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by

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Abstract

This report presents a new subspace-based method for bearing estimation in the presence of impulsive noise which can be modeled as a complex symmetric alpha-stable ($S\alpha S$) process. We define the covariation matrix of the array sensor outputs and show that eigendecomposition-based methods, such as the MUSIC algorithm, can be applied to the sample covariation matrix to extract the bearing information from the measurements. Consistent estimators for the covariation matrix are presented, and their asymptotic performance is studied through both theory and simulations. The improved performance of the proposed source localization method in the presence of a wide range of impulsive noise environments is demonstrated via Monte Carlo experiments.

Key Words: Stable processes, array processing, direction of arrival, subspace techniques

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I. INTRODUCTION

Statistical array processing based on the linear theory of random processes with finite second-order moments has been the focus of considerable academic research. Critical problems such as high-resolution direction finding, null- and beam-steering, and detection of the number of sources illuminating an array of sensors have been studied under the assumption of a Gaussian or second-order model. Many different classes of methods, compromising optimality for the sake of computational efficiency, have been proposed under the aforementioned statistical framework [1].

Looking toward real world applications, we are interested in developing array processing methods for a larger class of random processes which include the Gaussian processes as special elements. The availability of such methods would make it possible to operate in environments which differ from Gaussian environments in significant ways.

The class of stable distributions has some important characteristics which make it very attractive for modeling impulsive noise environments. Stable processes satisfy the stability property which states that linear combinations of jointly stable variables are indeed stable. They arise as limiting processes of sums of independent, identically-distributed random variables via the generalized central limit theorem. They are described by their characteristic exponent α, taking values $0 < \alpha \leq 2$. Gaussian processes are stable processes with $\alpha = 2$. Stable distributions have heavier tails than the normal distribution, possess finite pth order moments only for $p < \alpha$, and are appropriate for modeling noise with outliers. The main reason for the difficulty in developing signal processing methods based on stable processes is due to the fact that the linear space of a stable process is not a Hilbert space, as in the case of Gaussian processes, but either a Banach ($1 < \alpha < 2$) or a metric space ($0 < \alpha \leq 1$) both of which are more unyielding in their structure.

Recently, it has been shown that the improved performance gained by designing signal processing algorithms in an alpha-stable framework justifies the mathematical and computational complexity involved. In [2] we dealt with optimal approaches (optimal in the Maximum Likelihood (ML) sense) to the direction of arrival (DOA) problem in the presence of impulsive noise. The analysis was based on the assumption that the additive noise could be modeled as a complex *symmetric α-stable ($S\alpha S$)* process. The optimal ML techniques employed in [2] are often regarded as exceedingly complex due to the high computational load of the multivariate nonlinear optimization problem involved with these techniques. Hence, sub-optimal methods need to be developed for the solution of the DOA problem in the presence of impulsive noise, when reduced computational cost is a crucial design requirement.

In this report, we present subspace techniques for the source localization problem, techniques which are based on the geometrical properties of the data model. Considerable research has been done in this area under the framework of Gaussian distributed signals and/or noise. The better known of the so-called eigendecomposition-based methods are the MUSIC [3], Minimum Norm [4]-[5], and the ESPRIT method [6]. These methods estimate the bearings of the source signals by performing an eigendecomposition on the spatial covariance matrix of the array sensor outputs. Studies concerning the statistical efficiency of the most popular eigendecomposition-based method, namely the MUSIC algorithm, have been done in [7]-[8]. The relationship between the MUSIC and ML methods has also been studied in [7]. Since $S\alpha S$ processes do not possess finite pth order moments for $p \geq \alpha$, traditional subspace techniques employing second- and higher-order moments [9] cannot be applied in impulsive noise environments modeled under the stable law. Instead, properties of fractional lower-order moments (FLOM's) and covariations should be used.

This report extends the subspace-based techniques for bearing estimation to processes with finite moments of order p ($p < 2$) and to complex isotropic $S\alpha S$ processes. The report is organized as follows: In section II, we formulate the bearing estimation problem and we give a brief review of second-order subspace-based techniques. Also, we present some necessary introduction on α-stable processes. In section III, we discuss the development of subspace techniques in the presence of α-stable distributed signals and noise. Our analysis is based on the formulation of the covariation matrix of the array sensor outputs. Finally, simulation experiments are presented in section IV, and conclusions are drawn in section V.

II. Background

A. Problem Formulation

Consider an array of r sensors with arbitrary locations and arbitrary directional characteristics, that receive signals generated by q narrow-band sources with known center frequency ω and locations $\theta_1, \theta_2, \ldots, \theta_q$. Since the signals are narrow-band, the propagation delay across the array is much smaller than the reciprocal of the signal bandwidth, and it follows that, by using a complex envelop representation, the array output can be expressed as

$$\mathbf{x}(t) = \sum_{k=1}^{q} \mathbf{a}(\theta_k) s_k(t) + \mathbf{n}(t), \tag{1}$$

where

- $\mathbf{x}(t) = [x_1(t), \ldots, x_r(t)]^T$ is the vector of the signals received by the array sensors
- $s_k(t)$ is the signal emitted by the kth source as received at the reference sensor 1 of the array
- $\mathbf{a}(\theta_k) = [1, e^{-\jmath\omega\tau_2(\theta_k)}, \ldots, e^{-\jmath\omega\tau_r(\theta_k)}]^T$ is the steering vector of the array toward direction θ_k
- $\tau_i(\theta_k)$ is the propagation delay between the first and the ith sensor for a waveform coming from direction θ_k
- $\mathbf{n}(t) = [n_1(t), \ldots, n_r(t)]^T$ is the noise vector

(1) can be expressed in a compact form as

$$\mathbf{x}(t) = \mathbf{A}(\boldsymbol{\theta})\mathbf{s}(t) + \mathbf{n}(t), \tag{2}$$

where $\mathbf{A}(\boldsymbol{\theta})$ is the $r \times q$ matrix of the array steering vectors

$$\mathbf{A}(\boldsymbol{\theta}) = [\mathbf{a}(\theta_1), \ldots, \mathbf{a}(\theta_q)], \tag{3}$$

and $\mathbf{s}(t)$ is the $q \times 1$ vector of the signals

$$\mathbf{s}(t) = [s_1(t), \ldots, s_q(t)]^T. \tag{4}$$

Assuming that M snapshots are taken at time instants $t_1, \ldots, t_M$, the data can be expressed as

$$\mathbf{X} = \mathbf{A}(\boldsymbol{\theta})\mathbf{S} + \mathbf{N}, \tag{5}$$

where $\mathbf{X}$ and $\mathbf{N}$ are the $r \times M$ matrices

$$\mathbf{X} = [\mathbf{x}(t_1), \ldots, \mathbf{x}(t_M)], \tag{6}$$

$$\mathbf{N} = [\mathbf{n}(t_1), \ldots, \mathbf{n}(t_M)], \tag{7}$$

and $\mathbf{S}$ is the $q \times M$ matrix

$$\mathbf{S} = [\mathbf{s}(t_1), \ldots, \mathbf{s}(t_M)]. \tag{8}$$

Our objective is to estimate the directions of arrival $\theta_1, \ldots, \theta_q$ of the sources from the M snapshots of the array $\mathbf{x}(t_1), \ldots, \mathbf{x}(t_M)$.

B. Subspace-Based DOA Methods – The MUSIC Algorithm

In this section, assume that the q ($q < r$) signal waveforms are non-coherent, complex Gaussian random processes with covariance matrix $\mathbf{\Sigma}$. Also, the noise vector $\mathbf{n}(t)$ is a complex Gaussian random process, independent of the signals, with covariance matrix $\sigma^2\mathbf{I}$.

The covariance matrix of the observation vector $\mathbf{x}(t)$ is given by

$$\mathbf{R} = E\{\mathbf{x}(t)\mathbf{x}^H(t)\} = \mathbf{A}(\boldsymbol{\theta})\mathbf{\Sigma}\mathbf{A}^H(\boldsymbol{\theta}) + \sigma^2\mathbf{I}, \tag{9}$$

where $E\{\cdot\}$ is the expectation operator. The $q \times q$ matrix $\mathbf{\Sigma}$ has full rank since we assumed non-coherence of the q incoming plane waves. If the the steering vectors are linearly independent, it follows that matrix $\mathbf{A}(\boldsymbol{\theta})$ has full rank q.

MUSIC belongs to the class of spatial spectral estimation techniques which are based on the eigendecomposition of the covariance matrix $\mathbf{R}$. The rationale behind this approach is that one wants to emphasize the choices for the steering vector $\mathbf{a}(\theta)$, which correspond to signal directions. The method exploits the property that the directions of arrival determine the eigenstructure of the matrix.

