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Finite-Sample Covariances of Second-, Third-, and Fourth-Order Sample Cumulants in Narrowband Array Processing

by

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Abstract

Recently, fourth-order cumulants were successfully applied in the area of narrowband array signal processing. For example, the virtual- ESPRIT-Algorithm (VESPA) [8] for direction-finding and recovery of independent sources can also calibrate an array of unknown configuration. Furthermore with extended VESPA [16] direction-finding of highly correlated or coherent sources is possible. In addition, fourth-order cumulants can also be used for estimation of the range as well as the angle in the near-field case [2].

This large number of new algorithms motivates a performance analysis to compare the higher-order statistics based algorithms with the conventional second-order statistics based algorithms. Up to now, for higher-order statistics based algorithms only *asymptotic* results are available for direction-finding. These results are restricted to a certain class of communication signals [1].

In this technical report we avoid these restrictions by deriving the finite-sample covariance of

- the second-order sample cumulant (moment),
- the third-order sample cumulant (moment), and
- the fourth-order sample cumulant

for: *finite* data length, any kind of random signals, any kind of noises, any array shapes, and, arbitrary sensors. We do this, because most performance analyses are based on one of these covariances. Consequently, this report provides the fundamentals for very general performance analyses of second- and higher-order statistics based array-processing algorithms.

Contents

Chapter 1

Introduction

Since existing second-order statistics based methods cannot solve numerous problems in narrowband array signal processing, many higher-order cumulant based algorithms have been recently proposed. For example, the virtual-ESPRIT-Algorithm (VESPA) for direction-finding and recovery of independent sources can also calibrate an array of unknown configuration [8]. Also, VESPA has been extended to direction-finding of highly correlated or coherent sources [16], which happens in practice due to multipath propagation or smart jammer. In addition, higher-order cumulants can be used to increase the effective aperture of an arbitrary antenna array [8]. Since higher-order cumulants are insensitive to additive, white or colored Gaussian noise, Gaussian noise suppression is always accomplished by them. Non-Gaussian noise suppression using higher-order cumulants is often possible, too [9]. Furthermore, fourth-order cumulants can also be used for estimation of the range as well as the angle in the near-field case [2]. These new cumulant-based algorithms motivate a performance analysis not only among themselves, but also with second-order statistics based methods.

In the domain of second-order statistics, many performance analyses have been done. First approaches were for *deterministic* situations. COX [7] compared the effects of mismatch in the signal direction of some optimum beamformers. COMPTON [5] showed that the steering vector accuracy is essentially a question of the signal dynamic range. The greater the dynamic range, the more accurate the steering vector must be. ZAHM [27] calculated the output signal-to-noise ratio and showed the serious performance degradation due to steering vector mismatches.

Later approaches were for *stochastic* situations. COX ET AL. [6] investigated the effects of amplitude and phase errors in the steering vectors. They assumed that these errors can be modeled as zero-mean random variables that are independent among the sensors, and showed the impact of these errors on the power response of a linear predictive beamformer. Phase errors lead to a sharp peak but at the wrong position, whereas amplitude errors can lead to peak splitting. COMPTON [4] computed the output signal-to-interference-plus-noise ratio (SINR) in case of additive steering-vector errors. He showed the remarkable result that increasing the number of sensors can lead to a decrease of the SINR as long as the steering-vector error variance is high enough. Besides the steering-vector errors, GODARA [15] considered the impact of *weight vector errors*, modeled also as zero mean uncorrelated random variables, on some different beamformers. FRIEDLANDER AND PORAT [13] calculated the output-interference-to-signal ratio (OISR) of a particular class of null-steering algorithms. For a list of more error sources see VURAL [24], who investigated their impact on optimum beamformers, or NG [18], who investigated the impact of wavelength, gain, and steering-vector phase errors.

In the past decade, array signal processing methods based on eigen-decomposition of the (second-order) covariance matrix have received considerable attention, since they provide high resolution with low computational complexity. Many authors have analyzed the performance of eigendecomposition based algorithms. FRIEDLANDER [13] derived an expression for the sensitivity of the well-known MUSIC algorithm using a first-order Taylor expansion of the inverse of the MUSIC spectrum. He found that the sensitivity of MUSIC for linear arrays to phase errors is relatively small and is practically independent of source separation, whereas the sensitivity of MUSIC for circular arrays is inversely proportional to source separation. Furthermore, he observed that increasing the array aperture reduces the sensitivity of the system to modeling errors. XU AND BUCKLEY [25] presented a bias analysis of the MUSIC location estimator using a second-order Taylor expansion of the derivative of the MUSIC spectrum. In these results, all the performance analyses are based on the asymptotic case (infinite data case); XU AND BUCKLEY showed in their simulations that the bias expression can be accurately applied to a very limited number, N, of independent snapshots ($N = 20$). REED ET AL. [20] derived an exact output SINR formula as a function of N for an optimum beamformer. Unfortunately, their results are limited to Gaussian source signals and sensor noises. FELDMAN AND GRIFFITH [12] also investigated the SINR formula of Reed and they showed by simulation, that Reed's formula can be well approximated by a simple expression if N is larger than approximately fifty. These results suggest that perhaps a small number of samples is sufficient for practical use of many asymptotic performance analyses; this can only be proven if the finite-sample covariance can be calculated so that small-sample performance analyses can be performed.

Some performance analyses for higher-order cumulant based methods have already been done. CARDOSO AND MOULINES [1] have derived closed-form expressions of the asymptotic covariance of MUSIC-like direction-of-arrival (DOA) estimates based on two different fourth-order cumulant matrices, and have compared these results with the standard covariance-based MUSIC estimate. They showed that, in case of a single source and a linear array with sensors spaced at regular intervals of half a wavelength, fourth-order MUSIC achieves the same performance as second-order MUSIC, for a special class of signals and in the high-SNR limit. In the low-SNR limit and the single source case, fourth-order MUSIC outperforms second-order MUSIC, if the source kurtosis divided by the source variance is large enough. Furthermore, in the multiple independent sources case and for large SNR, the variance of the DOA of a weak source is significantly increased for fourth-order methods in contrast to second-order MUSIC, where this phenomenom does not occur. YUEN AND FRIEDLANDER [26] presented an asymptotic performance analysis of VESPA, and a second-order and another fourth-order ESPRIT algorithm (see CHIANG AND NIKIAS [3]). Despite the fact that VESPA requires less sensors, it can perform just as well as second-order ESPRIT in some cases and outperforms fourth-order ESPRIT in almost all cases. However, YUEN AND FRIEDLANDER only consider 4-QAM communication signals and additive white Gaussian noise.

Most all of the performance analyses just described are based on the *asymptotic* covariances of either second-order or fourth-order sample cumulants. In this report, we derive the *finite-sample* covariances of the second-, third- and fourth-order sample cumulants. This gives us more insight in the dependence of the model parameters; for example, we used this formula to derive an exact expression for the relative error between the asymptotic covariance and the finite-sample covariance [17]. This expression can be used to give good advice on how many data are necessary for practical use of an asymptotic performance analysis. Furthermore, based on the finite-sample covariance, rules can be given about which cumulants should be used for accurately estimating the array processing model parameters (for example, the steering vector).

