a concise mathematical formulation.

Step 1. In this step, we show that the extreme order statistics from a SαS parent pdf $(0 < \alpha < 2)$ are attracted to the Frechet family of distribution with parameter $\beta = \alpha$. To show this fact, we apply theorems 2.2 and 2.3. We have [36, 37]

$$F_\alpha(\gamma, \delta; x) < 1 \text{ for all } x$$

and

$$\lim_{t\to\infty} \frac{1 - F_\alpha(\gamma, \delta; tx)}{1 - F_\alpha(\gamma, \delta; t)} = \lim_{t\to\infty} \frac{\frac{1}{\pi}\frac{\Gamma(\alpha)\sin(\frac{\pi\alpha}{2})}{t^\alpha x^\alpha}}{\frac{1}{\pi}\frac{\Gamma(\alpha)\sin(\frac{\pi\alpha}{2})}{t^\alpha}} = x^{-\alpha}, \quad 0 < \alpha < 2.$$

In writing the above equation, we have made use of the asymptotic tail behavior of non–Gaussian SαS pdfs [39]. The conditions of theorem 2.2 are satisfied, therefore, the maxima of collections of non–Gaussian SαS random variables are attracted to the Frechet asymptotic distribution for maxima.

Similarly, we can use theorem 2.3 to show that the minima of collections of non–Gaussian SαS random variables are attracted to the Frechet asymptotic distribution for minima. Combining these two asymptotic distributions and using the fact that maxima and minima are asymptotically independent, we show that, for large segment length $K = N/L$ (N is the total number of observed data and L is the number of segments), the asymptotic joint pdf of the maxima and minima of each segment of the data will be of the form

$$f_{M,m}(\overline{X}_l, \underline{X}_l) = \begin{cases} [\frac{1}{b_K} f_M(\overline{X}_l/b_K)][\frac{1}{d_K} f_m(\underline{X}_l/d_K)], & \text{if } \overline{X}_l > 0, \underline{X}_l < 0 \\ 0, & \text{otherwise,} \end{cases}$$

where f_M and f_m are the Frechet distributions for maxima and minima in Eqs.(2-6) and (2-9), respectively, with shape parameter $\beta = \alpha$ and the constants b_K and d_K can be chosen as $b_K = F_a^{-1}[\gamma, \delta; (1 - 1/K)]$ and $d_K = |F_a^{-1}[\gamma, \delta; (1/K)]|$ [36, 37].

Step 2. The transformation

$$\begin{aligned} \overline{x}_l &= \log \overline{X}_l \\ \underline{x}_l &= -\log(-\underline{X}_l), \end{aligned}$$

converts Frechet distributed random variables to Gumbel distributed ones [37]. Therefore, for large K, $\overline{x}_l$ and $\underline{x}_l$ will be asymptotically Gumbel distributed, i.e. their asymptotic joint pdf will have the form

$$f_{M,m}^G(\overline{x}_l, \underline{x}_l) = \frac{\exp(-\frac{\overline{x}_l - \overline{\lambda}_G}{b_G}) \exp[-\exp(-\frac{\overline{x}_l - \overline{\lambda}_G}{b_G})]}{b_G} \cdot \frac{\exp(-\frac{\underline{\lambda}_G - \underline{x}_l}{b_G}) \exp[-\exp(-\frac{\underline{\lambda}_G - \underline{x}_l}{b_G})]}{b_G},$$

where $b_G = 1/\alpha$, $\overline{\lambda}_G = \log(b_K)$, and $\underline{\lambda}_G = -\log(d_K)$.

The variance of the Gumbel distribution of shape parameter b_G is [37]

$$\sigma_G^2 = \frac{\pi^2 b_G^2}{6} = \frac{\pi^2}{6\alpha^2}$$

and, therefore, its moment estimates from the sequencies $\{\overline{x}_l\}$ and $\{\underline{x}_l\}$ of transformed maxima and minima, respectively, can be combined to form the estimate

$$\hat{\alpha} = \frac{\pi}{2\sqrt{6}} \left(\frac{1}{\overline{s}} + \frac{1}{\underline{s}}\right)$$

as in Eq.(3-7).