Let $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_r$ denote the eigenvalues of matrix $\mathbf{R}$, and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r$ denote the eigenvalues of matrix $\mathbf{A}(\boldsymbol{\theta})\mathbf{\Sigma}\mathbf{A}^H(\boldsymbol{\theta})$, respectively. Since matrix $\mathbf{A}(\boldsymbol{\theta})$ is of full column rank q, the $(r-q)$ smallest eigenvalues of the matrix $\mathbf{A}(\boldsymbol{\theta})\mathbf{\Sigma}\mathbf{A}^H(\boldsymbol{\theta})$ are equal to zero, and the eigenvalue decomposition of the spatial covariance matrix $\mathbf{R}$ can be written as

$$\mathbf{R} = \sum_{i=1}^{q}[\lambda_i + \sigma^2]\mathbf{v}_i\mathbf{v}_i^H + \sum_{i=q+1}^{r} \sigma^2\mathbf{v}_i\mathbf{v}_i^H, \tag{10}$$

where $\{\mathbf{v}_i\}_{i=1}^r$ are the orthogonal eigenvectors of the matrix $\mathbf{R}$. It can be easily proven that the subspace spanned by the eigenvectors $\{\mathbf{v}_{q+1}, \mathbf{v}_{q+2}, \ldots, \mathbf{v}_r\}$ is the orthogonal complement of the subspace spanned by the steering vectors $\{\mathbf{a}(\theta_1), \mathbf{a}(\theta_2), \ldots, \mathbf{a}(\theta_q)\}$. Since the eigenvectors of the covariance matrix $\mathbf{R}$ are orthogonal to each other, then the subspace spanned by the eigenvectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_q\}$ is exactly the same as the subspace spanned by the vectors $\{\mathbf{a}(\theta_1), \mathbf{a}(\theta_2), \ldots, \mathbf{a}(\theta_q)\}$. It follows that if the field contains q distinct non-coherent propagating signals in a spatially white noise environment, then the eigenvalue decomposition of the spatial covariance matrix $\mathbf{R}$ results in the formation of two disjoint subspaces that are the orthogonal complement of each other. The first one, called the *signal plus noise subspace,* is spanned by the eigenvectors corresponding to the q largest eigenvalues of $\mathbf{R}$. The second, called the *noise subspace* is spanned by the eigenvectors corresponding to the $r - q$ smallest eigenvalues of $\mathbf{R}$. Given the eigenvectors of $\mathbf{R}$ we may determine the signal directions of arrival by searching for the steering vectors $\mathbf{a}(\theta)$ which are orthogonal to the noise subspace.

In practice, $\mathbf{R}$ is unknown, but can be consistently estimated from the available data as

$$\hat{\mathbf{R}} = \frac{1}{M} \sum_{t=1}^{M} \mathbf{x}(t)\mathbf{x}(t)^H. \tag{11}$$

Because of the uncertainty in the eigenvector estimates $\{\hat{\mathbf{v}}_{q+1}, \hat{\mathbf{v}}_{q+2}, \ldots, \hat{\mathbf{v}}_r\}$ introduced by the way the matrix $\mathbf{R}$ is estimated, one can only search for the steering vectors that are most closely orthogonal to the noise subspace. The MUSIC algorithm estimates the signal directions as the peaks of the MUSIC spatial spectrum estimator given by

$$P_{MUSIC}(\theta) = \frac{1}{\sum_{i=q+1}^{r} |\mathbf{a}(\theta)^H \hat{\mathbf{v}}_i|^2}. \tag{12}$$

C. Mathematical Preliminaries: Complex $S\alpha S$ Random Variables

A complex random variable (r.v.) $X = X_1 + \jmath X_2$ is symmetric α-stable ($S\alpha S$) if X_1 and X_2 are jointly $S\alpha S$ and then its characteristic function is written as

$$\begin{aligned} \varphi(\omega) &= E\{\exp[\jmath\Re(\omega X^*)]\} = E\{\exp[\jmath(\omega_1 X_1 + \omega_2 X_2)]\} \\ &= \exp\left[-\int_{S_2} |\omega_1 x_1 + \omega_2 x_2|^\alpha d\Gamma_{X_1,X_2}(x_1, x_2)\right], \end{aligned} \tag{13}$$

where $\omega = \omega_1 + \jmath\omega_2$, $\Re[\cdot]$ is the real part operator, and Γ_{X_1,X_2} is a symmetric measure on the unit sphere S_2, called the *spectral measure* of the random variable X. The *characteristic exponent* α is restricted to the values $0 < \alpha \leq 2$ and it determines the shape of the distribution. The smaller the characteristic exponent α, the heavier the tails of the density.

A complex random variable $X = X_1 + \jmath X_2$ is *isotropic* if and only if (X_1, X_2) has a uniform spectral measure. In this case, the characteristic function of X can be written as

$$\varphi(\omega) = E\{\exp(\jmath\Re[\omega X^*])\} = \exp(-\gamma|\omega|^\alpha), \tag{14}$$

where γ ($\gamma > 0$) is the *dispersion* of the distribution. The dispersion plays a role analogous to the role that the variance plays for second-order processes. Namely, it determines the spread of the probability density function around the origin. A method for generating complex isotropic $S\alpha S$ random variables is presented in Appendix A.

Several complex r.v.'s are jointly $S\alpha S$ if their real and imaginary parts are jointly $S\alpha S$. When $X = X_1 + \jmath X_2$ and $Y = Y_1 + \jmath Y_2$ are jointly $S\alpha S$ with $1 < \alpha \leq 2$, the *covariation* of X and Y is defined by

$$[X, Y]_\alpha = \int_{S_4} (x_1 + \jmath x_2)(y_1 + \jmath y_2)^{<\alpha-1>} d\Gamma_{X_1,X_2,Y_1,Y_2}(x_1, x_2, y_1, y_2), \tag{15}$$

where we use throughout the convention

$$Y^{<\beta>} = |Y|^{\beta-1} Y^*. \tag{16}$$

It can be shown that for every $1 \leq p < \alpha$, the covariation can be expressed as a function of moments [10]

$$[X, Y]_\alpha = \frac{E\{XY^{<p-1>}\}}{E\{|Y|^p\}} \gamma_Y, \tag{17}$$

where γ_Y is the dispersion of the r.v. Y given by

$$\gamma_Y^{p/\alpha} = \frac{E\{|Y|^p\}}{C(p,\alpha)} \qquad \text{for } 0 < p < \alpha, \tag{18}$$

with

$$C(p, \alpha) = \frac{2^{p+1}\Gamma(\frac{p+2}{2})\Gamma(-\frac{p}{\alpha})}{\alpha\Gamma(\frac{1}{2})\Gamma(-\frac{p}{2})}, \tag{19}$$

where $\Gamma(\cdot)$ is the gamma function defined by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt. \tag{20}$$

Obviously, from (17) it holds that

$$[X, X]_\alpha = \gamma_X. \tag{21}$$

Also, the *covariation coefficient* of X and Y is defined by

$$\lambda_{X,Y} = \frac{[X, Y]_\alpha}{[Y, Y]_\alpha}, \tag{22}$$

and by using (17) it can be expressed as

$$\lambda_{X,Y} = \frac{E\{XY^{<p-1>}\}}{E\{|Y|^p\}} \qquad \text{for } 1 \leq p < \alpha. \tag{23}$$

The covariation of complex jointly $S\alpha S$ r.v.'s is not generally symmetric and has the following properties [11]:

P1 If X_1, X_2 and Y are jointly $S\alpha S$, then

$$[aX_1 + bX_2, Y]_\alpha = a[X_1, Y]_\alpha + b[X_2, Y]_\alpha \tag{24}$$

for any complex constants a and b.

P2 If Y_1 and Y_2 are independent and X_1, X_2 and Y are jointly $S\alpha S$, then

$$[aX_1, bY_1 + cY_2]_\alpha = ab^{<\alpha-1>}[X_1, Y_1]_\alpha + ac^{<\alpha-1>}[X_1, Y_2]_\alpha \tag{25}$$

for any complex constants a, b and c.

P3 If X and Y are independent $S\alpha S$, then $[X, Y]_\alpha = 0$.

III. Subspace Techniques in the α-Stable Framework

A. The Array Covariation Matrix

In this section, we assume that the q signal waveforms are non-coherent, complex isotropic $S\alpha S$ ($1 < \alpha \leq 2$) random processes with covariation matrix $\mathbf{\Gamma}_S = \mathrm{diag}(\gamma_{s_1}, \ldots, \gamma_{s_q})$. Also, the noise vector $\mathbf{n}(t)$ is a complex isotropic $S\alpha S$ random process with the same characteristic exponent α as the signals. The noise is assumed to be independent of the signals with covariation matrix $\mathbf{\Gamma}_N = \gamma_n \mathbf{I}$.

