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Finite-Sample Covariances of Second-, Third-, and Fourth-Order Sample Cumulants in Narrowband Array Processing

by

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Abstract

Recently, fourth-order cumulants were successfully applied in the area of narrowband array signal processing. For example, the virtual-ESPRIT-Algorithm (VESPA) [8] for direction-finding and recovery of independent sources can also calibrate an array of unknown configuration. Furthermore, with extended VESPA [16] direction-finding of highly correlated or coherent sources is possible. In addition, fourth-order cumulants can also be used for estimation of the range as well as the angle in the near-field case [2].

This large number of new algorithms motivates a performance analysis to compare the higher-order statistics based algorithms with the conventional second-order statistics based algorithms. Up to now, for higher-order statistics based algorithms only asymptotic results are available for direction-finding. These results are restricted to a certain class of communication signals [1].

In this technical report we avoid these restrictions by deriving the finite-sample covariance of

- the second-order sample cumulant (moment),
- the third-order sample cumulant (moment), and
- the fourth-order sample cumulant

for: finite data length, any kind of random signals, any kind of noises, any array shapes, and, arbitrary sensors. We do this, because most performance analyses are based on one of these covariances. Consequently, this report provides the fundamentals for very general performance analyses of second- and higher-order statistics based array-processing algorithms.
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Chapter 1

Introduction

Since existing second-order statistics based methods cannot solve numerous problems in narrowband array signal processing, many higher-order cumulant based algorithms have been recently proposed. For example, the virtual-ESPRIT-Algorithm (VESPA) for direction-finding and recovery of independent sources can also calibrate an array of unknown configuration [8]. Also, VESPA has been extended to direction-finding of highly correlated or coherent sources [16], which happens in practice due to multipath propagation or smart jammer. In addition, higher-order cumulants can be used to increase the effective aperture of an arbitrary antenna array [8]. Since higher-order cumulants are insensitive to additive, white or colored Gaussian noise, Gaussian noise suppression is always accomplished by them. Non-Gaussian noise suppression using higher-order cumulants is often possible, too [9]. Furthermore, fourth-order cumulants can also be used for estimation of the range as well as the angle in the near-field case [2]. These new cumulant-based algorithms motivate a performance analysis not only among themselves, but also with second-order statistics based methods.

In the domain of second-order statistics, many performance analyses have
been done. First approaches were for deterministic situations. COX [7] compared the effects of mismatch in the signal direction of some optimum beamformers. COMPTON [5] showed that the steering vector accuracy is essentially a question of the signal dynamic range. The greater the dynamic range, the more accurate the steering vector must be. ZAHM [27] calculated the output signal-to-noise ratio and showed the serious performance degradation due to steering vector mismatches.

Later approaches were for stochastic situations. COX ET AL. [6] investigated the effects of amplitude and phase errors in the steering vectors. They assumed that these errors can be modeled as zero-mean random variables that are independent among the sensors, and showed the impact of these errors on the power response of a linear predictive beamformer. Phase errors lead to a sharp peak but at the wrong position, whereas amplitude errors can lead to peak splitting. COMPTON [4] computed the output signal-to-interference-plus-noise ratio (SINR) in case of additive steering-vector errors. He showed the remarkable result that increasing the number of sensors can lead to a decrease of the SINR as long as the steering-vector error variance is high enough. Besides the steering-vector errors, GODARA [15] considered the impact of weight vector errors, modeled also as zero mean uncorrelated random variables, on some different beamformers. FRIEDLANDER AND PORAT [13] calculated the output-interference-to-signal ratio (OISR) of a particular class of null-steering algorithms. For a list of more error sources see VURAL [24], who investigated their impact on optimum beamformers, or NG [18], who investigated the impact of wavelength, gain, and steering-vector phase errors.

In the past decade, array signal processing methods based on eigen-decomposition of the (second-order) covariance matrix have received considerable attention, since they provide high resolution with low computational
complexity. Many authors have analyzed the performance of eigendecomposition based algorithms. FRIEDLANDER [13] derived an expression for the sensitivity of the well-known MUSIC algorithm using a first-order Taylor expansion of the inverse of the MUSIC spectrum. He found that the sensitivity of MUSIC for linear arrays to phase errors is relatively small and is practically independent of source separation, whereas the sensitivity of MUSIC for circular arrays is inversely proportional to source separation. Furthermore, he observed that increasing the array aperture reduces the sensitivity of the system to modeling errors. Xu and Buckley [25] presented a bias analysis of the MUSIC location estimator using a second-order Taylor expansion of the derivative of the MUSIC spectrum. In these results, all the performance analyses are based on the asymptotic case (infinite data case). Xu and Buckley showed in their simulations that the bias expression can be accurately applied to a very limited number, \( N \), of independent snapshots (\( N = 20 \)). Reed et al. [20] derived an exact output SINR formula as a function of \( N \) for an optimum beamformer. Unfortunately, their results are limited to Gaussian source signals and sensor noises. Feldman and Griffith [12] also investigated the SINR formula of Reed and they showed by simulation, that Reed’s formula can be well approximated by a simple expression if \( N \) is larger than approximately fifty. These results suggest that perhaps a small number of samples is sufficient for practical use of many asymptotic performance analyses; this can only be proven if the finite-sample covariance can be calculated so that small-sample performance analyses can be performed.

Some performance analyses for higher-order cumulant based methods have already been done. Cardoso and Moulines [1] have derived closed-form expressions of the asymptotic covariance of MUSIC-like direction-of-arrival (DOA) estimates based on two different fourth-order cumulant ma-
trices, and have compared these results with the standard covariance-based MUSIC estimate. They showed that, in case of a single source and a linear array with sensors spaced at regular intervals of half a wavelength, fourth-order MUSIC achieves the same performance as second-order MUSIC, for a special class of signals and in the high-SNR limit. In the low-SNR limit and the single source case, fourth-order MUSIC outperforms second-order MUSIC, if the source kurtosis divided by the source variance is large enough. Furthermore, in the multiple independent sources case and for large SNR, the variance of the DOA of a weak source is significantly increased for fourth-order methods in contrast to second-order MUSIC, where this phenomenon does not occur. YUEN AND FRIEDLANDER [26] presented an asymptotic performance analysis of VESPA, and a second-order and another fourth-order ESPRIT algorithm (see CHIANG AND NIKIAS [3]). Despite the fact that VESPA requires less sensors, it can perform just as well as second-order ESPRIT in some cases and outperforms fourth-order ESPRIT in almost all cases. However, YUEN AND FRIEDLANDER only consider 4-QAM communication signals and additive white Gaussian noise.

Most all of the performance analyses just described are based on the asymptotic covariances of either second-order or fourth-order sample cumulants. In this report, we derive the finite-sample covariances of the second-, third- and fourth-order sample cumulants. This gives us more insight in the dependence of the model parameters; for example, we used this formula to derive an exact expression for the relative error between the asymptotic covariance and the finite-sample covariance [17]. This expression can be used to give good advice on how many data are necessary for practical use of an asymptotic performance analysis. Furthermore, based on the finite-sample covariance, rules can be given about which cumulants should be used for accurately estimating the array processing model parameters (for example,
the steering vector).

Because the formulas, especially the finite-sample covariance of the fourth-order cumulant are extremely tedious to implement, we have published all of them as Matlab\textsuperscript{\textcopyright} M-Files on the Internet "World Wide Web" under

\url{http://fb9nt-ln.uni-duisburg.de/mitarbeiter/kaiser/tech_rep.96/tech_rep.html}

The implementation of our formulas in other programming languages, like "C" and Mathematica\textsuperscript{\textcopyright}, is planned for the near future. Due to the fast growing Internet, you should contact the first author if you have problems receiving the files.

After the problem statement, we derive, in chapters 2–4, the finite-sample covariance of the second-, third-, and fourth-order sample cumulants, respectively. Chapter 5 contains all necessary M-Files.

1.1 Problem statement

The received $M \times 1$ signal vector $\mathbf{r}(t)$ of an array consisting of $M$ sensors can be modeled under the narrowband assumption as

$$
\mathbf{r}(t) = \mathbf{A}s(t) + \mathbf{n}(t),
$$

(1.1)

where $s(t)$ is a $P \times 1$ zero-mean vector which contains the independent source signals at time $t$, $\mathbf{A}$ is an $M \times P$ steering matrix, and $\mathbf{n}(t)$ is the $M \times 1$ independent, but not necessarily Gaussian, zero-mean measurement noise vector. Any noise signal $n_m(t)$, $m = 1(1)M$ is\footnote{$m = 1(1)M$ is a more compact form of $m = 1, 2, \ldots, M$. The number in parentheses means the increment of the sequence.} independent of any source.
signal $s_p(t)$, $p = 1(1)P$. Furthermore, any signal $s_p(t)$ ($n_m(t)$) is modeled as a sequence of independent and identically, but arbitrarily distributed (i.i.d.) complex random variables with finite moments up to the eighth-order. This model is commonly used in the far-field case, but it can be extended to coherent signals (see Gönen et al. [16]) and to the near-field case (see Challa and Shamsunder [2]).

The second-order cumulant (moment) is then defined as

$$ c_{k,l}^{(2)} = E \{ r_k(t)r^*_l(t) \}, \quad (1.2) $$

the third-order cumulant (moment) as

$$ c_{k,l,m}^{(3)} = E \{ r_k(t)r^*_l(t)r_m(t) \} \quad (1.3) $$

and the fourth-order cumulant as

$$ c_{k,l,m,n}^{(4)} = E \{ r_k(t)r^*_l(t)r^*_m(t)r_n(t) \} $$

$$ -E \{ r_k(t)r^*_l(t) \} E \{ r^*_m(t)r_n(t) \} $$

$$ -E \{ r_k(t)r^*_m(t) \} E \{ r^*_l(t)r_n(t) \} $$

$$ -E \{ r_k(t)r_n(t) \} E \{ r^*_l(t)r^*_m(t) \}, \quad (1.4) $$

since $r_m(t)$ is also zero-mean for $m = 1(1) M$. By replacing the expected values by time averages we obtain the second-order sample cumulant (moment)

$$ c_{k,l}^{(2)} = \frac{1}{N} \sum_{t=1}^{N} r_k(t)r^*_l(t), \quad (1.5) $$

the third-order sample cumulant (moment)

$$ c_{k,l,m}^{(3)} = \frac{1}{N} \sum_{t=1}^{N} r_k(t)r^*_l(t)r_m(t) \quad (1.6) $$
and the fourth-order sample cumulant

\[
\hat{c}_{k,l,m,n}^{(4)} = \frac{1}{\alpha} \sum_{t=1}^{N} r_k(t)r_t^*(t)r_m^*(t)r_n(t) \\
- \frac{1}{\beta} \sum_{t=1}^{N} r_k(t)r_t^*(t) \sum_{p=1}^{N} r_m^*(p)r_n(p) \\
- \frac{1}{\beta} \sum_{t=1}^{N} r_k(t)r_m^*(t) \sum_{p=1}^{N} r_t^*(p)r_n(p) \\
- \frac{1}{\beta} \sum_{t=1}^{N} r_k(t)r_n(t) \sum_{p=1}^{N} r_t^*(p)r_m^*(p),
\]  

(1.7)

where \( N \in \mathbb{N} \) is the data length and \( \alpha \) and \( \beta \) are functions of \( N \). If the fourth-order sample cumulant is an unbiased estimate,

\[
\alpha = \frac{N^2 - N}{N + 2}, \quad \forall N > 1
\]

(1.8)

\[
\beta = N^2 - N, \quad \forall N > 1
\]

(1.9)

otherwise, \( \alpha = N \) and \( \beta = N^2 \) is commonly used. Note that the second-and the third-order sample cumulants are always unbiased. In the following section we will derive the finite-sample covariance of all of these sample cumulants for the model defined in (1.1).

### 1.2 The finite-sample covariance of the second-order sample cumulant

The finite-sample covariance of the second-order sample cumulant can be written as

\[
\text{Cov} \left( \hat{c}_{k_1,d_1}^{(2)}, \hat{c}_{k_2,d_2}^{(2),H} \right) = \mathbb{E} \left\{ \hat{c}_{k_1,d_1}^{(2)} \hat{c}_{k_2,d_2}^{(2),H} \right\} - \mathbb{E} \left\{ \hat{c}_{k_1,d_1}^{(2)} \right\} \mathbb{E} \left\{ \hat{c}_{k_2,d_2}^{(2),H} \right\}
\]

(1.10)

with
\[
E \left\{ \hat{c}_{k_1,d_1}^{(2)} \hat{c}_{k_2,d_2}^{(2)H} \right\} \\
= \frac{1}{N^2} \sum_{t=1}^{N} \sum_{p=1}^{N} E \left\{ r_{k_1}(t)r_{k_2}^*(p)r_{k_2}(p) \right\} \\
= \frac{1}{N} E \left\{ r_{k_1}(t)r_{k_1}^*(t)r_{k_2}^*(t)r_{k_2}(t) \right\} + \frac{N^2 - N}{N^2} E \left\{ r_{k_1}(t)r_{k_1}^*(t) \right\} E \left\{ r_{k_2}^*(t)r_{k_2}(t) \right\}, 
\]

where the summations dissolve due to the i.i.d. property of \( r_m(t) \) \( \forall m = 1(1)M \).

Since \( E \left\{ \hat{c}_{k_1,d_1}^{(2)} \right\} = E \left\{ r_{k_1}(t)r_{k_1}^*(t) \right\}, \) (1.10) yields

\[
\text{Cov} \left( \hat{c}_{k_1,d_1}^{(2)}, \hat{c}_{k_2,d_2}^{(2)H} \right) = \frac{1}{N} \left( E \left\{ r_{k_1}(t)r_{k_1}^*(t)r_{k_2}^*(t)r_{k_2}(t) \right\} - E \left\{ r_{k_1}(t)r_{k_1}^*(t) \right\} E \left\{ r_{k_2}^*(t)r_{k_2}(t) \right\} \right). 
\]

In order to evaluate (1.12) we must calculate the fourth- and the second-order moments of the array output signal for the model defined in (1.1).

### 1.3 Fourth-order moment

In the following we often omit the time variable \( t \) in order to simplify the notation.

\[
E \left\{ r_{k_1}(t)r_{k_1}^H(t)r_{k_2}^H(t)r_{k_2}(t) \right\} \\
= E \left\{ (a_{k_1}^T s + n_{k_1})(s^H a_{k_1}^* + n_{k_1}^*)(s^H a_{k_2}^* + n_{k_2})(a_{k_2}^T s + n_{k_2}) \right\} \\
= E \left\{ a_{k_1}^T s^H a_{k_1}^* s^H a_{k_2}^* a_{k_2}^T s \right\} + E \left\{ a_{k_1}^T s^H a_{k_1}^* \right\} E \left\{ n_{k_2}^* n_{k_2} \right\} + E \left\{ a_{k_2}^T s^H a_{k_2}^* \right\} E \left\{ n_{k_1}^* n_{k_1} \right\} + E \left\{ a_{k_1}^T s a_{k_2}^T s \right\} E \left\{ n_{k_1}^* n_{k_2} \right\} + 
\]
\[
E \{ s^H a^*_k a^T_k \} \ E \{ n_k, n_l \} + \ E \{ s^H a^*_l a^T_l s \} \ E \{ n_k, n_k^* \} + \\
E \{ s^H a^*_k a^T_l s \} \ E \{ n_k, n_l^* \} + \ E \{ n_k, n_l^* n_k^* n_l^* \} 
\]

(1.13)

where \( a^T_k \) is a steering row vector, since it is the \( k_1 \)th-row vector of the steering matrix \( A \). For further calculating the expected value in eq. (1.13), the random variables \( s \) must be separated from the deterministic variables \( (a_k) \). Since we are faced with this problem in the following chapters again, we are looking for a general notation. Between the scalars \( s^H a^*_k \), \( a_k^T s \), we can simply introduce a Kronecker product and using the rule for the Kronecker-product (\( \otimes \))

\[
(AB) \otimes (CD) = ( A \otimes C ) ( B \otimes D )
\]

(1.14)
yields

\[
E \{ r_{k_1}(t) r_{l_1}^H(t) r_{l_2}(t) r_{l_2}(t) \} \\
= \left( a^T_k \otimes a^H_{l_1} \otimes a^T_{l_2} \right) E \{ s \otimes s^* \otimes s^* \otimes s \} + \\
\left( a^T_k \otimes a^H_{l_1} \right) E \{ s \otimes s^* \} E \{ n_{k_1} n_{l_2} \} + \\
\left( a^T_k \otimes a^H_{l_2} \right) E \{ s \otimes s^* \} E \{ n_{k_1} n_{l_2} \} + \\
\left( a^T_{l_1} \otimes a^T_{l_2} \right) E \{ s \otimes s^* \} E \{ n_{k_1} n_{k_2} \} + \\
\left( a^H_{l_1} \otimes a^T_{l_2} \right) E \{ s^* \otimes s^* \} E \{ n_{k_1} n_{k_2} \} + \\
\left( a^H_{l_1} \otimes a^T_{l_2} \right) E \{ s^* \otimes s \} E \{ n_{k_1} n_{l_2} \} + \\
\left( a^H_{l_2} \otimes a^T_{l_2} \right) E \{ s^* \otimes s \} E \{ n_{k_1} n_{l_2} \} + \\
E \{ n_{k_1} n_{l_1}^* n_{k_2} n_{l_2} \}.
\]

(1.15)

Therefore, given the statistics of the signals and the noise up to order 4, the fourth-order moment can be calculated. For example, the \( P^4 \times 1 \) vector
$E \{s \otimes s^* \otimes s^* \otimes s\}$ can be written for $P = 2$ as

$$
\begin{pmatrix}
E \{s_1 s_1^* s_2^* s_1\} \\
E \{s_1 s_1^* s_2^* s_2\} \\
E \{s_1 s_2^* s_2^* s_1\} \\
E \{s_1 s_2^* s_2^* s_2\} \\
E \{s_1 s_2^* s_2^* s_1\} \\
E \{s_2 s_2^* s_1^* s_1\} \\
E \{s_2 s_2^* s_1^* s_2\} \\
E \{s_2 s_2^* s_2^* s_2\} \\
E \{s_2 s_2^* s_2^* s_1\} \\
E \{s_2 s_2^* s_2^* s_2\}
\end{pmatrix}
= 
\begin{pmatrix}
E \{|s_1|^4\} \\
0 \\
0 \\
E \{|s_1|^2\} E \{|s_2|^2\} \\
0 \\
E \{|s_1|^2\} E \{|s_2|^2\} \\
E \{|s_1|^2\} E \{|s_2|^2\} \\
0 \\
E \{|s_1|^2\} E \{|s_2|^2\} \\
0 \\
E \{|s_2|^4\}
\end{pmatrix}.
$$

Note that $P^4 - P^3$ elements of this vector are always equal to zero due to the independence of sources and the zero-mean assumptions.
1.4 Second-order moment

Again, applying the Kronecker product leads to

\[ E \left\{ r_{k_1}(t) r_{i_1}^H(t) \right\} \quad = \quad \left( a_{k_1}^T \otimes a_{i_1}^H \right) E \{ s \otimes s^* \} + E \left\{ n_{k_1} n_{i_1}^* \right\} \]

\[ E \left\{ r_{k_2}^H(t) r_{i_2}(t) \right\} \quad = \quad \left( a_{k_2}^H \otimes a_{i_2}^T \right) E \{ s^* \otimes s \} + E \left\{ n_{k_2}^* n_{i_2} \right\} . \]

1.5 Final formula

Now, the finite-sample covariance of the second-order sample cumulant is given by

\[
\text{Cov} \left( \hat{c}_{k_1,i_1}, \hat{c}_{k_2,i_2}^{(2),H} \right) = \frac{1}{N} \left( \left( a_{k_1}^T \otimes a_{i_1}^H \otimes a_{k_2}^H \otimes a_{i_2}^T \right) \left( E \{ s \otimes s^* \otimes s^* \otimes s \} - E \{ s \otimes s^* \} \otimes E \{ s^* \otimes s \} \right) + \right.
\]
\[
\left. \left( a_{k_1}^T \otimes a_{k_2}^H \right) E \{ s \otimes s^* \} E \left\{ n_{i_1} n_{i_2} \right\} + \right.
\]
\[
\left. \left( a_{k_1}^T \otimes a_{i_2}^T \right) E \{ s \otimes s \} E \left\{ n_{i_1}^* n_{k_2} \right\} + \right.
\]
\[
\left. \left( a_{i_1}^H \otimes a_{k_2}^H \right) E \{ s^* \otimes s^* \} E \left\{ n_{k_1} n_{i_2} \right\} + \right.
\]
\[
\left. \left( a_{i_1}^H \otimes a_{i_2}^T \right) E \{ s^* \otimes s \} E \left\{ n_{k_1} n_{k_2}^* \right\} + \right.
\]
\[
\left. E \left\{ n_{k_1} n_{i_1}^* n_{k_2} n_{i_2} \right\} - E \left\{ n_{k_1} n_{i_1} \right\} E \left\{ n_{k_2}^* n_{i_2} \right\} \right). \tag{1.16}
\]
Chapter 2

The finite-sample covariance of the third-order sample cumulant

The finite-sample covariance of the third-order sample cumulant can be written as

\[
\text{Cov}\left(\hat{c}_{k_1,l_1,m_1}^{(3)}, \hat{c}_{k_2,l_2,m_2}^{(3),H}\right) = \\
E\left\{\hat{c}_{k_1,l_1,m_1}^{(3)}\right\} E\left\{\hat{c}_{k_2,l_2,m_2}^{(3),H}\right\} - E\left\{\hat{c}_{k_1,l_1,m_1}^{(3)}\right\} E\left\{\hat{c}_{k_2,l_2,m_2}^{(3),H}\right\}.
\]

(2.1)

with

\[
E\left\{\hat{c}_{k_1,l_1,m_1}^{(3)}\hat{c}_{k_2,l_2,m_2}^{(3),H}\right\} = \frac{1}{N^2} \sum_{t=1}^{N} \sum_{p=1}^{N} E\left\{r_{k_1}(t)r_{l_1}^{H}(t)r_{m_1}(t)r_{k_2}(t)r_{l_2}(t)r_{m_2}^{H}(t)p\right\}
\]

\[
= \frac{1}{N} E\left\{r_{k_1}(t)r_{l_1}^{H}(t)r_{m_1}(t)r_{k_2}(t)r_{l_2}(t)r_{m_2}^{H}(t)\right\} + \\
\frac{N^2 - N}{N^2} E\left\{r_{k_1}(t)r_{l_1}^{H}(t)r_{m_1}(t)\right\} E\left\{r_{k_2}(t)r_{l_2}(t)r_{m_2}^{H}(t)\right\}
\]

where the summations dissolve due to the i.i.d. property of \(r_m(t) \forall m = 1(1)M\).
Since \( E \{ \hat{a}_{k_1,d_1,m_1}^{(3)} \} = E \{ r_{k_1}^{(H)}(t)r_{m_1}^{(H)}(t) \} \), (2.1) yields

\[
\text{Cov} \left( \hat{a}_{k_1,d_1,m_1}^{(3)}, \hat{a}_{k_2,d_2,m_2}^{(3)H} \right) = \\
\frac{1}{N} \left( E \left\{ r_{k_1}(t)r_{i_1}^{(H)}(t)r_{m_1}(t)r_{k_2}(t)r_{i_2}^{(H)}(t)r_{m_2}^{(H)}(t) \right\} - \\
E \left\{ r_{k_1}(t)r_{i_1}^{(H)}(t)r_{m_1}(t) \right\} E \left\{ r_{k_2}(t)r_{i_2}^{(H)}(t)r_{m_2}^{(H)}(t) \right\} \right). \quad (2.2)
\]

In order to evaluate (2.2) we must calculate the sixth- and the third-order moments of the array output signal for the model defined in (1.1).

### 2.1 Sixth-order moment

Let \( b_k = \mathbf{a}_k^T \). Then

\[
E \left\{ r_{k_1}(t)r_{i_1}^{(H)}(t)r_{m_1}(t)r_{k_2}(t)r_{i_2}^{(H)}(t)r_{m_2}^{(H)}(t) \right\}
= E \left\{ (b_{k_1} + n_{k_1})(b_{i_1}^H + n_{i_1}^H)(b_{m_1} + n_{m_1})(b_{k_2}^H + n_{k_2}^H)(b_{i_2} + n_{i_2})(b_{m_2}^H + n_{m_2}^H) \right\}
\]

\[
= E \left\{ b_{k_1}b_{i_1}^Hb_{m_1}b_{k_2}^Hb_{i_2}b_{m_2}^H \right\} + E \left\{ b_{k_1}b_{i_1}b_{m_1}b_{k_2}^Hb_{i_2}b_{m_2}^Hn_{m_2}^H \right\} + \\
E \left\{ b_{k_1}b_{i_1}b_{m_1}b_{k_2}^Hb_{i_2}b_{m_2}^Hn_{k_2}^H \right\} + E \left\{ b_{k_1}b_{i_1}b_{m_1}n_{k_2}^Hb_{i_2}b_{m_2}^Hb_{k_2}^H \right\} + \\
E \left\{ b_{k_1}b_{i_1}b_{m_1}n_{k_2}^Hb_{i_2}n_{m_2}^Hb_{k_2}^H \right\} + E \left\{ b_{k_1}b_{i_1}b_{m_1}b_{k_2}^Hn_{i_2}n_{m_2}^Hb_{k_2}^H \right\} + \\
E \left\{ b_{k_1}b_{i_1}b_{m_1}b_{k_2}^Hn_{i_2}n_{m_2}^Hn_{k_2}^Hb_{k_2}^H \right\} + E \left\{ b_{k_1}b_{i_1}b_{m_1}n_{i_2}n_{m_2}^Hn_{k_2}^Hb_{k_2}^H \right\} + \\
E \left\{ b_{k_1}b_{i_1}b_{m_1}n_{i_2}n_{m_2}^Hn_{i_2}n_{m_2}^Hb_{k_2}^H \right\} + E \left\{ b_{k_1}b_{i_1}b_{m_1}b_{k_2}^Hn_{i_2}n_{m_2}^Hb_{k_2}^H \right\} + \\
E \left\{ b_{k_1}b_{i_1}b_{m_1}b_{k_2}^Hn_{i_2}n_{m_2}^Hn_{k_2}^Hb_{k_2}^H \right\} + E \left\{ b_{k_1}b_{i_1}b_{m_1}n_{i_2}n_{m_2}^Hb_{k_2}^Hn_{i_2}n_{m_2}^Hb_{k_2}^H \right\} + \\
E \left\{ b_{k_1}b_{i_1}b_{m_1}n_{i_2}n_{m_2}^Hn_{k_2}^Hb_{k_2}^Hn_{i_2}n_{m_2}^Hb_{k_2}^H \right\} + E \left\{ b_{k_1}b_{i_1}b_{m_1}b_{k_2}^Hn_{i_2}n_{m_2}^Hn_{k_2}^Hb_{k_2}^H \right\} + \\
E \left\{ b_{k_1}b_{i_1}b_{m_1}b_{k_2}^Hn_{i_2}n_{m_2}^Hn_{i_2}n_{m_2}^Hb_{k_2}^H \right\} + E \left\{ b_{k_1}b_{i_1}b_{m_1}n_{i_2}n_{m_2}^Hn_{k_2}^Hb_{k_2}^Hn_{i_2}n_{m_2}^Hb_{k_2}^H \right\}.
\]

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Close inspection of eq. (2.3) reveals that many terms are very similar to each other; hence, only the followings terms must be developed further:

a) \( E \left\{ b_{k_1} b_{n_1}^{H} b_{m_1} b_{k_2} b_{n_2}^{H} b_{m_2} \right\} \)

b) \( E \left\{ b_{k_1} b_{l_1}^{H} b_{m_1} b_{k_2} b_{n_2}^{H} b_{m_2} \right\} \)
c) \[ E \left\{ b_k b_i^H b_m^H n_{k_2}^H n_{l_2}^H n_{m_2}^H \right\} \]

d) \[ E \left\{ b_k b_i^H n_{m_1}^H n_{l_2}^H n_{m_2}^H \right\} \]

e) \[ E \left\{ n_{k_1}^H n_{l_1}^H n_{k_2}^H n_{l_2}^H n_{m_2}^H \right\} . \]

Note that
\[ E \left\{ b_k b_i^H b_{m_1} b_{k_2}^H b_{l_2} n_{m_2}^H \right\} = E \left\{ b_k n_{l_1}^H n_{m_1}^H n_{k_2}^H n_{l_2}^H n_{m_2}^H \right\} = 0 \]
due to the independence and zero-mean assumptions of all signals and noises.

The remaining terms in (2.3) can be easily obtained by proper change of indices in the formulas for a), b), c), d), e).