Because the formulas, especially the finite-sample covariance of the fourth-order cumulant are extremely tedious to implement, we have published all of them as Matlab™ M-Files on the Internet "World Wide Web" under

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http://fb9nt-ln.uni-duisburg.de/mitarbeiter/kaiser
        /tech_rep.96/tech_rep.html
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The implementation of our formulas in other programming languages, like "C" and Mathematica™, is planned for the near future. Due to the fast growing Internet, you should contact the first author if you have problems receiving the files.

After the problem statement, we derive, in chapters 2–4, the finite-sample covariance of the second-, third-, and fourth-order sample cumulants, respectively. Chapter 5 contains all necessary M-Files.

1.1 Problem statement

The received $M \times 1$ signal vector $\mathbf{r}(t)$ of an array consisting of M sensors can be modeled under the narrowband assumption as

$$\mathbf{r}(t) = \mathbf{A}\mathbf{s}(t) + \mathbf{n}(t), \tag{1.1}$$

where $\mathbf{s}(t)$ is a $P \times 1$ zero-mean vector which contains the independent source signals at time t, $\mathbf{A}$ is an $M \times P$ steering matrix, and $\mathbf{n}(t)$ is the $M \times 1$ independent, but *not necessarily Gaussian*, zero-mean measurement noise vector. Any noise signal $n_m(t)$, $m = 1(1)M$ is[1] independent of any source

[1] $m = 1(1)M$ is a more compact form of $m = 1, 2, ..., M$. The number in parentheses means the increment of the sequence.

signal $s_p(t)$, $p = 1(1)P$. Furthermore, any signal $s_p(t)$ ($n_m(t)$) is modeled as a sequence of independent and identically, but *arbitrarily* distributed (i.i.d.) complex random variables with finite moments up to the eighth-order. This model is commonly used in the far-field case, but it can be extended to coherent signals (see GŐNEN ET AL. [16]) and to the near-field case (see CHALLA AND SHAMSUNDER [2]).

The *second-order cumulant (moment)* is then defined as

$$c_{k,l}^{(2)} = \mathrm{E}\left\{r_k(t)r_l^*(t)\right\}, \tag{1.2}$$

the *third-order cumulant (moment)* as

$$c_{k,l,m}^{(3)} = \mathrm{E}\left\{r_k(t)r_l^*(t)r_m(t)\right\} \tag{1.3}$$

and the *fourth-order cumulant* as

$$\begin{aligned} c_{k,l,m,n}^{(4)} &= \mathrm{E}\left\{r_k(t)r_l^*(t)r_m^*(t)r_n(t)\right\} \\ &\quad -\mathrm{E}\left\{r_k(t)r_l^*(t\right\}\mathrm{E}\left\{r_m^*(t)r_n(t)\right\} \\ &\quad -\mathrm{E}\left\{r_k(t)r_m^*(t\right\}\mathrm{E}\left\{r_l^*(t)r_n(t)\right\} \\ &\quad -\mathrm{E}\left\{r_k(t)r_n(t\right\}\mathrm{E}\left\{r_l^*(t)r_m^*(t)\right\}, \end{aligned} \tag{1.4}$$

since $r_m(t)$ is also zero-mean for $m = 1(1)M$. By replacing the expected values by time averages we obtain the *second-order sample cumulant (moment)*

$$\hat{c}_{k,l}^{(2)} = \frac{1}{N}\sum_{t=1}^{N} r_k(t)r_l^*(t), \tag{1.5}$$

the *third-order sample cumulant (moment)*

$$\hat{c}_{k,l,m}^{(3)} = \frac{1}{N}\sum_{t=1}^{N} r_k(t)r_l^*(t)r_m(t) \tag{1.6}$$

and the *fourth-order sample cumulant*

$$\begin{aligned}
\hat{c}^{(4)}_{k,l,m,n} &= \frac{1}{\alpha}\sum_{t=1}^{N} r_k(t)r_l^*(t)r_m^*(t)r_n(t) \\
&\quad -\frac{1}{\beta}\sum_{t=1}^{N} r_k(t)r_l^*(t)\sum_{p=1}^{N} r_m^*(p)r_n(p) \\
&\quad -\frac{1}{\beta}\sum_{t=1}^{N} r_k(t)r_m^*(t)\sum_{p=1}^{N} r_l^*(p)r_n(p) \\
&\quad -\frac{1}{\beta}\sum_{t=1}^{N} r_k(t)r_n(t)\sum_{p=1}^{N} r_l^*(p)r_m^*(p),
\end{aligned} \tag{1.7}$$

where $N \in \mathbb{N}$ is the data length and α and β are functions of N. If the fourth-order sample cumulant is an *unbiased* estimate,

$$\alpha = \frac{N^2 - N}{N+2}, \qquad \forall N > 1 \tag{1.8}$$

$$\beta = N^2 - N, \qquad \forall N > 1 \tag{1.9}$$

otherwise, $\alpha = N$ and $\beta = N^2$ is commonly used. Note that the second- and the third-order sample cumulants are always unbiased. In the following section we will derive the finite-sample covariance of all of these sample cumulants for the model defined in (1.1).

1.2 The finite-sample covariance of the second-order sample cumulant

The finite-sample covariance of the second-order sample cumulant can be written as

$$\mathrm{Cov}\left(\hat{c}^{(2)}_{k_1,l_1}, \hat{c}^{(2),H}_{k_2,l_2}\right) = \mathrm{E}\left\{\hat{c}^{(2)}_{k_1,l_1}\hat{c}^{(2),H}_{k_2,l_2}\right\} - \mathrm{E}\left\{\hat{c}^{(2)}_{k_1,l_1}\right\}\mathrm{E}\left\{\hat{c}^{(2),H}_{k_2,l_2}\right\} \tag{1.10}$$

with

$$
\begin{aligned}
& \mathrm{E}\left\{\hat{c}^{(2)}_{k_1,l_1}\hat{c}^{(2),H}_{k_2,l_2}\right\} \\
& = \frac{1}{N^2}\sum_{t=1}^{N}\sum_{p=1}^{N}\mathrm{E}\left\{r_{k_1}(t)r^*_{l_1}(t)r^*_{k_2}(p)r_{l_2}(p)\right\} \\
& = \frac{1}{N}\mathrm{E}\left\{r_{k_1}(t)r^*_{l_1}(t)r^*_{k_2}(t)r_{l_2}(t)\right\} + \\
& \quad \frac{N^2-N}{N^2}\mathrm{E}\left\{r_{k_1}(t)r^*_{l_1}(t)\right\}\mathrm{E}\left\{r^*_{k_2}(t)r_{l_2}(t)\right\},
\end{aligned}
\tag{1.11}
$$

where the summations dissolve due to the i.i.d. property of $r_m(t)$ $\forall m = 1(1)M$.
Since $\mathrm{E}\left\{\hat{c}^{(2)}_{k_1,l_1}\right\} = \mathrm{E}\left\{r_{k_1}(t)r^*_{l_1}(t)\right\}$, (1.10) yields

$$
\begin{aligned}
& \mathrm{Cov}\left(\hat{c}^{(2)}_{k_1,l_1}, \hat{c}^{(2),H}_{k_2,l_2}\right) = \\
& \quad \frac{1}{N}\left(\mathrm{E}\left\{r_{k_1}(t)r^*_{l_1}(t)r^*_{k_2}(t)r_{l_2}(t)\right\} - \right. \\
& \quad\quad \left. \mathrm{E}\left\{r_{k_1}(t)r^*_{l_1}(t)\right\}\mathrm{E}\left\{r^*_{k_2}(t)r_{l_2}(t)\right\}\right).
\end{aligned}
\tag{1.12}
$$

In order to evaluate (1.12) we must calculate the fourth- and the second-order moments of the array output signal for the model defined in (1.1).