(2) can be written as

$$\mathbf{x}(t) = \mathbf{w}(t) + \mathbf{n}(t), \tag{26}$$

where $\mathbf{w}(t) = \mathbf{A}(\boldsymbol{\theta})\mathbf{s}(t)$. By the stability property, it follows that $\mathbf{w}(t)$ is also a complex isotropic $S\alpha S$ random vector with components

$$w_i(t) = \mathbf{A}_i(\boldsymbol{\theta})\mathbf{s}(t) = a_i(\theta_1)s_1(t) + \cdots + a_i(\theta_q)s_q(t) \quad i = 1, \ldots, r. \tag{27}$$

Also, it holds that $\mathbf{w}(t)$ is independent of $\mathbf{n}(t)$.

Now, we define the *covariation matrix* of the observation vector process $\mathbf{x}(t)$ as the matrix whose elements are the covariations $[x_i(t), x_j(t)]_\alpha$ of the components of $\mathbf{x}(t)$. We have that

$$\begin{aligned} [x_i(t), x_j(t)]_\alpha &= [w_i(t) + n_i(t), w_j(t) + n_j(t)]_\alpha \\ &= [w_i(t), w_j(t)]_\alpha + [w_i(t), n_j(t)]_\alpha + [n_i(t), w_j(t)]_\alpha + [n_i(t), n_j(t)]_\alpha. \end{aligned} \tag{28}$$

By the independence assumption of $\mathbf{w}(t)$ and $\mathbf{n}(t)$ and by property P3 we have that

$$[w_i(t), n_j(t)]_\alpha = 0, \tag{29}$$

and

$$[n_i(t), w_j(t)]_\alpha = 0. \tag{30}$$

Also, by using (27) and properties P1 and P2 it follows that

$$\begin{aligned} [w_i(t), w_j(t)]_\alpha &= [\sum_{k=1}^{q} a_i(\theta_k) s_k(t), w_j(t)]_\alpha \\ &= \sum_{k=1}^{q} a_i(\theta_k)[s_k(t), w_j(t)]_\alpha \\ &= \sum_{k=1}^{q} a_i(\theta_k)[s_k(t), \sum_{l=1}^{q} a_j(\theta_l) s_l(t)]_\alpha \\ &= \sum_{k=1}^{q} a_i(\theta_k) a_j^{<\alpha-1>}(\theta_k)\gamma_{s_k}, \end{aligned} \tag{31}$$

where $\gamma_{s_k} = [s_k, s_k]_\alpha$. Finally, due to the noise assumption made earlier, it holds that

$$[n_i(t), n_j(t)]_\alpha = \gamma_n \delta_{i,j}, \tag{32}$$

where $\delta_{i,j}$ is the Kronecker delta function. Combining (28)-(32) we obtain the following expression for the covariations of the sensor measurements:

$$[x_i(t), x_j(t)]_\alpha = \sum_{k=1}^{q} a_i(\theta_k) a_j^{<\alpha-1>}(\theta_k)\gamma_{s_k} + \gamma_n \delta_{i,j} \quad i, j = 1, \ldots, r. \tag{33}$$

Also, the dispersion and covariation coefficients of the array sensor measurements are given respectively by

$$\gamma_{x_j(t)} = \sum_{k=1}^{q} |a_j(\theta_k)|^\alpha \gamma_{s_k} + \gamma_n \quad j = 1, \ldots, r, \tag{34}$$

and

$$\lambda_{x_i(t), x_j(t)} = \frac{\sum_{k=1}^{q} a_i(\theta_k) a_j^{<\alpha-1>}(\theta_k)\gamma_{s_k} + \gamma_n \delta_{i,j}}{\sum_{k=1}^{q} |a_j(\theta_k)|^\alpha \gamma_{s_k} + \gamma_n} \quad i, j = 1, \ldots, r. \tag{35}$$

In matrix form, (33) gives the following expression for the covariation matrix of the observation vector:

$$[\mathbf{x}(t), \mathbf{x}(t)]_\alpha = \mathbf{A}(\boldsymbol{\theta})\boldsymbol{\Gamma}_S\mathbf{A}^{<\alpha-1>}(\boldsymbol{\theta}) + \gamma_n\mathbf{I}, \tag{36}$$

where the (i,j)th element of matrix $\mathbf{A}^{<\alpha-1>}(\boldsymbol{\theta})$ results from the (j,i)th element of $\mathbf{A}(\boldsymbol{\theta})$ according to the operation

$$[\mathbf{A}^{<\alpha-1>}(\boldsymbol{\theta})]_{i,j} = [\mathbf{A}(\boldsymbol{\theta})]_{j,i}^{<\alpha-1>} = |\,[\mathbf{A}(\boldsymbol{\theta})]_{j,i}\,|^{\alpha-2}\,[\mathbf{A}(\boldsymbol{\theta})]_{j,i}^{*} \tag{37}$$

Clearly, when $\alpha = 2$, i.e., for Gaussian distributed signals and noise, the expression for the covariation matrix is reduced to the form shown in (9).

When the amplitude response of the sensors equals unity, i.e., for steering vectors of the form $\mathbf{a}(\theta_k) = [1, e^{-\jmath\omega\tau_2(\theta_k)}, \ldots, e^{-\jmath\omega\tau_r(\theta_k)}]^T$, it follows that

$$[\mathbf{A}^{<\alpha-1>}(\boldsymbol{\theta})]_{i,j} = |e^{-\jmath\omega\tau_j(\theta_i)}|^{\alpha-2}e^{\jmath\omega\tau_j(\theta_i)} = [\mathbf{A}(\boldsymbol{\theta})]_{j,i}^{*}, \tag{38}$$

and thus the covariation matrix can be written as

$$[\mathbf{x}(t), \mathbf{x}(t)]_\alpha = \mathbf{A}(\boldsymbol{\theta})\boldsymbol{\Gamma}_S\mathbf{A}^{H}(\boldsymbol{\theta}) + \gamma_n\mathbf{I}. \tag{39}$$

Also, from (34) and (35) the dispersion and covariation coefficients of the array sensor measurements can be written as

$$\gamma_{x_j(t)} = \sum_{k=1}^{q}\gamma_{s_k} + \gamma_n \quad j = 1, \ldots, r, \tag{40}$$

and

$$\lambda_{x_i(t),x_j(t)} = \frac{\sum_{k=1}^{q} a_i(\theta_k)a_j^*(\theta_k)\gamma_{s_k} + \gamma_n\delta_{i,j}}{\sum_{k=1}^{q}\gamma_{s_k} + \gamma_n} \quad i,j = 1, \ldots, r. \tag{41}$$

Hence, in this case, we can apply the subspace techniques described in section II.B, to the covariation or the covariation coefficient matrices of the observation vector to extract the bearing information. Since several estimators have been proposed for the covariation coefficient of two $S\alpha S$ random variables, in the following we will use the covariation coefficient matrix of the array sensor measurements to estimate the bearings of the sources.

We will refer to the new algorithm resulting from the eigendecomposition of the array covariation coefficient matrix as the **Robust Covariation-Based MUSIC** or **ROC-MUSIC**. In practice, we have to estimate the covariation matrix from a finite number of array sensor measurements. The following section describes one such estimator, based on fractional lower order moments of the stable process, and studies its asymptotic statistical properties.

B. Covariation Estimators

A proposed estimator for the covariation coefficient $\lambda_{X,Y}$ defined in (22), is called the *fractional lower order (FLOM) estimator* and is given by [12]

$$\hat{\lambda}_{X,Y} = \frac{\sum_{i=1}^{n} X_i Y_i^{<p-1>}}{\sum_{i=1}^{n} |Y_i|^p} \tag{42}$$

for some $1 \leq p < \alpha$ and independent observations $(X_1, Y_1), \ldots, (X_n, Y_n)$. Unfortunately, this estimator, although it is unbiased, it has very large variance. To circumvent this difficulty we introduce the *modified covariation coefficient function*

$$\lambda_{X,Y}(p) = \frac{E\{XY^{<p-1>}\}}{E\{|Y|^p\}}, \tag{43}$$

for $1/2 < p < \alpha$ if X and Y are real $S\alpha S$ random variables, and $0 < p < \alpha$ if X and Y are complex isotropic $S\alpha S$ random variables. This modified covariation coefficient function is well defined (finite) for the aforementioned values of the parameter p as follows from the following proposition:

Proposition 1 *Let X and Y be jointly $S\alpha S$ random variables. It holds that $E\{|X||Y|^{p-1}\} < \infty$ and $E\{|X|^2|Y|^{2p-2}\} < \infty$ if $1/2 < p < \alpha/2$ $(0 < p < \alpha/2)$ for X and Y real (isotropic complex.)*

Proof See Appendix B.