Proceeding as we did on page 9, we obtain:

a) \[ E \left\{ b_k b_i^H b_{m_1} b_{k_2}^H b_{l_2}^H \right\} = \left( a_{k_1}^T \otimes a_{l_1}^H \otimes a_{m_1}^T \otimes a_{k_2}^H \otimes a_{l_2}^T \otimes a_{m_2}^H \right) E \left\{ s \otimes s^* \otimes s \otimes s^* \right\} E \left\{ s \otimes s^* \otimes s \otimes s^* \right\} E \left\{ n_{l_2}^H n_{m_2}^H \right\} \] (2.4)

b) \[ E \left\{ b_k b_i^H b_{m_1} b_{k_2}^H n_{l_2}^H n_{m_2}^H \right\} = \left( a_{k_1}^T \otimes a_{l_1}^H \otimes a_{m_1}^T \otimes a_{k_2}^H \right) E \left\{ s \otimes s^* \otimes s \otimes s^* \right\} E \left\{ n_{l_2}^H n_{m_2}^H \right\} \] (2.5)

c) \[ E \left\{ b_k b_i^H n_{m_1}^H n_{k_2}^H n_{l_2}^H n_{m_2}^H \right\} = \left( a_{k_1}^T \otimes a_{l_1}^H \otimes a_{m_1}^T \right) E \left\{ s \otimes s^* \otimes s \right\} E \left\{ n_{k_2}^H n_{l_2}^H n_{m_2}^H \right\} \] (2.6)

d) \[ E \left\{ b_k b_i^H n_{m_1}^H n_{k_2}^H n_{l_2}^H n_{m_2}^H \right\} = \left( a_{k_1}^T \otimes a_{l_1}^H \right) E \left\{ s \otimes s^* \right\} E \left\{ n_{m_1}^H n_{k_2}^H n_{l_2}^H n_{m_2}^H \right\} \] (2.7)

and e) can be calculated directly from the given noise moments.
2.2 Third-order moment

\[ E \left\{ r_{k_1}(t)r_{l_1}^H(t)r_{m_1}(t) \right\} = \]
\[ \left( a_{l_1}^T \otimes a_{l_1}^H \otimes a_{m_1}^T \right) E \{ s \otimes s^* \otimes s \} + E \left\{ n_{k_1}n_{l_1}^Hn_{m_1} \right\} \]  \hspace{1cm} (2.8)

\[ E \left\{ r_{k_2}(t)r_{l_2}(t)r_{m_2}^H(t) \right\} = \]
\[ \left( a_{l_2}^H \otimes a_{l_2}^T \otimes a_{m_2}^H \right) E \{ s^* \otimes s \otimes s^* \} + E \left\{ n_{k_2}^Hn_{l_2}n_{m_2}^H \right\} \]  \hspace{1cm} (2.9)

2.3 Final formula

The finite-sample covariance of the third-order sample cumulant (moment) can be finally calculated by substituting (2.4)-(2.7) into (2.3) and (2.3),(2.8) and (2.9) into (2.1).
Chapter 3

The finite-sample covariance of the fourth-order sample cumulant

The finite-sample covariance of the fourth-order sample cumulant can be written as

\[ \text{Cov} \left( \hat{c}_{k_1,i_1,m_1,n_1}^{(4)}, \hat{c}_{k_2,i_2,m_2,n_2}^{(4), H} \right) = 
\text{E} \left\{ \hat{c}_{k_1,i_1,m_1,n_1}^{(4)} \right\} - \text{E} \left\{ \hat{c}_{k_1,i_1,m_1,n_1}^{(4)} \right\} \text{E} \left\{ \hat{c}_{k_2,i_2,m_2,n_2}^{(4), H} \right\}. \] (3.1)

Using (1.7) in (3.1) we obtain

\[ \text{E} \left\{ \hat{c}_{k_1,i_1,m_1,n_1}^{(4)}, \hat{c}_{k_2,i_2,m_2,n_2}^{(4), H} \right\} = 
\frac{1}{\alpha^2} \sum_{t=1}^{N} \sum_{p=1}^{N} \text{E} \left\{ r_{k_1}(t) r_{m_1}(t) r_{n_1}(t) r_{k_2}(p) r_{m_2}(p) r_{n_2}(p) \right\} 
- \frac{2}{\alpha \beta} \text{Re} \left\{ \sum_{t=1}^{N} \sum_{p=1}^{N} \sum_{q=1}^{N} \text{E} \left\{ r_{k_1}(t) r_{m_1}(t) r_{n_1}(t) r_{k_2}(p) r_{m_2}(q) r_{n_2}(q) \right\} \right\} \]
\[-\frac{2}{\alpha \beta} \text{Re} \left\{ \sum_{t=1}^{N} \sum_{p=1}^{N} \sum_{q=1}^{N} E \left\{ r_{k_1}(t) r_{l_1}^H(t) r_{m_1}^H(t) r_{n_1}(t) r_{k_2}(p) r_{m_2}(p) r_{l_2}(q) r_{n_2}^H(q) \right\} \right\} \]

\[-\frac{2}{\alpha \beta} \text{Re} \left\{ \sum_{t=1}^{N} \sum_{p=1}^{N} \sum_{q=1}^{N} E \left\{ r_{k_1}(t) r_{l_1}^H(t) r_{m_1}(t) r_{n_1}(t) r_{k_2}(p) r_{m_2}(q) r_{l_2}(q) r_{n_2}(q) \right\} \right\} \]

\[+ \frac{1}{\beta^2} \sum_{t=1}^{N} \sum_{p=1}^{N} \sum_{q=1}^{N} \sum_{s=1}^{N} E \left\{ r_{k_1}(t) r_{l_1}^H(t) r_{m_1}(p) r_{n_1}(p) r_{k_2}(q) r_{l_2}(q) r_{m_2}(s) r_{n_2}(s) \right\} \]

\[+ \frac{2}{\beta^2} \text{Re} \left\{ \sum_{t=1}^{N} \sum_{p=1}^{N} \sum_{q=1}^{N} \sum_{s=1}^{N} E \left\{ r_{k_1}(t) r_{l_1}^H(t) r_{m_1}(p) r_{n_1}(p) r_{k_2}(q) r_{l_2}(s) r_{m_2}(s) r_{n_2}(s) \right\} \right\} \]

\[+ \frac{1}{\beta^2} \sum_{t=1}^{N} \sum_{p=1}^{N} \sum_{q=1}^{N} \sum_{s=1}^{N} E \left\{ r_{k_1}(t) r_{l_1}^H(t) r_{m_1}(p) r_{n_1}(p) r_{k_2}(q) r_{l_2}(q) r_{m_2}(q) r_{n_2}(s) \right\} \]

\[+ \frac{2}{\beta^2} \text{Re} \left\{ \sum_{t=1}^{N} \sum_{p=1}^{N} \sum_{q=1}^{N} \sum_{s=1}^{N} E \left\{ r_{k_1}(t) r_{l_1}^H(t) r_{m_1}(p) r_{n_1}(p) r_{k_2}(q) r_{l_2}(q) r_{m_2}(s) r_{n_2}(s) \right\} \right\} \]

\[+ \frac{1}{\beta^2} \sum_{t=1}^{N} \sum_{p=1}^{N} \sum_{q=1}^{N} \sum_{s=1}^{N} E \left\{ r_{k_1}(t) r_{n_1}(t) r_{l_1}^H(t) r_{m_1}(p) r_{k_2}(p) r_{m_2}(p) r_{l_2}(q) r_{n_2}^H(q) \right\} \]

\[+ \frac{2}{\beta^2} \text{Re} \left\{ \sum_{t=1}^{N} \sum_{p=1}^{N} \sum_{q=1}^{N} \sum_{s=1}^{N} E \left\{ r_{k_1}(t) r_{n_1}(t) r_{l_1}^H(t) r_{m_1}(p) r_{k_2}(p) r_{m_2}(q) r_{l_2}(q) r_{n_2}(s) \right\} \right\} \]

\[+ \frac{1}{\beta^2} \sum_{t=1}^{N} \sum_{p=1}^{N} \sum_{q=1}^{N} \sum_{s=1}^{N} E \left\{ r_{k_1}(t) r_{n_1}(t) r_{l_1}^H(t) r_{m_1}(p) r_{k_2}(p) r_{m_2}(q) r_{l_2}(q) r_{n_2}(s) \right\} \]

(3.2)

where the real operation comes from \((a + b)(a^* + b^*) = |a|^2 + 2 \text{Re} \, ab^* + |b|^2\).

Since many terms in (3.2) can be easily obtained by proper change of indices, only the following terms must be calculated:

\[a) \sum_{t=1}^{N} \sum_{p=1}^{N} E \left\{ r_{k_1}(t) r_{l_1}^H(t) r_{m_1}(t) r_{n_1}(t) r_{k_2}(p) r_{l_2}(p) r_{m_2}(p) r_{n_2}^H(p) \right\} \]

\[b) \sum_{t=1}^{N} \sum_{p=1}^{N} \sum_{q=1}^{N} E \left\{ r_{k_1}(t) r_{l_1}^H(t) r_{m_1}(t) r_{n_1}(t) r_{k_2}(p) r_{l_2}(p) r_{m_2}(q) r_{n_2}^H(q) \right\} \]

\[c) \sum_{t=1}^{N} \sum_{p=1}^{N} \sum_{q=1}^{N} \sum_{s=1}^{N} E \left\{ r_{k_1}(t) r_{l_1}^H(t) r_{m_1}(p) r_{n_1}(p) r_{k_2}(q) r_{l_2}(q) r_{m_2}(s) r_{n_2}^H(s) \right\} \]
First term a):

\[ \sum_{t=1}^{N} \sum_{p=1}^{N} \mathbb{E} \left\{ r_{k_1}(t)r_{l_1}(t)r_{m_1}(t)\mathbb{r}_{n_1}(t)r_{k_2}(p)r_{l_2}(p)\mathbb{r}_{m_2}(p)\mathbb{r}_{n_2}(p) \right\} \]

\[ = N \mathbb{E} \left\{ r_{k_1}(t)r_{l_1}(t)r_{m_1}(t)\mathbb{r}_{n_1}(t)r_{k_2}(t)r_{l_2}(t)\mathbb{r}_{m_2}(t)\mathbb{r}_{n_2}(t) \right\} + (N^2 - N) \]

\[ \mathbb{E} \left\{ r_{k_1}(t)r_{l_1}(t)r_{m_1}(t)\mathbb{r}_{n_1}(t) \right\} \mathbb{E} \left\{ r_{k_2}(t)r_{l_2}(t)r_{m_2}(t)\mathbb{r}_{n_2}(t) \right\}, \quad (3.3) \]

because \( r_k(t) \) is independent of \( r_l(p) \) for \( t \neq p \).

Second term b):

\[ \sum_{t=1}^{N} \sum_{p=1}^{N} \sum_{q=1}^{N} \mathbb{E} \left\{ r_{k_1}(t)r_{l_1}(t)r_{m_1}(t)\mathbb{r}_{n_1}(t)r_{k_2}(p)r_{l_2}(p)\mathbb{r}_{m_2}(q)\mathbb{r}_{n_2}(q) \right\} \]

\[ = N \mathbb{E} \left\{ r_{k_1}(t)r_{l_1}(t)r_{m_1}(t)\mathbb{r}_{n_1}(t)r_{k_2}(t)r_{l_2}(t)\mathbb{r}_{m_2}(t)\mathbb{r}_{n_2}(t) \right\} + (N^2 - N) \mathbb{E} \left\{ r_{k_1}(t)r_{l_1}(t)r_{m_1}(t)\mathbb{r}_{n_1}(t)r_{k_2}(t)\mathbb{r}_{m_2}(t)\mathbb{r}_{n_2}(t) \right\} \]

\[ + (N^2 - N) \mathbb{E} \left\{ r_{k_1}(t)r_{l_1}(t)r_{m_1}(t)\mathbb{r}_{n_1}(t)r_{k_2}(t)\mathbb{r}_{m_2}(t)\mathbb{r}_{n_2}(t) \right\} \mathbb{E} \left\{ r_{l_2}(t)\mathbb{r}_{m_2}(t)\mathbb{r}_{n_2}(t) \right\} \]

\[ + (N^2 - N) \mathbb{E} \left\{ r_{k_1}(t)r_{l_1}(t)r_{m_1}(t)\mathbb{r}_{n_1}(t)r_{m_2}(t)\mathbb{r}_{n_2}(t) \right\} \mathbb{E} \left\{ r_{k_2}(t)\mathbb{r}_{l_2}(t)\mathbb{r}_{n_2}(t) \right\} \]

\[ + (N^2 - N) \mathbb{E} \left\{ r_{k_1}(t)r_{l_1}(t)r_{m_1}(t)\mathbb{r}_{n_1}(t)\mathbb{r}_{m_2}(t)\mathbb{r}_{n_2}(t) \right\} \mathbb{E} \left\{ r_{k_2}(t)\mathbb{r}_{l_2}(t)\mathbb{r}_{n_2}(t) \right\} \]

\[ + (N^2 - N) \mathbb{E} \left\{ r_{k_1}(t)r_{l_1}(t)r_{m_1}(t)\mathbb{r}_{n_1}(t)\mathbb{r}_{m_2}(t)\mathbb{r}_{n_2}(t) \right\} \mathbb{E} \left\{ r_{l_2}(t)\mathbb{r}_{m_2}(t)\mathbb{r}_{n_2}(t) \right\} \]

\[ + (N^2 - 3N^2 + 2N) \]

\[ \mathbb{E} \left\{ r_{k_1}(t)r_{l_1}(t)r_{m_1}(t)\mathbb{r}_{n_1}(t) \right\} \mathbb{E} \left\{ r_{k_2}(t)\mathbb{r}_{l_2}(t)\mathbb{r}_{n_2}(t) \right\} \mathbb{E} \left\{ r_{m_2}(t)\mathbb{r}_{n_2}(t) \right\} \]
Third term c):

\[
\sum_{i=1}^{N} \sum_{p=1}^{N} \sum_{q=1}^{N} \sum_{s=1}^{N} E \left\{ r_{k_1}(t) r_{i_1}^H(t) r_{m_1}(t) r_{k_2}(q) r_{l_2}(q) r_{m_2}(s) r_{n_2}(s) \right\} \\
= NE \left\{ r_{k_1}(t) r_{i_1}^H(t) r_{m_1}(t) r_{k_2}(t) r_{l_2}(t) r_{m_2}(t) r_{n_2}(t) \right\} \\
+ (N^2 - N)E \left\{ r_{k_1}(t) r_{i_1}^H(t) \right\} E \left\{ r_{m_1}(t) r_{n_1}(t) r_{k_2}(t) r_{l_2}(t) r_{m_2}(t) r_{n_2}(t) \right\} \\
+ (N^2 - N)E \left\{ r_{m_1}(t) r_{n_1}(t) \right\} E \left\{ r_{k_1}(t) r_{i_1}^H(t) r_{k_2}(t) r_{l_2}(t) r_{m_2}(t) r_{n_2}(t) \right\} \\
+ (N^2 - N)E \left\{ r_{k_2}(t) r_{l_2}(t) \right\} E \left\{ r_{k_1}(t) r_{i_1}^H(t) r_{m_1}(t) r_{n_1}(t) r_{m_2}(t) r_{n_2}(t) \right\} \\
+ (N^2 - N)E \left\{ r_{l_2}(t) r_{l_2}(t) \right\} E \left\{ r_{k_1}(t) r_{i_1}^H(t) r_{m_1}(t) r_{n_1}(t) r_{m_2}(t) r_{n_2}(t) \right\} \\
+ (N^2 - N)E \left\{ r_{m_2}(t) r_{n_2}(t) \right\} E \left\{ r_{k_1}(t) r_{i_1}^H(t) r_{m_1}(t) r_{n_1}(t) r_{m_2}(t) r_{n_2}(t) \right\} \\
+ (N^2 - N)E \left\{ r_{k_1}(t) r_{i_1}^H(t) r_{m_1}(t) r_{n_1}(t) \right\} E \left\{ r_{k_2}(t) r_{l_2}(t) r_{m_2}(t) r_{n_2}(t) \right\} \\
+ (N^2 - N)E \left\{ r_{k_1}(t) r_{i_1}^H(t) r_{m_1}(t) r_{n_1}(t) \right\} E \left\{ r_{k_2}(t) r_{l_2}(t) r_{m_2}(t) r_{n_2}(t) \right\} \\
+ (N^2 - N)E \left\{ r_{k_1}(t) r_{i_1}^H(t) r_{m_1}(t) r_{n_1}(t) \right\} E \left\{ r_{k_2}(t) r_{l_2}(t) r_{m_2}(t) r_{n_2}(t) \right\} \\
+ (N^2 - N)E \left\{ r_{k_1}(t) r_{i_1}^H(t) r_{m_1}(t) r_{n_1}(t) \right\} E \left\{ r_{k_2}(t) r_{l_2}(t) r_{m_2}(t) r_{n_2}(t) \right\} \\
+ (N^2 - N)E \left\{ r_{k_1}(t) r_{i_1}^H(t) r_{m_1}(t) r_{n_1}(t) \right\} E \left\{ r_{k_2}(t) r_{l_2}(t) r_{m_2}(t) r_{n_2}(t) \right\} \\
+ (N^2 - N)E \left\{ r_{k_1}(t) r_{i_1}^H(t) r_{m_1}(t) r_{n_1}(t) \right\} E \left\{ r_{k_2}(t) r_{l_2}(t) r_{m_2}(t) r_{n_2}(t) \right\} \\
+ (N^2 - N)E \left\{ r_{k_1}(t) r_{i_1}^H(t) r_{m_1}(t) r_{n_1}(t) \right\} E \left\{ r_{k_2}(t) r_{l_2}(t) r_{m_2}(t) r_{n_2}(t) \right\} \\
E \left\{ r_{k_2}(t) r_{l_2}(t) \right\} E \left\{ r_{m_2}(t) r_{n_2}(t) \right\} .
\]

(3.5)
Again, many terms in $a$, $b$ and $c$ can be obtained by changing indices. The key terms are:

I \quad E \left\{ r_{k_1} r_{k_1}^H r_{m_1}^H r_{n_1} r_{k_2} r_{n_2}^H r_{m_2}^H \right\}

II \quad E \left\{ r_{k_1} r_{k_1}^H r_{m_1}^H r_{n_1} r_{k_2} r_{n_2} \right\}

III \quad E \left\{ r_{k_1} r_{k_1}^H r_{m_1}^H r_{n_1} \right\}

IV \quad E \left\{ r_{k_1} r_{k_1}^H \right\}.

Closed-form formulas for II, III, IV were derived in the previous sections, so only term I must be calculated for the model defined in (1.1).

### 3.1 Eighth-order moment

For the eighth-order moment, we obtain the following $2^8$ terms:

\[
E \left\{ r_{k_1} r_{k_1}^H r_{m_1}^H r_{n_1} r_{k_2} r_{n_2}^H r_{m_2} r_{n_2}^H \right\}
= E \left\{ (b_{k_1} + n_{k_1})(b_{k_1}^H + n_{k_1}^H)(n_{m_1} + n_{m_1}^H)(b_{n_1} + n_{n_1}) \right. \\
\quad \left. (b_{k_2} + n_{k_2})(b_{k_2}^H + n_{k_2}^H)(b_{n_2} + n_{n_2})(b_{n_2}^H + n_{n_2}^H) \right\}
= E \left\{ b_{k_1} b_{k_1}^H b_{m_1} b_{m_1}^H b_{k_2} b_{k_2}^H b_{n_2} b_{n_2}^H \right\} + E \left\{ b_{k_1} b_{k_1}^H b_{m_1} b_{m_1}^H b_{k_2} b_{k_2}^H b_{n_2} b_{n_2}^H \right\} + \\
E \left\{ b_{k_1} b_{k_1}^H b_{m_1} b_{m_1}^H b_{k_2} b_{k_2}^H b_{n_2} b_{n_2}^H \right\} + E \left\{ b_{k_1} b_{k_1}^H b_{m_1} b_{m_1}^H b_{k_2} b_{k_2}^H b_{n_2} b_{n_2}^H \right\} + \\
E \left\{ b_{k_1} b_{k_1}^H b_{m_1} b_{m_1}^H b_{k_2} b_{k_2}^H b_{n_2} b_{n_2}^H \right\} + E \left\{ b_{k_1} b_{k_1}^H b_{m_1} b_{m_1}^H b_{k_2} b_{k_2}^H b_{n_2} b_{n_2}^H \right\} + \\
E \left\{ b_{k_1} b_{k_1}^H b_{m_1} b_{m_1}^H b_{k_2} b_{k_2}^H b_{n_2} b_{n_2}^H \right\} + E \left\{ b_{k_1} b_{k_1}^H b_{m_1} b_{m_1}^H b_{k_2} b_{k_2}^H b_{n_2} b_{n_2}^H \right\} + \\
E \left\{ b_{k_1} b_{k_1}^H b_{m_1} b_{m_1}^H b_{k_2} b_{k_2}^H b_{n_2} b_{n_2}^H \right\} + E \left\{ b_{k_1} b_{k_1}^H b_{m_1} b_{m_1}^H b_{k_2} b_{k_2}^H b_{n_2} b_{n_2}^H \right\} + \\
\quad \vdots
\]
\[ E \{ b_{k_1} b_{l_1}^H n_{m_1} n_{n_1} b_{k_2}^H b_{l_2} n_{m_2} b_{n_2}^H \} + E \{ b_{k_1} b_{l_1}^H b_{m_1}^H n_{n_1} b_{k_2}^H b_{l_2} n_{m_2} n_{n_2} \} +
\]
\[ E \{ n_{k_1} n_{l_1} b_{m_1} b_{h_1} \} n_{n_1} n_{k_2} b_{l_2} n_{m_2} b_{h_2} \} + E \{ n_{k_1} n_{l_1} b_{m_1} b_{h_1} \} n_{n_1} n_{k_2} b_{l_2} n_{m_2} b_{h_2} \} + \]
\[ E \{ n_{k_1} n_{l_1} b_{m_1} b_{h_1} \} n_{n_1} n_{k_2} b_{l_2} n_{m_2} b_{h_2} \} + E \{ n_{k_1} n_{l_1} b_{m_1} b_{h_1} \} n_{n_1} n_{k_2} b_{l_2} n_{m_2} b_{h_2} \} + \]
\[ E \{ n_{k_1} n_{l_1} n_{m_1} b_{h_1} b_{l_1} b_{m_2} b_{n_2} \} + E \{ n_{k_1} n_{l_1} n_{m_1} b_{h_1} b_{l_1} b_{m_2} b_{n_2} \} + \]
\[ E \{ n_{k_1} n_{l_1} n_{m_1} b_{h_1} b_{l_1} b_{m_2} b_{n_2} \} + E \{ n_{k_1} n_{l_1} n_{m_1} b_{h_1} b_{l_1} b_{m_2} b_{n_2} \} + \]
\[ E \{ n_{k_1} n_{l_1} n_{m_1} b_{h_1} b_{l_1} b_{m_2} b_{n_2} \} + E \{ n_{k_1} n_{l_1} n_{m_1} b_{h_1} b_{l_1} b_{m_2} b_{n_2} \} + \]
\[ E \{ n_{k_1} n_{l_1} n_{m_1} b_{h_1} b_{l_1} b_{m_2} b_{n_2} \} + E \{ n_{k_1} n_{l_1} n_{m_1} b_{h_1} b_{l_1} b_{m_2} b_{n_2} \} + \]
\[ E \{ n_{k_1} n_{l_1} n_{m_1} b_{h_1} b_{l_1} b_{m_2} b_{n_2} \} + E \{ n_{k_1} n_{l_1} n_{m_1} b_{h_1} b_{l_1} b_{m_2} b_{n_2} \} + \]
\[ E \{ n_{k_1} n_{l_1} n_{m_1} b_{h_1} b_{l_1} b_{m_2} b_{n_2} \} + E \{ n_{k_1} n_{l_1} n_{m_1} b_{h_1} b_{l_1} b_{m_2} b_{n_2} \} + \]
\[ E \{ n_{k_1} n_{l_1} n_{m_1} b_{h_1} b_{l_1} b_{m_2} b_{n_2} \} + E \{ n_{k_1} n_{l_1} n_{m_1} b_{h_1} b_{l_1} b_{m_2} b_{n_2} \} + \]
\[ E \{ n_{k_1} n_{l_1} n_{m_1} b_{h_1} b_{l_1} b_{m_2} b_{n_2} \} + E \{ n_{k_1} n_{l_1} n_{m_1} b_{h_1} b_{l_1} b_{m_2} b_{n_2} \} + \]
\[ E \{ n_{k_1} n_{l_1} n_{m_1} b_{h_1} b_{l_1} b_{m_2} b_{n_2} \} + E \{ n_{k_1} n_{l_1} n_{m_1} b_{h_1} b_{l_1} b_{m_2} b_{n_2} \} + \]
\[ E \{ n_{k_1} n_{l_1} n_{m_1} b_{h_1} b_{l_1} b_{m_2} b_{n_2} \} + E \{ n_{k_1} n_{l_1} n_{m_1} b_{h_1} b_{l_1} b_{m_2} b_{n_2} \} + \]
\[ E \{ n_{k_1} n_{l_1} n_{m_1} b_{h_1} b_{l_1} b_{m_2} b_{n_2} \} + E \{ n_{k_1} n_{l_1} n_{m_1} b_{h_1} b_{l_1} b_{m_2} b_{n_2} \} + \]
\[ E \{ n_{k_1} n_{l_1} n_{m_1} b_{h_1} b_{l_1} b_{m_2} b_{n_2} \} + E \{ n_{k_1} n_{l_1} n_{m_1} b_{h_1} b_{l_1} b_{m_2} b_{n_2} \} + \]
\[ E \{ n_{k_1} n_{l_1} n_{m_1} b_{h_1} b_{l_1} b_{m_2} b_{n_2} \} + E \{ n_{k_1} n_{l_1} n_{m_1} b_{h_1} b_{l_1} b_{m_2} b_{n_2} \} + \]
\[ E \{ n_{k_1} n_{l_1} n_{m_1} b_{h_1} b_{l_1} b_{m_2} b_{n_2} \} + E \{ n_{k_1} n_{l_1} n_{m_1} b_{h_1} b_{l_1} b_{m_2} b_{n_2} \} + \]
\[ E \{ n_{k_1} n_{l_1} n_{m_1} b_{h_1} b_{l_1} b_{m_2} b_{n_2} \} + E \{ n_{k_1} n_{l_1} n_{m_1} b_{h_1} b_{l_1} b_{m_2} b_{n_2} \} + \]
\[ E \{ n_{k_1} n_{l_1} n_{m_1} b_{h_1} b_{l_1} b_{m_2} b_{n_2} \} + E \{ n_{k_1} n_{l_1} n_{m_1} b_{h_1} b_{l_1} b_{m_2} b_{n_2} \} + \]
\[ E \{ n_{k_1} n_{l_1} n_{m_1} b_{h_1} b_{l_1} b_{m_2} b_{n_2} \} + E \{ n_{k_1} n_{l_1} n_{m_1} b_{h_1} b_{l_1} b_{m_2} b_{n_2} \} + \]
\[ E \{ n_{k_1} n_{l_1} n_{m_1} b_{h_1} b_{l_1} b_{m_2} b_{n_2} \} + E \{ n_{k_1} n_{l_1} n_{m_1} b_{h_1} b_{l_1} b_{m_2} b_{n_2} \} + \]
\[ E \{ n_{k_1} n_{l_1} n_{m_1} b_{h_1} b_{l_1} b_{m_2} b_{n_2} \} + E \{ n_{k_1} n_{l_1} n_{m_1} b_{h_1} b_{l_1} b_{m_2} b_{n_2} \} + \]
\[ E \{ n_{k_1} n_{l_1} n_{m_1} b_{h_1} b_{l_1} b_{m_2} b_{n_2} \} + E \{ n_{k_1} n_{l_1} n_{m_1} b_{h_1} b_{l_1} b_{m_2} b_{n_2} \} + \]
\[ E \{ n_{k_1} n_{l_1} n_{m_1} b_{h_1} b_{l_1} b_{m_2} b_{n_2} \} + E \{ n_{k_1} n_{l_1} n_{m_1} b_{h_1} b_{l_1} b_{m_2} b_{n_2} \} + \]
\[ E \{ n_{k_1} n_{l_1} n_{m_1} b_{h_1} b_{l_1} b_{m_2} b_{n_2} \} + E \{ n_{k_1} n_{l_1} n_{m_1} b_{h_1} b_{l_1} b_{m_2} b_{n_2} \} + \]
\[ E \{ n_{k_1} n_{l_1} n_{m_1} b_{h_1} b_{l_1} b_{m_2} b_{n_2} \} + E \{ n_{k_1} n_{l_1} n_{m_1} b_{h_1} b_{l_1} b_{m_2} b_{n_2} \} + \]

(3.6)
Following the same idea of changing indices, only 7 Terms in (3.6) need further examination. They are:

1) \[ E \left\{ b_{k_1} b_{i_1}^{H} b_{m_1}^{H} b_{n_1} b_{k_2} b_{i_2} b_{m_2} b_{n_2}^{H} \right\} = \]
\[ (a_{k_1}^{T} \otimes a_{i_1}^{H} \otimes a_{m_1}^{H} \otimes a_{n_1}^{T} \otimes a_{k_2}^{H} \otimes a_{i_2}^{H} \otimes a_{m_2}^{T} \otimes a_{n_2}^{H}) E \{ s \otimes s^* \otimes s \otimes s^* \otimes s \otimes s \otimes s \} \]

2) \[ E \left\{ b_{k_1} b_{i_1}^{H} b_{m_1}^{H} b_{n_1} b_{k_2} b_{i_2} b_{n_2}^{H} n_{n_2}^{H} \right\} = \]
\[ (a_{k_1}^{T} \otimes a_{i_1}^{H} \otimes a_{m_1}^{H} \otimes a_{n_1}^{T} \otimes a_{k_2}^{H} \otimes a_{i_2}^{H} \otimes a_{n_2}^{H}) E \{ s \otimes s^* \otimes s \otimes s \otimes s \} E \{ n_{m_2} n_{n_2}^{H} \} \]

3) \[ E \left\{ b_{k_1} b_{i_1}^{H} b_{m_1}^{H} b_{n_1} b_{k_2} b_{l_2} b_{n_2} n_{n_2}^{H} \right\} = \]
\[ (a_{k_1}^{T} \otimes a_{i_1}^{H} \otimes a_{m_1}^{H} \otimes a_{n_1}^{T} \otimes a_{k_2}^{H} \otimes a_{l_2}^{H}) E \{ s \otimes s^* \otimes s \otimes s \} E \{ n_{m_2} n_{n_2}^{H} \} \]

4) \[ E \left\{ b_{k_1} b_{i_1}^{H} b_{m_1}^{H} b_{n_1} b_{i_2} b_{l_2} b_{n_2} n_{n_2}^{H} \right\} = \]
\[ (a_{k_1}^{T} \otimes a_{i_1}^{H} \otimes a_{m_1}^{H} \otimes a_{n_1}^{T} \otimes a_{i_2}^{H}) E \{ s \otimes s^* \otimes s \otimes s \} E \{ n_{l_2} n_{m_2} n_{n_2}^{H} \} \]