1.3 Fourth-order moment

In the following we often omit the time variable t in order to simplify the notation.

$$
\begin{aligned}
& \mathrm{E}\left\{r_{k_1}(t)r^H_{l_1}(t)r^H_{k_2}(t)r_{l_2}(t)\right\} \\
& = \mathrm{E}\left\{(\mathbf{a}^T_{k_1}\mathbf{s} + n_{k_1})(\mathbf{s}^H\mathbf{a}^*_{l_1} + n^*_{l_1})(\mathbf{s}^H\mathbf{a}^*_{k_2} + n^*_{k_2})(\mathbf{a}^T_{l_2}\mathbf{s} + n_{l_2})\right\} \\
& = \mathrm{E}\left\{\mathbf{a}^T_{k_1}\mathbf{s}\mathbf{s}^H\mathbf{a}^*_{l_1}\mathbf{s}^H\mathbf{a}^*_{k_2}\mathbf{a}^T_{l_2}\mathbf{s}\right\} + \mathrm{E}\left\{\mathbf{a}^T_{k_1}\mathbf{s}\mathbf{s}^H\mathbf{a}^*_{l_1}\right\}\mathrm{E}\left\{n^*_{k_2}n_{l_2}\right\} + \\
& \quad \mathrm{E}\left\{\mathbf{a}^T_{k_1}\mathbf{s}\mathbf{s}^H\mathbf{a}^*_{k_2}\right\}\mathrm{E}\left\{n^*_{l_1}n_{l_2}\right\} + \mathrm{E}\left\{\mathbf{a}^T_{k_1}\mathbf{s}\mathbf{a}^T_{l_2}\mathbf{s}\right\}\mathrm{E}\left\{n^*_{l_1}n^*_{k_2}\right\} +
\end{aligned}
$$

$$\mathrm{E}\left\{\mathbf{s}^H\mathbf{a}_{l_1}^*\mathbf{s}^H\mathbf{a}_{k_2}^*\right\}\mathrm{E}\left\{n_{k_1}n_{l_2}\right\}+\mathrm{E}\left\{\mathbf{s}^H\mathbf{a}_{l_1}^*\mathbf{a}_{l_2}^T\mathbf{s}\right\}\mathrm{E}\left\{n_{k_1}n_{k_2}^*\right\}+$$
$$\mathrm{E}\left\{\mathbf{s}^H\mathbf{a}*_{k_2}\mathbf{a}_{l_2}^T\mathbf{s}\right\}\mathrm{E}\left\{n_{k_1}n_{l_1}^*\right\}+\mathrm{E}\left\{n_{k_1}n_{l_1}^*n_{k_2}^*n_{l_2}\right\} \tag{1.13}$$

where $\mathbf{a}_{k_1}^T$ is a *steering row vector*, since it is the k_1th-row vector of the steering matrix A. For further calculating the expected value in eq. (1.13), the random variables ($\mathbf{s}$) must be separated from the deterministic variables ($\mathbf{a}_k$). Since we are faced with this problem in the following chapters again, we are looking for a general notation. Between the scalars $\mathbf{s}^H\mathbf{a}*_k$, $\mathbf{a}_k^T\mathbf{s}$, we can simply introduce a Kronecker product and using the rule for the Kronecker-product ($\otimes$)

$$(\mathbf{AB})\otimes(\mathbf{CD})=(\mathbf{A}\otimes\mathbf{C})(\mathbf{B}\otimes\mathbf{D}) \tag{1.14}$$

yields

$$\begin{aligned}
&\mathrm{E}\left\{r_{k_1}(t)r_{l_1}^H(t)r_{k_2}^H(t)r_{l_2}(t)\right\}\\
&= \left(\mathbf{a}_{k_1}^T\otimes\mathbf{a}_{l_1}^H\otimes\mathbf{a}_{k_2}^H\otimes\mathbf{a}_{l_2}^T\right)\mathrm{E}\left\{\mathbf{s}\otimes\mathbf{s}^*\otimes\mathbf{s}^*\otimes\mathbf{s}\right\}+\\
&\quad\left(\mathbf{a}_{k_1}^T\otimes\mathbf{a}_{l_1}^H\right)\mathrm{E}\left\{\mathbf{s}\otimes\mathbf{s}^*\right\}\mathrm{E}\left\{n_{k_2}^*n_{l_2}\right\}+\\
&\quad\left(\mathbf{a}_{k_1}^T\otimes\mathbf{a}_{k_2}^H\right)\mathrm{E}\left\{\mathbf{s}\otimes\mathbf{s}^*\right\}\mathrm{E}\left\{n_{l_1}^*n_{l_2}\right\}+\\
&\quad\left(\mathbf{a}_{k_1}^T\otimes\mathbf{a}_{l_2}^T\right)\mathrm{E}\left\{\mathbf{s}\otimes\mathbf{s}\right\}\mathrm{E}\left\{n_{l_1}^*n_{k_2}^*\right\}+\\
&\quad\left(\mathbf{a}_{l_1}^H\otimes\mathbf{a}_{k_2}^H\right)\mathrm{E}\left\{\mathbf{s}^*\otimes\mathbf{s}^*\right\}\mathrm{E}\left\{n_{k_1}n_{l_2}\right\}+\\
&\quad\left(\mathbf{a}_{l_1}^H\otimes\mathbf{a}_{l_2}^T\right)\mathrm{E}\left\{\mathbf{s}^*\otimes\mathbf{s}\right\}\mathrm{E}\left\{n_{k_1}n_{k_2}^*\right\}+\\
&\quad\left(\mathbf{a}_{k_2}^H\otimes\mathbf{a}_{l_2}^T\right)\mathrm{E}\left\{\mathbf{s}^*\otimes\mathbf{s}\right\}\mathrm{E}\left\{n_{k_1}n_{l_1}^*\right\}+\\
&\quad\mathrm{E}\left\{n_{k_1}n_{l_1}^*n_{k_2}^*n_{l_2}\right\}.
\end{aligned} \tag{1.15}$$

Therefore, given the statistics of the signals and the noise up to order 4, the fourth-order moment can be calculated. For example, the $P^4\times 1$ vector