Clearly, when $1 \leq p < \alpha$ the function $\lambda_{X,Y}(p)$ equals the covariation coefficient $\lambda_{X,Y}$ as defined in (22)-(23). The fractional lower order estimator (MFLOM) for the modified covariation coefficient function $\lambda_{X,Y}(p)$ is given by

$$\hat{\lambda}_{X,Y}(p) = \frac{\sum_{i=1}^{n} X_i Y_i^{<p-1>}}{\sum_{i=1}^{n} |Y_i|^p} \tag{44}$$

for independent observations $(X_1, Y_1), \ldots, (X_n, Y_n)$. The theoretical performance of this estimator is given by the following theorem:

Theorem 1 *The estimator $\hat{\lambda}_{X,Y}(p)$ given by (44) is consistent and asymptotically normal with mean equal to the true modified covariation coefficient function $\lambda_{X,Y}(p)$, and variance*

$$\begin{aligned}
\frac{1}{n}\sigma_{\hat{\lambda}}^2(p) \;=\; & \frac{1}{n}\Bigg[\left[E\{|X|^2|Y|^{2p-2}\} - |E\{XY^{<p-1>}\}|^2\right]\frac{1}{(E\{|Y|^p\})^2} - \\
& 2\Re\left\{\left[E\{XY^{<p-1>}|Y|^p\} - E\{XY^{<p-1>}\}E\{|Y|^p\}\right]\frac{E\{XY^{<p-1>}\}}{(E\{|Y|^p\})^3}\right\} + \\
& \left[E\{|X|^{2p}\} - (E\{|Y|^p\})^2\right]\left(\frac{|E\{XY^{<p-1>}\}|}{(E\{|Y|^p\})^2}\right)^2\Bigg],
\end{aligned} \tag{45}$$

where $\Re[z]$ denotes the real part of z and p varies in the range $0 < p < \alpha/2$ for the complex isotropic case. In notational form,

$$\sqrt{n}(\hat{\lambda}_{X,Y}(p) - \lambda_{X,Y}(p)) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \sigma_\lambda^2(p)), \tag{46}$$

where $\xrightarrow{\mathcal{L}}$ denotes convergence in distribution, and $\mathcal{N}(\mu, \sigma^2)$ is the normal distribution with mean μ and variance σ^2.

Proof See Appendix C.

For the case of real jointly $S\alpha S$ random variables the asymptotic variance of the estimator is finite for values of p in the range $1/2 < p < \alpha/2$. The simulation experiments in the following section give some significant insight on the performances of the MFLOM estimator and the proposed ROC-MUSIC algorithm.

IV. Simulations

We performed three simulation experiments to assess the performance of the MFLOM estimator and to compare the MUSIC and ROC-MUSIC algorithms. The first experiment illustrates the behavior of the MFLOM estimator of the covariation coefficient as a function of p and compares MFLOM with the least-squares and screened ratio estimators. The second and third experiments study the performance of the MUSIC and ROC-MUSIC algorithms in the presence of simulated $S\alpha S$ noise and real radar clutter, respectively. The improved performance of the ROC-MUSIC method in terms of both bias as well as mean square error is apparent in the simulation results.

A. Experiment #1

The purpose of this experiment is to study the influence of the parameter p to the performance of the MFLOM estimator of the covariation coefficient. Two real $S\alpha S$ ($1 < \alpha \leq 2$) random variables, X and Y, are defined as

$$X = a_{11}U_1 + a_{12}U_2,$$

$$Y = a_{21}U_1 + a_{22}U_2,$$

where U_1, and U_2 are independent, $S\alpha S$ random variables. The model coefficients $\{a_{ij};\ \ i,j = 1,2\}$ are given by

$$[a_{ij}] = \begin{bmatrix} -0.75 & 0.25 \\ 0.18 & 0.78 \end{bmatrix}. \tag{47}$$

It follows from (35) that the true covariation coefficient λ of X with Y is

$$\lambda = \frac{\sum_{j=1}^{2} a_{1j} a_{2j}^{<\alpha-1>}}{\sum_{j=1}^{2} |a_{2j}|^{\alpha}}. \tag{48}$$

We generated $n = 5,000$ independent samples of U_1, U_2 and U_3 and we calculated the MFLOM estimator by means of the expression

$$\hat{\lambda}_{MFLOM}(p) = \frac{\sum_{i=1}^{n} X_i Y_i^{<p-1>}}{\sum_{i=1}^{n} |Y_i|^{p}}$$

for different values of p in the range [0,2]. We run $K = 1,000$ Monte Carlo experiments and compared the performance of the MFLOM estimator to that of the least-squares and the screened ratio estimators. The two later estimators are defined as follows:

Least-squares estimator:

$$\hat{\lambda}_{LS} = \frac{\sum_{i=1}^{n} X_i Y_i}{\sum_{i=1}^{n} |Y_i|^2}$$

Screened ratio estimator [13]:

$$\hat{\lambda}_{SR} = \frac{\sum_{i=1}^{n} X_i Y_i^{-1} \chi_{Y_i}}{\sum_{i=1}^{n} \chi_{Y_i}},$$

where

$$\chi_Y = \begin{cases} 1 & \text{if } c_1 < |Y| < c_2 \\ 0 & \text{otherwise} \end{cases}$$

In [13] it is shown that the least-squares estimator $\hat{\lambda}_{LS}$ is not consistent for $1 < \alpha < 2$ while the screened ratio estimator $\hat{\lambda}_{SR}$ is strongly consistent.

Figure 1 shows the standard deviation of the MFLOM estimator of the modified covariation coefficient as a function of the parameter p and for different values of the characteristic exponent α. As we can see, for the case of non-Gaussian stable signals ($1 < \alpha < 2$), the values of p in the range $(1/2, \alpha/2)$ result into the smallest standard deviations. For Gaussian signals the optimal value of p is 2 and the resulting MFLOM estimator is simply the least-squares estimator, as expected.

In Figures 2-3 we plot the MFLOM estimates for $\alpha = 1.5$ and several values of p. We also plot the running sample variance estimate S_k^2 defined as follows:

$$S_k^2 = \frac{1}{k-1}\sum_{i=1}^{k}(\hat{\lambda}_{MFLOM}^{(i)}(p) - \bar{\hat{\lambda}}_{MFLOM}^{(k)}(p))^2 \quad k = 1, \ldots, K, \tag{49}$$

where

$$\bar{\hat{\lambda}}_{MFLOM}^{(k)}(p) = \frac{1}{k}\sum_{i=1}^{k}\hat{\lambda}_{MFLOM}^{(i)}(p). \tag{50}$$

If the population of the MFLOM estimates $\{\hat{\lambda}_{MFLOM}^{(i)}(p)\}_{i=1}^{K}$ has finite variance, S_k^2 will converge to a finite value (converging variance test.) As we can see, only for the value of p in the range $(1/2, \alpha/2)$, i.e., for $p = 0.7$, is the MFLOM estimator normally distributed with finite variance, which supports the theoretical analysis of section III.B. Figure 4 shows the screened ratio estimates and the corresponding running sample variance.

Table 1 shows comparative results of the three estimators. We include the mean of the estimators, the standard deviation in parentheses, and the value of p for which the smallest standard deviation is achieved by the MFLOM estimator. For the screened ratio estimator, we chose $c_1 = 0.1$ and $c_2 = \infty$. Clearly, for $\alpha \neq 2$, the least-squares estimator has large standard deviation compared to the other two estimators. For the Gaussian case ($\alpha = 2$), the least-squares estimator is optimum. The screened ratio estimator exhibits small bias and standard deviation. On the other hand, the MFLOM estimator gives the smallest bias and standard deviation of all three estimators for a choice of the parameter p in the range $(1/2, \alpha/2)$. For such values of p the MFLOM estimator of the modified covariation coefficient function converges to the true covariation coefficient calculated by (48). Hence, we can conjecture that (22) holds for $1/2 < p < \alpha$ in the case of real $S\alpha S$ random variables.

B. Experiment #2

In this experiment we compare the performances of the proposed ROC-MUSIC algorithm to MUSIC. The sample covariation coefficient matrix (SCCM), as estimated by (42), is not symmetric and hence it has complex eigenvalues in general. The more snapshots are available at the array sensors, the more nearly symmetric SCCM becomes. We come around this problem by performing the eigenvalue decomposition to the sum of the sample covariation coefficient matrix and its conjugate.

The array is linear with five sensors spaced a half-wavelength apart. A single source is positioned at 20°. The noise is assumed to follow the complex isotropic $S\alpha S$ distribution with dispersion $\gamma = 1$. The number of snapshots available to the algorithms is $M = 50$. In every experiment we perform 100 Monte-Carlo runs and compute the bias and the standard deviation of the direction-of-arrival estimates. We use two values of the parameter p in the estimation of the covariation (c.f. (44)): $p = 0.8$ and $p = 0.4$.