5) \[ E \left\{ b_{k_1} b_{i_1}^{H} b_{m_1}^{H} b_{n_1} n_{k_2} n_{l_2} b_{n_2} n_{n_2}^{H} \right\} = \]
\[ (a_{k_1}^{T} \otimes a_{i_1}^{H} \otimes a_{m_1}^{H}) E \{ s \otimes s^* \otimes s \} E \{ n_{k_2} n_{l_2} n_{m_2} n_{n_2}^{H} \} \]

6) \[ E \left\{ b_{k_1} b_{i_1}^{H} n_{m_1} n_{k_2} n_{l_2} b_{n_2} n_{n_2}^{H} \right\} = \]
\[ (a_{k_1}^{T} \otimes a_{i_1}^{H}) E \{ s \otimes s \} E \{ n_{m_1} n_{k_2} n_{l_2} n_{m_2} n_{n_2}^{H} \} \]

7) \[ E \left\{ n_{k_1} n_{i_1} n_{m_1} n_{k_2} n_{l_2} b_{m_2} b_{n_2} n_{n_2}^{H} \right\} \]

where 7) can be directly obtained from the given noise moments. To complete the work, we must compute the expected value of the fourth-order sample cumulant.

\[ E \left\{ \hat{c}_{k_1,l_1,m_1,n_1}^{(4)} \right\} = \frac{1}{\alpha} \sum_{t=1}^{N} E \left\{ r_{k_1}(t) r_{l_1}^{H}(t) r_{m_1}^{H}(t) r_{n_1}(t) \right\} - \frac{1}{\beta} \sum_{t=1}^{N} \sum_{p=1}^{N} E \left\{ r_{k_1}(t) r_{l_1}^{H}(t) r_{m_1}^{H}(p) r_{n_1}(p) \right\} \]
\[-\frac{1}{\beta} \sum_{i=1}^{N} \sum_{p=1}^{N} \mathbb{E} \left\{ r_{k_1}(t) r_{l_i}^H(t) r_{m_1}^H(t) r_{n_1}(t) \right\} \]
\[-\frac{1}{\beta} \sum_{i=1}^{N} \sum_{p=1}^{N} \mathbb{E} \left\{ r_{k_1}(t) r_{l_i}^H(t) r_{m_1}^H(p) r_{n_1}(t) \right\} \]

\[= \frac{N}{\alpha} \mathbb{E} \left\{ r_{k_1}(t) r_{l_i}^H(t) r_{m_1}^H(t) r_{n_1}(t) \right\} \]
\[-\frac{N}{\beta} \mathbb{E} \left\{ r_{k_1}(t) r_{l_i}^H(t) r_{m_1}^H(t) r_{n_1}(t) \right\} - \frac{N^2}{\beta} \frac{N}{\beta} \mathbb{E} \left\{ r_{k_1}(t) r_{l_i}^H(t) \right\} \mathbb{E} \left\{ r_{m_1}^H(t) r_{n_1}(t) \right\} \]
\[-\frac{N}{\beta} \mathbb{E} \left\{ r_{k_1}(t) r_{l_i}^H(t) r_{m_1}^H(t) r_{n_1}(t) \right\} - \frac{N^2}{\beta} \frac{N}{\beta} \mathbb{E} \left\{ r_{k_1}(t) r_{l_i}^H(t) \right\} \mathbb{E} \left\{ r_{m_1}^H(t) r_{n_1}(t) \right\} \]
\[-\frac{N}{\beta} \mathbb{E} \left\{ r_{k_1}(t) r_{l_i}^H(t) r_{m_1}^H(t) r_{n_1}(t) \right\} - \frac{N^2}{\beta} \frac{N}{\beta} \mathbb{E} \left\{ r_{k_1}(t) r_{n_1}(t) \right\} \mathbb{E} \left\{ r_{l_i}^H(t) r_{m_1}^H(t) \right\} \]
\[
= \left( \frac{N}{\alpha} - \frac{3N}{\beta} \right) \mathbb{E} \left\{ r_{k_1}(t) r_{l_i}^H(t) r_{m_1}^H(t) r_{n_1}(t) \right\} \\
- \left( \frac{N^2}{\beta} - \frac{N}{\beta} \right) \left[ \mathbb{E} \left\{ r_{k_1}(t) r_{l_i}^H(t) \right\} \mathbb{E} \left\{ r_{m_1}^H(t) r_{n_1}(t) \right\} + \right. \\
\left. \mathbb{E} \left\{ r_{k_1}(t) r_{l_i}^H(t) \right\} \mathbb{E} \left\{ r_{m_1}^H(t) r_{n_1}(t) \right\} + \right. \\
\left. \mathbb{E} \left\{ r_{k_1}(t) r_{n_1}(t) \right\} \mathbb{E} \left\{ r_{l_i}^H(t) r_{m_1}^H(t) \right\} \right].
\]

Note, that if
\[\alpha = \frac{N^2 - N}{N + 2} \quad \text{and} \quad \beta = N^2 - N\]

the fourth-order sample cumulant is an unbiased estimator.
3.2 Finite-sample covariance of the fourth-order sample cumulant as a function of \( N \)

Since we deal with the finite-sample case, the principal behaviour of the finite-sample covariance of the fourth-order sample cumulant as a function of \( N \) is interesting. Using eqs. (3.2), (3.3), (3.4), (3.5), it can be shown that

\[
E \{ \hat{c}_{k_1,l_1,m_1,n_1}^{(4)} \hat{c}_{k_2,l_2,m_2,n_2}^{(4),H} \} = \frac{1}{\alpha^2} \left( N^2 c_1 + N c_2 \right) + \frac{1}{\alpha \beta} \left( N^3 c_3 + N^2 c_4 + N c_5 \right) + \frac{1}{\beta^2} \left( N^4 c_6 + N^3 c_7 + N^2 c_8 + N c_9 \right)
\]

and (3.7) can be written as

\[
E \{ \hat{c}_{k_1,l_1,m_1,n_1}^{(4)} \} = \frac{1}{\alpha} N c_{10} + \frac{1}{\beta} \left( N^2 c_{11} + N c_{12} \right), \tag{3.8}
\]

where \( c_1, ..., c_{12} \) are independent of \( N \). This leads to

\[
\text{Cov} \left( \hat{c}_{k_1,l_1,m_1,n_1}^{(4)}, \hat{c}_{k_2,l_2,m_2,n_2}^{(4),H} \right) = \frac{1}{\alpha^2} \left( N^2 b_1 + N b_2 \right) + \frac{1}{\alpha \beta} \left( N^3 b_3 + N^2 b_4 + N b_5 \right) + \frac{1}{\beta^2} \left( N^4 b_6 + N^3 b_7 + N^2 b_8 + N b_9 \right), \tag{3.9}
\]

where \( b_1, ..., b_9 \) are independent of \( N \). By some tedious calculation it can be shown, that

\[ b_1 = b_3 = b_6 = 0, \]

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which confirms that $\text{Cov} \left( \hat{c}_{k_1,l_1,m_1,n_1}, \hat{c}_{k_2,l_2,m_2,n_2}^{(i),H} \right)$ is $O(N^{-1})$ if $\alpha \sim N$ and $\beta \sim N^2$. Hence, the functional behaviour of the finite-sample covariance for the biased estimator ($\alpha = N$, $\beta = N^2$) is given by

$$\text{Cov} \left( \hat{c}_{k_1,l_1,m_1,n_1}, \hat{c}_{k_2,l_2,m_2,n_2}^{(i),H} \right) = \frac{a_1}{N} + \frac{a_2}{N^2} + \frac{a_3}{N^3}$$

where $a_1, a_2, a_3$ are independent of $N$. 
Chapter 4

M-Files

In this section we show how the finite-sample covariance can be implemented under MATLAB$^\text{TM}$ in so-called *M-Files*. We provide the code for:

1. "**cov2all.m**": Calculation of the finite-sample covariance of the second-order sample cumulant (moment).

2. "**cov3all.m**": Calculation of the finite-sample covariance of the third-order sample cumulant (moment).

3. "**cov4all.m**": Calculation of the finite-sample covariance of the fourth-order sample cumulant.

4. "**xmain.m**": Calculation of the expected value

   \[
   E \left\{ \prod_{j=1}^{J} X_j \right\}
   \]

   of the product of \( J \) complex random variables \( X_j \). The moments of these random variables must be given up to \( J \). The only purpose of this M-File is to simplify the notation in the other M-Files.
For reasons of clearness, the important equations in the M-Files are cross-referenced with the former sections and chapters.

### 4.1 cov2all.m

```matlab
function [cov2,E2all1,E2all2]=cov2all(N,index,A,gs,gn)

%COV2ALL Calculation of the covariance of the correlation estimator
%
c^2 = 1/N *sum_t=1^N r_h(t) * r_l^H(t)
%
by an analytic formula in case of the usual model in narrowband
array processing:
%
   r_vec(t) = A_mat s_vec(t) + n_vec(t)
%
where s_vec(t) includes the independent source signals, matrix A_mat
consists in the steering vectors and n_vec(t) is measurement noise
of the N sensors with arbitrary distribution. This M-File can be
also used if some or all of the signals are coherent by a proper
modification of the matrix A_mat (see "Applications of Cumulants
to Array Processing: Direction Finding in Coherent Signal
Environment", E. Goosen, M. C. Dogan, J. M. Mendel, 20th Asilomar
Conference, October 31-November 2, 1994, Pacific Grove, CA.).
Furthermore the near-field case can be also investigated by matrix
modification (see "Higher-Order Subspace Algorithms for Passive
Localization of Near-Field Sources", R. N. Challa and S. Shamander,
29th Annual Asilomar Conference on Signals, Systems and Computers,
Pacific Grove, CA, October 30 - November 1, 1995.

Syntax:

[cov2,E2all1,E2all2]=cov2all(N,index,A,gs,gn)

where

- `cov2` = Covariance of the correlation estimator (as a function of N).
- `E2all1` = Expected value of the correlation estimator for the first
two indices [k1 11] (as a function of N).
- `E2all2` = Expected value of the correlation estimator for the second
two indices [k2 12] (as a function of N).
- `N` = vector with data lengths.
- `index` = signal indices, index=[k1 11 k2 12], if k2=k1,12=11, then
```
calculation of the variance otherwise of the covariance

% A = Steering vector matrix
% gs = [gRs1,2 gRs1,4
gIs1,2 gIs1,4
   ...
gRsP,2 gRsP,4
gIsP,2 gIsP,4 ]

% Matrix that contains the second and the fourth of the
% source signals. The following example explains the
% abbreviations:
% gRs1,4 = 4th moment of the real part of signal 1
%
% gn = [gRn1,2 gRn1,4
gIn1,2 gIn1,4
   ...
gRnM,2 gRnM,4
gInM,2 gInM,4]

% Matrix that contains the second and the fourth moment of the
% noise signals. The abbreviations are analogous to "gs".

% Inserting the first- and third order moment only for a more systematic
% M-file regarding "cov3all.m" and "cov4all.m".
%H,gs_cols=sizeof(gs);
gs=[zeros(H,1) gs(:,1) zeros(H,1) gs(:,2)]; gs_cols=gs_cols+2;

% Inserting the third order moment only for a more systematic
% M-file regarding "cov3all.m" and "cov4all.m" (The first order moment
% is setting to zero by the M-file "xmaln.m".
%[H1,gn_cols]=sizeof(gn); gn_cols=gn_cols+1;
gn=[gn(:,1) zeros(H1,1) gn(:,2)];

% Checking for correct parameter dimensions
P=cell(H/2); % P=Number of sources
if (gs_cols^=4)
disp('gs-Matrix not valid')
return
end
M=cell(H1/2); % M=Number of Sensors
if (gn_cols^=3)
disp('gn-Matrix not valid')

return
end

[A_rows,A_cols]=size(A);
if (A_cols~=P) || (A_rows~=H)
disp('A-Matrix not valid')
return
end

% Index abbreviations
k1=index(1); l1=index(2); k2=index(3); l2=index(4);

% Noise abbreviations: E_{k11H} means E(n_{k1} n_{1H}) and so on ...
E111H=xma(n(k1 j*11),gn);
E1k2H=xma(n(k1 j*k2),gn);
E1l2=xma(n(k1 12),gn);
E11k2H=xma(n(j*11 j*k2),gn);
E11l2=xma(n(j*11 12),gn);
E12H=xma(n(j*k2 12),gn);
E111k2H=xma(n(k1 j*11 j*k2 12),gn);

% Calculation of modified steering vectors
A=[A sqrt((-1)*A)]; % a_{k(1:H)} = A(k,:)  

% Calculation of the source signal based Kronecker-products:
% E_{2m1S} means E(s \kron s*), E_{4m1S} means E(s \kron s* \kron s* \kron s*) ...
\z2=1:2;\z4=1:
E2m1S=ones(1,H^2);E4m1S=ones(1,H^4);

% Steering vector abbreviations
ak1=A(k1,:);
all1=conj(A(l1,:));
ak2H=conj(A(k2,:));
al2=A(l2,:));

for a=1:H
for b=1:H
% Begin 2nd signal moment vector
sig_ind=[sort([a b]) -1];
mom2=1;
for i=2:2+1
if sig_ind(i)==sig_ind(i-1)
mom2=mom2+1;
else

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E2malS(z2)=E2malS(z2)*gs(sig_ind(i-1),momz2);
    momz2=1;
    end
end
z2=z2+1;

%End 2nd signal moment vector
for c=1:H
    for d=1:H
        %Begin 4th signal moment vector
        sig_ind=[sort([a b c d]) -1];
        momz4=1;
        for i=2:4+1
            if sig_ind(i)==sig_ind(i-1)
                momz4=momz4+1;
            else
                E4malS(z4)=E4malS(z4)*gs(sig_ind(i-1),momz4);
                momz4=1;
            end
        end
        z4=z4+1;
        %End 4th signal moment vector
    end
end

%Kronecker-product abbreviations: K111h means (a_k1 \ kron a_1H) and so on...
K111H=kron(a_k1,a_1H);
K1k2H=kron(a_k1,a_k2H);
Kk112=kron(a_k1,a_12);
K11Hk2H=kron(a_1H,a_k2H);
K11Hk12=kron(a_1H,a_12);
Kk2H12=kron(a_k2H,a_12);
Kk11Hk2H12=kron(a_k1,kron(a_1H,kron(a_k2H,a_12)));

%Calculation of a) See Section "Fourth-order Moment"
%  
%  E(\{rk1 rl1H rk2H rl12\})
%  
%  Rk111Hk2H12=E4malS\prime+E4malS\prime+Ek111H*E2malS\prime*Ek2H12+...
%  Ek1k2H*E2malS\prime*E11H12+Kk112*E2malS\prime*E11Hk2H+...
%  K11Hk2H*E2malS\prime*Ek112+Kk111H*E2malS\prime*Ek1k2H+...
%  Ek2H12*E2malS\prime*Ek11H+Ek111Hk2H12;
%CALCULATION OF b1) See Section "Second-order Moment"
%  
%  E(rk1 r11H)
%
Rk11H=kk11H*E2ma15^*Ek11H;

%CALCULATION OF b1) See Section "Second-order Moment"
%
%  E(rk2H r12)
%
Rk2H12=kk2H12*E2ma15^*Ek2H12;

E2all1 = Rk11H;
E2all2 = Rk2H12;

cov2=(Rk11Hkk2H12-Ek11H*Rk2H12)/N;

4.2 cov3all.m

function [cov3,E3all1,E3all2]=cov3all(N,index,A,gs,gn)
%CUV3ALL Calculation of the covariance of the cumulant estimator
%
c = 1/alpha*sum_t=1^N r_k(t) * r_l"H(t) * r_m"H(t)
%
by an analytic formula in case of the usual model in narrowband
array processing:

    r_vec(t) = A_mat s_vec(t) + n_vec(t)
%
where s_vec(t) includes the independent source signals, matrix A_mat
consists in the steering vectors and n_vec(t) is measurement noise
of the M sensors with arbitrary distribution. This M-File can be
also used if some or all of the signals are coherent by a proper
modification of the matrix A_mat (see "Applications of Cumulants
to Array Processing: Direction Finding in Coherent Signal
Environment", E. Goemen, M. C. Bogan, J. M. Mendel, 28th. Asilomar
Furthermore the near-field case can be also investigated by matrix
modification (see "Higher-Order Subspace Algorithms for Passive
Localization of Near-Field Sources", R. H. Challa and S. Shamsunder,
29th Annual Asilomar Conference on Signals, Systems and Computers,
Syntax:
[cov3,E3all1,E3all2]=cov3all(n,index,A,gs,gn)

where

cov3 = Covariance of the cumulant estimator (as a function of n).
E3all1 = Expected value of the cumulant estimator for the first
three indices [k1 l1 m1] (as a function of n).
E3all2 = Expected value of the cumulant estimator for the second
three indices [k2 l2 m2] (as a function of n).
N = vector with data lengths.
index = signal indices, index=[k1 l1 m1 k2 l2 m2], if k2=k1,12=11
and m1=m2, then calculation of the variance otherwise of
the covariance.
A = Steering vector matrix
gs = [gs1,2 gs1,3 gs1,4 gs1,6
     gs1,2 gs1,3 gs1,4 gs1,6
     :     :     :     :
     gsP,2 gsP,3 gsP,4 gsP,6
     gsP,2 gsP,3 gsP,4 gsP,6 ]

Matrix that contains the second, third, fourth and sixth
moment of the source signals. The following example
explains the abbreviations:

gs1,4 = 4th moment of the real part of signal 1

gn = [gn1,2 gn1,3 gn1,4 gn1,6
     gn1,2 gn1,3 gn1,4 gn1,6
     :     :     :     :
     gnM,2 gnM,3 gnM,4 gnM,6
     gnM,2 gnM,3 gnM,4 gnM,6]

Matrix that contains the second, third, fourth and sixth-
moment of the noise signals. The abbreviations are
analogous to "gs".

% Inserting the first- and fifth-order moment only for a more systematic
% M-file regarding "cov2all.m" and "cov4all.m".
[N,gs_cols]=size(gs);
gs=[zeros(N,1) gs(:,1:3) zeros(N,1) gs(:,4)]; gs_cols=gs_cols+2;
% Inserting the fifth order moment only for a more systematic
% M-file regarding "cov3all.m" and "cov4all.m" (The first order moment
% is setting to zero by the M-file "xmnl.m".
[H1, gn_cols]=size(gn);
gn=[gn(:,1:3) zeros(H1,1) gn(:,4)]; gn_cols=gn_cols+1;

% Checking for correct parameter dimensions
P=ceil(H/2); % P=Number of sources
if (gn_cols==0)
    disp('gn-Matrix not valid')
    return
end
N=ceil(H1/2); % N=Number of sensors
if (gn_cols==5)
    disp('gn-Matrix not valid')
    return
end
[A_rows, A_cols]=size(A);
if (A_cols==P) || (A_rows==N)
    disp('A-Matrix not valid')
    return
end

% Index abbreviations
k1=index(1); l1=index(2); m1=index(3);
k2=index(4); l2=index(5); m2=index(6);

% Noise abbreviations: E(k1,l1,m1) means E(n_k1 n_l1 n_m1) and so on ...
% Sixth moment
E(k1,l1,m1,k2,l2,m2)=xmnln([k1 j=l1 m1 j=k2 l2 j=m2].gn);

% Fourth moments
E(k1,l1,m1,k2,l2)=xmnln([k1 j=l1 m1 j=k2 l2].gn);
E(k1,l1,m1,k2)=xmnln([k1 j=l1 m1 j=k2].gn);
E(k1,l1,m1)=xmnln([k1 j=l1 m1].gn);
E(k1,k2,l2)=xmnln([k1 j=k1 j=k2 l2].gn);
E(k1,k2)=xmnln([k1 j=k1 j=k2].gn);
E(k1)=xmnln([k1 j=k1].gn);
E(k2)=xmnln([k2 j=k2].gn);
E(m1,m2)=xmnln([m1 j=m1 j=m2].gn);
E(m2)=xmnln([m2 j=m2].gn);
E(k1,k2,m1,m2)=xmnln([k1 j=k1 j=k2 m1 m2].gn);
E(k1,k2,m2)=xmnln([k1 j=k1 j=k2 j=m2].gn);
E(k1,k2)=xmnln([k1 j=k1 j=k2].gn);
E(k2)=xmnln([k2 j=k2].gn);
E(k1)=xmnln([k1 j=k1].gn);
E(k2)=xmnln([k2 j=k2].gn);
E(m1,m2)=xmnln([m1 j=m1 j=m2].gn);
E(m2)=xmnln([m2 j=m2].gn);
E(k1,k2,m1,m2)=xmnln([k1 j=k1 j=k2 m1 m2].gn);
E(k1,k2)=xmnln([k1 j=k1 j=k2].gn);
E(k2)=xmnln([k2 j=k2].gn);
E(k1)=xmnln([k1 j=k1].gn);
E(k2)=xmnln([k2 j=k2].gn);
E(m1,m2)=xmnln([m1 j=m1 j=m2].gn);
E(m2)=xmnln([m2 j=m2].gn);
E(k1,k2,m1,m2)=xmnln([k1 j=k1 j=k2 m1 m2].gn);
E(k1,k2)=xmnln([k1 j=k1 j=k2].gn);
E(k2)=xmnln([k2 j=k2].gn);
E(k1)=xmnln([k1 j=k1].gn);
E(k2)=xmnln([k2 j=k2].gn);
E(m1,m2)=xmnln([m1 j=m1 j=m2].gn);
E(m2)=xmnln([m2 j=m2].gn);
E(k1,k2,m1,m2)=xmnln([k1 j=k1 j=k2 m1 m2].gn);
E(k1,k2)=xmnln([k1 j=k1 j=k2].gn);
E(k2)=xmnln([k2 j=k2].gn);
E(k1)=xmnln([k1 j=k1].gn);
E(k2)=xmnln([k2 j=k2].gn);
E(m1,m2)=xmnln([m1 j=m1 j=m2].gn);
E(m2)=xmnln([m2 j=m2].gn);
E(k1,k2,m1,m2)=xmnln([k1 j=k1 j=k2 m1 m2].gn);
E(k1,k2)=xmnln([k1 j=k1 j=k2].gn);
E(k2)=xmnln([k2 j=k2].gn);
E(k1)=xmnln([k1 j=k1].gn);
E(k2)=xmnln([k2 j=k2].gn);
E(m1,m2)=xmnln([m1 j=m1 j=m2].gn);
E(m2)=xmnln([m2 j=m2].gn);
E(k1,k2,m1,m2)=xmnln([k1 j=k1 j=k2 m1 m2].gn);
E(k1,k2)=xmnln([k1 j=k1 j=k2].gn);
E(k2)=xmnln([k2 j=k2].gn);
E(k1)=xmnln([k1 j=k1].gn);
E(k2)=xmnln([k2 j=k2].gn);
E(m1,m2)=xmnln([m1 j=m1 j=m2].gn);
E(m2)=xmnln([m2 j=m2].gn);
E(k1,k2,m1,m2)=xmnln([k1 j=k1 j=k2 m1 m2].gn);
E(k1,k2)=xmnln([k1 j=k1 j=k2].gn);
E(k2)=xmnln([k2 j=k2].gn);
Third moments

\begin{align*}
E_{1111} & = \text{xmain}((j=11 k=1 m=12, gn)); \\
E_{1112} & = \text{xmain}((j=11 k=2 m=12, gn)); \\
E_{1122} & = \text{xmain}((j=12 k=2 m=12, gn)); \\
E_{1113} & = \text{xmain}((j=11 k=3 m=12, gn)); \\
E_{1123} & = \text{xmain}((j=12 k=3 m=12, gn)); \\
E_{1133} & = \text{xmain}((j=13 k=3 m=12, gn)); \\
E_{1222} & = \text{xmain}((j=12 k=2 m=12, gn)); \\
E_{1223} & = \text{xmain}((j=12 k=2 m=12, gn)); \\
E_{1233} & = \text{xmain}((j=12 k=3 m=12, gn));
\end{align*}

Second moments

\begin{align*}
E_{1111} & = \text{xmain}((j=11 m=1, gn)); \\
E_{1112} & = \text{xmain}((j=11 k=2 m=12, gn)); \\
E_{1122} & = \text{xmain}((j=12 k=2 m=12, gn)); \\
E_{1113} & = \text{xmain}((j=11 k=3 m=12, gn)); \\
E_{1123} & = \text{xmain}((j=12 k=3 m=12, gn)); \\
E_{1133} & = \text{xmain}((j=13 k=3 m=12, gn)); \\
E_{1222} & = \text{xmain}((j=12 k=2 m=12, gn)); \\
E_{1223} & = \text{xmain}((j=12 k=2 m=12, gn)); \\
E_{1233} & = \text{xmain}((j=12 k=3 m=12, gn));
\end{align*}
E12m2H=xmalf([l2 j*m2].gn);

%Calculation of the modified steering matrix
A=[A sqrt(-1)*A];          %a_k(1:H)= A(k,:)

%Calculation of the source signal based Kronecker-products:
'E2malS means E(s \ kron s*), E4malS means E(s \ kron s* \ kron s) ...
 z2=1; z3=1; z4=1; z6=1; E2malS=ones(1,H*2); E3malS=ones(1,H*3); E4malS=ones(1,H*4); E6malS=ones(1,H*6);

%Steering vector abbreviations
ak1=A(k1,:);
allH=conj(A(l1,:));
am1=A(m1,:);
ak2H=conj(A(k2,:));
all2=A(l2,:);
am2H=conj(A(m2,:));

for a=1:H
    for b=1:H
        %Begin 2nd signal moment vector
        sig_ind=[sort([a b]) -1];
        momz2=1;
        for i=2:2+1
            if sig_ind(i)==sig_ind(i-1)
                momz2=momz2+1;
            else
                E2malS(z2)=E2malS(z2)+g0(sig_ind(i-1),momz2);
                momz2=1;
            end
        end
        z2=z2+1;
        %End 2nd signal moment vector
        for c=1:H
            %Begin 3rd signal moment vector
            sig_ind=[sort([a b c]) -1];
            momz3=1;
            for i=2:3+1
                if sig_ind(i)==sig_ind(i-1)
                    momz3=momz3+1;
                else
                    E3malS(z3)=E3malS(z3)+g0(sig_ind(i-1),momz3);
                    momz3=1;
                end
            end
            z3=z3+1;
            %End 3rd signal moment vector
        end
    end
end
E3ma1S(z3)=E3ma1S(z3)*gs(sig_ind(i-1),momz3);
momz3=1;
end
end
z3=z3+1;
%
End 3rd signal moment vector

for d=1:N
%Begin 4th signal moment vector
sig_ind=[sort([a b c d]) -1];
momz4=1;
for i=2:4+1
if sig_ind(i)==sig_ind(i-1)
momz4=momz4+1;
else
E4ma1S(z4)=E4ma1S(z4)*gs(sig_ind(i-1),momz4);
momz4=1;
end
end
z4=z4+1;
%
End 4th signal moment vector
for e=1:N
for f=1:N
%Begin 6th signal moment vector
sig_ind=[sort([a b c d e f]) -1];
momz6=1;
for i=2:6+1
if sig_ind(i)==sig_ind(i-1)
momz6=momz6+1;
else
E6ma1S(z6)=E6ma1S(z6)*gs(sig_ind(i-1),momz6);
momz6=1;
end
end
z6=z6+1;
%
End 6th signal moment vector
end
end
\(\text{Kronecker product abbreviations: } E_{k_{1}}^{1} \text{ means } (a_{k1} \otimes \text{kron } a_{l1H}) \text{ and so on...} \)

\[
\text{Kt11Hm1k2H12m2H} = \text{kron}(ak1, \text{kron}(al1H, \text{kron}(am1, \text{kron}(ak2H, \text{kron}(al2, am2H)))))
\]

\[
\text{Kt11Hm1k2H12m2H} = \text{kron}(ak1, \text{kron}(al1H, \text{kron}(am1, \text{kron}(ak2H, \text{kron}(am2H)))))
\]

\[
\text{Kt11Hm1k2H12m2H} = \text{kron}(ak1, \text{kron}(al1H, \text{kron}(ak2H, am2H)))
\]

\[
\text{Kt11Hm1k2H12m2H} = \text{kron}(ak1, \text{kron}(al1H, \text{kron}(al2, am2H)))
\]

\[
\text{Kt11Hm1k2H12m2H} = \text{kron}(ak1, \text{kron}(al1H, \text{kron}(al2, am2H)))
\]

\[
\text{Kt11Hm1k2H12m2H} = \text{kron}(ak1, \text{kron}(alam1, \text{kron}(ak2H, al2)))
\]

\[
\text{Kt11Hm1k2H12m2H} = \text{kron}(ak1, \text{kron}(alam1, \text{kron}(ak2H, al2)))
\]

\[
\text{Kt11Hm1k2H12m2H} = \text{kron}(ak1, \text{kron}(alam1, \text{kron}(ak2H, am2H)))
\]

\[
\text{Kt11Hm1k2H12m2H} = \text{kron}(ak1, \text{kron}(alam1, \text{kron}(ak2H, am2H)))
\]

\[
\text{Kt11Hm1k2H12m2H} = \text{kron}(ak1, \text{kron}(alam1, \text{kron}(ak2H, am2H)))
\]

\[
\text{Kt11Hm1k2H12m2H} = \text{kron}(ak1, \text{kron}(alam1, \text{kron}(ak2H, am2H)))
\]

\[
\text{Kt11Hm1k2H12m2H} = \text{kron}(ak1, \text{kron}(alam1, \text{kron}(ak2H, am2H)))
\]

\[
\text{Kt11Hm1k2H12m2H} = \text{kron}(ak1, \text{kron}(alam1, \text{kron}(ak2H, am2H)))
\]

\[
\text{Kt11Hm1k2H12m2H} = \text{kron}(ak1, \text{kron}(alam1, \text{kron}(ak2H, am2H)))
\]

\[
\text{Kt11Hm1k2H12m2H} = \text{kron}(ak1, \text{kron}(alam1, \text{kron}(ak2H, am2H)))
\]

\[
\text{Kt11Hm1k2H12m2H} = \text{kron}(ak1, \text{kron}(alam1, \text{kron}(ak2H, am2H)))
\]
% Two
E(k11H=kron(a11H, a11H));
E(k1m1=kron(a11, a1m1));
E(k1k2H=kron(a1k, a2H));
E(k1k2H=kron(a1k, a12));
E(k1m2H=kron(a1k, a1m2H));
E(k11Hm1=kron(a11H, a1m1));
E(k11Hk2H=kron(a11H, a2H));
E(k11Hm12=kron(a11H, a12));
E(k1m2H=kron(a1m2H, a2H));
E(k11Hm2H=kron(a11H, a2m2H));
E(k1m2H=kron(a1m2H, a2m2H));
E(k1k2H=kron(a2H, a12));
E(k1k2H=kron(a2H, a2m2H));
E(k12m2H=kron(a12, a2m2H));

% CALCULATION OF a) See Section "Sixth-order Moment"

TERM1 = ...
E(k11Hm1k2H12m2H=E6ma15sync11Hm1k2H=E6ma15 *E12m2H+ ... 
E(k11Hm12=E4ma15 sync1k2Hm12=E4ma15 *Ek2H12+ ... 
E(k11Hk2H12=E4ma15 *E1m12k2H=E4ma15 *Em12+ ... 
E(k11Hk2m2H=E4ma15 *Ek1m2Hk2H=E4ma15 *Ek1Hm2H+ ... 
E(k1k2H12m2H=E4ma15 *Ek1Hm1k2H12=E4ma15 *Ek1Hm12+ ... 
E(k1k2m2H=E4ma15 *Ek1Hm1k2m2H=E4ma15 *Ek1Hm12+ ... 
E(k1k2m2H=E4ma15 *Ek1Hm1k2m2H=E4ma15 *Ek1Hm12+ ... 
E(k11Hm1,E3ma15 syncE2k2Hm2H=E3ma15 *Em12m2H+ ... 
E(k11Hm12,E3ma15 syncE2k2Hm2H=E3ma15 *Em12m2H+ ... 
E(k11Hm1k2H=E3ma15 syncE1Hm12k2H=E3ma15 *E11Hm2H+ ... 
E(k11Hm12=E3ma15 syncE1Hm12k2H=E3ma15 *E11Hm2H+ ... 
E(k1k2Hm2H=E3ma15 *E11Hk2H2H=E3ma15 *E11Hk2H+ ... 
E(k1k2m2H=E3ma15 *E11Hk2m2H=E3ma15 *E11Hk2m2H+ ... 
E(k1k2m2H=E3ma15 *E11Hk2m2H=E3ma15 *E11Hk2m2H+ ... 
E(k11Hm1k2H=E3ma15 *E11Hm1k2H=E3ma15 *E11Hm1k2H+ ... 
E(k11Hm12=E3ma15 *E11Hm12k2H=E3ma15 *E11Hm12k2H+ ... 
E(k11Hm1k2H=E3ma15 *E11Hm1k2H=E3ma15 *E11Hm1k2H+ ... 