$\mathrm{E}\left\{\mathbf{s}\otimes\mathbf{s}^*\otimes\mathbf{s}^*\otimes\mathbf{s}\right\}$ can be written for $P = 2$ as

$$\begin{pmatrix} \mathrm{E}\{s_1 s_1^* s_1^* s_1\} \\ \mathrm{E}\{s_1 s_1^* s_1^* s_2\} \\ \mathrm{E}\{s_1 s_1^* s_2^* s_1\} \\ \mathrm{E}\{s_1 s_1^* s_2^* s_2\} \\ \mathrm{E}\{s_1 s_2^* s_1^* s_1\} \\ \mathrm{E}\{s_1 s_2^* s_1^* s_2\} \\ \mathrm{E}\{s_1 s_2^* s_2^* s_1\} \\ \mathrm{E}\{s_1 s_2^* s_2^* s_2\} \\ \mathrm{E}\{s_2 s_1^* s_1^* s_1\} \\ \mathrm{E}\{s_2 s_1^* s_1^* s_2\} \\ \mathrm{E}\{s_2 s_1^* s_2^* s_1\} \\ \mathrm{E}\{s_2 s_1^* s_2^* s_2\} \\ \mathrm{E}\{s_2 s_2^* s_1^* s_1\} \\ \mathrm{E}\{s_2 s_2^* s_1^* s_2\} \\ \mathrm{E}\{s_2 s_2^* s_2^* s_1\} \\ \mathrm{E}\{s_2 s_2^* s_2^* s_2\} \end{pmatrix} = \begin{pmatrix} \mathrm{E}\{|s_1|^4\} \\ 0 \\ 0 \\ \mathrm{E}\{|s_1|^2\}\,\mathrm{E}\{|s_2|^2\} \\ 0 \\ \mathrm{E}\{|s_1|^2\}\,\mathrm{E}\{|s_2|^2\} \\ \mathrm{E}\{s_1^2\}\,\mathrm{E}\{(s_2^*)^2\} \\ 0 \\ 0 \\ \mathrm{E}\{(s_1^*)^2\}\,\mathrm{E}\{s_2^2\} \\ \mathrm{E}\{|s_1|^2\}\,\mathrm{E}\{|s_2|^2\} \\ 0 \\ \mathrm{E}\{|s_1|^2\}\,\mathrm{E}\{|s_2|^2\} \\ 0 \\ 0 \\ \mathrm{E}\{|s_2|^4\} \end{pmatrix}.$$

Note that $P^4 - P^3$ elements of this vector are always equal to zero due to the independence of sources and the zero-mean assumptions.

1.4 Second-order moment

Again, applying the Kronecker product leads to

$$\begin{aligned}
\mathrm{E}\left\{r_{k_1}(t) r_{l_1}^H(t)\right\} &= \left(\mathbf{a}_{k_1}^T \otimes \mathbf{a}_{l_1}^H\right) \mathrm{E}\left\{\mathbf{s} \otimes \mathbf{s}^*\right\}+\mathrm{E}\left\{n_{k_1} n_{l_1}^*\right\} \\
\mathrm{E}\left\{r_{k_2}^H(t) r_{l_2}(t)\right\} &= \left(\mathbf{a}_{k_2}^H \otimes \mathbf{a}_{l_2}^T\right) \mathrm{E}\left\{\mathbf{s}^* \otimes \mathbf{s}\right\}+\mathrm{E}\left\{n_{k_2}^* n_{l_2}\right\} .
\end{aligned}$$

1.5 Final formula

Now, the finite-sample covariance of the second-order sample cumulant is given by

$$\begin{aligned}
&\operatorname{Cov}\left(\hat{c}_{k_1, l_1}^{(2)}, \hat{c}_{k_2, l_2}^{(2), H}\right) \\
&= \frac{1}{N}\left(\left(\mathbf{a}_{k_1}^T \otimes \mathbf{a}_{l_1}^H \otimes \mathbf{a}_{k_2}^H \otimes \mathbf{a}_{l_2}^T\right)\left(\mathrm{E}\left\{\mathbf{s} \otimes \mathbf{s}^* \otimes \mathbf{s}^* \otimes \mathbf{s}\right\}-\mathrm{E}\left\{\mathbf{s} \otimes \mathbf{s}^*\right\} \otimes \mathrm{E}\left\{\mathbf{s}^* \otimes \mathbf{s}\right\}\right)+\right. \\
&\quad \left(\mathbf{a}_{k_1}^T \otimes \mathbf{a}_{k_2}^H\right) \mathrm{E}\left\{\mathbf{s} \otimes \mathbf{s}^*\right\} \mathrm{E}\left\{n_{l_1}^* n_{l_2}\right\}+ \\
&\quad \left(\mathbf{a}_{k_1}^T \otimes \mathbf{a}_{l_2}^T\right) \mathrm{E}\{\mathbf{s} \otimes \mathbf{s}\} \mathrm{E}\left\{n_{l_1}^* n_{k_2}^*\right\}+ \\
&\quad \left(\mathbf{a}_{l_1}^H \otimes \mathbf{a}_{k_2}^H\right) \mathrm{E}\left\{\mathbf{s}^* \otimes \mathbf{s}^*\right\} \mathrm{E}\left\{n_{k_1} n_{l_2}\right\}+ \\
&\quad \left(\mathbf{a}_{l_1}^H \otimes \mathbf{a}_{l_2}^T\right) \mathrm{E}\left\{\mathbf{s}^* \otimes \mathbf{s}\right\} \mathrm{E}\left\{n_{k_1} n_{k_2}^*\right\}+ \\
&\quad \left.\mathrm{E}\left\{n_{k_1} n_{l_1}^* n_{k_2}^* n_{l_2}\right\}-\mathrm{E}\left\{n_{k_1} n_{l_1}^*\right\} \mathrm{E}\left\{n_{k_2}^* n_{l_2}\right\}\right) . \qquad (1.16)
\end{aligned}$$

Chapter 2

The finite-sample covariance of the third-order sample cumulant

The finite-sample covariance of the third-order sample cumulant can be written as

$$\begin{aligned}
&\mathrm{Cov}\left(\hat{c}^{(3)}_{k_1,l_1,m_1}, \hat{c}^{(3),H}_{k_2,l_2,m_2}\right) = \\
&\qquad \mathrm{E}\left\{\hat{c}^{(3)}_{k_1,l_1,m_1}\hat{c}^{(3),H}_{k_2,l_2,m_2}\right\} - \mathrm{E}\left\{\hat{c}^{(3)}_{k_1,l_1,m_1}\right\}\mathrm{E}\left\{\hat{c}^{(3),H}_{k_2,l_2,m_2}\right\}.
\end{aligned} \tag{2.1}$$

with

$$\begin{aligned}
\mathrm{E}\left\{\hat{c}^{(3)}_{k_1,l_1,m_1}\hat{c}^{(3),H}_{k_2,l_2,m_2}\right\} &= \frac{1}{N^2}\sum_{t=1}^{N}\sum_{p=1}^{N}\mathrm{E}\left\{r_{k_1}(t)r^H_{l_1}(t)r_{m_1}(t)r^H_{k_2}(p)r_{l_2}(p)r^H_{m_2}(p)\right\} \\
&= \frac{1}{N}\mathrm{E}\left\{r_{k_1}(t)r^H_{l_1}(t)r_{m_1}(t)r^H_{k_2}(t)r_{l_2}(t)r^H_{m_2}(t)\right\} + \\
&\quad \frac{N^2-N}{N^2}\mathrm{E}\left\{r_{k_1}(t)r^H_{l_1}(t)r_{m_1}(t)\right\}\mathrm{E}\left\{r^H_{k_2}(t)r_{l_2}(t)r^H_{m_2}(t)\right\}
\end{aligned}$$

where the summations dissolve due to the i.i.d. property of $r_m(t)$ $\forall m = 1(1)M$.