The results are depicted in Figure 5. All the curves are functions of the characteristic exponent α of the noise. Clearly, for very impulsive environments ($\alpha < 1.6$) MUSIC exhibits very large bias and standard deviation. On the other hand, for less impulsive noise ($\alpha > 1.8$), MUSIC and ROC-MUSIC have comparable performances. Comparing the ROC-MUSIC curves in Figure 5, we see that for $\alpha < 1.4$ the choice of $p = 0.4$ gives smaller standard deviation while for $\alpha > 1.4$ the choice of $p = 0.8$ gives better results. This is consistent with the results of the previous experiment and shows that values of p in the range $(1/2, \alpha/2)$ give estimators with smaller variances.

C. Experiment #3

The proposed algorithm has been tested with real radar sea clutter data provided by the Naval Surface Warfare Center, Carderock Division, Bethesda, Maryland. Clutter is a group of unwanted radar returns due to scattering centers such as precipitation, birds, and ocean waves. The received clutter signal can be represented in terms of its *in-phase* (I) and *quadrature* (Q) components. A typical sample set of the sea clutter data is shown in Figure 6. The spiky nature of these radar returns is obvious, and it has been shown, using the algorithms developed in [14], that they can be modeled as $S\alpha S$ processes with $\alpha = 1.85$ and $\gamma = 0.19$.

We repeated the second experiment in the presence of radar clutter and the results are shown in Figure 7. All the curves are functions of the signal power over the noise dispersion in dB, i.e., they are functions of a so-called pseudo signal-to-noise ratio $PSNR = 10\log(\frac{1}{\gamma M}\sum_{t=1}^{M}|s(t)|^2)$. For the ROC-MUSIC we used $p = 0.85$ in the expression for the FLOM estimator of the covariations. As we can see, the ROC-MUSIC slightly outperforms the MUSIC. The figure also shows the performances of the Maximum Likelihood method based on the Gaussian noise assumption (MAX. LIK. - Gaussian), and the Maximum Likelihood method based on the Cauchy noise assumption (MAX. LIK. - Cauchy) introduced in [2]. As demonstrated in [2], the Cauchy beamformer exhibits very small bias and estimation variance even for very low PSNR values. Figure 7 demonstrates the major trade-off between optimal maximum likelihood-based and sub-optimal subspace-based techniques. Namely, it demonstrates the trade-off between statistical efficiency and computational complexity: The subspace-based methods involve less computation than the ML methods but exhibit very large bias and variance for low signal-to-noise ratios.

V. Concluding Remarks

We have formulated the covariation matrix of the array outputs for the case of complex isotropic $S\alpha S$ signals and noise. We showed that for the special case of array sensors with unit amplitude response, the covariation matrix has similar form to the covariance matrix of Gaussian distributed signals. Therefore, subspace-based bearing estimation techniques can be applied to the covariation matrix resulting to improved direction of arrival estimates in the presence of impulsive noise environments. These techniques assume knowledge of the number of the sources generating the signals that illuminate the antenna array. In many practical situations, this knowledge may not be available a priori. Hence, future research includes the development of methods for the detection of the number of signals in the presence of impulsive noise based on application of information-theoretic criteria. Finally, we will address the problem of detecting and localizing multiple wide-band sources in impulsive noise environments.

Appendix A. Generation of Complex Isotropic $S\alpha S$ Random Variables

The generation of complex isotropic $S\alpha S$ deviates of characteristic exponent α is based on the following proposition [15]:

Proposition 2 *A complex $S\alpha S$ ($\alpha < 2$) random variable $X = X_1 + \jmath X_2$ is isotropic if and only if there exist two i.i.d. zero-mean Gaussian variables G_1 and G_2 and a real stable random variable A of characteristic exponent $\alpha/2$, dispersion $\cos^2(\pi\alpha/4)$ and skewness $\beta = 1$, independent of (G_1, G_2) such that $(X_1, X_2) \stackrel{d}{=} (A^{1/2}G_1, A^{1/2}G_2)$.*

We say that the vector (X_1, X_2) is *sub-Gaussian* with underlying vector (G_1, G_2). It can be shown that the real and imaginary parts of X are always dependent, unless G_1 and G_2 are degenerate. Hence, every complex isotropic $S\alpha S$ random variable with $\alpha < 2$ can be expressed as

$$X = A^{1/2}(G_1 + \jmath G_2), \tag{A.1}$$

and its generation involves the generation of a real, totally skewed stable random variable. The problem of generating a real stable deviate is studied in [16]. Here we present the result.

We want to generate a real standard stable random variable $S(\alpha, \beta, 1)$ of characteristic exponent α, skewness β and unit dispersion $\gamma = 1$. The following representations can be deduced:

$$S(\alpha, \beta, 1) = \frac{\sin \alpha(U - U_0)}{(\cos U)^{1/\alpha}} \left(\frac{\cos(U - \alpha(U - U_0))}{W} \right)^{\frac{1-\alpha}{\alpha}}, \quad \text{for } \alpha \neq 1, \tag{A.2}$$

and

$$S(1, \beta, 1) = \frac{2}{\pi} \left[(\frac{\pi}{2} + \beta U) \tan U - \beta \ln(\frac{\frac{\pi}{2} W \cos U}{\frac{\pi}{2} + \beta U}) \right], \tag{A.3}$$

where W is standard exponential with $Pr\{W > w\} = e^{-w}$, $w > 0$, and U is uniform on $(-\frac{\pi}{2}, \frac{\pi}{2})$. Also, $U_0 = -\frac{\pi}{2}\beta[k(\alpha)/\alpha]$ with $k(\alpha) = 1 - |1 - \alpha|$. Then, a stable variate of dispersion γ can be generated as $S(\alpha, \beta, \gamma) = \gamma^{1/\alpha} S(\alpha, \beta, 1)$.

Appendix B. Fractional Lower Order Moments of Products of $S\alpha S$ Random Variables

It is known that if X is a $S\alpha S$ random variable and $p > 0$, then $E\{|X|^p\} < \infty$ if and only if $p < \alpha$ [11]. Also, if $X_1, \ldots, X_n$ are n-fold dependent $S\alpha S$ random variables and $p_1, \ldots, p_n$ are positive numbers, then

$$E\{|X_1|^{p_1} \cdots |X_n|^{p_n}\} < \infty \text{ if and only if } p_i + \cdots + p_n < \alpha. \tag{B.1}$$

Recently, it was proven in [17] that for a real $S\alpha S$ random variable X it holds

$$E\{|X|^p\} < \infty \quad \text{for } -1 < p < \alpha. \tag{B.2}$$

Similarly, for an isotropic complex $S\alpha S$ random variable X it holds

$$E\{|X|^p\} < \infty \quad \text{for } -2 < p < \alpha. \tag{B.3}$$

We now consider the problem of determining a range of values of the parameter p for which $E\{|X||Y|^{p-1}\} < \infty$ and $E\{|X|^2|Y|^{2p-2}\} < \infty$.

Proof of Proposition 1 First consider $E\{|X|^2|Y|^{2p-2}\}$. It follows from (B.1) that $E\{|X|^2|Y|^{2p-2}\} < \infty$ for $p < \alpha/2$, when X and Y are jointly $S\alpha S$ random variables. If μ is the measure induced on R^2 by X and Y, then $E\{|X|^2|Y|^{2p-2}\}$ can be written as

$$E\{|X|^2|Y|^{2p-2}\} = \int_{R^2} |x|^2|y|^{2p-2}d\mu(x,y) =: I_1. \tag{B.4}$$

But $I_1 < \infty$ if and only if [18]

$$I_1' := \int_{R^2} |y|^{2p-2}d\mu(x,y) < \infty \tag{B.5}$$

By using (B.2-B.3) we get that $I_1' < \infty$ if $p > 1/2$ ($p > 0$) when X and Y are real (complex isotropic) $S\alpha S$ random variables. Hence it follows that $E\{|X|^2|Y|^{2p-2}\} < \infty$ if and only if $1/2 < p < \alpha/2$ ($0 < p < \alpha/2$) when X and Y are real (complex isotropic.) Finally, since $E\{|X|^2|Y|^{2p-2}\} < \infty$ implies $E\{|X||Y|^{p-1}\} < \infty$, it follows that for the aforementioned values of p, $E\{|X||Y|^{p-1}\}$ is also finite. ∎

Appendix C. Asymptotic Performance of the MFLOM Estimator

The fractional lower-order (MFLOM) estimator for the modified covariation coefficient function $\lambda_{X,Y}(p)$ given in (44) can be written as the quotient of two statistics:

$$\hat{\lambda}_{X,Y}(p) = \frac{T_{1n}}{T_{2n}} = \frac{\frac{1}{n}\sum_{i=1}^{n} X_i Y_i^{<p-1>}}{\frac{1}{n}\sum_{i=1}^{n} |Y_i|^p}. \tag{C.1}$$

First, we prove the following Lemma for the case of complex isotropic $S\alpha S$ random variables.