Step 3. We can now formalize steps 1 and 2 and derive the asymptotic performance of the estimator $\hat{\alpha}$ as in theorem 3.1.

Let $\sigma_M^2 = \int_{-\infty}^{\infty} u^2 \overline{f}_K(u)\, du - [\int_{-\infty}^{\infty} u \overline{f}_K(u)\, du]^2$ be the exact variance of the transformed maxima $\overline{x}_l$. Similarly, let $\sigma_m^2 = \int_{-\infty}^{\infty} u^2 \underline{f}_K(u)\, du - [\int_{-\infty}^{\infty} u \underline{f}_K(u)\, du]^2$ be the exact variance of the transformed minima $\underline{x}_l$. We have denoted the exact pdfs for the transformed maxima and minima, respectively, with $\overline{f}_K(\cdot)$ and $\underline{f}_K(\cdot)$. Then [40, 42],

$$(\overline{s}^2 - \sigma_M^2) \to \mathcal{N}(0, \frac{1}{L}[\mu_{4,M} - \frac{L-3}{L-1}\sigma_M^4]) \quad \text{as } L \to \infty$$

and

$$(\underline{s}^2 - \sigma_m^2) \to \mathcal{N}(0, \frac{1}{L}[\mu_{4,m} - \frac{L-3}{L-1}\sigma_m^4]) \quad \text{as } L \to \infty.$$

In the above, $\mu_{4,M}$ and $\mu_{4,m}$ are the exact fourth–order central moments of the transformed maxima and minima, respectively, and $\mathcal{N}(\xi, \eta^2)$ denotes the Gaussian pdf of mean ξ and variance η^2. The convergence is in distribution.

According to [40, 42], we will then have

$$(\frac{\pi}{\sqrt{6}\overline{s}} - \frac{\pi}{\sqrt{6}\sigma_M}) \to \mathcal{N}(0, \frac{1}{L}[\mu_{4,M} - \frac{L-3}{L-1}\sigma_M^4]\frac{\pi^2}{24\sigma_M^6}) \quad \text{as } L \to \infty$$

and

$$(\frac{\pi}{\sqrt{6}\underline{s}} - \frac{\pi}{\sqrt{6}\sigma_m}) \to \mathcal{N}(0, \frac{1}{L}[\mu_{4,m} - \frac{L-3}{L-1}\sigma_m^4]\frac{\pi^2}{24\sigma_m^6}) \quad \text{as } L \to \infty.$$

As $K = N/L \to \infty$, the distribution of the maxima and minima converges asymptotically to the Gumbel distribution of step 2. Therefore,

$$\begin{aligned} \sigma_M^2, \sigma_m^2 &\to \frac{\pi^2}{6\alpha^2} \\ \mu_{4,M}, \mu_{4,m} &\to \mu_{4,G}, \end{aligned}$$

where $\mu_{4,G}$ is the fourth–order central moment of the Gumbel distribution (to be given momentarily). Moreover, the transformed maxima and minima are asymptotically independent. We, thus, conclude that

$$(\frac{\pi}{2\sqrt{6}}[\frac{1}{\overline{s}} + \frac{1}{\underline{s}}] - \alpha) \to \mathcal{N}(0, \frac{9}{2L\pi^4}(\mu_{4,G} - \frac{L-3}{L-1}\frac{\pi^4}{36\alpha^4})\alpha^6.$$

We, finally, compute the fourth–order central moment $\mu_{4,G}$ of the Gumbel distribution. Let $\mu_{j,G}$ and $c_{j,G}$, $j = 2, 3, 4, \ldots$, be the central moments and the cumulants, respectively.