Since $\mathrm{E}\left\{\hat{c}^{(3)}_{k_1,l_1,m_1}\right\} = \mathrm{E}\left\{r_{k_1}(t)r^H_{l_1}(t)r_{m_1}(t)\right\}$, (2.1) yields

$$\begin{aligned}
&\mathrm{Cov}\left(\hat{c}^{(3)}_{k_1,l_1,m_1}, \hat{c}^{(3),H}_{k_2,l_2,m_2}\right) = \\
&\quad \frac{1}{N}\left(\mathrm{E}\left\{r_{k_1}(t)r^H_{l_1}(t)r_{m_1}(t)r^H_{k_2}(t)r_{l_2}(t)r^H_{m_2}(t)\right\} - \right.\\
&\qquad \left. \mathrm{E}\left\{r_{k_1}(t)r^H_{l_1}(t)r_{m_1}(t)\right\}\mathrm{E}\left\{r^H_{k_2}(t)r_{l_2}(t)r^H_{m_2}(t)\right\}\right). \qquad (2.2)
\end{aligned}$$

In order to evaluate (2.2) we must calculate the sixth- and the third-order moments of the array output signal for the model defined in (1.1).

2.1 Sixth-order moment

Let $\mathbf{b}_k = \mathbf{a}^T_k\mathbf{s}$. Then

$$\begin{aligned}
&\mathrm{E}\left\{r_{k_1}(t)r^H_{l_1}(t)r_{m_1}(t)r^H_{k_2}(t)r_{l_2}(t)r^H_{m_2}(t)\right\} \\
&= \mathrm{E}\left\{(b_{k_1}+n_{k_1})(b^H_{l_1}+n^H_{l_1})(b_{m_1}+n_{m_1})(b^H_{k_2}+n^H_{k_2})(b_{l_2}+n_{l_2})(b^H_{m_2}+n^H_{m_2})\right\} \\
&= \mathrm{E}\left\{b_{k_1}b^H_{l_1}b_{m_1}b^H_{k_2}b_{l_2}b^H_{m_2}\right\} + \mathrm{E}\left\{b_{k_1}b^H_{l_1}b_{m_1}b^H_{k_2}b_{l_2}n^H_{m_2}\right\} + \\
&\quad \mathrm{E}\left\{b_{k_1}b^H_{l_1}b_{m_1}b^H_{k_2}n_{l_2}b^H_{m_2}\right\} + \mathrm{E}\left\{b_{k_1}b^H_{l_1}b_{m_1}b^H_{k_2}n_{l_2}n^H_{m_2}\right\} + \\
&\quad \mathrm{E}\left\{b_{k_1}b^H_{l_1}b_{m_1}n^H_{k_2}b_{l_2}b^H_{m_2}\right\} + \mathrm{E}\left\{b_{k_1}b^H_{l_1}b_{m_1}n^H_{k_2}b_{l_2}n^H_{m_2}\right\} + \\
&\quad \mathrm{E}\left\{b_{k_1}b^H_{l_1}b_{m_1}n^H_{k_2}n_{l_2}b^H_{m_2}\right\} + \mathrm{E}\left\{b_{k_1}b^H_{l_1}b_{m_1}n^H_{k_2}n_{l_2}n^H_{m_2}\right\} + \\
&\quad \mathrm{E}\left\{b_{k_1}b^H_{l_1}n_{m_1}b^H_{k_2}b_{l_2}b^H_{m_2}\right\} + \mathrm{E}\left\{b_{k_1}b^H_{l_1}n_{m_1}b^H_{k_2}b_{l_2}n^H_{m_2}\right\} + \\
&\quad \mathrm{E}\left\{b_{k_1}b^H_{l_1}n_{m_1}b^H_{k_2}n_{l_2}b^H_{m_2}\right\} + \mathrm{E}\left\{b_{k_1}b^H_{l_1}n_{m_1}b^H_{k_2}n_{l_2}n^H_{m_2}\right\} + \\
&\quad \mathrm{E}\left\{b_{k_1}b^H_{l_1}n_{m_1}n^H_{k_2}b_{l_2}b^H_{m_2}\right\} + \mathrm{E}\left\{b_{k_1}b^H_{l_1}n_{m_1}n^H_{k_2}b_{l_2}n^H_{m_2}\right\} + \\
&\quad \mathrm{E}\left\{b_{k_1}b^H_{l_1}n_{m_1}n^H_{k_2}n_{l_2}b^H_{m_2}\right\} + \mathrm{E}\left\{b_{k_1}b^H_{l_1}n_{m_1}n^H_{k_2}n_{l_2}n^H_{m_2}\right\} + \\
&\quad \mathrm{E}\left\{b_{k_1}n^H_{l_1}b_{m_1}b^H_{k_2}b_{l_2}b^H_{m_2}\right\} + \mathrm{E}\left\{b_{k_1}n^H_{l_1}b_{m_1}b^H_{k_2}b_{l_2}n^H_{m_2}\right\} + \\
&\quad \mathrm{E}\left\{b_{k_1}n^H_{l_1}b_{m_1}b^H_{k_2}n_{l_2}b^H_{m_2}\right\} + \mathrm{E}\left\{b_{k_1}n^H_{l_1}b_{m_1}b^H_{k_2}n_{l_2}n^H_{m_2}\right\} + \\
&\quad \mathrm{E}\left\{b_{k_1}n^H_{l_1}b_{m_1}n^H_{k_2}b_{l_2}b^H_{m_2}\right\} + \mathrm{E}\left\{b_{k_1}n^H_{l_1}b_{m_1}n^H_{k_2}b_{l_2}n^H_{m_2}\right\} + \\
&\quad \mathrm{E}\left\{b_{k_1}n^H_{l_1}b_{m_1}n^H_{k_2}n_{l_2}b^H_{m_2}\right\} + \mathrm{E}\left\{b_{k_1}n^H_{l_1}b_{m_1}n^H_{k_2}n_{l_2}n^H_{m_2}\right\} +
\end{aligned}$$