Lemma 1 *Given the two-dimensional statistic (T_{1n}, T_{2n}) described in (C.1) with $0 < p < \alpha/2$, the asymptotic distribution (a.d.) of $\sqrt{n}(T_{1n} - \theta_1, T_{2n} - \theta_2)$ is bivariate normal with mean zero and covariance matrix*

$$\mathbf{S} = \frac{1}{n}\begin{bmatrix} E\{|X|^2|Y|^{2p-2}\} - |E\{XY^{<p-1>}\}|^2 & E\{XY^{<p-1>}|Y|^p\} - E\{XY^{<p-1>}\}E\{|Y|^p\} \\ E\{X^*Y^{*<p-1>}|Y|^p\} - E\{X^*Y^{*<p-1>}\}E\{|Y|^p\} & E\{|Y|^{2p}\} - (E\{|Y|^p\})^2 \end{bmatrix}, \tag{C.2}$$

where

$$\theta_1 = E\{XY^{<p-1>}\}, \tag{C.3}$$

and

$$\theta_2 = E\{|Y|^p\}. \tag{C.4}$$

Proof Clearly,

$$E\{T_{1n}\} = \frac{1}{n}\sum_{i=1}^{n} E\{X_i Y_i^{<p-1>}\} = E\{XY^{<p-1>}\}, \tag{C.5}$$

and

$$E\{T_{2n}\} = \frac{1}{n}\sum_{i=1}^{n} E\{|Y_i|^p\} = E\{|Y|^p\}. \tag{C.6}$$

Also,

$$\begin{aligned} E\{T_{1n}T_{1n}^*\} &= \frac{1}{n^2}E\{\sum_{i=1}^{n}\sum_{j=1}^{n} X_i Y_i^{<p-1>} X_j^* Y_j^{*<p-1>}\} \\ &= \frac{1}{n^2}\sum_{i=1}^{n} E\{|X_i Y_i^{<p-1>}|^2\} + \frac{1}{n^2}\underbrace{\sum_{i=1}^{n}\sum_{j=1}^{n}}_{i\neq j} E\{X_i Y_i^{<p-1>}\}E\{X_j^* Y_j^{*<p-1>}\} \\ &= \frac{1}{n}E\{|X|^2|Y|^{2p-2}\} + \frac{n-1}{n}|E\{XY^{<p-1>}\}|^2, \end{aligned} \tag{C.7}$$

and

$$\begin{aligned} E\{T_{2n}T_{2n}^*\} &= \frac{1}{n^2}E\{\sum_{i=1}^{n}\sum_{j=1}^{n} |Y_i|^p|Y_j|^p\} \\ &= \frac{1}{n^2}\sum_{i=1}^{n} E\{|Y_i|^{2p}\} + \frac{1}{n^2}\underbrace{\sum_{i=1}^{n}\sum_{j=1}^{n}}_{i\neq j} E\{|Y_i|^p\}E\{|Y_j|^p\} \\ &= \frac{1}{n}E\{|Y|^{2p}\} + \frac{n-1}{n}(E\{|Y|^p\})^2, \end{aligned} \tag{C.8}$$

where it follows from Proposition 1 that it must be $0 < p < \alpha/2$ so that $E\{|X|^2|Y|^{2p-2}\} < \infty$ and $E\{|Y|^{2p}\} < \infty$. By using (C.5)-(C.8) we have that

$$\begin{aligned} \mathrm{var}(T_{1n}) &= E\{T_{1n}T_{1n}^*\} - |E\{T_{1n}\}|^2 \\ &= \frac{1}{n}\sigma_{T_{1n}}^2 = \frac{1}{n}\left[E\{|X|^2|Y|^{2p-2}\} - |E\{XY^{<p-1>}\}|^2\right] < \infty, \quad 0 < p < \frac{\alpha}{2}, \end{aligned} \tag{C.9}$$

and

$$\begin{aligned} \mathrm{var}(T_{2n}) &= E\{T_{2n}T_{2n}^*\} - |E\{T_{2n}\}|^2 \\ &= \frac{1}{n}\sigma_{T_{2n}}^2 = \frac{1}{n}\left[E\{|Y|^{2p}\} - (E\{|Y|^p\})^2\right] < \infty, \quad 0 < p < \frac{\alpha}{2}. \end{aligned} \tag{C.10}$$

Hence by means of the central limit theorem, we conclude that the statistics T_{1n} and T_{2n} are asymptotically normal:

$$\sqrt{n}(T_{1n} - \theta_1) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \sigma^2_{T_{1n}}), \tag{C.11}$$

and

$$\sqrt{n}(T_{2n} - \theta_2) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \sigma^2_{T_{2n}}). \tag{C.12}$$

Now, for every complex numbers a and b define the statistic $T_n = aT_{1n} + bT_{2n}$. Then, it can be easily shown that the asymptotic distribution of $\sqrt{n}(T_n - aE\{XY^{<p-1>}\} - bE\{|Y|^p\})$ is normal with variance

$$\begin{aligned} &|a|^2 \left[E\{|X|^2|Y|^{2p-2}\} - |E\{XY^{<p-1>}\}|^2\right] \\ &-2\Re\left\{ab^* \left[E\{XY^{<p-1>}|Y|^p\} - E\{XY^{<p-1>}\}E\{|Y|^p\}\right]\right\} + \\ &|b|^2 \left[E\{|Y|^{2p}\} - (E\{|Y|^p\})^2\right] < \infty. \end{aligned} \tag{C.13}$$

It follows that the two statistics T_{1n} and T_{2n} are asymptotically jointly normal. Finally, the covariance of T_{1n} and T_{2n} can be shown to be

$$\mathrm{cov}(T_{1n}, T_{2n}) = \mathrm{cov}^*(T_{2n}, T_{1n}) = \frac{1}{n}[E\{XY^{<p-1>}|Y|^p\} - E\{XY^{<p-1>}\}E\{|Y|^p\}] < \infty, \tag{C.14}$$

and the proof of Lemma 1 is complete. ∎

Now, let g be the totally differentiable function of the two statistics T_{1n} and $T_{2n} \neq 0$ with

$$g(T_{1n}, T_{2n}) = \frac{T_{1n}}{T_{2n}}. \tag{C.15}$$

Then, from convergence theory [19, p. 321] and Lemma 1, it follows that the asymptotic distribution of

$$\sqrt{n}[g(T_{1n}, T_{2n}) - g(\theta_1, \theta_2)]$$

is normal with mean zero and variance

$$\frac{1}{n}\sigma^2_{\lambda}(p) = \sum_{i=1}^{2}\sum_{j=1}^{2} \mathbf{S}_{i,j} \frac{\partial g}{\partial \theta_i} \left(\frac{\partial g}{\partial \theta_j}\right)^*. \tag{C.16}$$

Combining (C.2-C.4), (C.9-C.10), and (C.14-C.16) we get that

$$\sqrt{n}\left[\frac{\frac{1}{n}\sum_{i=1}^{n} X_i Y_i^{<p-1>}}{\frac{1}{n}\sum_{i=1}^{n} |Y_i|^p} - \frac{E\{XY^{<p-1>}\}}{E\{|Y|^p\}}\right] \xrightarrow{\mathcal{L}} \mathcal{N}(0, \sigma^2_{\lambda}(p)), \tag{C.17}$$

with $\sigma^2_{\lambda}(p)$ as shown in (45). ∎

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Table 1: Performance of the covariation coefficient estimators

α	LS	SR	MFLOM	True λ
1.1	-0.4240	-0.4263	-0.4245 ($p = 0.55$)	-0.4252
	(1.1065)	(0.0651)	(0.0636)	
1.2	-0.3371	-0.3396	-0.3380 ($p = 0.58$)	-0.3384
	(1.0010)	(0.0603)	(0.0572)	
1.3	-0.2641	-0.2595	-0.2596 ($p = 0.60$)	-0.2601
	(0.9008)	(0.0574)	(0.0490)	
1.4	-0.1984	-0.1873	-0.1874 ($p = 0.65$)	-0.1900
	(0.8007)	(0.0565)	(0.0432)	
1.5	-0.1386	-0.1250	-0.1258 ($p = 0.68$)	-0.1273
	(0.6986)	(0.0536)	(0.0385)	
1.6	-0.0840	-0.0706	-0.0709 ($p = 0.70$)	-0.0715
	(0.5956)	(0.0522)	(0.0345)	
1.7	-0.0342	-0.0215	-0.0215 ($p = 0.75$)	-0.0222
	(0.4913)	(0.0501)	(0.0303)	
1.8	0.0113	0.0226	0.0225 ($p = 0.80$)	0.0215
	(0.3794)	(0.0472)	(0.0261)	
1.9	0.0537	0.0601	0.0600 ($p = 0.90$)	0.0599
	(0.2407)	(0.0456)	(0.0223)	
2.0	0.0932	0.0917	0.0932 ($p = 2.0$)	0.0936
	(0.0141)	(0.0365)	(0.0141)	

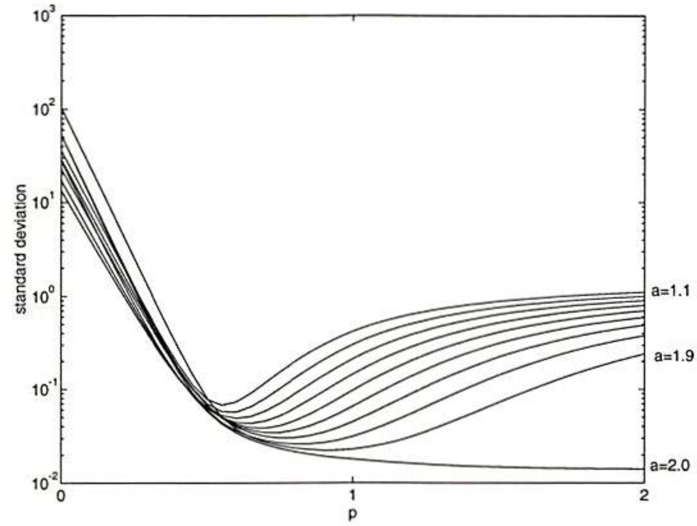


Figure 1: Standard deviation of the MFLOM estimates of the modified covariation coefficient as a function of the parameter p.