TERM2 = ...
E(k11Hm12=E3ma15 *Ek1k2H12=E3ma15 *Ek11Hk2H12=E3ma15 *E1k1m2H+ ... 
E(k11Hk2m2H=E3ma15 *Ek1m12k2H=E3ma15 *Ek11Hk2m2H+ ... 
E(k11Hm12k2H=E3ma15 *Ek1H1m2H=E3ma15 *Ek11H12+ ...
Km12m2H*E3ma1S'='Ek111Hk2H+Kk2H12m2H*E3ma1S'='Ek111Hm1+...
Kk11Hm*E2ma1S'='E3m1k2H12m2H*Kk11Hm*E2ma1S'='E11Hk2H12m2H+...
Kk1k2H*E2ma1S'='E3m1k2H12m2H*Kk112+Kk12m1S'='E11Hm1k2H2m2H+...
Kk1m2H*E2ma1S'='E11Hm1k2H12+Kk11Hm1*E2ma1S'='E11k2H12m2H+...
K11Hk2H*E2ma1S'='E3m112m2H*Kk11H12+Kk2ma1S'='E3ma1k2Hm1+...
K11Hm1*E3ma1S'='E11k2H12m2H*Kk1k2H12m2H*+E3ma1S'='E11k112m2H+...
K112m1S'='E2ma1S'='E3ma1k2H12m2H*Kk11Hm1*E2ma1S'='E11k1H12m2H+...
K12H12m2H*E2ma1S'='E3m112m2H*Kk12Hm2H*E3ma1S'='E1111m112+...
K12m2H*E2ma1S'='E1111H11k2H12m2H*Kk111Hm1k2H12m2H;

Rk111H1m1k2H12m2H=TERM1+TERM2;
%\text{CALCULATION OF b1) See Section "Third-order Moment"}
%\text{E(rk1 r11 H r11)}\text{=}Rk111Hm1=Ek111Hm1*E3ma1S'='Ek111Hm1;
%\text{CALCULATION OF b2) See Section "Third-order Moment"}
%\text{E(rk2 r12 r2H)}\text{=}Rk2H12m2H=Kk2H12m2H*E3ma1S'='Ek2H12m2H;

E3all1 = Rk111Hm1;
E3all2 = Rk2H12m2H;

\text{cov3 = (Rk111Hm1Kk2H12m2H-Rk111Hm1*Rk2H12m2H)./N;}

4.3 \text{cov4all.m}

function [cov4,E4all1,E4all2,cov4biased,relerror,relerrorbiased]=
cov4all(n,index,A,gs,gn,alpha,beta)
%COV4ALL Calculation of the covariance of the cumulant estimator
%\text{c = 1/alpha}*\text{sum}_t=1^n r_k(t) * r_l^-H(t) * r_m^-H(t) * r_n(t)
%\text{-1/beta}*(\text{sum}_t=1^n r_k(t)*r_l^-H(t))*(\text{sum}_t=1^n r_m^-H(t)*r_n(t))
%\text{-1/beta}*(\text{sum}_t=1^n r_k(t)*r_m^-H(t))*(\text{sum}_t=1^n r_l^-H(t)*r_n(t))
%\text{-1/beta}*(\text{sum}_t=1^n r_k(t)*r_n(t))*(\text{sum}_t=1^n r_l^-H(t)*r_m^-H(t))

% by an analytic formula in case of the usual model in narrowband
% array processing:

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\[ r_{vec}(t) = A_{mat} s_{vec}(t) + n_{vec}(t) \]

where \( s_{vec}(t) \) includes the independent source signals, matrix \( A_{mat} \) consists in the steering vectors and \( n_{vec}(t) \) is measurement noise of the \( N \) sensors with arbitrary distribution. This M-File can be also used if some or all of the signals are coherent by a proper modification of the matrix \( A_{mat} \) (see "Applications of Cumulants to Array Processing: Direction Finding in Coherent Signal Environment", E. Geelen, M. C. Egen, J. M. Mendel, 28th. Asilomar Conference, October 31-November 2, 1994, Pacific Grove, CA.). Furthermore the near-field case can be also investigated by matrix localization of Near-Field Sources", R. M. Challa and S. Sharma, 29th Annual Asilomar Conference on Signals, Systems and Computers, Pacific Grove, CA, October 30 - November 1, 1996.

Syntax:

\[
[cov4,E4all1,E4all2,cov4biased,relerror,relerrorbiased] = \text{cov4all}(N,index,A,gs,gn,alphn,beta,r)
\]

where

- \( cov4 = \) Covariance of the cumulant estimator (as a function of \( N \)).
- \( E4all1 = \) Expected value of the cumulant estimator for the first four indices \([k_1 l_1 m_1 n_1]\) (as a function of \( N \)).
- \( E4all2 = \) Expected value of the cumulant estimator for the second four indices \([k_2 l_2 m_2 n_2]\) (as a function of \( N \)).
- \( cov4biased = \) If more than three output arguments exists, then the covariance of the "Biased Estimator" is also calculated.
- \( relerror = \) Calculation of the relativ error between \( N \ast \text{Cov}(c^\ast,N) \) and \( \text{infy} \ast \text{Cov}(c^\ast,\text{infy}) \) as a function of \( N \).
- \( \text{relerrorbiased} = \) If more than five output arguments exists, then relativ error in case of the "Biased Estimator" is also calculated.

\( N = \) vector with data lengths.

\( index = \) signal indices, \( index = [k_1 l_1 m_1 n_1 k_2 l_2 m_2 n_2] \), if \( k_2 = k_1 \), \( l_2 = l_1 \), \( m_2 = m_1 \) and \( n_2 = n_1 \), then calculation of the variance otherwise of the covariance.

\( A = \) Steering vector matrix

\[
gs = [gRs1,2 gRs1,3 gRs1,4 gRs1,5 gRs1,6 gRs1,7 gRs1,8 gIs1,2 gIs1,3 gIs1,4 gIs1,5 gIs1,6 gIs1,7 gIs1,8 \\
gRsP,2 gRsP,3 gRsP,4 gRsP,5 gRsP,6 gRsP,7 gRsP,8 gIsP,2 gIsP,3 gIsP,4 gIsP,5 gIsP,6 gIsP,7 gIsP,8]
\]
Matrix that contains the second, third, fourth, fifth, sixth, seventh and eighth-moment of the source signals.
The following example explains the abbreviations:

gRn1,4 = 4th moment of the real part of signal 1

gn = [gRn1,2 gRn1,3 gRn1,4 gRn1,5 gRn1,6 gRn1,7 gRn1,8
gin1,2 gin1,3 gin1,4 gin1,5 gin1,6 gin1,7 gin1,8
  :   :   :   :   :   :   :
gRnM,2 gRnM,3 gRnM,4 gRnM,5 gRnM,6 gRnM,7 gRnM,8
ginM,2 ginM,3 ginM,4 ginM,5 ginM,6 ginM,7 ginM,8]

Matrix that contains the second, third, fourth, fifth, sixth, seventh and eighth-moment of the noise signals.
The abbreviations are analogous to "gs".

alpha = If alpha=(N^2-M)/(M^2) and beta=M^2-1, than the "Unbiased"
beta = Estimator is chosen and otherwise a "Biased Estimator"
(Usual alpha=M and beta=M^2).

Please note, that P>2 needs a huge amount of memory since vectors of the
size (2*P)^2 are used.

Inserting the first-order moment only for a more systematic
XM-file regarding "cov3all.m" and "cov4all.m".
[H,gm_cols]=size(gs);
gs=[zeros(H,1) gs]; gm_cols=gs_cols+1;

[H1,gm_cols]=size(gn);

Checking for correct parameter dimensions
P=ceil(H/2);                          % P=Number of sources
if (gs_cols<8)
disp('gs-Matrix not valid')
return
end
M=ceil(H1/2);                         % M=Number of sensors
if (gn_cols<7)
disp('gn-Matrix not valid')
return
end
[A_rows,A_cols]=size(A);
if (A_{cols}^{=P})(A_{rows}^{=N})
    disp('A-Matrix not valid')
    return
end

if nargout>4 %Calculation of the asymptotic solution
    Nasymp=10^{-6}*N(length(N));
    N=[N Nasymp];
    if nargin>5, alpha=[alpha Nasymp]; end;
    if nargin>6, beta=[beta Nasymp^2]; end;
end

%Index-abbreviations
k1=index(1); l1=index(2); m1=index(3); n1=index(4);
k2=index(5); l2=index(6); m2=index(7); n2=index(8);

%Noise-abbreviations: E_mn2H means E(n,m2 n,n2H) and so on ...

%T11111
Em2n2H=xmaln([m2 j*n2],gn);
E12n2H=xmaln([12 j*n2],gn);
E12m2=xmaln([12 m2],gn);
E12m2n2H=xmaln([12 m2 j*n2],gn);

%T11112
Ek2H2nH=xmaln([j*k2 j*n2],gn);
Ek2H2m2=xmaln([j*k2 m2],gn);
Ek2H2m2n2H=xmaln([j*k2 m2 j*n2],gn);
Ek2H12=xmaln([j*k2 12],gn);
Ek2H12n2H=xmaln([j*k2 12 j*n2],gn);
Ek2H12m2=xmaln([j*k2 12 m2],gn);
Ek2H12m2n2H=xmaln([j*k2 12 m2 j*n2],gn);

%T11121
En1n2H=xmaln([n1 j*n2],gn);
En1m2=xmaln([n1 m2],gn);
En1m2n2H=xmaln([n1 m2 j*n2],gn);
En112=xmaln([n1 12],gn);
En112n2H=xmaln([n1 12 j*n2],gn);
En112m2=xmaln([n1 12 m2],gn);
En112m2n2H=xmaln([n1 12 m2 j*n2],gn);

%T11122
En1k2H=xmaln([n1 j*k2],gn);

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En1kH2nH=xmaeln([m j=k2 j=n2], gn);
En1kH2m2=xmaeln([m j=k2 m=2], gn);
En1kH2m2nH=xmaeln([m j=k2 m=2 j=n2], gn);
En1kH2H2=xmaeln([m j=k2 12], gn);
En1kH2H2n2H=xmaeln([m j=k2 12 j=n2], gn);
En1kH2H2m2=xmaeln([m j=k2 12 m=2], gn);
En1kH2H2m2n2H=xmaeln([m j=k2 12 m=2 j=n2], gn);

XT1121
Em1Hn2H=xmaeln([j m=1 j=n2], gn);
Em1Hm2=xmaeln([j m=1 m=2], gn);
Em1Hm2nH=xmaeln([j m=1 m=2 j=n2], gn);
Em1Hl2=xmaeln([j m=1 12], gn);
Em1Hl2nH=xmaeln([j m=1 12 j=n2], gn);
Em1Hl2m2=xmaeln([j m=1 12 m=2], gn);
Em1Hl2m2n2H=xmaeln([j m=1 12 m=2 j=n2], gn);

XT1122
Em1HkH2=xmaeln([j m=1 j=k2], gn);
Em1HkH2nH=xmaeln([j m=1 j=k2 j=n2], gn);
Em1HkH2m2=xmaeln([j m=1 j=k2 m=2], gn);
Em1HkH2m2nH=xmaeln([j m=1 j=k2 m=2 j=n2], gn);
Em1HkH2H2=xmaeln([j m=1 j=k2 12], gn);
Em1HkH2H2n2H=xmaeln([j m=1 j=k2 12 j=n2], gn);
Em1HkH2H2m2=xmaeln([j m=1 j=k2 12 m=2], gn);
Em1HkH2H2m2n2H=xmaeln([j m=1 j=k2 12 m=2 j=n2], gn);

XT1121
Em1Hn1=xmaeln([j m=1 n=1], gn);
Em1Hn1n2H=xmaeln([j m=1 n=1 j=n2], gn);
Em1Hn1m2=xmaeln([j m=1 n=1 m=2], gn);
Em1Hn1m2n2H=xmaeln([j m=1 n=1 m=2 j=n2], gn);
Em1Hn1l2=xmaeln([j m=1 n=1 12], gn);
Em1Hn1l2n2H=xmaeln([j m=1 n=1 12 j=n2], gn);
Em1Hn1l2m2=xmaeln([j m=1 n=1 12 m=2], gn);
Em1Hn1l2m2n2H=xmaeln([j m=1 n=1 12 m=2 j=n2], gn);

XT1122
Em1Hn1kH2=xmaeln([j m=1 n=1 j=k2], gn);
Em1Hn1kH2n2H=xmaeln([j m=1 n=1 j=k2 j=n2], gn);
Em1Hn1kH2m2=xmaeln([j m=1 n=1 j=k2 m=2], gn);
Em1Hn1kH2m2n2H=xmaeln([j m=1 n=1 j=k2 m=2 j=n2], gn);
Em1Hn1kH2H2=xmaeln([j m=1 n=1 j=k2 12], gn);

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E11Hn1k2H12n2H=xmaeln([j*m1 n1 j*k2 12 j*n2],gn);
E11Hn1k2H12m2=xmaeln([j*m1 n1 j*k2 12 m2],gn);
E11Hn1k2H12m2n2H=xmaeln([j*m1 n1 j*k2 12 m2 j*n2],gn);

XT12111
E11Hn2H=xmaeln([j*n1 j*n2],gn);
E11Hn2m2=xmaeln([j*n1 m2],gn);
E11Hn2n2H=xmaeln([j*n1 m2 j*n2],gn);
E11H12=xmaeln([j*n1 12],gn);
E11H12n2H=xmaeln([j*n1 12 j*n2],gn);
E11H12m2=xmaeln([j*n1 12 m2],gn);
E11H12m2n2H=xmaeln([j*n1 12 m2 j*n2],gn);

XT12112
E11Hk2H=xmaeln([j*k1 j*k2],gn);
E11Hk2Hn2H=xmaeln([j*k1 j*k2 j*n2],gn);
E11Hk2Hm2=xmaeln([j*k1 j*k2 m2],gn);
E11Hk2Hm2n2H=xmaeln([j*k1 j*k2 m2 j*n2],gn);
E11Hk2H12=xmaeln([j*k1 j*k2 12],gn);
E11Hk2H12n2H=xmaeln([j*k1 j*k2 12 j*n2],gn);
E11Hk2H12m2=xmaeln([j*k1 j*k2 12 m2],gn);
E11Hk2H12m2n2H=xmaeln([j*k1 j*k2 12 m2 j*n2],gn);

XT12121
E11Hn1=xmaeln([j*n1],gn);
E11Hn1n2H=xmaeln([j*n1 n1 j*n2],gn);
E11Hn1n2H=xmaeln([j*n1 n1 j*n2],gn);
E11Hn1n2H=xmaeln([j*n1 n1 j*n2],gn);
E11Hn1n2H=xmaeln([j*n1 n1 j*n2],gn);
E11Hn1n2H=xmaeln([j*n1 n1 j*n2],gn);
E11Hn1n2H=xmaeln([j*n1 n1 j*n2],gn);

XT12122
E11Hn1k2H=xmaeln([j*n1 j*k2],gn);
E11Hn1k2Hn2H=xmaeln([j*n1 j*k2 j*n2],gn);
E11Hn1k2Hm2=xmaeln([j*n1 j*k2 m2],gn);
E11Hn1k2Hm2n2H=xmaeln([j*n1 j*k2 m2 j*n2],gn);
E11Hn1k2H12=xmaeln([j*n1 j*k2 12],gn);
E11Hn1k2H12n2H=xmaeln([j*n1 j*k2 12 j*n2],gn);
E11Hn1k2H12m2=xmaeln([j*n1 j*k2 12 m2],gn);
E1Hm1k2H12m2n2H=xma1n([j*11 n1 j*k2 12 m2 j*n2],gn);

%T12211
E1Hm1H=xma1n([j*11 j*m1],gn);
E1Hm1Hn2H=xma1n([j*11 j*m1 j*n2],gn);
E1Hm1Hm2=xma1n([j*11 j*m1 m2],gn);
E1Hm1Hm2n2H=xma1n([j*11 j*m1 m2 j*n2],gn);
E1Hm1Hl2=xma1n([j*11 j*m1 12],gn);
E1Hm1Hl2n2H=xma1n([j*11 j*m1 12 j*n2],gn);
E1Hm1Hn2=xma1n([j*11 j*m1 12 m2],gn);
E1Hm1Hn2n2H=xma1n([j*11 j*m1 12 m2 j*n2],gn);

%T12212
E1Hm1Hk2H=xma1n([j*11 j*m1 j*k2],gn);
E1Hm1Hk2Hn2H=xma1n([j*11 j*m1 j*k2 j*n2],gn);
E1Hm1Hk2Hm2=xma1n([j*11 j*m1 j*k2 m2],gn);
E1Hm1Hk2Hm2n2H=xma1n([j*11 j*m1 j*k2 m2 j*n2],gn);
E1Hm1Hk2Hl2=xma1n([j*11 j*m1 j*k2 12],gn);
E1Hm1Hk2Hl2n2H=xma1n([j*11 j*m1 j*k2 12 j*n2],gn);
E1Hm1Hk2Hl2m2=xma1n([j*11 j*m1 j*k2 12 m2],gn);
E1Hm1Hk2Hl2m2n2H=xma1n([j*11 j*m1 j*k2 12 m2 j*n2],gn);

%T12221
E1Hm1Hn1=xma1n([j*11 j*m1 n1],gn);
E1Hm1Hn1n2H=xma1n([j*11 j*m1 n1 j*n2],gn);
E1Hm1Hn1m2=xma1n([j*11 j*m1 n1 m2],gn);
E1Hm1Hn1m2n2H=xma1n([j*11 j*m1 n1 m2 j*n2],gn);
E1Hm1Hn1l2=xma1n([j*11 j*m1 n1 12],gn);
E1Hm1Hn1l2n2H=xma1n([j*11 j*m1 n1 12 j*n2],gn);
E1Hm1Hn1l2m2=xma1n([j*11 j*m1 n1 12 m2],gn);
E1Hm1Hn1l2m2n2H=xma1n([j*11 j*m1 n1 12 m2 j*n2],gn);

%T12222
E1Hm1Hn1k2H=xma1n([j*11 j*m1 n1 j*k2],gn);
E1Hm1Hn1k2Hn2H=xma1n([j*11 j*m1 n1 j*k2 j*n2],gn);
E1Hm1Hn1k2Hm2=xma1n([j*11 j*m1 n1 j*k2 m2],gn);
E1Hm1Hn1k2Hm2n2H=xma1n([j*11 j*m1 n1 j*k2 m2 j*n2],gn);
E1Hm1Hn1k2Hl2=xma1n([j*11 j*m1 n1 j*k2 12],gn);
E1Hm1Hn1k2Hl2n2H=xma1n([j*11 j*m1 n1 j*k2 12 j*n2],gn);
E1Hm1Hn1k2Hl2m2=xma1n([j*11 j*m1 n1 j*k2 12 m2],gn);

%T21111
Ek12H=xma1n([k*1 j*n2],gn);

50
Ek1m2=xmain([(k1 m2), gn]);
Ek1m2n2H=xmain([(k1 m2 j*n2), gn]);
Ek112=xmain([(k1 12), gn]);
Ek112n2H=xmain([(k1 12 j*n2), gn]);
Ek112m2=xmain([(k1 12 m2), gn]);
Ek112m2n2H=xmain([(k1 12 m2 j*n2), gn]);

XT21112
Ek1k2H=xmain([(k1 j*k2), gn]);
Ek1k2Hn2H=xmain([(k1 j*k2 j*n2), gn]);
Ek1k2Hm2=xmain([(k1 j*k2 m2), gn]);
Ek1k2Hm2n2H=xmain([(k1 j*k2 m2 j*n2), gn]);
Ek1k2H12=xmain([(k1 j*k2 12), gn]);
Ek1k2H12n2H=xmain([(k1 j*k2 12 j*n2), gn]);
Ek1k2H12m2=xmain([(k1 j*k2 12 m2)], gn));
Ek1k2H12m2n2H=xmain([(k1 j*k2 12 m2 j*n2), gn]);

XT21121
Ek1n1=xmain([(k1 n1), gn]);
Ek1n1n2H=xmain([(k1 n1 j*n2), gn]);
Ek1n1m2=xmain([(k1 n1 m2), gn]);
Ek1n1m2n2H=xmain([(k1 n1 m2 j*n2), gn]);
Ek1n112=xmain([(k1 n1 12), gn]);
Ek1n112n2H=xmain([(k1 n1 12 j*n2), gn]);
Ek1n112m2=xmain([(k1 n1 12 m2), gn]);
Ek1n112m2n2H=xmain([(k1 n1 12 m2 j*n2), gn]);

XT21122
Ek1n1k2H=xmain([(k1 n1 j*k2), gn]);
Ek1n1k2Hn2H=xmain([(k1 n1 j*k2 j*n2), gn]);
Ek1n1k2Hm2=xmain([(k1 n1 j*k2 m2), gn]);
Ek1n1k2Hm2n2H=xmain([(k1 n1 j*k2 m2 j*n2), gn]);
Ek1n1k2H12=xmain([(k1 n1 j*k2 12), gn]);
Ek1n1k2H12n2H=xmain([(k1 n1 j*k2 12 j*n2), gn]);
Ek1n1k2H12m2=xmain([(k1 n1 j*k2 12 m2), gn]);
Ek1n1k2H12m2n2H=xmain([(k1 n1 j*k2 12 m2 j*n2), gn]);

XT211211
Ek1m1=xmain([(k1 j*m1), gn]);
Ek1m1n2H=xmain([(k1 j*m1 j*n2), gn]);
Ek1m1m2=xmain([(k1 j*m1 m2), gn]);
Ek1m1m2n2H=xmain([(k1 j*m1 m2 j*n2), gn]);
Ek1m112=xmain([(k1 j*m1 12), gn]);

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Ek1m1H12n2H=xmaln([k1 j*m1 12 j*n2], gn);
Ek1m1H12m2=xmaln([k1 j*m1 12 m2], gn);
Ek1m1H12m2n2H=xmaln([k1 j*m1 12 m2 j*n2], gn);

\%T21212
Ek1m1Hk2H=xmaln([k1 j*m1 j*k2], gn);
Ek1m1Hk2Hn2H=xmaln([k1 j*m1 j*k2 j*n2], gn);
Ek1m1Hk2Hm2=xmaln([k1 j*m1 j*k2 m2], gn);
Ek1m1Hk2Hm2n2H=xmaln([k1 j*m1 j*k2 m2 j*n2], gn);
Ek1m1Hk2Hl2=xmaln([k1 j*m1 j*k2 l2], gn);
Ek1m1Hk2Hl2n2H=xmaln([k1 j*m1 j*k2 l2 j*n2], gn);
Ek1m1Hk2Hl2m2=xmaln([k1 j*m1 j*k2 l2 m2], gn);
Ek1m1Hk2Hl2m2n2H=xmaln([k1 j*m1 j*k2 l2 m2 j*n2], gn);

\%T21221
Ek1m1Hn1=xmaln([k1 j*m1 n1], gn);
Ek1m1Hn1n2H=xmaln([k1 j*m1 n1 j*n2], gn);
Ek1m1Hn1m2=xmaln([k1 j*m1 n1 m2], gn);
Ek1m1Hn1m2n2H=xmaln([k1 j*m1 n1 m2 j*n2], gn);
Ek1m1Hn1l2=xmaln([k1 j*m1 n1 l2], gn);
Ek1m1Hn1l2n2H=xmaln([k1 j*m1 n1 l2 j*n2], gn);
Ek1m1Hn1l2m2=xmaln([k1 j*m1 n1 l2 m2], gn);
Ek1m1Hn1l2m2n2H=xmaln([k1 j*m1 n1 l2 m2 j*n2], gn);