$$
\begin{aligned}
&\mathrm{E}\left\{b_{k_1} n_{l_1}^H n_{m_1} b_{k_2}^H b_{l_2} b_{m_2}^H\right\} + \mathrm{E}\left\{b_{k_1} n_{l_1}^H n_{m_1} b_{k_2}^H b_{l_2} n_{m_2}^H\right\} + \\
&\mathrm{E}\left\{b_{k_1} n_{l_1}^H n_{m_1} b_{k_2}^H n_{l_2} b_{m_2}^H\right\} + \mathrm{E}\left\{b_{k_1} n_{l_1}^H n_{m_1} b_{k_2}^H n_{l_2} n_{m_2}^H\right\} + \\
&\mathrm{E}\left\{b_{k_1} n_{l_1}^H n_{m_1} n_{k_2}^H b_{l_2} b_{m_2}^H\right\} + \mathrm{E}\left\{b_{k_1} n_{l_1}^H n_{m_1} n_{k_2}^H b_{l_2} n_{m_2}^H\right\} + \\
&\mathrm{E}\left\{b_{k_1} n_{l_1}^H n_{m_1} n_{k_2}^H n_{l_2} b_{m_2}^H\right\} + \mathrm{E}\left\{b_{k_1} n_{l_1}^H n_{m_1} n_{k_2}^H n_{l_2} n_{m_2}^H\right\} + \\
&\mathrm{E}\left\{n_{k_1} b_{l_1}^H b_{m_1} b_{k_2}^H b_{l_2} b_{m_2}^H\right\} + \mathrm{E}\left\{n_{k_1} b_{l_1}^H b_{m_1} b_{k_2}^H b_{l_2} n_{m_2}^H\right\} + \\
&\mathrm{E}\left\{n_{k_1} b_{l_1}^H b_{m_1} b_{k_2}^H n_{l_2} b_{m_2}^H\right\} + \mathrm{E}\left\{n_{k_1} b_{l_1}^H b_{m_1} b_{k_2}^H n_{l_2} n_{m_2}^H\right\} + \\
&\mathrm{E}\left\{n_{k_1} b_{l_1}^H b_{m_1} n_{k_2}^H b_{l_2} b_{m_2}^H\right\} + \mathrm{E}\left\{n_{k_1} b_{l_1}^H b_{m_1} n_{k_2}^H b_{l_2} n_{m_2}^H\right\} + \\
&\mathrm{E}\left\{n_{k_1} b_{l_1}^H b_{m_1} n_{k_2}^H n_{l_2} b_{m_2}^H\right\} + \mathrm{E}\left\{n_{k_1} b_{l_1}^H b_{m_1} n_{k_2}^H n_{l_2} n_{m_2}^H\right\} + \\
&\mathrm{E}\left\{n_{k_1} b_{l_1}^H n_{m_1} b_{k_2}^H b_{l_2} b_{m_2}^H\right\} + \mathrm{E}\left\{n_{k_1} b_{l_1}^H n_{m_1} b_{k_2}^H b_{l_2} n_{m_2}^H\right\} + \\
&\mathrm{E}\left\{n_{k_1} b_{l_1}^H n_{m_1} b_{k_2}^H n_{l_2} b_{m_2}^H\right\} + \mathrm{E}\left\{n_{k_1} b_{l_1}^H n_{m_1} b_{k_2}^H n_{l_2} n_{m_2}^H\right\} + \\
&\mathrm{E}\left\{n_{k_1} b_{l_1}^H n_{m_1} n_{k_2}^H b_{l_2} b_{m_2}^H\right\} + \mathrm{E}\left\{n_{k_1} b_{l_1}^H n_{m_1} n_{k_2}^H b_{l_2} n_{m_2}^H\right\} + \\
&\mathrm{E}\left\{n_{k_1} b_{l_1}^H n_{m_1} n_{k_2}^H n_{l_2} b_{m_2}^H\right\} + \mathrm{E}\left\{n_{k_1} b_{l_1}^H n_{m_1} n_{k_2}^H n_{l_2} n_{m_2}^H\right\} + \\
&\mathrm{E}\left\{n_{k_1} n_{l_1}^H b_{m_1} b_{k_2}^H b_{l_2} b_{m_2}^H\right\} + \mathrm{E}\left\{n_{k_1} n_{l_1}^H b_{m_1} b_{k_2}^H b_{l_2} n_{m_2}^H\right\} + \\
&\mathrm{E}\left\{n_{k_1} n_{l_1}^H b_{m_1} b_{k_2}^H n_{l_2} b_{m_2}^H\right\} + \mathrm{E}\left\{n_{k_1} n_{l_1}^H b_{m_1} b_{k_2}^H n_{l_2} n_{m_2}^H\right\} + \\
&\mathrm{E}\left\{n_{k_1} n_{l_1}^H b_{m_1} n_{k_2}^H b_{l_2} b_{m_2}^H\right\} + \mathrm{E}\left\{n_{k_1} n_{l_1}^H b_{m_1} n_{k_2}^H b_{l_2} n_{m_2}^H\right\} + \\
&\mathrm{E}\left\{n_{k_1} n_{l_1}^H b_{m_1} n_{k_2}^H n_{l_2} b_{m_2}^H\right\} + \mathrm{E}\left\{n_{k_1} n_{l_1}^H b_{m_1} n_{k_2}^H n_{l_2} n_{m_2}^H\right\} + \\
&\mathrm{E}\left\{n_{k_1} n_{l_1}^H n_{m_1} b_{k_2}^H b_{l_2} b_{m_2}^H\right\} + \mathrm{E}\left\{n_{k_1} n_{l_1}^H n_{m_1} b_{k_2}^H b_{l_2} n_{m_2}^H\right\} + \\
&\mathrm{E}\left\{n_{k_1} n_{l_1}^H n_{m_1} b_{k_2}^H n_{l_2} b_{m_2}^H\right\} + \mathrm{E}\left\{n_{k_1} n_{l_1}^H n_{m_1} b_{k_2}^H n_{l_2} n_{m_2}^H\right\} + \\
&\mathrm{E}\left\{n_{k_1} n_{l_1}^H n_{m_1} n_{k_2}^H b_{l_2} b_{m_2}^H\right\} + \mathrm{E}\left\{n_{k_1} n_{l_1}^H n_{m_1} n_{k_2}^H b_{l_2} n_{m_2}^H\right\} + \\
&\mathrm{E}\left\{n_{k_1} n_{l_1}^H n_{m_1} n_{k_2}^H n_{l_2} b_{m_2}^H\right\} + \mathrm{E}\left\{n_{k_1} n_{l_1}^H n_{m_1} n_{k_2}^H n_{l_2} n_{m_2}^H\right\}.
\end{aligned}
\tag{2.3}
$$

Close inspection of eq. (2.3) reveals that many terms are very similar to each other; hence, only the followings terms must be developed further:

a) $\mathrm{E}\left\{b_{k_1} b_{l_1}^H b_{m_1} b_{k_2}^H b_{l_2} b_{m_2}^H\right\}$

b) $\mathrm{E}\left\{b_{k_1} b_{l_1}^H b_{m_1} b_{k_2}^H n_{l_2} n_{m_2}^H\right\}$

c) $\mathrm{E}\left\{b_{k_1} b_{l_1}^H b_{m_1} n_{k_2}^H n_{l_2} n_{m_2}^H\right\}$

d) $\mathrm{E}\left\{b_{k_1} b_{l_1}^H n_{m_1} n_{k_2}^H n_{l_2} n_{m_2}^H\right\}$

e) $\mathrm{E}\left\{n_{k_1} n_{l_1}^H n_{m_1} n_{k_2}^H n_{l_2} n_{m_2}^H\right\}$.

Note that

$$\mathrm{E}\left\{b_{k_1} b_{l_1}^H b_{m_1} b_{k_2}^H b_{l_2} n_{m_2}^H\right\} = \mathrm{E}\left\{b_{k_1} n_{l_1}^H n_{m_1} n_{k_2}^H n_{l_2} n_{m_2}^H\right\} = 0$$

due to the independence and zero-mean assumptions of all signals and noises. The remaining terms in (2.3) can be easily obtained by proper change of indices in the formulas for **a)**, **b)**, **c)**, **d)**, **e)**.