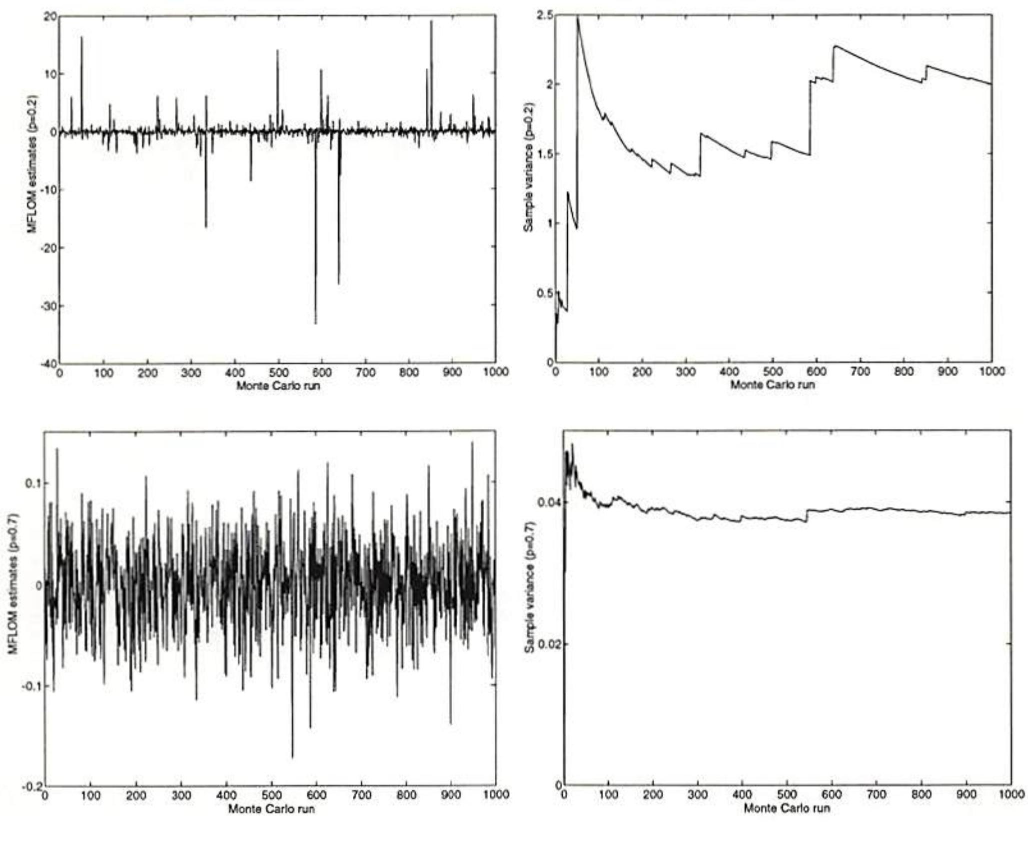


Figure 2: MFLOM estimates and running variances for $\alpha = 1.5$ and $p = 0.2, 0.7$.

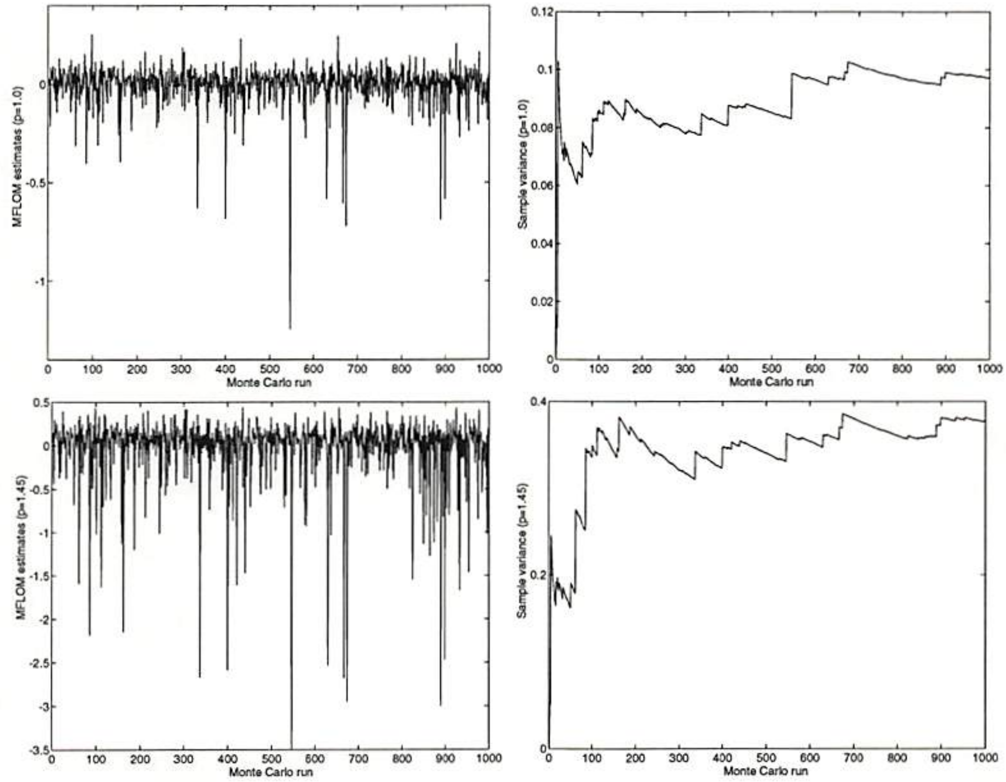


Figure 3: MFLOM estimates and running variances for $\alpha = 1.5$ and $p = 1.0, 1.45$.

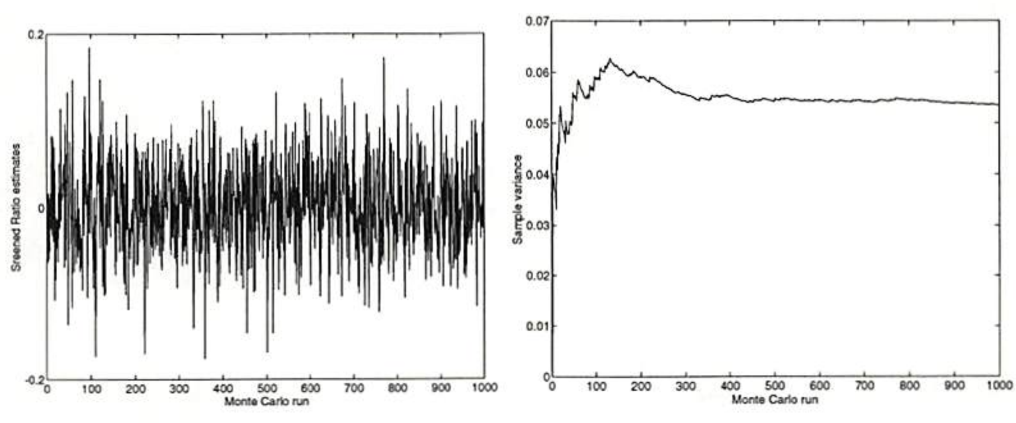


Figure 4: Screened ratio estimates and running variances for $\alpha = 1.5$.