of the Gumbel distribution of order j. Then, [45, p. 542]

$$c_{j,G} = (-b_G)^j \psi^{(j-1)}(1) = (-\frac{1}{\alpha})^j \psi^{(j-1)}(1), \quad j = 2, 3, 4, \ldots,$$

where $\psi(\cdot)$ is the digamma function. From known relations between moments and cumulants [2], we can now compute the fourth order moment $\mu_{4,G}$. Indeed:

$$\mu_{4,G} = \sum_{j=0}^{4} (-1)^j \mu_{1,G}^j m_{4-j,G},$$

where

$$\begin{aligned}
m_{1,G} &= \mu_{1,G} \approx \lambda_G + 0.577216\, b_G = \lambda_G + \frac{0.577216}{\alpha} \\
m_{2,G} &= \mu_{2,G} + \mu_{1,G}^2 = \frac{\pi^2}{6\alpha^2} + \mu_{1,G}^2 \\
m_{3,G} &= c_{3,G} + 3m_{2,G}m_{1,G} - 2m_{1,G}^3 \\
m_{4,G} &= c_{4,G} + 4m_{3,G}m_{1,G} + 3m_{2,G}^2 - 12m_{2,G}m_{1,G}^2 + 6m_{1,G}^4,
\end{aligned}$$

where λ_G is either $\overline{\lambda_G}$ or $\underline{\lambda_G}$ as they were defined earlier. Given the form of the cumulants $c_{j,G}$, $j = 2, 3, 4, \ldots$, it is clear that $\mu_{4,G}$ is on the order of $\frac{1}{\alpha^4}$, which shows that the variance of the error in the estimator $\hat{\alpha}$ is asymptotically on the order of α^2.

2. Maximum likelihood estimation of the parameters of the α–stable impulsive interference model

The key result regarding the restricted maximum likelihood estimates of the parameters α, γ, and δ of a SαS distribution from N independent realizations can be summarized in the following theorem (adapted from the corresponding theorem in [27, p. 952])

Theorem B.1 *When sampling from a SαS distribution, the maximum likelihood estimate ($\hat{\alpha}_{N,ml}$, $\hat{\gamma}_{N,ml}$, $\hat{\delta}_{N,ml}$) for the parameters (α, γ, δ), based on the first N observations and* *restricted so that $\alpha_{N,ml} > \epsilon$, where ϵ is arbitrarily small and positive, is consistent and asymptotically normal, as long as $(\alpha,\ \gamma,\ \delta)$, the true value of the parameters, are in the interior of the parameter space, i.e., $\epsilon < \alpha < 2$.*

With the above theorem in mind, we can now proceed to compute the Cramér–Rao lower bounds for the error in any unbiased estimate of the parameters $(\alpha,\ \gamma,\ \delta)$ of a SαS distribution. In particular, we find the Fisher information matrix $\mathbf{J}$ to be of the form:

$$\mathbf{J} = N \int_{-\infty}^{\infty} \begin{bmatrix} \frac{[\partial/\partial\alpha f_\alpha(\gamma,\delta;u)]^2}{f_\alpha(\gamma,\delta;u)} & \frac{[\partial/\partial\alpha f_\alpha(\gamma,\delta;u)][\partial/\partial\gamma f_\alpha(\gamma,\delta;u)]}{f_\alpha(\gamma,\delta;u)} & \frac{[\partial/\partial\alpha f_\alpha(\gamma,\delta;u)][\partial/\partial\delta f_\alpha(\gamma,\delta;u)]}{f_\alpha(\gamma,\delta;u)} \\ \frac{[\partial/\partial\gamma f_\alpha(\gamma,\delta;u)][\partial/\partial\alpha f_\alpha(\gamma,\delta;u)]}{f_\alpha(\gamma,\delta;u)} & \frac{[\partial/\partial\gamma f_\alpha(\gamma,\delta;u)]^2}{f_\alpha(\gamma,\delta;u)} & \frac{[\partial/\partial\gamma f_\alpha(\gamma,\delta;u)][\partial/\partial\delta f_\alpha(\gamma,\delta;u)]}{f_\alpha(\gamma,\delta;u)} \\ \frac{[\partial/\partial\delta f_\alpha(\gamma,\delta;u)][\partial/\partial\alpha f_\alpha(\gamma,\delta;u)]}{f_\alpha(\gamma,\delta;u)} & \frac{[\partial/\partial\delta f_\alpha(\gamma,\delta;u)][\partial/\partial\gamma f_\alpha(\gamma,\delta;u)]}{f_\alpha(\gamma,\delta;u)} & \frac{[\partial/\partial\delta f_\alpha(\gamma,\delta;u)]^2}{f_\alpha(\gamma,\delta;u)} \end{bmatrix} du$$