Proceeding as we did on page 9, we obtain:

a)

$$\mathrm{E}\left\{b_{k_1} b_{l_1}^H b_{m_1} b_{k_2}^H b_{l_2} b_{m_2}^H\right\} = \left(\mathbf{a}_{k_1}^T \otimes \mathbf{a}_{l_1}^H \otimes \mathbf{a}_{m_1}^T \otimes \mathbf{a}_{k_2}^H \otimes \mathbf{a}_{l_2}^T \otimes \mathbf{a}_{m_2}^H\right) \mathrm{E}\left\{\mathbf{s} \otimes \mathbf{s}^* \otimes \mathbf{s} \otimes \mathbf{s}^* \otimes \mathbf{s} \otimes \mathbf{s}^*\right\} \quad (2.4)$$

b)

$$\mathrm{E}\left\{b_{k_1} b_{l_1}^H b_{m_1} b_{k_2}^H n_{l_2} n_{m_2}^H\right\} = \left(\mathbf{a}_{k_1}^T \otimes \mathbf{a}_{l_1}^H \otimes \mathbf{a}_{m_1}^T \otimes \mathbf{a}_{k_2}^H\right) \mathrm{E}\left\{\mathbf{s} \otimes \mathbf{s}^* \otimes \mathbf{s} \otimes \mathbf{s}^*\right\} \mathrm{E}\left\{n_{l_2} n_{m_2}^H\right\} \quad (2.5)$$

c)

$$\mathrm{E}\left\{b_{k_1} b_{l_1}^H b_{m_1} n_{k_2}^H n_{l_2} n_{m_2}^H\right\} = \left(\mathbf{a}_{k_1}^T \otimes \mathbf{a}_{l_1}^H \otimes \mathbf{a}_{m_1}^T\right) \mathrm{E}\left\{\mathbf{s} \otimes \mathbf{s}^* \otimes \mathbf{s}\right\} \mathrm{E}\left\{n_{k_2}^H n_{l_2} n_{m_2}^H\right\} \quad (2.6)$$

d)

$$\mathrm{E}\left\{b_{k_1} b_{l_1}^H n_{m_1} n_{k_2}^H n_{l_2} n_{m_2}^H\right\} = \left(\mathbf{a}_{k_1}^T \otimes \mathbf{a}_{l_1}^H\right) \mathrm{E}\left\{\mathbf{s} \otimes \mathbf{s}^*\right\} \mathrm{E}\left\{n_{m_1} n_{k_2}^H n_{l_2} n_{m_2}^H\right\} \quad (2.7)$$

and **e)** can be calculated directly from the given noise moments.

2.2 Third-order moment

$$
\begin{aligned}
&\mathrm{E}\left\{r_{k_1}(t)r_{l_1}^H(t)r_{m_1}(t)\right\} = \\
&\qquad \left(\mathbf{a}_{k_1}^T \otimes \mathbf{a}_{l_1}^H \otimes \mathbf{a}_{m_1}^T\right) \mathrm{E}\left\{\mathbf{s}\otimes\mathbf{s}^*\otimes\mathbf{s}\right\} + \mathrm{E}\left\{n_{k_1} n_{l_1}^H n_{m_1}\right\} \qquad (2.8) \\
&\mathrm{E}\left\{r_{k_2}^H(t)r_{l_2}(t)r_{m_2}^H(t)\right\} = \\
&\qquad \left(\mathbf{a}_{k_2}^H \otimes \mathbf{a}_{l_2}^T \otimes \mathbf{a}_{m_2}^H\right) \mathrm{E}\left\{\mathbf{s}^*\otimes\mathbf{s}\otimes\mathbf{s}^*\right\} + \mathrm{E}\left\{n_{k_2}^H n_{l_2} n_{m_2}^H\right\} \qquad (2.9)
\end{aligned}
$$

2.3 Final formula

The finite-sample covariance of the third-order sample cumulant (moment) can be finally calculated by substituting (2.4)- (2.7) into (2.3) and (2.3),(2.8) and (2.9) into (2.1).

Chapter 3

The finite-sample covariance of the fourth-order sample cumulant

The finite-sample covariance of the fourth-order sample cumulant can be written as

$$\begin{aligned}\operatorname{Cov}\left(\hat{c}^{(4)}_{k_1,l_1,m_1,n_1}, \hat{c}^{(4),H}_{k_2,l_2,m_2,n_2}\right) = \\ \mathrm{E}\left\{\hat{c}^{(4)}_{k_1,l_1,m_1,n_1}\hat{c}^{(4),H}_{k_2,l_2,m_2,n_2}\right\} - \mathrm{E}\left\{\hat{c}^{(4)}_{k_1,l_1,m_1,n_1}\right\}\mathrm{E}\left\{\hat{c}^{(4),H}_{k_2,l_2,m_2,n_2}\right\}.\end{aligned} \tag{3.1}$$

Using (1.7) in (3.1) we obtain

$$\mathrm{E}\left\{\hat{c}^{(4)}_{k_1,l_1,m_1,n_1}\hat{c}^{(4),H}_{k_2,l_2,m_2,n_2}\right\} =$$

$$\frac{1}{\alpha^2}\sum_{t=1}^{N}\sum_{p=1}^{N}\mathrm{E}\left\{r_{k_1}(t)r^H_{l_1}(t)r^H_{m_1}(t)r_{n_1}(t)r^H_{k_2}(p)r_{l_2}(p)r_{m_2}(p)r^H_{n_2}(p)\right\}$$

$$-\frac{2}{\alpha\beta}\operatorname{Re}\left\{\sum_{t=1}^{N}\sum_{p=1}^{N}\sum_{q=1}^{N}\mathrm{E}\left\{r_{k_1}(t)r^H_{l_1}(t)r^H_{m_1}(t)r_{n_1}(t)r^H_{k_2}(p)r_{l_2}(p)r_{m_2}(q)r^H_{n_2}(q)\right\}\right\}$$