\%T21222
Ek1m1Hn1k2H=xmaln([k1 j*m1 n1 j*k2], gn);
Ek1m1Hn1k2Hn2H=xmaln([k1 j*m1 n1 j*k2 j*n2], gn);
Ek1m1Hn1k2Hm2=xmaln([k1 j*m1 n1 j*k2 m2], gn);
Ek1m1Hn1k2Hm2n2H=xmaln([k1 j*m1 n1 j*k2 m2 j*n2], gn);
\[ E_{k1} I_{m1} k_{2} I_{L2} = \text{xma1n}([k1 \{ j=m1 \ n1 \ j \ k2 \ 12 \}], \text{gn}) \]
\[ E_{k1} I_{m1} k_{2} I_{H2} = \text{xma1n}([k1 \ j=m1 \ n1 \ j \ k2 \ 12 \ j \ n2] \text{,} \text{gn}) \]
\[ E_{k1} I_{m1} k_{2} I_{L2} = \text{xma1n}([k1 \ n1 \ j \ k2 \ 12 \ m2] \text{,} \text{gn}) \]

\[ \text{X}22111 \]
\[ E_{k1} I_{m1} n1 \text{H}_2 = \text{xma1n}([k1 \ j=m1 \ j \ n1] \text{,} \text{gn}) \]
\[ E_{k1} I_{m1} n2 \text{H}_2 = \text{xma1n}([k1 \ j=m1 \ j \ n2] \text{,} \text{gn}) \]
\[ E_{k1} I_{m1} n2 \text{H}_2 = \text{xma1n}([k1 \ j=m1 \ m2] \text{,} \text{gn}) \]
\[ E_{k1} I_{m1} n2 \text{H}_2 = \text{xma1n}([k1 \ j=m1 \ m2 \ j \ n2] \text{,} \text{gn}) \]
\[ E_{k1} I_{m1} n2 \text{H}_2 = \text{xma1n}([k1 \ j=m1 \ 12] \text{,} \text{gn}) \]
\[ E_{k1} I_{m1} n2 \text{H}_2 = \text{xma1n}([k1 \ j=m1 \ 12 \ j \ n2] \text{,} \text{gn}) \]
\[ E_{k1} I_{m1} n2 \text{H}_2 = \text{xma1n}([k1 \ j=m1 \ 12 \ m2] \text{,} \text{gn}) \]
\[ E_{k1} I_{m1} n2 \text{H}_2 = \text{xma1n}([k1 \ j=m1 \ 12 \ m2 \ j \ n2] \text{,} \text{gn}) \]

\[ \text{X}22212 \]
\[ E_{k1} I_{m1} k_{2} \text{H}_2 = \text{xma1n}([k1 \ j=m1 \ j \ k2] \text{,} \text{gn}) \]
\[ E_{k1} I_{m1} k_{2} \text{H}_2 = \text{xma1n}([k1 \ j=m1 \ j \ k2 \ j \ n2] \text{,} \text{gn}) \]
\[ E_{k1} I_{m1} k_{2} \text{H}_2 = \text{xma1n}([k1 \ j=m1 \ j \ k2 \ m2] \text{,} \text{gn}) \]
\[ E_{k1} I_{m1} k_{2} \text{H}_2 = \text{xma1n}([k1 \ j=m1 \ j \ k2 \ m2 \ j \ n2] \text{,} \text{gn}) \]
\[ E_{k1} I_{m1} k_{2} \text{H}_2 = \text{xma1n}([k1 \ j=m1 \ j \ k2 \ 12] \text{,} \text{gn}) \]
\[ E_{k1} I_{m1} k_{2} \text{H}_2 = \text{xma1n}([k1 \ j=m1 \ j \ k2 \ 12 \ j \ n2] \text{,} \text{gn}) \]
\[ E_{k1} I_{m1} k_{2} \text{H}_2 = \text{xma1n}([k1 \ j=m1 \ j \ k2 \ 12 \ m2] \text{,} \text{gn}) \]
\[ E_{k1} I_{m1} k_{2} \text{H}_2 = \text{xma1n}([k1 \ j=m1 \ j \ k2 \ 12 \ m2 \ j \ n2] \text{,} \text{gn}) \]

\[ \text{X}22211 \]
\[ E_{k1} I_{m1} n1 \text{H}_1 = \text{xma1n}([k1 \ j=m1 \ n1] \text{,} \text{gn}) \]
\[ E_{k1} I_{m1} n1 \text{H}_1 = \text{xma1n}([k1 \ j=m1 \ n1 \ j \ n2] \text{,} \text{gn}) \]
\[ E_{k1} I_{m1} n2 \text{H}_2 = \text{xma1n}([k1 \ j=m1 \ n1 \ m2] \text{,} \text{gn}) \]
\[ E_{k1} I_{m1} n2 \text{H}_2 = \text{xma1n}([k1 \ j=m1 \ n1 \ m2 \ j \ n2] \text{,} \text{gn}) \]
\[ E_{k1} I_{m1} n2 \text{H}_2 = \text{xma1n}([k1 \ j=m1 \ 12] \text{,} \text{gn}) \]
\[ E_{k1} I_{m1} n2 \text{H}_2 = \text{xma1n}([k1 \ j=m1 \ 12 \ j \ n2] \text{,} \text{gn}) \]
\[ E_{k1} I_{m1} n2 \text{H}_2 = \text{xma1n}([k1 \ j=m1 \ 12 \ m2] \text{,} \text{gn}) \]
\[ E_{k1} I_{m1} n2 \text{H}_2 = \text{xma1n}([k1 \ j=m1 \ 12 \ m2 \ j \ n2] \text{,} \text{gn}) \]

\[ \text{X}22222 \]
\[ E_{k1} I_{m1} k_{2} \text{H}_2 = \text{xma1n}([k1 \ j=m1 \ n1 \ j \ k2] \text{,} \text{gn}) \]
\[ E_{k1} I_{m1} k_{2} \text{H}_2 = \text{xma1n}([k1 \ j=m1 \ n1 \ j \ k2 \ j \ n2] \text{,} \text{gn}) \]
\[ E_{k1} I_{m1} k_{2} \text{H}_2 = \text{xma1n}([k1 \ j=m1 \ n1 \ j \ k2 \ m2] \text{,} \text{gn}) \]
\[ E_{k1} I_{m1} k_{2} \text{H}_2 = \text{xma1n}([k1 \ j=m1 \ n1 \ j \ k2 \ m2 \ j \ n2] \text{,} \text{gn}) \]
\[ E_{k1} I_{m1} k_{2} \text{H}_2 = \text{xma1n}([k1 \ j=m1 \ n1 \ j \ k2 \ 12] \text{,} \text{gn}) \]
\[ E_{k1} I_{m1} k_{2} \text{H}_2 = \text{xma1n}([k1 \ j=m1 \ n1 \ j \ k2 \ 12 \ j \ n2] \text{,} \text{gn}) \]
\[ E_{k1} I_{m1} k_{2} \text{H}_2 = \text{xma1n}([k1 \ j=m1 \ n1 \ j \ k2 \ 12 \ m2] \text{,} \text{gn}) \]
%T22211
Ek11Hm1H=xmaln([k1 j=11 j*m1],gn);
Ek11Hm1Hn2H=xmaln([k1 j=11 j*m1 j*n2],gn);
Ek11Hm1Hm2=xmaln([k1 j=11 j*m1 m2],gn);
Ek11Hm1Hn2n2H=xmaln([k1 j=11 j*m1 m2 j*n2],gn);
Ek11Hm1H12=xmaln([k1 j=11 j*m1 12],gn);
Ek11Hm1H12n2H=xmaln([k1 j=11 j*m1 12 j*n2],gn);
Ek11Hm1H12m2=xmaln([k1 j=11 j*m1 12 m2],gn);
Ek11Hm1H12m2n2H=xmaln([k1 j=11 j*m1 12 m2 j*n2],gn);

%T22212
Ek11Hm1Hk2H=xmaln([k1 j=11 j*m1 j*k2],gn);
Ek11Hm1Hk2Hn2H=xmaln([k1 j=11 j*m1 j*k2 j*n2],gn);
Ek11Hm1Hk2Hm2=xmaln([k1 j=11 j*m1 j*k2 m2],gn);
Ek11Hm1Hk2Hn2n2H=xmaln([k1 j=11 j*m1 j*k2 m2 j*n2],gn);
Ek11Hm1Hk2H12=xmaln([k1 j=11 j*m1 j*k2 12],gn);
Ek11Hm1Hk2H12n2H=xmaln([k1 j=11 j*m1 j*k2 12 j*n2],gn);
Ek11Hm1Hk2H12m2=xmaln([k1 j=11 j*m1 j*k2 12 m2],gn);

%T22221
Ek11Hm1Hn1=xmaln([k1 j=11 j*m1 n1],gn);
Ek11Hm1Hn1n2H=xmaln([k1 j=11 j*m1 n1 j*n2],gn);
Ek11Hm1Hn1m2=xmaln([k1 j=11 j*m1 n1 m2],gn);
Ek11Hm1Hn1n2n2H=xmaln([k1 j=11 j*m1 n1 m2 j*n2],gn);
Ek11Hm1Hn112=xmaln([k1 j=11 j*m1 n1 12],gn);
Ek11Hm1Hn112n2H=xmaln([k1 j=11 j*m1 n1 12 j*n2],gn);
Ek11Hm1Hn112m2=xmaln([k1 j=11 j*m1 n1 12 m2],gn);

%T22222
Ek11Hm1Hnk2H=xmaln([k1 j=11 j*m1 n1 j*k2],gn);
Ek11Hm1Hnk2Hn2H=xmaln([k1 j=11 j*m1 n1 j*k2 j*n2],gn);
Ek11Hm1Hnk2Hm2=xmaln([k1 j=11 j*m1 n1 j*k2 m2],gn);
Ek11Hm1Hnk2H12=xmaln([k1 j=11 j*m1 n1 j*k2 12],gn);
Ek11Hm1Hnk2H12n2H=xmaln([k1 j=11 j*m1 n1 j*k2 12 j*n2],gn);

%Calculation of the modified steering matrix
A=[A sqrt((-1)*A)]; %a_k(1:H)= A(k,:)

% Calculation of the source signal based Kronecker-products
E2m1S means E(s \kron s*), E4m1S means E(s \kron s* \kron s* \kron s* \kron s) ...
E2m1S=ones(1,1)'; E3m1S=ones(1,1)'; E4m1S=ones(1,1)'; E5m1S=ones(1,1)';
E6m1S=ones(1,1)'; E7m1S=ones(1,1)'; E8m1S=ones(1,1)'; E9m1S=ones(1,1)';
E10m1S=ones(1,1)'; E11m1S=ones(1,1)'; E12m1S=ones(1,1)';
54
%Steering vector abbreviations
ak1=A(k1,:);
al1H=conj(A(l1,:));
am1H=conj(A(m1,:));
an1=A(n1,:);

ak2H=conj(A(k2,:));
al2=A(l2,:); 
am2=A(m2,:);
an2H=conj(A(n2,:));

for a=1:H
for b=1:H
  %Begin 2nd signal moment vector
  sig_ind=[sort([a b]) -1];
  momz2=1;
  for i=2:2+1
    if sig_ind(i)==sig_ind(i-1)
      momz2=momz2+1;
    else
      E2mnlS(z2)=E2mnlS(z2)+ge(sig_ind(i-1),momz2);
      momz2=1;
    end
  end
  z2=z2+1;
  %End 2nd signal moment vector
  for c=1:H
    [a b c]
  %Begin 3rd signal moment vector
    sig_ind=[sort([a b c]) -1];
    momz3=1;
    for i=2:3+1
      if sig_ind(i)==sig_ind(i-1)
        momz3=momz3+1;
      else
        E3mnlS(z3)=E3mnlS(z3)+ge(sig_ind(i-1),momz3);
        momz3=1;
      end
    end
    z3=z3+1;
    %End 3rd signal moment vector
  for d=1:H
    %Begin 4th signal moment vector

sig_ind=[sort([a b c d]) -1];
momz4=1;
for i=2:4+1
    if sig_ind(i)==sig_ind(i-1)
        momz4=momz4+1;
    else
        E4ma1S(z4)=E4ma1S(z4)*gs(sig_ind(i-1),momz4);
        momz4=1;
    end
end
z4=z4+1;
%
End 4th signal moment vector

for e=1:N
    %Begin 5th signal moment vector
    sig_ind=[sort([a b c d e]) -1];
momz5=1;
for i=2:5+1
    if sig_ind(i)==sig_ind(i-1)
        momz5=momz5+1;
    else
        E5ma1S(z5)=E5ma1S(z5)*gs(sig_ind(i-1),momz5);
        momz5=1;
    end
end
z5=z5+1;
%
End 5th signal moment vector

for f=1:N
    %Begin 6th signal moment vector
    sig_ind=[sort([a b c d e f]) -1];
momz6=1;
for i=2:6+1
    if sig_ind(i)==sig_ind(i-1)
        momz6=momz6+1;
    else
        E6ma1S(z6)=E6ma1S(z6)*gs(sig_ind(i-1),momz6);
        momz6=1;
    end
end
z6=z6+1;
%
End 6th signal moment vector

for g=1:N
    %Begin 7th signal moment vector
    for h=1:N
        %Begin 7th signal moment vector

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sig.ind=sort([a b c d e f g h]) -1;  
momz8=1;  
E8malS=1;  
for i=2:8+1  
if sig.ind(i)==sig.ind(i-1)  
momz8=momz8+1;  
else  
E8malS=E8malS*gs(sig.ind(i-1),momz8);  
momz8=1;  
end  
end  
E8malS(z8)=E8malS*ak1(a)*al1(b)*am1(c)*am1(d)*ak2(e)*al2(f)*  
*am2(g)*am2(h);  
z8=z8+1;  
%End 8th signal moment vector  
end  
end  
end  
end  
E8malb=sum(E8malS);  

%Kronecker product abbreviations: Kk1iH means (a_k1 \ kron a_l1H) and so on...

%T11111  
Kk11H11H1k2H12=kron(ak1,kron(al1H,kron(am1H,kron(an1,kron(ak2H,al2)))));  
Kk11H11H1nk2Hm2=kron(ak1,kron(al1H,kron(am1H,kron(an1,kron(ak2H,am2)))));  
Kk11H11H1nk2Hk2H=kron(ak1,kron(al1H,kron(am1H,kron(an1,kron(ak2H,am2))));  
Kk11H11H1nk2Hk2H=kron(ak1,kron(al1H,kron(am1H,kron(an1,kron(ak2H,am2))));

%T11112  
Kk11H11H112m2=kron(ak1,kron(al1H,kron(am1H,kron(an1,kron(ak2,am2)))));  
Kk11H11H112n2H=kron(ak1,kron(al1H,kron(am1H,kron(an1,kron(ak2,al2)))));  
Kk11H11H112n2H=kron(ak1,kron(al1H,kron(am1H,kron(an1,al2))));  
Kk11H11H11m2n2H=kron(ak1,kron(al1H,kron(am1H,kron(an1,am2))));  
Kk11H11H11m2n2H=kron(ak1,kron(al1H,kron(am1H,kron(an1,am2))));  
Kk11H11H11m2n2H=kron(ak1,kron(al1H,kron(am1H,kron(an1,am2))));  
Kk11H11H11m2n2H=kron(ak1,kron(al1H,kron(am1H,kron(an1,am2))));

%T111121  
Kk11H11Hk2H12m2=kron(ak1,kron(al1H,kron(am1H,kron(ak2H,kron(al2,am2)))));
KK11H2H=Kron(ak1,kron(al1H,kron(ak2H,am2H)));
KK11H2H=Kron(ak1,kron(al1H,ak2H));

TZ11H22
KK11H2m2n2H=Kron(ak1,kron(al1H,kron(al2,kron(am2H,am2H))));
KK11H2m2n2H=Kron(ak1,kron(al1H,kron(al2,am2)));
KK11H2m2n2H=Kron(ak1,kron(al1H,kron(al2,am2H)));
KK11H2m1n2H=Kron(ak1,kron(al1H,al2)));
KK11H2m2n2H=Kron(ak1,kron(al1H,kron(am2,am2H)));
KK11H2m2n2H=Kron(ak1,kron(al1H,kron(am2,am2)));
KK11H2m2n2H=Kron(ak1,kron(al1H,am2));
KK11H2m2n2H=Kron(ak1,kron(al1H,am2H));
KK11H2m2n2H=Kron(ak1,kron(al1H,am2));

TZ11H24
KK11H2nH2H=Kron(ak1,kron(amiH,kron(amiH,kron(ak2H,ak2H))));
KK11H2nH2H=Kron(ak1,kron(amiH,kron(amiH,kron(ak2H,ak2H))));
KK11H2nH2H=Kron(ak1,kron(amiH,kron(amiH,kron(ak2H,ak2H))));
KK11H2nH2H=Kron(ak1,kron(amiH,kron(amiH,kron(ak2H,ak2H))));
KK11H2nH2H=Kron(ak1,kron(amiH,kron(amiH,kron(ak2H,ak2H))));
KK11H2nH2H=Kron(ak1,kron(amiH,kron(amiH,ak2H)));

TZ11H25
KK11H2nH2H=Kron(ak1,kron(amiH,kron(amiH,kron(ak2H,ak2H))));
KK11H2nH2H=Kron(ak1,kron(amiH,kron(amiH,ak2H)));
KK11H2nH2H=Kron(ak1,kron(amiH,ak2H));
KK11H2nH2H=Kron(ak1,kron(amiH,ak2H));
KK11H2nH2H=Kron(ak1,kron(amiH,ak2H));

TZ1211
KK1211

TZ1212

TZ1213

TZ1214

TZ1215

TZ1216

TZ1217

TZ1218

TZ1219

TZ1220
\[ K_{11} \text{H1}_{2m2n2H} = \text{kron}(a_{11}, \text{kron}(a_{m1H}, \text{kron}(a_{12}, \text{kron}(a_{m2}, a_{2H})))) \]
\[ K_{11} \text{H1}_{2m2} = \text{kron}(a_{11}, \text{kron}(a_{m1H}, \text{kron}(a_{12}, a_{2H}))) \]
\[ K_{11} \text{H1}_{2n2H} = \text{kron}(a_{11}, \text{kron}(a_{m1H}, \text{kron}(a_{12}, a_{n2H}))) \]
\[ K_{11} \text{H1}_{2} = \text{kron}(a_{11}, \text{kron}(a_{m1H}, a_{12})) \]
\[ K_{11} \text{H1}_{m2n2H} = \text{kron}(a_{11}, \text{kron}(a_{m1H}, \text{kron}(a_{m2}, a_{2H}))) \]
\[ K_{11} \text{H1}_{m2} = \text{kron}(a_{11}, \text{kron}(a_{m1H}, a_{m2})) \]
\[ K_{11} \text{H2} = \text{kron}(a_{11}, \text{kron}(a_{m1H}, a_{2H})) \]

\[ \text{Z12211} \]
\[ K_{11} \text{H1}_{2m2n2H} = \text{kron}(a_{11}, \text{kron}(a_{m1H}, \text{kron}(a_{2H}, \text{kron}(a_{12}, \text{kron}(a_{m2}, a_{2H})))))) \]
\[ K_{11} \text{H1}_{2m2} = \text{kron}(a_{11}, \text{kron}(a_{m1H}, \text{kron}(a_{2H}, \text{kron}(a_{12}, a_{m2})))) \]
\[ K_{11} \text{H1}_{2n2H} = \text{kron}(a_{11}, \text{kron}(a_{m1H}, \text{kron}(a_{2H}, \text{kron}(a_{12}, a_{n2H})))) \]
\[ K_{11} \text{H1}_{2} = \text{kron}(a_{11}, \text{kron}(a_{m1H}, \text{kron}(a_{2H}, a_{12}))) \]
\[ K_{11} \text{H1}_{m2n2H} = \text{kron}(a_{11}, \text{kron}(a_{m1H}, \text{kron}(a_{2H}, \text{kron}(a_{m2}, a_{n2H})))) \]
\[ K_{11} \text{H1}_{m2} = \text{kron}(a_{11}, \text{kron}(a_{m1H}, \text{kron}(a_{2H}, a_{m2}))) \]
\[ K_{11} \text{H2} = \text{kron}(a_{11}, \text{kron}(a_{m1H}, a_{2H})) \]

\[ \text{Z12212} \]
\[ K_{11} \text{H1}_{2m2n2H} = \text{kron}(a_{11}, \text{kron}(a_{m1H}, \text{kron}(a_{12}, \text{kron}(a_{m2}, a_{2H})))) \]
\[ K_{11} \text{H1}_{2m2} = \text{kron}(a_{11}, \text{kron}(a_{m1H}, \text{kron}(a_{12}, a_{m2}))) \]
\[ K_{11} \text{H1}_{2n2H} = \text{kron}(a_{11}, \text{kron}(a_{m1H}, \text{kron}(a_{12}, a_{n2H}))) \]
\[ K_{11} \text{H1}_{2} = \text{kron}(a_{11}, \text{kron}(a_{m1H}, a_{12})) \]
\[ K_{11} \text{H1}_{m2n2H} = \text{kron}(a_{11}, \text{kron}(a_{m1H}, \text{kron}(a_{12}, a_{n2H}))) \]
\[ K_{11} \text{H1}_{m2} = \text{kron}(a_{11}, \text{kron}(a_{m1H}, a_{m2})) \]
\[ K_{11} \text{H2} = \text{kron}(a_{11}, \text{kron}(a_{m1H}, a_{2H})) \]

\[ \text{Z12221} \]
\[ K_{11} \text{H1}_{2m2n2H} = \text{kron}(a_{11}, \text{kron}(a_{2H}, \text{kron}(a_{12}, \text{kron}(a_{m2}, a_{2H})))) \]
\[ K_{11} \text{H1}_{2m2} = \text{kron}(a_{11}, \text{kron}(a_{2H}, \text{kron}(a_{12}, a_{m2}))) \]
\[ K_{11} \text{H1}_{2n2H} = \text{kron}(a_{11}, \text{kron}(a_{2H}, \text{kron}(a_{12}, a_{n2H}))) \]
\[ K_{11} \text{H1}_{2} = \text{kron}(a_{11}, \text{kron}(a_{2H}, a_{12})) \]
\[ K_{11} \text{H1}_{m2n2H} = \text{kron}(a_{11}, \text{kron}(a_{2H}, \text{kron}(a_{12}, a_{n2H}))) \]
\[ K_{11} \text{H1}_{m2} = \text{kron}(a_{11}, \text{kron}(a_{2H}, a_{m2})) \]
\[ K_{11} \text{H2} = \text{kron}(a_{11}, \text{kron}(a_{2H}, a_{2H})) \]

\[ \text{Z12222} \]
\[ K_{11} \text{H1}_{2m2n2H} = \text{kron}(a_{11}, \text{kron}(a_{12}, \text{kron}(a_{m2}, a_{2H}))) \]
\[ K_{11} \text{H1}_{2m2} = \text{kron}(a_{11}, \text{kron}(a_{12}, a_{m2})) \]
\[ K_{11} \text{H1}_{2n2H} = \text{kron}(a_{11}, \text{kron}(a_{12}, a_{2H})) \]

60
Kk112=\text{kron}(a_{11}, a_{12});
Kk1m2n2H=\text{kron}(a_{11}, \text{kron}(a_{m2}, a_{2H}));
Kk1m2=\text{kron}(a_{11}, a_{m2});
Kk1n2H=\text{kron}(a_{11}, a_{n2H});

\%211
K1l1m1Hn1k2Hl12m2=\text{kron}(a_{11}, \text{kron}(a_{m1H}, \text{kron}(a_{n1}, \text{kron}(a_{k2H}, \text{kron}(a_{l2}, a_{m2}))));
K1l1m1Hn1k2Hl2n2H=\text{kron}(a_{11}, \text{kron}(a_{m1H}, \text{kron}(a_{n1}, \text{kron}(a_{k2H}, \text{kron}(a_{l2}, a_{n2H}))));
K1l1m1Hn1k2Hl2=\text{kron}(a_{11}, \text{kron}(a_{m1H}, \text{kron}(a_{n1}, \text{kron}(a_{k2H}, a_{l2}))));
K1l1m1Hn1k2Hm2n2H=\text{kron}(a_{11}, \text{kron}(a_{m1H}, \text{kron}(a_{n1}, \text{kron}(a_{k2H}, \text{kron}(a_{m2}, a_{2H}))));
K1l1m1Hn1k2Hm2=\text{kron}(a_{11}, \text{kron}(a_{m1H}, \text{kron}(a_{n1}, \text{kron}(a_{k2H}, a_{m2})));
K1l1m1Hn1k2H=\text{kron}(a_{11}, \text{kron}(a_{m1H}, \text{kron}(a_{n1}, \text{kron}(a_{k2H}, a_{2H}))));
K1l1m1Hn1k2H=\text{kron}(a_{11}, \text{kron}(a_{m1H}, \text{kron}(a_{n1}, a_{k2H})));

\%211
K1l1m1Hn1l12m2n2H=\text{kron}(a_{11}, \text{kron}(a_{m1H}, \text{kron}(a_{n1}, \text{kron}(a_{k2H}, \text{kron}(a_{l2}, a_{m2})))));
K1l1m1Hn1l12m2=\text{kron}(a_{11}, \text{kron}(a_{m1H}, \text{kron}(a_{n1}, \text{kron}(a_{k2H}, a_{l2}))));
K1l1m1Hn1l12n2H=\text{kron}(a_{11}, \text{kron}(a_{m1H}, \text{kron}(a_{n1}, \text{kron}(a_{k2H}, a_{2H}))));
K1l1m1Hn1l12=\text{kron}(a_{11}, \text{kron}(a_{m1H}, \text{kron}(a_{n1}, a_{k2H})));}
K1l1m1Hn1l12=\text{kron}(a_{11}, \text{kron}(a_{m1H}, \text{kron}(a_{n1}, a_{k2H})));}
K1l1m1Hn1l12=\text{kron}(a_{11}, \text{kron}(a_{m1H}, \text{kron}(a_{n1}, a_{k2H})));}
K1l1m1Hn1l12=\text{kron}(a_{11}, \text{kron}(a_{m1H}, \text{kron}(a_{n1}, a_{k2H})));}
K1l1m1Hn1l12=\text{kron}(a_{11}, \text{kron}(a_{m1H}, \text{kron}(a_{n1}, a_{k2H})));}

\%211
K1l1m1Hk2Hl12m2n2H=\text{kron}(a_{11}, \text{kron}(a_{m1H}, \text{kron}(a_{k2H}, \text{kron}(a_{l2}, a_{m2}))));
K1l1m1Hk2Hl2n2H=\text{kron}(a_{11}, \text{kron}(a_{m1H}, \text{kron}(a_{k2H}, a_{l2})));}
K1l1m1Hk2Hl2=\text{kron}(a_{11}, \text{kron}(a_{m1H}, \text{kron}(a_{k2H}, a_{2H})));}
K1l1m1Hk2Hm2n2H=\text{kron}(a_{11}, \text{kron}(a_{m1H}, \text{kron}(a_{k2H}, \text{kron}(a_{m2}, a_{2H}))));
K1l1m1Hk2Hm2=\text{kron}(a_{11}, \text{kron}(a_{m1H}, \text{kron}(a_{k2H}, a_{m2})));
K1l1m1Hk2Hn2H=\text{kron}(a_{11}, \text{kron}(a_{m1H}, \text{kron}(a_{k2H}, a_{2H})));}
K1l1m1Hk2H=\text{kron}(a_{11}, \text{kron}(a_{m1H}, a_{k2H}));

\%211
K1l1m1Hl12m2n2H=\text{kron}(a_{11}, \text{kron}(a_{m1H}, \text{kron}(a_{l2}, \text{kron}(a_{m2}, a_{2H}))));
K1l1m1Hl12m2=\text{kron}(a_{11}, \text{kron}(a_{m1H}, \text{kron}(a_{l2}, a_{m2})));}
K1l1m1Hl12n2H=\text{kron}(a_{11}, \text{kron}(a_{m1H}, \text{kron}(a_{l2}, a_{n2H})));}
K1l1m1Hl12=\text{kron}(a_{11}, \text{kron}(a_{m1H}, \text{kron}(a_{l2}, a_{2H})));}
K1l1m1Hl12=\text{kron}(a_{11}, \text{kron}(a_{m1H}, \text{kron}(a_{l2}, a_{2H})));}
K1l1m1Hl12=\text{kron}(a_{11}, \text{kron}(a_{m1H}, \text{kron}(a_{l2}, a_{2H})));}
K1l1m1Hn2H=\text{kron}(a_{11}, \text{kron}(a_{m1H}, a_{n2H}));}
K1l1m1H=\text{kron}(a_{11}, a_{m1H});
Kn1k2Hn2H=kron(an1,kron(ak2H,an2H));
Kn1k2H=kron(an1,ak2H);

%T22212
Kn112m2n2H=kron(an1,kron(al2,kron(am2,an2H)));
Kn112m2=kron(an1,kron(al2,am2));
Kn112n2H=kron(an1,kron(al2,an2H));
Kn112=kron(an1,al2);
Kn1m2n2H=kron(an1,kron(am2,an2H));
Kn1m2=kron(an1,am2);
Kn1n2H=kron(an1,an2H);

%T22221
Kk2H12m2n2H=kron(ak2H,kron(al2,kron(am2,an2H)));
Kk2H12m2=kron(ak2H,kron(al2,am2));
Kk2H12n2H=kron(ak2H,kron(al2,an2H));
Kk2H12=kron(ak2H,al2);
Kk2Hm2n2H=kron(ak2H,kron(am2,an2H));
Kk2Hm2=kron(ak2H,am2);
Kk2Hn2H=kron(ak2H,an2H);

%T22222
Kl12m2n2H=kron(al2,kron(am2,an2H));
Kl12m2=kron(al2,am2);
Kl12n2H=kron(al2,an2H);
Kl1n2H=kron(am2,an2H);

%CALCULATION OF a) See Section "Eighth-order Moment"
%
%  E(rk1 rl1H rm1H rn1 rk2H rl2 rm2 rn2H)
%
T11111=. . .
   E5ma1b+Kk111Hm1Hn1k2H12+Esma1S'+Em2n2H+. . .
Kk111Hm1Hn1k2H2+Esma1S'*Kk12n2H+Kk111Hm1Hn1k2Hn2H+Esma1S'*E12m2+. . .
Kk111Hm1Hn1k2H+Esma1S'*E12m2n2H;

T11112=. . .
Kk111Hm1Hn1l2m2+Esma1S'*Ek2Hn2H+Kk111Hm1Hn1l2m2H+Esma1S'*Ek2Hm2+. . .
Kk111Hm1Hn1l2+Esma1S'*Ek2Hm2n2H+Kk111Hm1Hn1m2n2H+Esma1S'*Ek2H12+. . .
Kk111Hm1Hnl2+Esma1S'*Ek2Hl2n2H+Kk111Hm1Hnl2n2H+Esma1S'*Ek2Hl2m2+. . .
Kk111Hm1Hnl1+Esma1S'*Ek2H12m2n2H;
T11121...
Kk111Hm1Hk2H12m2*E6m1aS' *En1n2H*Kk111Hm1Hk2H12n2H*E6m1aS' *En1n2+...
Kk111Hm1Hk2H12*E6m1aS' *En1m2n2H*Kk111Hm1Hk2H12m2*E6m1aS' *En1n2+...
Kk111Hm1Hk2H12m2*E6m1aS' *En1n2n2H*Kk111Hm1Hk2H12n2H*E6m1aS' *En1n2+...
Kk111Hm1Hk2H12*E6m1aS' *En1m2n2H;  

T11122...
Kk111Hm1H12m2n2H*E6m1aS' *En1k2H*Kk111Hm1H12n2*E6m1aS' *En1k2Hn2H+
Kk111Hm1H12n2H*E6m1aS' *En1k2Hm2+Kk111Hm1H12*E6m1aS' *En1k2Hm2n2H+
Kk111Hm1Hn2n2H*E6m1aS' *En1k2H12+Kk111Hm1Hn2*E6m1aS' *En1k2H12n2H+
Kk111Hm1Hn2H*E6m1aS' *En1k2H12m2+Kk111Hm1H*E6m1aS' *En1k2H12m2n2H;  

T11211...
Kk111Hn1k2H12m2*E6m1aS' *En1n1H*Kk111Hn1k2H12n2H*E6m1aS' *En1n1Hn2+
Kk111Hn1k2H12*E6m1aS' *En1n1Hm2n2H*Kk111Hn1k2H12n2H*E6m1aS' *En1n1Hn2+
Kk111Hn1k2H12m2*E6m1aS' *En1n1H12n2+Kk111Hn1k2H12*E6m1aS' *En1n1H12n2+
Kk111Hn1k2H12*E6m1aS' *En1n1H12m2*E6m1aS' *En1k2H12n2H+
Kk111Hn1k2Hn2H*E6m1aS' *En1k2H12m2n2H;  

T11212...
Kk111Hn1l2m2n2H*E6m1aS' *En1k2H*Kk111Hn1l2n2H*E6m1aS' *En1n1H12n2+
Kk111Hn1l2n2H*E6m1aS' *En1n1Hm2n2H*Kk111Hn1l2n2H*E6m1aS' *En1n1Hn2+
Kk111Hn1l2m2n2H*E6m1aS' *En1n1H12n2+Kk111Hn1l2n2H*E6m1aS' *En1n1H12n2+
Kk111Hn1l2n2H*E6m1aS' *En1k2H12n2+Kk111Hn1l2n2H*E6m1aS' *En1n1H12n2+
Kk111Hn1l2m2n2H*E6m1aS' *En1n1H12m2+Kk111Hn1l2*E6m1aS' *En1k2H12n2H;  