The corresponding Cramér–Rao bounds for the parameters $(\alpha,\ \gamma,\ \delta)$ can now be computed as the first, second, and third diagonal element in the matrix $\mathbf{J}^{-1}$. Taking into account the fact that $f_\alpha(\gamma,\delta;u) = \gamma^{-\frac{1}{\alpha}} f_\alpha[(u-\delta)\gamma^{-\frac{1}{\alpha}}]$, we see clearly that the Cramér–Rao bounds for the parameters $(\alpha,\ \gamma,\ \delta)$ are independent of the true location parameter δ.

Unfortunately, numerical methods need to be employed in the evaluation of the above Cramér–Rao bounds. It is instructive, however, to consider the case of known $\alpha = 1$ and examine the Cramér–Rao bounds in the estimation of the location parameter δ and the dispersion γ of the a SαS pdf. In this case, the corresponding Fisher information matrix becomes:

$$\mathrm{j} = N \int_{-\infty}^{\infty} \begin{bmatrix} \frac{[\partial/\partial\gamma f_1(\gamma,\delta;u)]^2}{f_1(\gamma,\delta;u)} & \frac{[\partial/\partial\gamma f_1(\gamma,\delta;u)][\partial/\partial\delta f_1(\gamma,\delta;u)]}{f_1(\gamma,\delta;u)} \\ \frac{[\partial/\partial\delta f_1(\gamma,\delta;u)][\partial/\partial\gamma f_1(\gamma,\delta;u)]}{f_1(\gamma,\delta;u)} & \frac{[\partial/\partial\delta f_1(\gamma,\delta;u)]^2}{f_1(\gamma,\delta;u)} \end{bmatrix} du,$$

where $f_1(\gamma,\delta;u) = \frac{(\gamma/\pi)}{\gamma^2+(u-\delta)^2}$. The above integrals can be computed either analytically or numerically and the matrix is easily inverted to give the corresponding Cramér–Rao bounds.

In the following Table 4, we compare the Cramér–Rao bounds, the variance (in brackets) of the maximum likelihood estimators, and the variance (in brackets) of the proposed estimators, as the latter two have been computed via 1,000 Monte–Carlo runs. For the maximum likelihood and the proposed estimators, we also give the mean. As test parameters, we assumed $\gamma = 1$ and 10, $\delta = 1$, and total number of observations $N = 100$ and the fractional moment in the dispersion estimator was of order $p = 1/4$. From Table 4, it becomes clear that certain difficulties that were encountered in Section 3 in the Monte–Carlo simulations are *not* due to a low efficiency in the proposed estimators, but are inherent in the nature of this parameter estimation problem. In fact, the proposed estimators have an efficiency quite close to either the efficiency of maximum likelihood estimators or the corresponding Cramér–Rao bounds.

Table 4:

	Performance of Estimator $\hat{\delta}$		**Performance of Estimator $\hat{\gamma}$**	
	$\gamma = 1$	$\gamma = 10$	$\gamma = 1$	$\gamma = 10$
C–R bound	(0.0200)	(2.0001)	(0.0209)	(2.0085)
ML	0.9968 (0.0205)	0.9276 (2.0881)	1.0054 (0.0207)	9.9691 (2.0925)
Proposed	0.9969 (0.0251)	0.9539 (2.2374)	0.9996 (0.0334)	9.9400 (3.3871)

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