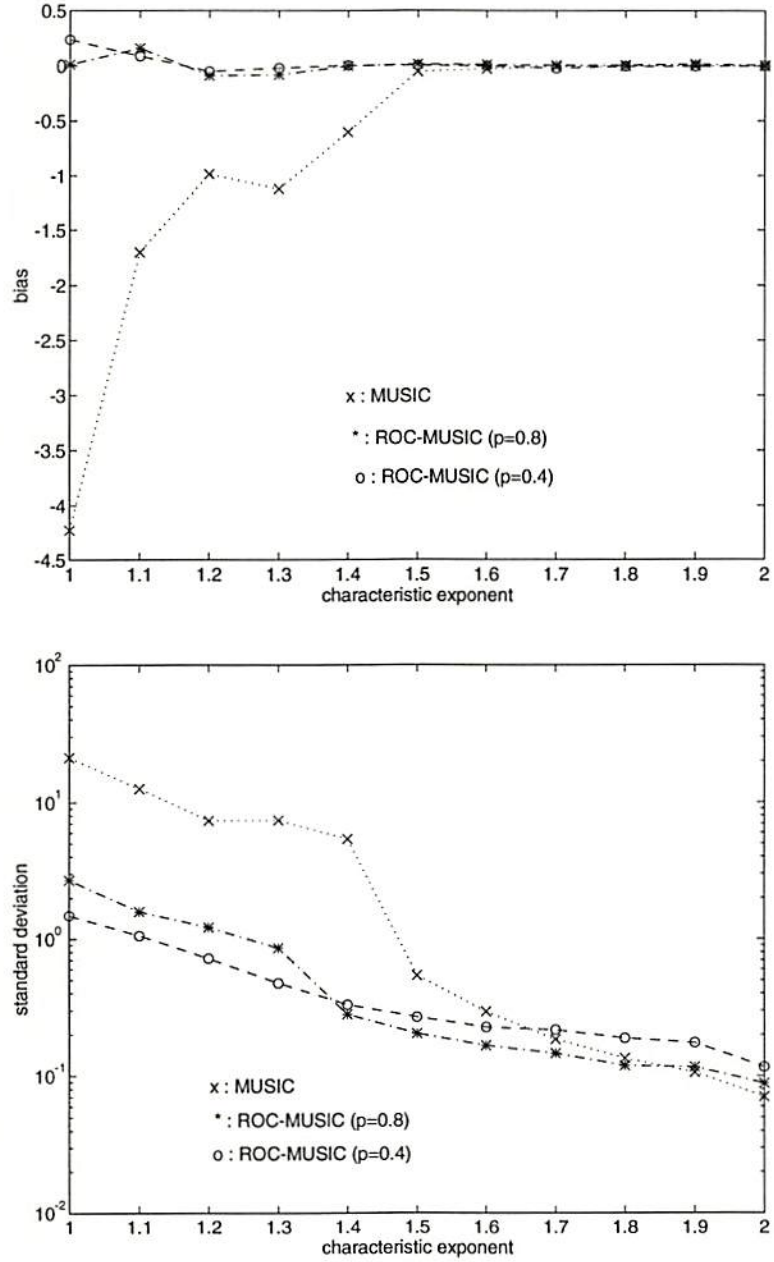


Figure 5: Bias and standard deviation of the estimators of DOA as a function of the characteristic exponent α.

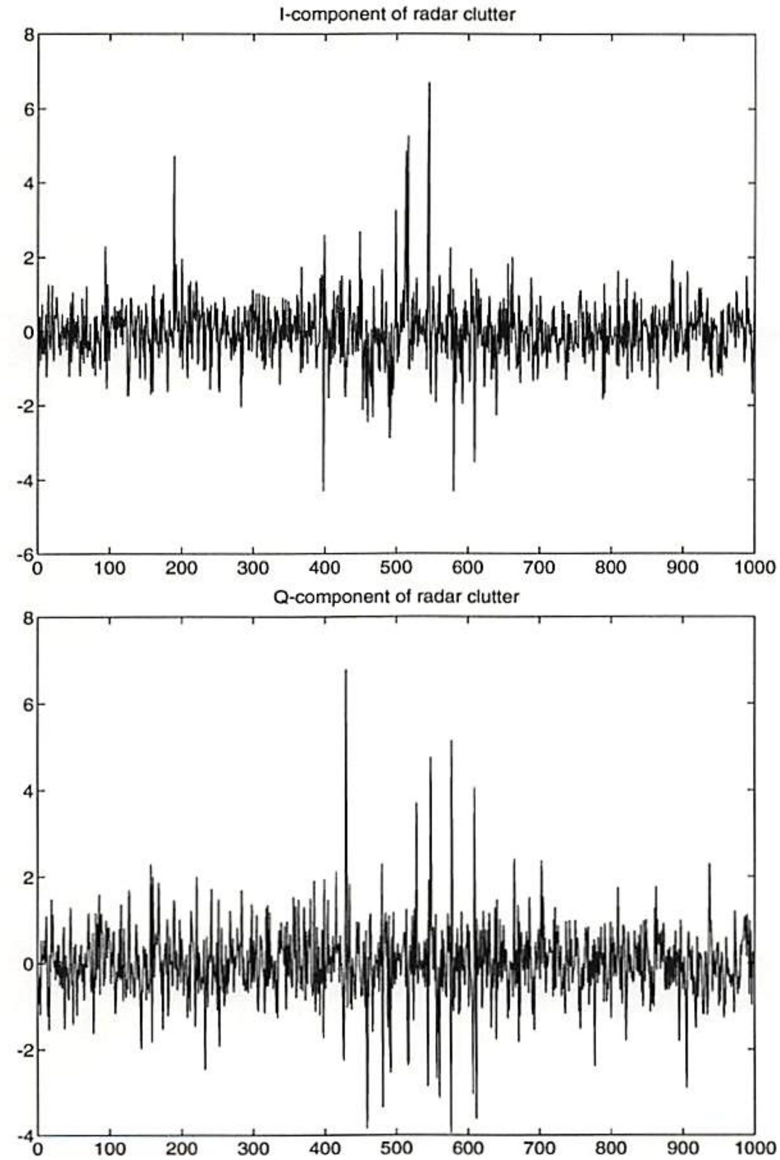
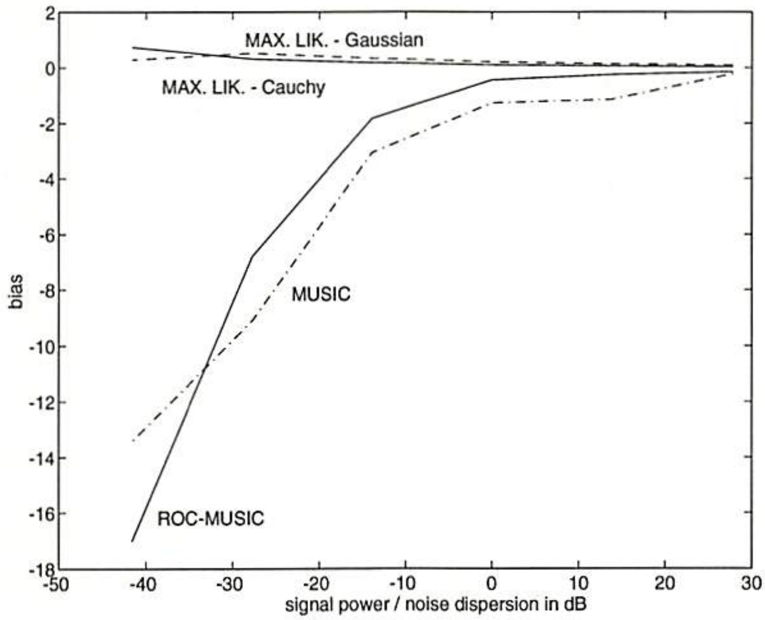


Figure 6: In-phase (I) and Quadrature (Q) components of radar clutter.



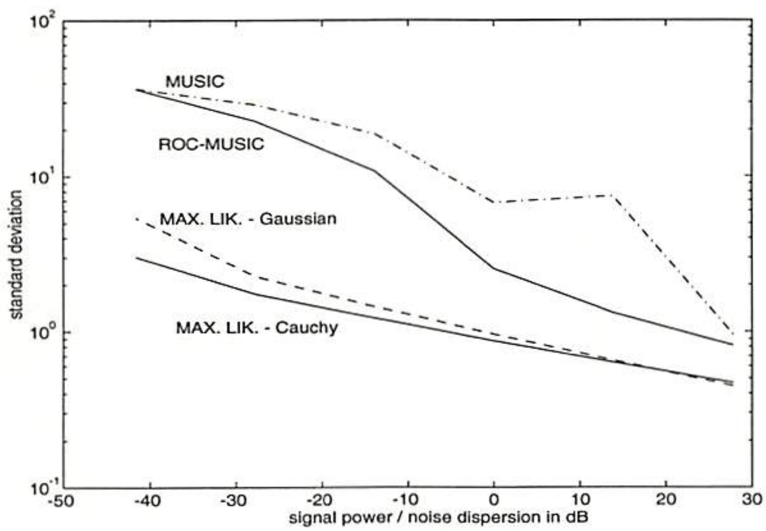


Figure 7: Bias and standard deviation of the estimators of DOA as a function of $PSNR = 10\log(\frac{1}{\gamma M}\sum_{t=1}^{M}|s(t)|^2)$.