T11221...
Kk111Hk2H12m2n2H*E6m1aS' *En1Hn1+Kk111Hk2H12m2*E6m1aS' *En1H12n2+
Kk111Hk2H12n2H*E6m1aS' *En1H12m2+Kk111Hk2H12n2*E6m1aS' *En1n1Hm2n2H+
Kk111Hk2H12m2n2H*E6m1aS' *En1H12n2+Kk111Hk2H12n2*E6m1aS' *En1n1H12n2+
Kk111Hk2H12n2H*E6m1aS' *En1H12m2+Kk111Hk2H12*E6m1aS' *En1n1H12n2+
Kk111Hk2H12*E6m1aS' *En1H12m2n2H*E6m1aS' *En1H12n2H+

T11222...
Kk111H2m2n2H*E6m1aS' *En1Hn1+Kk111Hk12H12m2*E6m1aS' *En1H12n2+
Kk111Hk2H12n2H*E6m1aS' *En1H12m2+Kk111Hk12H12*E6m1aS' *En1H12n2+
Kk111Hk2H12n2H*E6m1aS' *En1H12m2+Kk111Hk12H12*E6m1aS' *En1H12n2+
Kk111Hk12H12*E6m1aS' *En1H12m2+Kk111Hk12H12*E6m1aS' *En1H12n2+
Kk111Hk12H12*E6m1aS' *En1H12m2n2H*E6m1aS' *En1H12n2H;  

T12111...
Kk111Hn1k2H12m2*E6m1aS' *En1Hn2H*Kk111Hn1k2H12n2H*E6m1aS' *En1H12n2+
Kk111Hn1k2H12*E6m1aS' *En1Hn2n2H*Kk111Hn1k2H12n2H*E6m1aS' *En1H12n2+
Kk111Hn1k2Hn2H*E6m1aS' *En1H12n2+Kk111Hn1k2H12n2H*E6m1aS' *En1H12n2+
Kk111Hn1k2H12*E6m1aS' *En1H12m2n2H;  

T12112...
K11Hn2n2H+E3ma1S'*Ek11Hn1k2H12+K11Hm2+Ek11Hm1k2H12n2H+... 
K11Hn2H+E2ma1S'*Ek11Hn1k2H12m2;

T2211=... 
Km1Hn1k2H12m2n2H+E6ma1S'*Ek11Hn+Km1Hn1k2H12m2+Ek11Hn1k2H12n2H+... 
Km1Hn1k2H12n2H+E5ma1S'*Ek11Hn2m2+Ek11Hn1k2H12+Ek6ma1S'*Ek11Hm2+Ek4ma1S'*Ek11Hm1k2H12n2H+... 
Km1Hn1k2Hm2n2H+E5ma1S'*Ek11Hn1k2H12+Ek6ma1S'*Ek11Hn1k2H12n2H+... 
Km1Hn1k2Hn2m2n2H+E2ma1S'*Ek11Hn1k2H12m2+Ek5ma1S'*Ek11Hn1k2H12n2H+... 
Km1Hn1m2n2H+E6ma1S'*Ek11Hn2m2+Ek11Hn1k2H12n2H+... 
Km1Hn1m2n2H+E5ma1S'*Ek11Hn2m2+Ek11Hn1k2H12+Ek6ma1S'*Ek11Hn1k2H12n2H+... 
Km1Hn1m2n2H+E2ma1S'*Ek11Hn1k2H12m2+Ek5ma1S'*Ek11Hn1k2H12n2H+... 
Km1Hn1m2n2H+E6ma1S'*Ek11Hn2m2+Ek11Hn1k2H12+Ek6ma1S'*Ek11Hn1k2H12n2H+... 
Km1Hn1m2n2H+E5ma1S'*Ek11Hn2m2+Ek11Hn1k2H12+Ek6ma1S'*Ek11Hn1k2H12n2H+... 
Km1Hn1m2n2H+E2ma1S'*Ek11Hn1k2H12m2+Ek5ma1S'*Ek11Hn1k2H12n2H+... 
Km1Hn1m2n2H+E6ma1S'*Ek11Hn2m2+Ek11Hn1k2H12+Ek6ma1S'*Ek11Hn1k2H12n2H+... 
Km1Hn1m2n2H+E5ma1S'*Ek11Hn2m2+Ek11Hn1k2H12+Ek6ma1S'*Ek11Hn1k2H12n2H+... 
Km1Hn1m2n2H+E2ma1S'*Ek11Hn1k2H12m2+Ek5ma1S'*Ek11Hn1k2H12n2H+... 
Km1Hn1m2n2H+E6ma1S'*Ek11Hn2m2+Ek11Hn1k2H12+Ek6ma1S'*Ek11Hn1k2H12n2H+... 
Km1Hn1m2n2H+E5ma1S'*Ek11Hn2m2+Ek11Hn1k2H12+Ek6ma1S'*Ek11Hn1k2H12n2H+... 
Km1Hn1m2n2H+E2ma1S'*Ek11Hn1k2H12m2+Ek5ma1S'*Ek11Hn1k2H12n2H+... 
Km1Hn1m2n2H+E6ma1S'*Ek11Hn2m2+Ek11Hn1k2H12+Ek6ma1S'*Ek11Hn1k2H12n2H+... 
Km1Hn1m2n2H+E5ma1S'*Ek11Hn2m2+Ek11Hn1k2H12+Ek6ma1S'*Ek11Hn1k2H12n2H+... 
Km1Hn1m2n2H+E2ma1S'*Ek11Hn1k2H12m2+Ek5ma1S'*Ek11Hn1k2H12n2H+... 
Km1Hn1m2n2H+E6ma1S'*Ek11Hn2m2+Ek11Hn1k2H12+Ek6ma1S'*Ek11Hn1k2H12n2H+... 
Km1Hn1m2n2H+E5ma1S'*Ek11Hn2m2+Ek11Hn1k2H12+Ek6ma1S'*Ek11Hn1k2H12n2H+... 
Km1Hn1m2n2H+E2ma1S'*Ek11Hn1k2H12m2+Ek5ma1S'*Ek11Hn1k2H12n2H+... 
Km1Hn1m2n2H+E6ma1S'*Ek11Hn2m2+Ek11Hn1k2H12+Ek6ma1S'*Ek11Hn1k2H12n2H+... 
Km1Hn1m2n2H+E5ma1S'*Ek11Hn2m2+Ek11Hn1k2H12+Ek6ma1S'*Ek11Hn1k2H12n2H+... 
Km1Hn1m2n2H+E2ma1S'*Ek11Hn1k2H12m2+Ek5ma1S'*Ek11Hn1k2H12n2H+... 
Km1Hn1m2n2H+E6ma1S'*Ek11Hn2m2+Ek11Hn1k2H12+Ek6ma1S'*Ek11Hn1k2H12n2H+... 
Km1Hn1m2n2H+E5ma1S'*Ek11Hn2m2+Ek11Hn1k2H12+Ek6ma1S'*Ek11Hn1k2H12n2H+... 
Km1Hn1m2n2H+E2ma1S'*Ek11Hn1k2H12m2+Ek5ma1S'*Ek11Hn1k2H12n2H+... 
Km1Hn1m2n2H+E6ma1S'*Ek11Hn2m2+Ek11Hn1k2H12+Ek6ma1S'*Ek11Hn1k2H12n2H+... 
Km1Hn1m2n2H+E5ma1S'*Ek11Hn2m2+Ek11Hn1k2H12+Ek6ma1S'*Ek11Hn1k2H12n2H+... 
Km1Hn1m2n2H+E2ma1S'*Ek11Hn1k2H12m2+Ek5ma1S'*Ek11Hn1k2H12n2H+... 
Km1Hn1m2n2H+E6ma1S'*Ek11Hn2m2+Ek11Hn1k2H12+Ek6ma1S'*Ek11Hn1k2H12n2H+... 
Km1Hn1m2n2H+E5ma1S'*Ek11Hn2m2+Ek11Hn1k2H12+Ek6ma1S'*Ek11Hn1k2H12n2H+... 
Km1Hn1m2n2H+E2ma1S'*Ek11Hn1k2H12m2+Ek5ma1S'*Ek11Hn1k2H12n2H+... 
Km1Hn1m2n2H+E6ma1S'*Ek11Hn2m2+Ek11Hn1k2H12+Ek6ma1S'*Ek11Hn1k2H12n2H+... 
Km1Hn1m2n2H+E5ma1S'*Ek11Hn2m2+Ek11Hn1k2H12+Ek6ma1S'*Ek11Hn1k2H12n2H+... 
Km1Hn1m2n2H+E2ma1S'*Ek11Hn1k2H12m2+Ek5ma1S'*Ek11Hn1k2H12n2H+... 
Km1Hn1m2n2H+E6ma1S'*Ek11Hn2m2+Ek11Hn1k2H12+Ek6ma1S'*Ek11Hn1k2H12n2H+... 
Km1Hn1m2n2H+E5ma1S'*Ek11Hn2m2+Ek11Hn1k2H12+Ek6ma1S'*Ek11Hn1k2H12n2H+... 
Km1Hn1m2n2H+E2ma1S'*Ek11Hn1k2H12m2+Ek5ma1S'*Ek11Hn1k2H12n2H+... 
Km1Hn1m2n2H+E6ma1S'*Ek11Hn2m2+Ek11Hn1k2H12+Ek6ma1S'*Ek11Hn1k2H12n2H+... 
Km1Hn1m2n2H+E5ma1S'*Ek11Hn2m2+Ek11Hn1k2H12+Ek6ma1S'*Ek11Hn1k2H12n2H+... 
Km1Hn1m2n2H+E2ma1S'*Ek11Hn1k2H12m2+Ek5ma1S'*Ek11Hn1k2H12n2H+... 
Km1Hn1m2n2H+E6ma1S'*Ek11Hn2m2+Ek11Hn1k2H12+Ek6ma1S'*Ek11Hn1k2H12n2H+... 
Km1Hn1m2n2H+E5ma1S'*Ek11Hn2m2+Ek11Hn1k2H12+Ek6ma1S'*Ek11Hn1k2H12n2H+... 
Km1Hn1m2n2H+E2ma1S'*Ek11Hn1k2H12m2+Ek5ma1S'*Ek11Hn1k2H12n2H+...
Kk2Hn2H*E2ma1S''*Ek111Hm1Hn1l2m2;

T22222=...  
Kl2m2n2H*E3ma1S'*Ek111Hm1Hn1k2H+Kl2m2*E2ma1S'*Ek111Hm1Hn1k2Hn2H*...  
Kl2n2H*E2ma1S'*Ek111Hm1Hn1k2Hm2+Km2n2H*E2ma1S'*Ek111Hm1Hn1k2H12*...  
Ek111Hm1Hn1k2H12m2n2H;

Rk111Hm1Hn1k2H12m2n2H= T11111+T11112+T11121+T11122*...  
T11211+T11212+T11221+T11222*...  
T12111+T12112+T12121+T12122*...  
T12211+T12212+T12221+T12222*...  
T21111+T21112+T21121+T21122*...  
T21211+T21212+T21221+T21222*...  
T22111+T22112+T22121+T22122*...  
T22211+T22212+T22221+T22222;

%CALCULATION OF b) See Section "Sixth-order Moment"
%
% E{rk1 r11H r11H r11H r12 r12}
%
T111=...
Kk111Hm1Hn1k2H12+E6ma1S''+Ek111Hm1Hn1*E6ma1S'*Ek2H12 +...  
Kk111Hm1Hk2H*E4ma1S'*En12+Kk111Hm1H12+E4ma1S'*En12H*...  
Kk111Hm1H+E3ma1S'*En12H12;

T112=...
Kk111Hn1k2H*E4ma1S'*E11H12*Kk111Hn1*E4ma1S'*E11Hk2H +...  
Kk111Hn1+E3ma1S'*E11Hk2H12*Kk111Hk2H*E4ma1S'*E11Hn1 +...  
Kk111Hk2H*E3ma1S'*E11Hn1+Kk111H12+Kk111H2+*E3ma1S'*E11Hn1k2H +...  
Kk111H+E2ma1S'*E11Hnk12H12;

T121=...
Kk1m1Hn1k2H*E4ma1S'*E11H12+Kk1m1Hn1*E4ma1S'*E11Hk2H +...  
Kk1m1Hn1+E3ma1S'*E11Hk2H12*Kk1m1Hk2H*E4ma1S'*E11Hn1 +...  
Kk1m1Hk2H*E3ma1S'*E11Hn1+Kk1m1H12+E3ma1S'*E11Hn1k2H +...  
Kk1m1H+E2ma1S'*E11Hnk12H12;

T122=...
Kk11k2H12*E4ma1S'*E11Hm1H+Kk1n1k2H*E3ma1S'*E11Hm1H12 +...  
Kk11n2*E3ma1S'*E11Hm1Hk2H+Kk1n1w*E2ma1S'*E11Hm1H 2H12 +...  
Kk1k2_12HE3 a1S'*E11Hm1H1+(k1k2*E2ma1S'*E11Hm1H112 +...  
Ek112*E2ma1S'*E11Hm1Hn1k2H;
\begin{verbatim}
T211=...  
K11Hm1Hn1k2H+E4m1s'*E11l2*K11Hm1Hn1l2+E4m1s'*E1k1k2H+...  
K11Hm1Hn1+E3m1s'*E1k1k2H+Kl1Hm1Hk2Hl2+E4m1s'*E1k1l1+...  
K11Hm1Hk2H+E3m1s'*E1k1l2*K11Hm1Hn1l2+E3m1s'*E1k1k2H+...  
K11Hm1H+E2m1s'*E1k1l1k2H;

T212=...  
K11Hm1k2Hl2+E4m1s'*E1k1l2*K11Hm1k2H+E3m1s'*E1k1l1H+...  
K11Hm1l2+E3m1s'*E1k1l1H+K11Hm1+E2m1s'*E1k1k2Hl2+...  
K11Hk2Hl2+E3m1s'*E1k1l1H1+K11Hk2H+E2m1s'*E1k1l1l2+...  
K11H2+E2m1s'*E1k1Hn1k2H;

T221=...  
Km1Hn1k2Hl2+E4m1s'*E1k1l1H*Kn1k2H+E3m1s'*E1k1l1H+...  
Km1Hn1l2+E3m1s'*E1k1l1H+Kn1l2+E2m1s'*E1k1l1H+...  
K11Hk2Hl2+E3m1s'*E1k1l1H1+Kn1k2H+E2m1s'*E1k1l1l2+...  
K11H2+E2m1s'*E1k1Hn1k2H;

T222=...  
Kn1k2Hl2+E3m1s'*E1k1l1H+Kn1k2H+E2m1s'*E1k1l1H+...  
Kn1l2+E2m1s'*E1k1l1H+Kn1k2Hl2+E2m1s'*E1k1l1l1+...  
Ek1l1Hm1k2Hl2;

Rk11Hm1Hn1k2Hl2=T111+T112mT121+T122+T211+T212+T221+T222;

%CALCULATION OF b2) See Section "Sixth-order Moment"
%
% E{r(k1 r1lH r1lH r1lH r2l r2lH}  
%
T111=...  
Kk11Hm1l1Hn1m2h2+E6m1s'*Kk11Hm1Hn1l1Hm2+...  
Kk11Hm1l1Hn2+E4m1s'*Em1n2H*Kk11Hm1Hn2+4m1s'*Em1m2+...  
Kk11Hm1l1H+E3m1s'*Em1m2n2H;

T112=...  
Kk11Hn1m2+E4m1s'*Em1Hn1h2+Kk11Hn1n2H+E4m1s'*Em1Hn2+...  
Kk11Hn1+E3m1s'*Em1Hn1n2H+Kk11Hn1n2H+E4m1s'*Em1Hn1+...  
Kk11Hn2+E3m1s'*Em1Hn1n2H+Kk11Hn2n2H+E3m1s'*Em1Hn1m2+...  
Kk11H+E2m1s'*Em1Hn2n2H;

T121=...  
Kk11Hm1m2+E4m1s'*El1Hn2H*Kk11Hm1n2H+E4m1s'*El1Hn2+...  
Kk11Hn1+E3m1s'*El1Hn2n2H*Kk11Hn1n2H+4m1s'*El1Hn1+...  
Kk11Hm1+3m1s'*El1Hn2n2H*Kk11Hm1n2H+4m1s'*El1Hn1+...
\end{verbatim}
KK1m1Hm2*E3ma15*E11Hn1n2H+KK1m1Hn2H*E3ma15*E11Hn1m2+
KK1m1H*E2ma15*E11Hn1m2n2H;

T122=...
KK1m1n2n2H*E4ma15*E11Hn1m1H+KK1m1n2m2*E3ma15*E11Hm1Hn2H+

KK1m1n2H*E3ma15*E11Hm1Hn2m2*KK1m1n*E2ma15*E11Hm1n2n2H+

KK1m2n2H*E3ma15*E11Hm1Hn1H+KK1m2m2*E2ma15*E11Hm1m1n2H+

KK1m2n2H*E2ma15*E11Hm1m1n2;

T211=...
K11Hm1n1n2m2*E4ma15*E1m1n2H+K11Hm1n1n2n2H*E4ma15*E1m1n2+

K11Hm1n1n1*E3ma15*E1m1n2m2+K11Hm1n1n2H*E4ma15*E1m1n1+

K11Hm1n1n2n2H*E3ma15*E1m1n1H+K11Hm1n1n2H*E3ma15*E1m1n1+

K11Hm1n1n2n2H*E2ma15*E1m1n1n2H;

T212=...
K11Hm1n2n2H*E4ma15*E1m1n2H+K11Hm1n2n2H*E3ma15*E1m1n2H+

K11Hm1n2H*E3ma15*E1m1n2m2+K11Hm1n2H*E2ma15*E1m1n2n2H+

K11Hm1n2H*E3ma15*E1m1n1H+K11Hm1n2H*E3ma15*E1m1n1H+

K11Hm1n2n2H*E2ma15*E1m1n1n2H;

T221=...
Km1Hm1n2n2H*E4ma15*E11Hn1m2*E3ma15*E11Hn2H+

Km1Hm1n2H*E3ma15*E11Hn2m2+Km1Hm2n2H*E2ma15*E11Hn2n2H+

Km1Hm1n2H*E3ma15*E11Hn1H+Km1Hm2n2H*E2ma15*E11Hn1n2H+

Km1Hm2n2H*E2ma15*E11Hn1n2;

T222=...
Knm1n2n2H*E3ma15*E11Hm1m1H+Knm1n2m2*E2ma15*E11Hm1Hn2H+

Knm1n2H*E2ma15*E11Hm1Hn1+Knm2n2H*E2ma15*E11Hm1Hn1+

Ek11Hm1Hn1n2m2H;

Rk11Hm1Hn1m1n2H=T111+T112+T121+T122+T211+T212+T221+T222;

%CALCULATION OF b3) See Section "Sixth-order Moment"
%
% E(rk1 r11H 1m1H rnl rk2H rm2}
%
T111=...
Kk11Hm1n1m2Hm2*E6ma15*+Kk11Hm1n1m2*E4ma15*E2Hm2+

Kk11Hm1Hk2H*E4ma15*E11m1+Kk11Hm1m2*E4ma15*E11k2H+

Kk11Hm11H*E3ma15*E11k2Hm2;
T112...
Kk11Hn1k2H*E4ma1S'*Em11Hn2+Kk11Hn1m2+E4ma1S'*Em11Hk2H + ...
Kk11Hn1*E3ma1S'*Em11Hk2Hm2+Kk11Hk2Hm2+E4ma1S'*Em11Hn1 + ...
Kk11Hk2H*E3ma1S'*Em11Hn1m2+Kk11Hm2+E3ma1S'*Em11Hn1k2H + ...
Kk11Hn*E2ma1S'*Em11Hn1k2Hm2;

T121...
Kk1m1Hn1k2H*E4ma1S'*El1Hn2+Kk1m1Hn1m2+E4ma1S'*El1Hk2H + ...
Kk1m1Hn1*E3ma1S'*El1Hk2Hm2+Kk1m1Hk2Hm2+E4ma1S'*El1Hn1 + ...
Kk1m1Hk2H*E3ma1S'*El1Hn1m2+Kk1m1Hm2+E3ma1S'*El1Hn1k2H + ...
Kk1m1H*E2ma1S'*El1Hn1k2Hm2;

T122...
Kk1n1k2Hm2+E4ma1S'*El1Hm1H+Kk1n1k2H*E3ma1S'*El1Hm1m2 + ...
Kk1n1m2*E3ma1S'*El1Hn1k2H*Kn1m1+K2ma1S'*El1Hn1k2Hm2 + ...
Kk1k2Hm2*E3ma1S'*El1Hn1m1+Kk1k2H*E2ma1S'*El1Hn1m2 + ...
Kk1m2*E2ma1S'*El1Hn1k2Hn2;

T211...
Kk11Hm1k2H*E4ma1S'*Ek1m2+Kk11Hm1m2+E4ma1S'*Ek1k2H + ...
Kk11Hm1n1*E3ma1S'*Ek1k2H*Kk11Hm2+E4ma1S'*Ek1n1 + ...
Kk11Hm1H*E3ma1S'*Ek1n1m2+Kk11Hm2+E3ma1S'*Ek1n1k2H + ...
Kk11Hm*E2ma1S'*Ek1n1k2Hm2;

T212...
Kk11Hn1k2Hm2+E4ma1S'*Ek1m1H+Kk11Hn1k2H*E3ma1S'*Ek1m1m2 + ...
Kk11Hn1m2*E3ma1S'*Ek1m1H*Kk11Hn2+E4ma1S'*Ek1m1Hm2 + ...
Kk11Hk2Hm2+E3ma1S'*Ek1m1n1+Kk11Hk2H*E2ma1S'*Ek1m1n2 + ...
Kk11Hm2*E2ma1S'*Ek1m1n1k2H;

T221...
Kk11Hn1k2Hm2+E4ma1S'*Ek11Hn2+Kk11Hn1k2H*E3ma1S'*Ek11Hm2 + ...
Kk11Hn1m2*E3ma1S'*Ek11Hk2H+Kk11Hn1*E2ma1S'*Ek11Hk2Hm2 + ...
Kk11Hk2Hm2+E3ma1S'*Ek11Hn1+Kk11Hk2H*E2ma1S'*Ek11Hn1m2 + ...
Kk11Hn2*E2ma1S'*Ek11Hn1k2H;

T222...
Kk11Hn1k2Hm2*E3ma1S'*Ek11Hm1H+Kk11Hn1k2H*E2ma1S'*Ek11Hm1Hm2 + ...
Kk11Hn1m2*E2ma1S'*Ek11Hn1k2H+Kk11Hm2*E2ma1S'*Ek11Hm1Hn1 + ...
Kk11Hn1H*E2ma1S'*Ek11Hn1k2Hm2;

Rk11Hn1Hn1k2Hm2=T111+T112+T121+T211+T212+T221+T222;
%CALCULATION OF b4) See Section "Sixth-order Moment"
%
% \( E(rl1H \text{ rm1H } rn1 \text{ rl2 } rn2H) \)
%
T111=...
Kk11Hm1Hn12n2H+E6ma1S'+Kk11Hm1Hn1+E4ma1S'\*E12n2H +...
Kk11Hm1Hn12+6ma1S'\*Em1Hn2H+Kk11Hm1Hn2H+E4ma1S'\*En112 +...
Kk11Hm1H+E3ma1S'\*En112n2H;

T112=...
Kk11Hn12+E4ma1S'\*Em1Hn2H+Kk11Hn1n2H+E6ma1S'\*Em1H22 +...
Kk11Hn1+E3ma1S'\*Em1H12n2H+Kk11H12n2H+E6ma1S'\*Em1Hn1 +...
Kk11H12+E3ma1S'\*Em1H1n2H+Kk11Hn2H+E3ma1S'\*Em1Hn12 +...
Kk11H+E2ma1S'\*Em1Hn12n2H;

T121=...
Kk1m1Hn12+E4ma1S'\*El1Hn2H+Kk1m1Hn1n2H+E6ma1S'\*El1H12 +...
Kk1m1Hn1+E3ma1S'\*El1H12n2H+Kk1m1H12n2H+E6ma1S'\*El1Hn1 +...
Kk1m1H12+E3ma1S'\*El1H1n2H+Kk1m1Hn2H+E3ma1S'\*El1Hn12 +...
Kk1m1H+E2ma1S'\*El1Hn12n2H;

T122=...
Kk1n1l2n2H+E4ma1S'\*El1Hm1H+Kk1n1l2+6ma1S'\*El1Hm1Hn2H +...
Kk1n1H+6ma1S'\*El1Hm1H12+Kk1n1+E2ma1S'\*El1Hm1H12n2H +...
Kk1n12n2H+E3ma1S'\*El1Hm1Hn1+Kk1l2+E2ma1S'\*El1Hm1Hn12 +...
Kk1n1H+E2ma1S'\*El1Hm1Hn12;

T211=...
Kl1Hm1Hn12+E4ma1S'\*Ekm1Hn2H+Kl1Hm1Hn1n2H+E6ma1S'\*Ek1l2 +...
Kl1Hm1Hn1+E3ma1S'\*Ekm1H12n2H+Kl1Hm1H12n2H+E6ma1S'\*Ekm1n1 +...
Kl1H1H12+E3ma1S'\*Ekm1n2H+Kl1Hm1Hn2H+E3ma1S'\*Ekm1n12 +...
Kl1Hm1H+E2ma1S'\*Ekm1n12n2H;

T212=...
Kl1Hn1n2n2H+E4ma1S'\*Ekm1nH+Kl1Hn1n1n2H+E6ma1S'\*Ek1lHn2H +...
Kl1Hn1n2H+E3ma1S'\*Ekm1n1H+Kl1Hn1n2H+E6ma1S'\*Ekm1n1Hn2H +...
Kl1H1H+2ma1S'\*Ekm1n1Hn1+Kl1Hn2H+E3ma1S'\*Ekm1n1Hn12 +...
Kl1Hn2H+E2ma1S'\*Ekm1n1Hn12;

T221=...
Km1Hn12n2H+E4ma1S'\*Ek1lHn12+6ma1S'\*Ek1lHn12H +...
Km1Hn1n2H+E3ma1S'\*Ek1lH12n2H+Km1Hn1+E2ma1S'\*Ek1lH12n2H +...
Km1H12n2H+E3ma1S'\*Ek1lHn1+Km1H12n2H+E3ma1S'\*Ek1lHn12H +...
K_{11}H_{n2H+E2ma15}^{*}E_{k11}H_{n12}^{*};

T_{122}= \ldots
K_{11}L_{n2H+E3ma15}^{*}E_{k11}H_{m1H+K_{11}L_{2H+E2ma15}^{*}E_{k11}H_{m1H2H}^{*}+\ldots}
K_{11}n_{2H+E2ma15}^{*}E_{k11}H_{m1H12+K_{11}n_{2H+E2ma15}^{*}E_{k11}H_{m1H11}^{*}+\ldots}
E_{k11}H_{m1H112n2H};

R_{k11}H_{m1H112n2H}=T_{111+T_{112+T_{121}+T_{122}+T_{211}+T_{212}+T_{221}+T_{222}};

\%CALCULATION OF b5) See Section "Sixth-order Moment"

\%
\%
E(r_{k1} r_{111} r_{m1H} r_{n1} r_{k2H} r_{n2H})
%
T_{111}= \ldots
K_{11}L_{1Hm1H1k2Hn2H+E6ma15}^{*}E_{k11}H_{m1H1n1H1E4ma15}^{*}E_{k2H2n2H}^{*}+\ldots
K_{11}L_{1Hm1H1k2H+Em1Hn2H+E4ma15}^{*}E_{k11}H_{m1H1n2H}^{*}E_{En1k2H}^{*}+\ldots
K_{11}L_{1Hm1H1Em1H}^{*}E_{Em1H2n2Hn2H};

T_{112}= \ldots
K_{11}L_{1Hm1H1k2H+Em1Hn1n2H+E4ma15}^{*}Em1Hn2H+K_{11}L_{1Hm1H1n1H1E6ma15}^{*}Em1Hk2H+\ldots
K_{11}L_{1Hm1H1Em1H1k2Hn2H+E4ma15}^{*}Em1Hn1+\ldots
K_{11}L_{1H1k2H+Em1H1m1H1n2H+E3ma15}^{*}Em1Hn1+\ldots
K_{11}L_{1Hm1H2n2Hn2H};

T_{121}= \ldots
K_{11}L_{1Hm1k2Hn2H+E4ma15}^{*}E_{11Hn2H+K_{11}m1H1n1H1E4ma15}^{*}E_{11Hk2H}+\ldots
K_{11}L_{1Hm1H1E3ma15}^{*}E11Hk2Hn2H+K_{11}m1H1k2Hn2H+E4ma15^{*}E_{11Hn1}+\ldots
K_{11}L_{1Hm1H2H+E3ma15}^{*}E_{11Hn12H+K_{11}m1H1n1H1E3ma15}^{*}E_{11Hn1k2H}+\ldots
K_{11}L_{1Hm1H+E2ma15}^{*}E_{11Hn1k2H2n2H};

T_{122}= \ldots
K_{11}L_{1Hm1k2Hn2H+E4ma15}^{*}E_{11Hn1H1H12H+K_{11}m1k2Hn2H+E3ma15}^{*}E_{11Hn1H2H}+\ldots
K_{11}L_{1Hm1n1H1E3ma15}^{*}E_{11Hn1Hk2H+K_{11}m1n2H+E2ma15}^{*}E_{11Hn1Hk2H}+\ldots
K_{11}m1k2Hn2H+E3ma15^{*}E_{11Hn1Hn1H1k2H}+\ldots
K_{11}L_{1Hm1k2Hn1H1k2H}+\ldots
K_{11}L_{1Hm1H2Hn2H+E2ma15}^{*}E_{11Hm1H1k2H};

T_{211}= \ldots
K_{11}L_{1Hm1H1k2Hn2H+E4ma15}^{*}E_k1Hn1H+K_{11}Hm1H1n1H2H+E4ma15^{*}E_{k1k2H}+\ldots
K_{11}L_{1Hm1H1n1H1E3ma15}^{*}E_{k1k2H}+K_{11}Hm1H1k2Hn2H+E3ma15^{*}E_{k1n1}+\ldots
K_{11}L_{1Hm1k2Hn1H1H1k2H}+\ldots
K_{11}L_{1Hm1H2Hn1H1k2H};

T_{212}= \ldots

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\[ \begin{align*}
K_{11Hn1k2Hn2H} & \times E_{d1a1s} \times E_{k1m1H} + K_{11Hn1k2Hn2H} \times E_{d2a1s} \times E_{k1m2H} + \ldots \\
K_{11Hn1k2Hn2H} & \times E_{d3a1s} \times E_{k1m1H} + K_{11Hn1k2Hn2H} \times E_{d2a1s} \times E_{k1m2H} + \ldots \\
K_{11Hk2Hn2H} & \times E_{d1a1s} \times E_{k1m1H} + K_{11Hn1k2Hn2H} \times E_{d2a1s} \times E_{k1m2H} + \ldots \\
K_{11Hn1k2Hn2H} & \times E_{d1a1s} \times E_{k1m1H} + K_{11Hn1k2Hn2H} \times E_{d2a1s} \times E_{k1m2H} + \ldots
\end{align*} \]

T221 = \ldots

K_{11Hn1k2Hn2H} \times E_{d1a1s} \times E_{k1m1H} + K_{11Hn1k2Hn2H} \times E_{d2a1s} \times E_{k1m2H} + \ldots

K_{11Hn1k2Hn2H} \times E_{d3a1s} \times E_{k1m1H} + K_{11Hn1k2Hn2H} \times E_{d2a1s} \times E_{k1m2H} + \ldots

K_{11Hk2Hn2H} \times E_{d1a1s} \times E_{k1m1H} + K_{11Hn1k2Hn2H} \times E_{d2a1s} \times E_{k1m2H} + \ldots

K_{11Hn1k2Hn2H} \times E_{d1a1s} \times E_{k1m1H} + K_{11Hn1k2Hn2H} \times E_{d2a1s} \times E_{k1m2H} + \ldots

T222 = \ldots

K_{11Hn1k2Hn2H} \times E_{d1a1s} \times E_{k1m1H} + K_{11Hn1k2Hn2H} \times E_{d2a1s} \times E_{k1m2H} + \ldots

K_{11Hn1k2Hn2H} \times E_{d3a1s} \times E_{k1m1H} + K_{11Hn1k2Hn2H} \times E_{d2a1s} \times E_{k1m2H} + \ldots

K_{11Hn1k2Hn2H} \times E_{d1a1s} \times E_{k1m1H} + K_{11Hn1k2Hn2H} \times E_{d2a1s} \times E_{k1m2H} + \ldots

K_{11Hn1k2Hn2H} \times E_{d1a1s} \times E_{k1m1H} + K_{11Hn1k2Hn2H} \times E_{d2a1s} \times E_{k1m2H} + \ldots

\%
\%
E_{(r1k1H \ r1m1H \ r1n1 \ r1l2 \ r1m2)}
\%

T111 = \ldots

K_{11Hn1Hn1k2Hn2H} \times E_{d1a1s} \times E_{k1m1H} + K_{11Hn1Hn1k2Hn2H} \times E_{d2a1s} \times E_{k1m2H} + \ldots

K_{11Hn1Hn1k2Hn2H} \times E_{d3a1s} \times E_{k1m1H} + K_{11Hn1Hn1k2Hn2H} \times E_{d2a1s} \times E_{k1m2H} + \ldots

K_{11Hn1Hn1k2Hn2H} \times E_{d1a1s} \times E_{k1m1H} + K_{11Hn1Hn1k2Hn2H} \times E_{d2a1s} \times E_{k1m2H} + \ldots

T112 = \ldots

K_{11Hn1Hn1k2Hn2H} \times E_{d1a1s} \times E_{k1m1H} + K_{11Hn1Hn1k2Hn2H} \times E_{d2a1s} \times E_{k1m2H} + \ldots

K_{11Hn1Hn1k2Hn2H} \times E_{d3a1s} \times E_{k1m1H} + K_{11Hn1Hn1k2Hn2H} \times E_{d2a1s} \times E_{k1m2H} + \ldots

K_{11Hn1Hn1k2Hn2H} \times E_{d1a1s} \times E_{k1m1H} + K_{11Hn1Hn1k2Hn2H} \times E_{d2a1s} \times E_{k1m2H} + \ldots

T121 = \ldots

K_{11Hn1Hn1k2Hn2H} \times E_{d1a1s} \times E_{k1m1H} + K_{11Hn1Hn1k2Hn2H} \times E_{d2a1s} \times E_{k1m2H} + \ldots

K_{11Hn1Hn1k2Hn2H} \times E_{d3a1s} \times E_{k1m1H} + K_{11Hn1Hn1k2Hn2H} \times E_{d2a1s} \times E_{k1m2H} + \ldots

K_{11Hn1Hn1k2Hn2H} \times E_{d1a1s} \times E_{k1m1H} + K_{11Hn1Hn1k2Hn2H} \times E_{d2a1s} \times E_{k1m2H} + \ldots

T122 = \ldots

K_{11Hn1Hn1k2Hn2H} \times E_{d1a1s} \times E_{k1m1H} + K_{11Hn1Hn1k2Hn2H} \times E_{d2a1s} \times E_{k1m2H} + \ldots

K_{11Hn1Hn1k2Hn2H} \times E_{d3a1s} \times E_{k1m1H} + K_{11Hn1Hn1k2Hn2H} \times E_{d2a1s} \times E_{k1m2H} + \ldots

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Kk1m2*E2ma1S'*e11H1mH1n1l2;

T211=...
K11H1n1H1l2*E4ma1S'*e1k1m2+K11H1m1H1m2*E4ma1S'*e1k12+...
K11H1n1H1n1*E3ma1S'*e1k1l2+K11H1m1H1l2m2*E4ma1S'*e1k1n1+...
K11H1m1H12*E3ma1S'*e1k1n1m2+K11H1m1H1m2*E3ma1S'*e1k1n1l2+...
K11H1H1*E2ma1S'*e1k1n1l2m2;

T212=...
K11H1n1H1l2m2*E4ma1S'*e1k1m1H1+K11H1n1l2+E3ma1S'*e1k1m1H1n2+...
K11H1n1m2*E3ma1S'*e1k1l1H1+K11H1n1*E2ma1S'*e1k1m1H1l2m2+...
K11H1l2m2*E3ma1S'*e1k1n1H1+K11H1l2*E2ma1S'*e1k1n1m2+...
K11H1n2+*E2ma1S'*e1k1n1l2;

T221=...
K11H1n1H12m2*E4ma1S'*e1k1l1H1+K11H1n1l2+*E3ma1S'*e1k1l1H1n2+...
K11H1n1m2*E3ma1S'*e1k1l1H1+K11H1n1*E2ma1S'*e1k1l1H1l2m2+...
K11H1l2m2*E3ma1S'*e1k1n1H1+K11H1l2*E2ma1S'*e1k1n1l2m2+...
K11H1n2+*E2ma1S'*e1k1n1l2;

T222=...
K11l2m2*E3ma1S'*e1k1l1H1H1+K11n1l2*E2ma1S'*e1k1l1H1m2+...
K11m2*E2ma1S'*e1k1l1H1l2+K11m2*E2ma1S'*e1k1l1H1n1+...
K11l1l1+H1m1H1l2m2;

Rk11H1m1H1n1l2m2=T111+T112+T121+T122+T211+T212+T221+T222;

%CALCULATION OF b7) See Section "Sixth-order Moment"
%
% E((r1m1 H r1 n1 r2 H r12 r2 m2 r2 n2 H)
%
% T111=...
K11H1k12H1l2m2*E6ma1S'*e1k1n1k12H1l2+*E4ma1S'*e1m2n2H+...
K11H1k12H1m2*E4ma1S'*e1l12H1n1k12H1m2+*E4ma1S'*e1l12m2+...
K11H1k12H1*E3ma1S'*e1l12m2+2H;

T112=...
K11H1n1l2m2*E4ma1S'*e1k2H1n1l2*E4ma1S'*e1k2H1m2+...
K11H1l2*E4ma1S'*e1k2H1n1l2m2+*E4ma1S'*e1k2H1l2+...
K11H1n1l2*E4ma1S'*e1k2H1n1l2H*E3ma1S'*e1k2H1l2m2+...
K11H1n1*E2ma1S'*e1k2H1l2m2H;

T121=...
%CALCULATION OF b6) See Section "Sixth-order Moment"
%  
%  E(rkl r1H r2H r12 rm2 rn2H)  
%
K12n2H*E2ma1S*Ek11Hk2Hm2*Kn2n2H*E2ma1S*Ek011Hk2Hl2 + ... 
Ek11Hk2Hl2m2n2H;

Rk11Hk2Hl2m2n2H=T111*T112*T121+T122+T211+T212*T22m*T222;

%CALCULATION OF b9) See Section "Sixth-order Moment"
%
%
T111=...
K11Hn1k2H12m2n2H*Ek6ma1S*Ek11Hn1k2H12*Ek4ma1S*Ek12n2H + ...
K11Hn1k2H12m2n2H*Ek6ma1S*Ek12n2H+Ek11Hn1k2H12*Ek4ma1S*Ek12n2H + ...
K11Hn1k2H12m2n2H*Ek6ma1S*Ek12n2H+Ek11Hn12n2H+Ek12n2H;

T112=...
K11Hn112n2H*Ek2Hn2H+Ek11Hn12n2H+Ek4ma1S*Ek2Hn2H + ...
K11Hn112n2H*Ek2Hn2H+Ek11Hn12n2H+Ek4ma1S*Ek2Hn2H + ...
K11Hn112n2H*Ek2Hn2H+Ek11Hn12n2H+Ek4ma1S*Ek2Hn2H; 

T121=...
K11Hk2H12m2n2H*Ek6ma1S*Ek12n2H+Ek11Hk2H12m2n2H*Ek4ma1S*Ek12n2H + ...
K11Hk2H12m2n2H*Ek6ma1S*Ek12n2H+Ek11Hk2H12m2n2H*Ek4ma1S*Ek12n2H + ...
K11Hk2H12m2n2H*Ek6ma1S*Ek12n2H+Ek11Hk2H12m2n2H*Ek4ma1S*Ek12n2H; 

T122=...
K11Hk2H12m2n2H*Ek6ma1S*Ek12n2H+Ek11Hk2H12m2n2H*Ek4ma1S*Ek12n2H + ...
K11Hk2H12m2n2H*Ek6ma1S*Ek12n2H+Ek11Hk2H12m2n2H*Ek4ma1S*Ek12n2H + ...
K11Hk2H12m2n2H*Ek6ma1S*Ek12n2H+Ek11Hk2H12m2n2H*Ek4ma1S*Ek12n2H; 

T211=...
Kn1k2H12m2n2H*Ek6ma1S*Ek11Hn12n2H+Kn1k2H12m2n2H*Ek4ma1S*Ek11Hn2H + ...
Kn1k2H12m2n2H*Ek6ma1S*Ek11Hn12n2H+Kn1k2H12m2n2H*Ek4ma1S*Ek11Hn2H + ...
Kn1k2H12m2n2H*Ek6ma1S*Ek11Hn12n2H+Kn1k2H12m2n2H*Ek4ma1S*Ek11Hn2H; 

T212=...
Kn112n2H*Ek4ma1S*Ek11Hk2Hn2H+Kn112n2H*Ek4ma1S*Ek11Hk2Hn2H + ...
Kn112n2H*Ek4ma1S*Ek11Hk2Hn2H+Kn112n2H*Ek4ma1S*Ek11Hk2Hn2H + ...
Kn112n2H*Ek4ma1S*Ek11Hk2Hn2H+Kn112n2H*Ek4ma1S*Ek11Hk2Hn2H; 

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\[ K_{1n2H} \cdot E_{2n18} \cdot E_{11Hn2H} \cdot 2m2; \]

\[ \text{T221=} \]
\[ K_{2n12n2H} \cdot E_{2n18} \cdot E_{11Hn2H} \cdot 2m2 \cdot E_{3n18} \cdot E_{11Hn2H} \cdot 2m2 + \]
\[ K_{1n2n12H} \cdot E_{3n18} \cdot E_{11Hn2H} + \]
\[ K_{2n12n2H} \cdot E_{2n18} \cdot E_{11Hn2H} \cdot 2m2 \cdot E_{3n18} \cdot E_{11Hn2H} \cdot 2m2 + \]
\[ K_{1n2n12H} \cdot E_{3n18} \cdot E_{11Hn2H} \cdot 2m2 \cdot E_{3n18} \cdot E_{11Hn2H} \cdot 2m2 + \]
\[ K_{1n2n12H} \cdot E_{3n18} \cdot E_{11Hn2H} \cdot 2m2; \]

\[ \text{T222=} \]
\[ K_{1n2n12H} \cdot E_{2n18} \cdot E_{11Hn2H} \cdot 2m2 \cdot E_{2n18} \cdot E_{11Hn2H} + \]
\[ K_{1n2n12H} \cdot E_{2n18} \cdot E_{11Hn2H} \cdot 2m2 \cdot E_{2n18} \cdot E_{11Hn2H} + \]
\[ E_{11Hn2H} \cdot E_{11Hn2H} \cdot 2m2 + \]
\[ R_{11Hn2H} \cdot 2m2 = T_{111} + T_{112} + T_{121} + T_{212} + T_{221} + T_{222}; \]

\%CALCULATION OF b10) See Section "Sixth-order Moment"

\% \[ E \{ r_k1 \cdot r_m1 \cdot H \cdot r_{k2} \cdot r_{l2} \cdot r_{m2} \cdot r_{n2H} \} \]

\% \[ T_{111} \]
\[ K_{1n2n12H} \cdot E_{2n18} \cdot E_{11Hn2H} \cdot 2m2 \cdot E_{3n18} \cdot E_{11Hn2H} \cdot 2m2 + \]
\[ K_{1n2n12H} \cdot E_{2n18} \cdot E_{11Hn2H} \cdot 2m2 \cdot E_{2n18} \cdot E_{11Hn2H} \cdot 2m2 + \]
\[ K_{1n2n12H} \cdot E_{2n18} \cdot E_{11Hn2H} \cdot 2m2 \cdot E_{2n18} \cdot E_{11Hn2H} \cdot 2m2 + \]
\[ K_{1n2n12H} \cdot E_{2n18} \cdot E_{11Hn2H} \cdot 2m2; \]

\[ T_{112} \]
\[ K_{1n2n12H} \cdot E_{2n18} \cdot E_{11Hn2H} + \]
\[ K_{1n2n12H} \cdot E_{2n18} \cdot E_{11Hn2H} \cdot 2m2 \cdot E_{2n18} \cdot E_{11Hn2H} \cdot 2m2 + \]
\[ K_{1n2n12H} \cdot E_{2n18} \cdot E_{11Hn2H} \cdot 2m2 \cdot E_{2n18} \cdot E_{11Hn2H} \cdot 2m2 + \]
\[ K_{1n2n12H} \cdot E_{2n18} \cdot E_{11Hn2H} \cdot 2m2 \cdot E_{2n18} \cdot E_{11Hn2H} \cdot 2m2; \]

\[ T_{121} \]
\[ K_{1n2n12H} \cdot E_{2n18} \cdot E_{11Hn2H} + \]
\[ K_{1n2n12H} \cdot E_{2n18} \cdot E_{11Hn2H} \cdot 2m2 \cdot E_{2n18} \cdot E_{11Hn2H} \cdot 2m2 + \]
\[ K_{1n2n12H} \cdot E_{2n18} \cdot E_{11Hn2H} \cdot 2m2 \cdot E_{2n18} \cdot E_{11Hn2H} \cdot 2m2 + \]
\[ K_{1n2n12H} \cdot E_{2n18} \cdot E_{11Hn2H} \cdot 2m2; \]

\[ T_{122} \]
\[ K_{1n2n12H} \cdot E_{2n18} \cdot E_{11Hn2H} + \]
\[ K_{1n2n12H} \cdot E_{2n18} \cdot E_{11Hn2H} \cdot 2m2 \cdot E_{2n18} \cdot E_{11Hn2H} \cdot 2m2 + \]
\[ K_{1n2n12H} \cdot E_{2n18} \cdot E_{11Hn2H} \cdot 2m2 \cdot E_{2n18} \cdot E_{11Hn2H} \cdot 2m2 + \]
\[ K_{1n2n12H} \cdot E_{2n18} \cdot E_{11Hn2H} \cdot 2m2; \]
T211=...
Km1Hk2H2n2m2+E4ma1S'*Ek1n2H+Km1Hk2H12n2H+E4ma1S'*Ek1n2 +...
Km1Hk2H12=E3ma1S'*Ek1m2n2H+Km1Hk2H2m2n2H+E4ma1S'*Ek1l2 +...
Km1Hk2H12=E3ma1S'*Er1l2n2H+Km1Hk2H12n2H+E3ma1S'*Ek1l3n2 +...
Km1Hk2H=E2ma1S'*Ek1l2m2n2H;

T212=...
Km1H12n2m2n2H=E4ma1S'*Ek1k2H+Km1H12m2n2+*E4ma1S'*Ek1k2Hn2H +...
Km1H12n2H+E3ma1S'*Ek1k2m2n2n2H +...
Km1H12n2H=E3ma1S'*Ek1k2m2n2n2H +...
Km1H12n2n2H=E3ma1S'*Ek1k2H12+Km1H12m2n2+*E4ma1S'*Ek1k2H12n2H +...
Km1H12n2H=E3ma1S'*Ek1k2H12m2n2H;

T221=...
Kk2H12n2m2n2H=E4ma1S'*Ek1m1K+Ek1H12m2n2+E3ma1S'*Ek1m1H1n2H +...
Kk2H12n2n2H=E3ma1S'*Ek1m1H12m2n2+*Ek1h12n2H +...
Kk2H12n2n2H=E3ma1S'*Ek1m1H12m2n2 +...
Kk2H12n2n2H=E3ma1S'*Ek1m1H12m2n2H +...
Kk1H12n2H+E2ma1S'*Ek1m1H12m2n2 +...
Kk1H12n2H=E2ma1S'*Ek1m1H12m2n2H;

T222=...
Kl1H12n2n2n2H=E3ma1S'*Ek1m1Hk2H+Kl1m12n2+E2ma1S'*Ek1m1Hk2Hn2H +...
Kl1H12n2n2n2H=E3ma1S'*Ek1m1Hk2Hm2n2 +...
Kl1H12n2n2n2H=E3ma1S'*Ek1m1Hk2Hm2n2H +...
Kl1H12n2n2n2H=E3ma1S'*Ek1m1Hk2Hm2n2H +...
Ekm1H12n2n2n2H;

RK1m1Hk2H12n2n2H=T111+T112=T121+T122+T211+T212+T221+T222;

$\text{CALCULATION OF b11) See Section "Sixth-order Moment"}$

% E(r11H rm1H rk2H rl2 rm2 n2H)
%

T111=...
Kl1Hm1Hk2H12n2n2H=E4ma1S'*Ekm1Hm1Hk2H12+*Ekm1Hm1Hk2H+E4ma1S'*Em1n2n2H +...
Kl1Hm1Hk2H12n2n2H=E4ma1S'*El1n2n2+Kl1Hm1Hk2H12n2H+E4ma1S'*El1m2 +...
Kl1Hm1Hk2H12n2n2H=E4ma1S'*El1m2n2H;

T112=...
Kl1Hm1H12n2n2+*E4ma1S'*Ek2H12n2+Kl1Hm1H12n2H=E4ma1S'*Ek2H12 +...
Kl1Hm1H12n2n2+*E4ma1S'*Ek2Hn2n2H +...
Kl1Hm1H12n2n2+*E4ma1S'*Ek2H12n2H +...
Kl1Hm1H12n2n2+*E4ma1S'*Ek2H12n2H +...
Kl1Hm1H12n2n2+*E4ma1S'*Ek2H12n2H;

T121=...
Kl1Hk2H12n2n2+*E4ma1S'*Em1Hn2H+Kl1Hk2H12n2H=E4ma1S'*Em1Hn2 +...
\begin{align*}
K11H2H12 \times E3ma15^1 \times E4H1H12 \times E5ma15^1 \times E6H1H2 \times E7ma15^1 \times E8H1H12 \times E9ma15^1 \times E10H1H2 \\
K11H2H12 \times E4ma15^1 \times E5H1H12 \times E6ma15^1 \times E7H1H2 \times E8ma15^1 \times E9H1H12 \times E10ma15^1 \times E11H1H2 \\
K11H2H12 \times E2ma15^1 \times E3H1H12 \times E4ma15^1 \times E5H1H2 \times E6ma15^1 \times E7H1H12 \times E8ma15^1 \times E9H1H2 \\
K11H2H12 \times E2ma15^1 \times E3H1H12 \times E4ma15^1 \times E5H1H2 \times E6ma15^1 \times E7H1H12 \times E8ma15^1 \times E9H1H2 \\
T122 = \ldots \\
K11H2H12 \times E4ma15^1 \times E5H1H12 \times E6ma15^1 \times E7H1H2 \times E8ma15^1 \times E9H1H12 \times E10ma15^1 \times E11H1H2 \\
K11H2H12 \times E3ma15^1 \times E4H1H12 \times E5ma15^1 \times E6H1H2 \times E7ma15^1 \times E8H1H12 \times E9ma15^1 \times E10H1H2 \\
K11H2H12 \times E2ma15^1 \times E3H1H12 \times E4ma15^1 \times E5H1H2 \times E6ma15^1 \times E7H1H12 \times E8ma15^1 \times E9H1H2 \\
K11H2H12 \times E2ma15^1 \times E3H1H12 \times E4ma15^1 \times E5H1H2 \times E6ma15^1 \times E7H1H12 \times E8ma15^1 \times E9H1H2 \\
T211 = \ldots \\
K11H2H12 \times E4ma15^1 \times E5H1H12 \times E6ma15^1 \times E7H1H2 \times E8ma15^1 \times E9H1H12 \times E10ma15^1 \times E11H1H2 \\
K11H2H12 \times E3ma15^1 \times E4H1H12 \times E5ma15^1 \times E6H1H2 \times E7ma15^1 \times E8H1H12 \times E9ma15^1 \times E10H1H2 \\
K11H2H12 \times E2ma15^1 \times E3H1H12 \times E4ma15^1 \times E5H1H2 \times E6ma15^1 \times E7H1H12 \times E8ma15^1 \times E9H1H2 \\
K11H2H12 \times E2ma15^1 \times E3H1H12 \times E4ma15^1 \times E5H1H2 \times E6ma15^1 \times E7H1H12 \times E8ma15^1 \times E9H1H2 \\
T212 = \ldots \\
K11H2H12 \times E4ma15^1 \times E5H1H12 \times E6ma15^1 \times E7H1H2 \times E8ma15^1 \times E9H1H12 \times E10ma15^1 \times E11H1H2 \\
K11H2H12 \times E3ma15^1 \times E4H1H12 \times E5ma15^1 \times E6H1H2 \times E7ma15^1 \times E8H1H12 \times E9ma15^1 \times E10H1H2 \\
K11H2H12 \times E2ma15^1 \times E3H1H12 \times E4ma15^1 \times E5H1H2 \times E6ma15^1 \times E7H1H12 \times E8ma15^1 \times E9H1H2 \\
K11H2H12 \times E2ma15^1 \times E3H1H12 \times E4ma15^1 \times E5H1H2 \times E6ma15^1 \times E7H1H12 \times E8ma15^1 \times E9H1H2 \\
T221 = \ldots \\
K11H2H12 \times E4ma15^1 \times E5H1H12 \times E6ma15^1 \times E7H1H2 \times E8ma15^1 \times E9H1H12 \times E10ma15^1 \times E11H1H2 \\
K11H2H12 \times E3ma15^1 \times E4H1H12 \times E5ma15^1 \times E6H1H2 \times E7ma15^1 \times E8H1H12 \times E9ma15^1 \times E10H1H2 \\
K11H2H12 \times E2ma15^1 \times E3H1H12 \times E4ma15^1 \times E5H1H2 \times E6ma15^1 \times E7H1H12 \times E8ma15^1 \times E9H1H2 \\
K11H2H12 \times E2ma15^1 \times E3H1H12 \times E4ma15^1 \times E5H1H2 \times E6ma15^1 \times E7H1H12 \times E8ma15^1 \times E9H1H2 \\
T222 = \ldots \\
K11H2H12 \times E4ma15^1 \times E5H1H12 \times E6ma15^1 \times E7H1H2 \times E8ma15^1 \times E9H1H12 \times E10ma15^1 \times E11H1H2 \\
K11H2H12 \times E3ma15^1 \times E4H1H12 \times E5ma15^1 \times E6H1H2 \times E7ma15^1 \times E8H1H12 \times E9ma15^1 \times E10H1H2 \\
K11H2H12 \times E2ma15^1 \times E3H1H12 \times E4ma15^1 \times E5H1H2 \times E6ma15^1 \times E7H1H12 \times E8ma15^1 \times E9H1H2 \\
K11H2H12 \times E2ma15^1 \times E3H1H12 \times E4ma15^1 \times E5H1H2 \times E6ma15^1 \times E7H1H12 \times E8ma15^1 \times E9H1H2 \\
R11H1H2H12 \times T111 \times T121 \times T211 \times T212 \times T221 \times T222 \times T222;
\end{align*}

\textit{\%CALCULATION OF bi2) See Section "Sixth-order Moment"}
\textit{\%}
\begin{align*}
\% \quad E(\text{rk1 rni rk2H rl2 rm2 rn2H}) \\
\%
\end{align*}

\textit{\%}
\begin{align*}
\% T111 = \ldots \\
K11k1k2H12 \times E4ma15^1 \times E5H1H12 \times E6ma15^1 \times E7H1H2 \times E8ma15^1 \times E9H1H12 \times E10ma15^1 \times E11H1H2 \\
K11k1k2H12 \times E4ma15^1 \times E5H1H12 \times E6ma15^1 \times E7H1H2 \times E8ma15^1 \times E9H1H12 \times E10ma15^1 \times E11H1H2 \\
\%
\end{align*}

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Rk1n1k2H12m2n2H=T111+T112+T121+T122+T211+T212+T221+T222;

%CALCULATION OF c) See Section "Fourth-order Moment"
%
%   E(rk1 r1H r1H r1H)
%

    Rk111Hm1Hn1=...
    Kk111Hm1Hn1+E4m1s'*Kk111H+E2ma1s'*Em1Hn1 +...
    Kk1m1H+E2ma1s'*El1Hn1+Kk1n1+E2ma1s'*El1Hm1H +...
    Kk1Hm1H+E2ma1s'*Ek1n1+Kk1Hn1+E2ma1s'*Ek1m1H +...
    Km1Hn1+E2ma1s'*Ek111H+Ek111Hm1Hn1;

%CALCULATION OF c2) See Section "Fourth-order Moment"
%
%   E(rk2H r12 r12 r2H)
%

    Rk2H12m2n2H=...
    Kk2H12m2n2H+E4m1s'*Kk2H12+E2ma1s'*Em2n2H +...
    Kk2Hm2+E2ma1s'*El2n2H+Kk2Hn2H+E2ma1s'*El2m2 +...
    Kk2m2+E2ma1s'*Ek2Hn2H+Kk2n2H+E2ma1s'*Ek2Hm2 +...
    Km2n2H+E2ma1s'*Ek2H12+Ek2H12m2n2H;

%CALCULATION OF c3) See Section "Fourth-order Moment"
%
%   E(rk1 r11H rk2H r12)
%

    Rk111Hk2H12=...
    Kk111Hk2H12+E4m1s'*Kk111H+E2ma1s'*Ek2H12 +...
    Kk1k2H+E2ma1s'*El1H12+Kk1k12+E2ma1s'*El1Hk2H +...
    Kk1Hk2H+E2ma1s'*Ek112+Kk1H12+E2ma1s'*Ek1k2H +...
    Kk2H12+E2ma1s'*Ek111H+Ek111Hk2H12;

%CALCULATION OF c4) See Section "Fourth-order Moment"
%
%   E(rm1H r1i H r2 i r2H)
%

    Rm1Hm1m2n2H=...
    Km1Hm1m2n2H+E4m1s'*Km1Hm1+E2ma1s'*Em2n2H +...
    Km1m2+E2ma1s'*Em1n2H+Km1m2H+E2ma1s'*Em1m2 +...
    Kn1m2+E2ma1s'*Em1Hn2H+Kn1n2H+E2ma1s'*Em1Hm2 +...
    Km2n2H+E2ma1s'*Em1Hn1+EmlHn1m2n2H;
\% \text{CALCULATION OF c6) See Section "Fourth-order Moment"}
\%
\% \quad E_{\text{rk1 r11H rm2 rn2H}}
\%
R_{k111Hm2n2H} = \ldots
K_{k111Hm2n2H} + E_{m1s} + E_{k111H} + E_{E2m1s} + E_{Em2n2H} + \ldots
K_{k12} + E_{E2m1s} + E_{k111H} + E_{Em2} + \ldots
K_{k111Hm2n2H} + E_{E2m1s} + E_{k111H} + E_{Em2} + \ldots
K_{m2n2H} + E_{E2m1s} + E_{k111H} + E_{Em2n2H} = \ldots
\%
\% \text{CALCULATION OF c6) See Section "Fourth-order Moment"}
\%
\% \quad \quad E_{\text{r1H rn1 rk2H r12}}
\%
R_{m1Hn1k2Hn2H} = \ldots
K_{m1Hn1k2Hn2H} + E_{m1s} + E_{k1n1} + E_{E2m1s} + E_{Ek2Hn2H} + \ldots
K_{m1Hk2H} + E_{E2m1s} + E_{En112} + E_{k1Hn1} + E_{E2m1s} + E_{Em1k2H} + \ldots
K_{n1k2H} + E_{E2m1s} + E_{k1Hn1} + E_{k1n1} + E_{Em1k2H} + \ldots
K_{k2Hn1} + E_{E2m1s} + E_{k1Hn1} + E_{Em1k2H} = \ldots
\%
\% \text{CALCULATION OF c7) See Section "Fourth-order Moment"}
\%
\% \quad E_{\text{rk1 r11H rk2H rm2}}
\%
R_{k111Hk2Hn2H} = \ldots
K_{k111Hk2Hn2H} + E_{m1s} + E_{k111H} + E_{E2m1s} + E_{Ek2Hn2H} + \ldots
K_{k11k2H} + E_{E2m1s} + E_{k111H} + E_{Em2} + E_{E2m1s} + E_{Ek1Hk2H} + \ldots
K_{k11Hk2H} + E_{E2m1s} + E_{k11Hn1} + E_{E2m1s} + E_{Ek1k2H} + \ldots
K_{k2Hn2H} + E_{E2m1s} + E_{k111H} + E_{Ek11Hk2Hn2H} = \ldots
\%
\% \text{CALCULATION OF c8) See Section "Fourth-order Moment"}
\%
\% \quad E_{\text{r1H rn1 rl2 rn2H}}
\%
R_{m1Hn112n2H} = \ldots
K_{m1Hn12n2H} + E_{m1s} + E_{k1n1} + E_{E2m1s} + E_{Ek2H} + \ldots
K_{m1H} + E_{E2m1s} + E_{k11n2H} + E_{k1n2H} + E_{E2m1s} + E_{Em112} + \ldots
K_{n112} + E_{E2m1s} + E_{k11Hn1} + E_{Em2} + E_{Em1H12} + \ldots
K_{k12n2H} + E_{E2m1s} + E_{k1n1} + E_{Em1n12n2H} = \ldots
\%
\% \text{CALCULATION OF c9) See Section "Fourth-order Moment"}

\% E(rk1 \ r1H \ r12 \ rn2H) \\
% \% Rk11H12n2H=...
Kk11H12n2H*E4ma1S'*Ek11H1*E2ma1S'*El2n2H +...
Kk12*E2ma1S'*El1Hn2H*Kk1n2H*E2ma1S'*El1H12 +...
K11H12*E2ma1S'*Ek1n2H*K11Hn2H*E2ma1S'*Ek112 +...
K12n2H*E2ma1S'*Ek11H1*Ek11H12n2H; \\
\%CALCULATION OF c10) See Section "Fourth-order Moment" \\
% \% E(rm1H \ rn1 \ rk2H \ rm2) \\
% \% Rm1Hn1k2Hn2=...
Km1Hn1k2Hn2*E4ma1S'*Em1Hn1*E2ma1S'*Ek2Hn2 +...
Km1Hk2H*E2ma1S'*Em1m2*Em1Hn2*E2ma1S'*Em1k2H +...
Kn1k2H*E2ma1S'*Em1Hm2*Em1m2*E2ma1S'*Em1Hk2H +...
Kk2Hn2*E2ma1S'*Em1Hn1*Em1Hn1k2Hn2; \\
\%CALCULATION OF c11) See Section "Fourth-order Moment" \\
% \% E(rk1 \ r1H \ rk2H \ rn2H) \\
% \% Rk11Hk2Hn2=...
Kk11Hk2Hn2*E4ma1S'*Kk11H1*E2ma1S'*Ek2Hn2 +...
Kk1k2H*E2ma1S'*El1Hn2H*Kk1n2H*E2ma1S'*El1Hk2H +...
K11Hk2H*E2ma1S'*Ek1n2H*K11Hn2H*E2ma1S'*Ek1k2H +...
Kk2Hn2*E2ma1S'*Ek11H*Ek11Hk2Hn2H; \\
\%CALCULATION OF c12) See Section "Fourth-order Moment" \\
% \% E(rm1H \ rn1 \ r12 \ rm2) \\
% \% Rm1Hn112m2=...
Km1Hn112m2*E4ma1S'*Em1Hn1*E2ma1S'*El2m2 +...
Km1H12*E2ma1S'*Em1m2*Em1Hm2*E2ma1S'*Em1l2 +...
Kn1l2*E2ma1S'*Em1Hm2*Kn1m2*E2ma1S'*Em1H12 +...
K12m2*E2ma1S'*Em1Hn1*Em1Hn1l2m2; \\
\%CALCULATION OF c13) See Section "Fourth-order Moment" \\
%
\% \ E(rk1 \ r11H \ r12 \ rm2) \\
\%
Rk11H12m2=... \\
Kk11H12m2=E4ma1S'*Kk11H1+*E2ma1S'*E12m2 +... \\
Kk112=E2ma1S'*E11Hm2*Kk1m2*E2ma1S'*E11H12 +... \\
K11H12=E2ma1S'*Ek1m2*Kk11Hm2*E2ma1S'*Ek112 +... \\
K12m2=E2ma1S'*Ek11H*Ek11H12m2; \\

\%CALCULATION OF c14) See Section "Fourth-order Moment" 
\%
\% \ E(rm1H \ rni \ rk2H \ rn2H) \\
\%
Rm1Hm1k2Hm2H=... \\
Km1Hm1k2Hm2H=E4ma1S'*Km1Hn1*E2ma1S'*Ek2Hn2H +... \\
Km1Hk2H=E2ma1S'*Em1n2H*Km1Hn2H*E2ma1S'*Em1k2H +... \\
Kn1k2H=E2ma1S'*Em1n2H*Kn1n2H*E2ma1S'*Em1Hk2H +... \\
Kk2Hn2H=E2ma1S'*Em1nH1*Em1Hn1k2Hn2H; \\

\%CALCULATION OF c15) See Section "Fourth-order Moment" 
\%
\% \ E(rk1 \ rm1H \ rk2H \ rm2) \\
\%
Rk1m1Hk2Hm2=... \\
Kk1m1Hk2Hm2=E4ma1S'*Kk1m1H*E2ma1S'*Ek2Hm2 +... \\
Kk1k2H=E2ma1S'*Em1Hm2*Kk1m2*E2ma1S'*Em1Hk2H +... \\
Km1Hk2H=E2ma1S'*Ek1m2*Km1Hm2*E2ma1S'*Ek1k2H +... \\
Kk2Hm2=E2ma1S'*Ek1mH*Ek1mHk2Hm2; \\

\%CALCULATION OF c16) See Section "Fourth-order Moment" 
\%
\% \ E(r11H \ rni \ r12 \ rn2H) \\
\%
R11Hn1l2n2H=... \\
K11Hn1l2n2H=E4ma1S'*K11Hn1*E2ma1S'*E12n2H +... \\
K11H12=E2ma1S'*En1n2H*K11Hn2H*E2ma1S'*En112 +... \\
Kn1l2=E2ma1S'*E11Hn2H*Kn1n2H*E2ma1S'*E11H12 +... \\
K12n2H=E2ma1S'*E11Hni1*E11Hn1l2n2H;
Calculation of c17) See Section "Fourth-order Moment"

% E(rk1 rm1N r12 rN2N)
%
Rk1m1H12n2H = ...
Kk1m1H2n2H + E4ma1S' + Ekm1H + E2ma1S' + E12n2H + ...
Kk112 + E2ma1S' + En1Hn2H + Kk1n2H + E2ma1S' + En1H12 + ...
Kn1H12 + E2ma1S' + Ekn1H2H + Kn1Hn2H + E2ma1S' + E1k112 + ...
K12n2H + E2ma1S' + Ekm1H + Ekm1H12n2H;

Calculation of c18) See Section "Fourth-order Moment"

% E(r11H r1N r1k2H rN2)
%
Rl1Hm1H1k2Hn2H = ...
Kl1Hm1H1k2Hm2 + E4ma1S' + Kl1Hm1 + E2ma1S' + Ekm2 + ...
Kl1H1k2H + E2ma1S' + En1Hn2H + Kl1Hm2 + E2ma1S' + En1Hk2H + ...
Kn1Hk2H + E2ma1S' + Ekn1Hm2 + Kn1Hm2 + E2ma1S' + E11Hk2H + ...
Kk1k2H + E2ma1S' + Ekm1H + E11Hm1 + E11Hm1k2Hn2H;

Calculation of c19) See Section "Fourth-order Moment"

% E(rk1 rm1H r2H rN2H)
%
Rk1m1Hk2Hn2H = ...
Kk1m1Hk2Hn2H + E4ma1S' + Kk1m1H + E2ma1S' + Ekm2H + ...
Kk1k2H + E2ma1S' + En1Hn2H + Kk1n2H + E2ma1S' + En1Hk2H + ...
Kn1Hk2H + E2ma1S' + Ekn1Hn2H + Kn1Hn2H + E2ma1S' + E1k12H + ...
Kk2Hn2H + E2ma1S' + Ekm1H + Ekm1Hk12Hn2H;

Calculation of c20) See Section "Fourth-order Moment"

% E(r11H r1N r12 rN2)
%
Rl1Hn112n2 = ...
Kl1Hn112n2 + E4ma1S' + Kl1Hn1 + E2ma1S' + El1N2 + ...
Kl11H + E2ma1S' + En1m2 + Kl1Hn2 + E2ma1S' + En112 + ...
Kn112 + E2ma1S' + El1N2 + Kn1m2 + E2ma1S' + El1H12 + ...
K12n2 + E2ma1S' + El1N1 + El1Hn112n2;

Calculation of c21) See Section "Fourth-order Moment"

% E(rk1 rm1H r12 rN2)

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% Rk1m1H12m2=...
Kk1m1H12m2*E4ma1s'+Ek1m1H=E2ma1s'*El1m2 +...
Kk1l2*E2ma1s'*Em1l2+Ek1m2+E2ma1s'*Em1l12 +...
Km1l12*E2ma1s'*Ek1m2+Km1H2+E2ma1s'*Ek1l12 +...
Kl12m2*E2ma1s'*Ek1m1H+Ek1m1l12m2;

%CALCULATION OF c22) See Section “Fourth-order Moment”
% 
%   E{r11H r1n r2H r2H}
% 
% Rk1Hn1k2Hn2H=...
Kk1Hn1k2Hn2H+E4ma1s'*Kk1Hn1+Ek2Hn2H +...
Kk1l1k2H*E2ma1s'*Em1l2+Kk1Hn2H+E2ma1s'*Ek1k2H +...
Kn1k2H+E2ma1s'*Ek1l2+Kl1Hn2H+E2ma1s'*Ek1k2H +...
Kk2Hn2H*E2ma1s'*Ek1Hn1+Ek1Hn1k2Hn2H;

%CALCULATION OF c23) See Section “Fourth-order Moment”
% 
%   E{r1k r1n r2H r2H}
% 
% Rk1n1k2Hn2H=...
Kk1nk2Hn2H+E4ma1s'*Kk1nk1+E2ma1s'*Ek2Hn2H +...
Kk1k2H*E2ma1s'*Em1k2+Kk1n2H+E2ma1s'*Ek1k2H +...
Kn1k2H+E2ma1s'*Ek1n2+Kl1n2H+E2ma1s'*Ek1k2H +...
Kk2Hn2H*E2ma1s'*Ek1n1+Ek1nk2Hn2H;

%CALCULATION OF c24) See Section “Fourth-order Moment”
% 
%   E{r11H r1m1H r12 r2m2}
% 
% Rk1Hm1H12m2=...
K1Hm1H12m2*E4ma1s'*K1Hm1H+E2ma1s'*El1m2 +...
K1H12*E2ma1s'*Em1H2+K1Hm2+E2ma1s'*Em1H12 +...
Km1H12*E2ma1s'*El1m2+Km1Hm2+E2ma1s'*El1H12 +...
K1m2*E2ma1s'*El1Hm1H+K1Hm1H12m2;

%CALCULATION OF c25) See Section “Fourth-order Moment”
% 
%   E{r1k r1n r12 r2m2}
% 
% Rk1n1l12m2=...
Kk1n1l12m2*E4ma1s'*Kk1n1+E2ma1s'*El1m2 +...
Kk112*E2ma1S'*En1m2*Kk1m2*E2ma1S'*En112 +...
Kn112*E2ma1S'*Ekin2*Knn1m2*E2ma1S'*Ek112 +...
Kl2m2*E2ma1S'*Ekin1+Ekin112m2;

%CALCULATION OF c26) See Section "Fourth-order Moment"
%  E(r1rH rm1H rk2H rn2H)
%
Kl1Hm1Hk2Hn2H=...
Kl1Hm1Hk2Hn2H=E4ma1S'+Kl1Hm1H+E2ma1S'*Ek2Hn2H +...
Kl1Hk2H*E2ma1S'+EmlHn2H+Kl1Hn2H*E2ma1S'*EmlHk2H +...
Kn1Hk2H*E2ma1S'*E11Hn2H+Km1Hn2H*E2ma1S'*E11Hk2H +...
Kk2Hn2H*E2ma1S'*E11Hm1H+Kl1Hm1Hk2Hn2H;

%CALCULATION OF d) See Section "Second-order Moment"
%  E(rk1 r11H)
%
Rk111H= Kk111H*E2ma1S'+Ek111H;

%CALCULATION OF d2) See Section "Second-order Moment"
%  E(rm2 rn2H)
%
Rn2n2H= Km2n2H*E2ma1S'+Em2n2H;

%CALCULATION OF d3) See Section "Fourth-order Moment"
%  E(rk2H r12)
%
Rk2H12= Kk2H12*E2ma1S'+Ek2H12;

%CALCULATION OF d4) See Section "Fourth-order Moment"
%  E(r12 rn2H)
%
Rl2n2H= Kl2n2H*E2ma1S'+E12n2H;

%CALCULATION OF d5) See Section "Fourth-order Moment"
%  E(rk2H rm2)
%
Rk2Hm2= Kk2Hm2*E2ma1S'+Ek2Hm2;
CALCULATION OF d6) See Section "Fourth-order Moment"

\[ E(r_{12} \cdot r_{2m}) \]

\[ R_{12m2} = k_{12m2} \cdot E_{2m}a_1s' + E_{12m2}; \]

CALCULATION OF d7) See Section "Fourth-order Moment"

\[ E(r_{k2}r_{n2}) \]

\[ R_{k2Hn2N} = k_{k2Hn2N} \cdot E_{2m}a_1s' + E_{k2Hn2}; \]

CALCULATION OF d8) See Section "Fourth-order Moment"

\[ E(r_{m1}r_{n1}) \]

\[ R_{m1hn1} = k_{m1hn1} \cdot E_{2ma1}s' + E_{m1hn1}; \]

CALCULATION OF d9) See Section "Fourth-order Moment"

\[ E(r_{k1}r_{m1}) \]

\[ R_{k1m1h} = k_{k1m1h} \cdot E_{2ma1}s' + E_{k1m1h}; \]

CALCULATION OF d10) See Section "Fourth-order Moment"

\[ E(r_{l1}r_{n1}) \]

\[ R_{l1hn1} = k_{l1hn1} \cdot E_{2ma1}s' + E_{l1hn1}; \]

CALCULATION OF d11) See Section "Fourth-order Moment"

\[ E(r_{k1}r_{n1}) \]

\[ R_{k1n1} = k_{k1n1} \cdot E_{2ma1}s' + E_{k1n1}; \]

CALCULATION OF d12) See Section "Fourth-order Moment"

\[ E(r_{l1}r_{m1}) \]

\[ E_{l1hm1} = k_{l1hm1} \cdot E_{2ma1}s' + E_{l1hm1}; \]
%CALCULATION OF 1)
% See Chapter "The finite-sample covariance of the fourth-order sample cumulant"
% sum(sum(E(rk1(t) r1H(t) rni1(t) rni2(p) r12(p) r2m(p) r2n(p))))
% t p
% sum1=N*Rk11H1H1n1k2H12m2n2H+(N."2-N)*Rk11H1H1n1*Rk2H12m2n2H;

%CALCULATION OF 2a)
% See Chapter "The finite-sample covariance of the fourth-order sample cumulant"
% sum(sum(E(rk1(t) r1H(t) rni1(t) rni2(p) r12(p) r2m(q) r2n(q))))
% t p q
% sum2a= N *Rk11H1H1n1k2H12m2n2H+...
  (N."2-N)*Rk11H1H1n1*Rk2H12m2n2H+...
  (N."2-N)*Rk11H1H1n1k2H12*Rm2n2H+...
  (N."2-N)*Rk11H1H1n1m2n2H*Rk2H12+...
  (N."3-3+N."2+2*N)*Rk11H1H1n1*Rk2H12*Rm2n2H;

%CALCULATION OF 2b)
% See Chapter "The finite-sample covariance of the fourth-order sample cumulant"
% sum(sum(E(rk1(t) r1H(t) rni1(t) rni2(p) r12(q) r2n(q))))
% t p q
% sum2b= N *Rk11H1H1n1k2H12m2n2H+...
  (N."2-N)*Rk11H1H1n1*Rk2H12m2n2H+...
  (N."2-N)*Rk11H1H1n1k2Hm2*Rl2n2H+...
  (N."2-N)*Rk11H1H1n1l2n2H*Rk2Hm2+...
  (N."3-3+N."2+2*N)*Rk11H1H1n1*Rk2Hm2*Rl2n2H;

%CALCULATION OF 2c)
% See Chapter "The finite-sample covariance of the fourth-order sample cumulant"
% sum(sum(E(rk1(t) r1H(t) rni1(t) rni2(p) r2n(q) r12(q))))
% t p q
% sum2c= N *Rk11H1H1n1k2H12m2n2H+...
  (N."2-N)*Rk11H1H1n1*Rk2H12m2n2H+...
  (N."2-N)*Rk11H1H1n1k2Hn2*Rl2m2+...
  (N."2-N)*Rk11H1H1n1l2n2*Rk2Hn2+...
  (N."3-3+N."2+2*N)*Rk11H1H1n1*Rk2Hn2*Rl2m2;
\%CALCULATION OF 3a)
\%See Chapter "The finite-sample covariance of the fourth-order sample cumulant"
\%
\%sum(sum(sum(E(rk1(t) r1H(t) rm1H(p) rm1(p) rk2H(q) r12(q) rm2(s) rm2H(s))))))
\% t p q s
\%
\%sum3a= \# *E(k11H1H1H1k2H12m2n2H+ \
(\# "2-0") *rk11H1H1H1k2H12m2n2H+ \
(\# "2-0") *rm1H1H1k2H12m2n2H+ \
(\# "2-0") *rk2H12m2n2H+ \
(\# "2-0") *rm1H1H1k2H12m2n2H+ \
(\# "2-0") *r2H12m2n2H+ \
(\# "2-0") *rm1H1H1k2H12m2n2H+ \
(\# "2-0") *rk11H1H1H1k2H12m2n2H+ \
(\# "2-0") *rm1H1H1k2H12m2n2H+ \
(\# "2-0") *rk2H12m2n2H+ \
(\# "2-0") *rm1H1H1k2H12m2n2H+ \
(\# "2-0") *rk2H12m2n2H+ 
\%

\%CALCULATION OF 3b)
\%See Chapter "The finite-sample covariance of the fourth-order sample cumulant"
\%
\%sum(sum(sum(E(rk1(t) r1H(t) rm1H(p) rm1(p) rk2H(q) r12(q) rm2(s) rm2H(s))))))
\% t p q s
\%
\%sum3b= \# *E(k11H1H1H1k2H12m2n2H+ \
(\# "2-0") *rk11H1H1H1k2H12m2n2H+ \
(\# "2-0") *rm1H1H1k2H12m2n2H+ \
(\# "2-0") *rk2H12m2n2H+ \
(\# "2-0") *r12H12m2n2H+ \
(\# "2-0") *r12H12m2n2H+ 
\%

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%CALCULATION OF 3c)
%See Chapter "The finite-sample covariance of the fourth-order sample cumulant"

%sum(sum(sum(E(r_k1(t) r_1H(t) r_m1H(p) r_n1(p) r_k2H(q) r_m2H(q) r_12(s) r_m2(s))))))
% t p q s

sum3c = \sum_{N} E(r_{k1}^{2} + r_{1H}^{2} + r_{m1H}^{2} + r_{k2H}^{2} + r_{m2H}^{2} + r_{12}^{2})

%CALCULATION OF 3d)
%See Chapter "The finite-sample covariance of the fourth-order sample cumulant"

%sum(sum(sum(sum(E(r_k1(t) r_1H(t) r_m1H(p) r_n1(p) r_k2H(q) r_m2H(q) r_12(s) r_m2(s))))))
% t p q s

sum3d = \sum_{N} E(r_{k1}^{2} + r_{1H}^{2} + r_{m1H}^{2} + r_{k2H}^{2} + r_{m2H}^{2} + r_{12}^{2})
(N.*4-6*N.*3+11*N.*2-6*N.*) * Rk1m1H* Rl1Hm1* Rk2Hm2* Rl2m2H;

% CALCULATION OF 3e
% See Chapter "The finite-sample covariance of the fourth-order sample cumulant"
% sum(sum(sum(E(rk1(t) rm1H(t) r1H(p) rl1H(p) r2H(q) r2H(q) r12(s) rm2(s)))))
% t p q s
% sum3e = N * Rk11Hm1Hm1k2H12m2n2H+...
(N.*2-N.*) * Rk1m1H* Rl1Hm1k2H12m2n2H+...
(N.*2-N.*) * Rl1Hm1* Rk1m1Hk2H12m2n2H+...
(N.*2-N.*) * Rk2H2n2H* Rk11Hm1Hm1k12m2+...
(N.*2-N.*) * Rl1Hm1* Rk11Hm1k2H12m2n2H+...
(N.*2-N.*) * Rk11Hm1Hm1* Rk2H12m2n2H+...
(N.*2-N.*) * Rk1m1Hk2H2n2H* Rl1Hm1* Rl12m2+...
(N.*3-N.*) * Rk1m1Hk2H2n2H* Rl1Hm1* Rk2H2n2H+...
(N.*3-N.*) * Rk1m1H* Rk11Hm1k2H2n2H+...
(N.*3-N.*) * Rk2H2n2H* Rk11Hm1Hm1k12m2+...
(N.*3-N.*) * Rl1Hm1* Rk11Hm1k2H2n2H+...
(N.*3-N.*) * Rk1m1H* Rk11Hm1k2H2n2H+...
(N.*3-N.*) * Rk1m1Hk2H2n2H* Rl1Hm1* Rl12m2+...
(N.*4-6*N.*3+11*N.*2-6*N.*) * Rk1m1H* Rl1Hm1* Rk2H2n2H* Rl12m2;

% CALCULATION OF 3f
% See Chapter "The finite-sample covariance of the fourth-order sample cumulant"
% sum(sum(sum(E(rk1(t) rm1(t) r1H(p) rm1H(p) r2H(q) r2H(q) r12(s) rm2(s)))))
% t p q s
% sum3f = N * Rk11Hm1Hm1k2H12m2n2H+...
(N.*2-N.*) * Rk1m1* Rl1Hm1k2H12m2n2H+...
(N.*2-N.*) * Rl1Hm1* Rk1m1k2H12m2n2H+...
(N.*2-N.*) * Rk2H2n2H* Rk11Hm1Hm1k12m2+...
(N.*2-N.*) * Rl1Hm1* Rk11Hm1k2H2n2H+...
(N.*2-N.*) * Rk11Hm1Hm1* Rk2H12m2n2H+...
(N.*2-N.*) * Rk1m1Hk2H2n2H* Rl1Hm1* Rl12m2+...
(N.*3-N.*) * Rk1m1Hk2H2n2H* Rl1Hm1* Rk2H2n2H+...
(N.*3-N.*) * Rk1m1H* Rk11Hm1k2H2n2H+...
(N.*3-N.*) * Rk2H2n2H* Rk11Hm1Hm1k12m2+...
(N.*3-N.*) * Rl1Hm1* Rk11Hm1k2H2n2H+...
(N.*3-N.*) * Rk1m1H* Rk11Hm1k2H2n2H+...
(N.*3-N.*) * Rk1m1Hk2H2n2H* Rl1Hm1* Rl12m2+...
(N.*3-N.*) * Rk1m1H* Rk11Hm1k2H2n2H+...

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if nargin<6, alpha=(N.^2-M.)/(N+2); end;
if nargin<7, beta=N.^2-M; end;

\%
\% Calculation of the expected value \( E(c') \)
\%

N=H(:); 
I=ones(1,length(H));
Ecall1 = N.*(I./alpha-3*I./beta)*Rkll1Hm1H1n1...
\% \( -N. \cdot \frac{2-M}{\beta} \cdot \frac{Rk111H1+Rm1Hn1+Rk111H1+Rk111H1+Rk111H1} {N.} \);
Ecall2 = N.*(I./alpha-3*I./beta)*Rk2H12m2n2H...
\% \( -N. \cdot \frac{2-M}{\beta} \cdot \frac{Rk2H12+Rm2n2H+Rk2Hn2+Rk2Hn2+Rk2Hn2+Rk2Hn2} {N.} \);

\%
\% ...finally the last step!!!!
\%

cov4=sum1./alpha.'^2-2*real(sum2a+sum2b+sum2c)./(alpha.*beta) +...
\% \( \frac{\text{sum3a+sum3d+sum3f}}{\beta} \cdot \frac{2}{N} + \frac{\text{sum3b+sum3c+sum3e}}{\beta} \cdot \frac{2}{N} \)
Ecall1.*Ecall2;

if nargout>4
cov4asympcov4(length(cov4));
cov4=cov4(1:length(cov4)-1);
Ecall1=Ecall1(1:length(Ecall1)-1);
Ecall2=Ecall2(1:length(Ecall2)-1);
end

\%Calculation of the covariance of the "Biased Estimator"
if nargout>3
Ecall1b = (I^2-3*I./N)*Rk111Hm1H1n1...
\% \( -(I-3)/(N) \cdot \frac{(Rk111H1+Rm1Hn1+Rk111H1+Rk111H1+Rk111H1) \cdot \frac{Rk111H1+Rk111H1+Rk111H1} {N} \)}{N.
Ecall2b = (I^2-3*I./N)*Rk2H12m2n2H...
\% \( -(I-3)/(N) \cdot \frac{(Rk2H12+Rm2n2H+Rk2Hn2+Rk2Hn2+Rk2Hn2+Rk2Hn2) \cdot \frac{Rk2H12+Rm2n2H+Rk2Hn2+Rk2Hn2+Rk2Hn2+Rk2Hn2} {N} \)}{N.
cov4biased = sum1./N.'^2-2*real(sum2a+sum2b+sum2c)./(N.3) +...
\% \( \frac{\text{sum3a+sum3d+sum3f}}{\beta} \cdot \frac{2}{N} + \frac{\text{sum3b+sum3c+sum3e}}{\beta} \cdot \frac{2}{N} \)
Ecall1b.*Ecall2b;
if nargin>4
    cov4biasedasymp=cov4biased(length(cov4biased));
    cov4biased=cov4biased(1:length(cov4biased)-1);
    Ecallib=Ecallib(1:length(Ecallib)-1);
    Ecall2b=Ecall2b(1:length(Ecall2b)-1);
end
end

% Calculation of the relative error between the asymptotic and the finite
% data length covariance

if nargin>4
    N=N(1:length(N)-1);
    relerror=(N.*cov4-Nasymp.*cov4asymp)./(N.*cov4);
    if nargin>5
        relerrorbiased=(N.*cov4biased-Nasymp.*cov4biasedasymp)./(N.*cov4biased);
    end
end

4.4 xmaln.m

function xmn=xmaln(index,gn)

% XMALN M-File for simplifying the notation in covxall.m (x=2,3,4).
% Syntax:
%  xmn = xmaln(index,gn)
% where
%    index = contains the indices of the moment, eventually
%    multiplied with an "j" to indicate that the
%    corresponding random variable is conjugated.
% Example: For calculating E(n1 n2 n6 n2''n7''n)
% index = [1 2 5 j*2 j*7]
% gn = [gRn1,2 gRn1,3 .... gRn1,K
% gIn1,2 gIn1,3 .... gIn1,K
% :   :   :
% gRnM,2 gRnM,3 .... gRnM,K
% gInM,2 gInM,3 .... gInM,K]
% Matrix that contains all moments of the real- and
% imaginary signal part from the second to the Kth-

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% moment, where K=length(index).
%
% The following example explains the abbreviations:
% ghn1,3 = 3rd moment of the real part of signal 1
%
% Extracting of the important information in the index-vector
index1=abs(index);
index2=2*sign(abs(index-real(index)))+1; %index2=1 if random variable is real
% =-1 if " " " imag.
K=length(index);

% Insertion of the first moment
[H,gn_cols]=size(gn);
gn=zeros(H,1) gn;

[sig_ind,1]=sort(index1);
sig_ind=[sig_ind [-1];
index2=index2(1); % Proper sorting of index2

xmn=1;
momz=1;
for i=2:K+1 % Loop for determining equal random variables
    if sig_ind(i)==sig_ind(i-1)
        momz=momz+1;
    else
        sum=0;
        hindex2=index2(1:-momz:i-1);
        for m=1:2:momz % Loop on all 2^momz summands
            hindex2=[hindex2(-1)];
            momz2=1;
            hif=1;
            hj=1;
            for l=2:momz+1
                if sig_ind2(l)==sig_ind2(l-1)
                    momz2=momz2+1;
                    Ind2=2*sig_ind2(l-1)+sig_ind2(l-1)-2;
                    if gerode(Ind)
                        hj=hj*sqrt(-1)*hindex2(l-1);
                    end
                end
```
else
  Ind=2*\text{sig}_{\text{ind}(i-1)}+\text{sig}_{\text{ind2}(i-1)}-2;
  if \text{gerade}(\text{Ind})
    \text{hilf}=\text{hilf}*\text{gn}(\text{Ind},\text{momz2})\*\text{sqrt}(-1)*\text{khindex2}(i-1)*h_j;
  else
    \text{hilf}=\text{hilf}*\text{gn}(\text{Ind},\text{momz2});
  end
  \text{momz2}=1;
end
end
\text{sum}=\text{sum}+\text{hilf};
end
\text{xmn}=\text{xmn}\*\text{sum};
\text{momz}=1;
end
end
Bibliography