$$
\begin{aligned}
&-\frac{2}{\alpha\beta}\operatorname{Re}\left\{\sum_{t=1}^{N}\sum_{p=1}^{N}\sum_{q=1}^{N}\mathrm{E}\left\{r_{k_1}(t)r_{l_1}^H(t)r_{m_1}^H(t)r_{n_1}(t)r_{k_2}^H(p)r_{m_2}(p)r_{l_2}(q)r_{n_2}^H(q)\right\}\right\}\\
&-\frac{2}{\alpha\beta}\operatorname{Re}\left\{\sum_{t=1}^{N}\sum_{p=1}^{N}\sum_{q=1}^{N}\mathrm{E}\left\{r_{k_1}(t)r_{l_1}^H(t)r_{m_1}^H(t)r_{n_1}(t)r_{k_2}^H(p)r_{n_2}^H(p)r_{m_2}(q)r_{l_2}(q)\right\}\right\}\\
&+\frac{1}{\beta^2}\sum_{t=1}^{N}\sum_{p=1}^{N}\sum_{q=1}^{N}\sum_{s=1}^{N}\mathrm{E}\left\{r_{k_1}(t)r_{l_1}^H(t)r_{m_1}^H(p)r_{n_1}(p)r_{k_2}^H(q)r_{l_2}(q)r_{m_2}(s)r_{n_2}^H(s)\right\}\\
&+\frac{2}{\beta^2}\operatorname{Re}\left\{\sum_{t=1}^{N}\sum_{p=1}^{N}\sum_{q=1}^{N}\sum_{s=1}^{N}\mathrm{E}\left\{r_{k_1}(t)r_{l_1}^H(t)r_{m_1}^H(p)r_{n_1}(p)r_{k_2}^H(q)r_{m_2}(q)r_{l_2}(s)r_{n_2}^H(s)\right\}\right\}\\
&+\frac{2}{\beta^2}\operatorname{Re}\left\{\sum_{t=1}^{N}\sum_{p=1}^{N}\sum_{q=1}^{N}\sum_{s=1}^{N}\mathrm{E}\left\{r_{k_1}(t)r_{l_1}^H(t)r_{m_1}^H(p)r_{n_1}(p)r_{k_2}^H(q)r_{n_2}^H(q)r_{m_2}(s)r_{l_2}(s)\right\}\right\}\\
&+\frac{1}{\beta^2}\sum_{t=1}^{N}\sum_{p=1}^{N}\sum_{q=1}^{N}\sum_{s=1}^{N}\mathrm{E}\left\{r_{k_1}(t)r_{m_1}^H(t)r_{l_1}^H(p)r_{n_1}(p)r_{k_2}^H(q)r_{m_2}(q)r_{l_2}(s)r_{n_2}^H(s)\right\}\\
&+\frac{2}{\beta^2}\operatorname{Re}\left\{\sum_{t=1}^{N}\sum_{p=1}^{N}\sum_{q=1}^{N}\sum_{s=1}^{N}\mathrm{E}\left\{r_{k_1}(t)r_{m_1}^H(t)r_{l_1}^H(p)r_{n_1}(p)r_{k_2}^H(q)r_{n_2}^H(q)r_{m_2}(s)r_{l_2}(s)\right\}\right\}\\
&+\frac{1}{\beta^2}\sum_{t=1}^{N}\sum_{p=1}^{N}\sum_{q=1}^{N}\sum_{s=1}^{N}\mathrm{E}\left\{r_{k_1}(t)r_{n_1}(t)r_{l_1}^H(p)r_{m_1}^H(p)r_{k_2}^H(q)r_{n_2}^H(q)r_{m_2}(s)r_{l_2}(s)\right\},
\end{aligned}
\tag{3.2}
$$

where the real operation comes from $(a+b)(a^*+b^*) = |a|^2 + 2\operatorname{Re} ab^* + |b|^2$.

Since many terms in (3.2) can be easily obtained by proper change of indices, only the following terms must be calculated:

a) $\sum_{t=1}^{N}\sum_{p=1}^{N}\mathrm{E}\left\{r_{k_1}(t)r_{l_1}^H(t)r_{m_1}^H(t)r_{n_1}(t)r_{k_2}^H(p)r_{l_2}(p)r_{m_2}(p)r_{n_2}^H(p)\right\}$

b) $\sum_{t=1}^{N}\sum_{p=1}^{N}\sum_{q=1}^{N}\mathrm{E}\left\{r_{k_1}(t)r_{l_1}^H(t)r_{m_1}^H(t)r_{n_1}(t)r_{k_2}^H(p)r_{l_2}(p)r_{m_2}(q)r_{n_2}^H(q)\right\}$

c) $\sum_{t=1}^{N}\sum_{p=1}^{N}\sum_{q=1}^{N}\sum_{s=1}^{N}\mathrm{E}\left\{r_{k_1}(t)r_{l_1}^H(t)r_{m_1}^H(p)r_{n_1}(p)r_{k_2}^H(q)r_{l_2}(q)r_{m_2}(s)r_{n_2}^H(s)\right\}$

First term a):

$$\sum_{t=1}^{N}\sum_{p=1}^{N} \mathrm{E}\left\{r_{k_1}(t)r_{l_1}^H(t)r_{m_1}^H(t)r_{n_1}(t)r_{k_2}^H(p)r_{l_2}(p)r_{m_2}(p)r_{n_2}^H(p)\right\}$$

$$= \; N\mathrm{E}\left\{r_{k_1}(t)r_{l_1}^H(t)r_{m_1}^H(t)r_{n_1}(t)r_{k_2}^H(t)r_{l_2}(t)r_{m_2}(t)r_{n_2}^H(t)\right\}$$

$$+(N^2-N)$$

$$\mathrm{E}\left\{r_{k_1}(t)r_{l_1}^H(t)r_{m_1}^H(t)r_{n_1}(t)\right\}\mathrm{E}\left\{r_{k_2}^H(t)r_{l_2}(t)r_{m_2}(t)r_{n_2}^H(t)\right\}, \quad (3.3)$$

because $r_k(t)$ is independent of $r_l(p)$ for $t \neq p$.

Second term b):

$$\sum_{t=1}^{N}\sum_{p=1}^{N}\sum_{q=1}^{N} \mathrm{E}\left\{r_{k_1}(t)r_{l_1}^H(t)r_{m_1}^H(t)r_{n_1}(t)r_{k_2}^H(p)r_{l_2}(p)r_{m_2}(q)r_{n_2}^H(q)\right\}$$

$$= N\mathrm{E}\left\{r_{k_1}(t)r_{l_1}^H(t)r_{m_1}^H(t)r_{n_1}(t)r_{k_2}^H(t)r_{l_2}(t)r_{m_2}(t)r_{n_2}^H(t)\right\}$$

$$+(N^2-N)\mathrm{E}\left\{r_{k_1}(t)r_{l_1}^H(t)r_{m_1}^H(t)r_{n_1}(t)\right\}\mathrm{E}\left\{r_{k_2}^H(t)r_{l_2}(t)r_{m_2}(t)r_{n_2}^H(t)\right\}$$

$$+(N^2-N)\mathrm{E}\left\{r_{k_1}(t)r_{l_1}^H(t)r_{m_1}^H(t)r_{n_1}(t)r_{k_2}^H(t)r_{l_2}(t)\right\}\mathrm{E}\left\{r_{m_2}(t)r_{n_2}^H(t)\right\} \quad (3.4)$$

$$+(N^2-N)\mathrm{E}\left\{r_{k_1}(t)r_{l_1}^H(t)r_{m_1}^H(t)r_{n_1}(t)r_{m_2}(t)r_{n_2}^H(t)\right\}\mathrm{E}\left\{r_{k_2}^H(t)r_{l_2}(t)\right\}$$

$$+(N^3-3N^2+2N)$$

$$\mathrm{E}\left\{r_{k_1}(t)r_{l_1}^H(t)r_{m_1}^H(t)r_{n_1}(t)\right\}\mathrm{E}\left\{r_{k_2}^H(t)r_{l_2}(t)\right\}\mathrm{E}\left\{r_{m_2}(t)r_{n_2}^H(t)\right\}$$

Third term c):

$$\sum_{t=1}^{N}\sum_{p=1}^{N}\sum_{q=1}^{N}\sum_{s=1}^{N} \mathrm{E}\left\{r_{k_1}(t)r_{l_1}^H(t)r_{m_1}^H(p)r_{n_1}(p)r_{k_2}^H(q)r_{l_2}(q)r_{m_2}(s)r_{n_2}^H(s)\right\}$$

$$= N\mathrm{E}\left\{r_{k_1}(t)r_{l_1}^H(t)r_{m_1}^H(t)r_{n_1}(t)r_{k_2}^H(t)r_{l_2}(t)r_{m_2}(t)r_{n_2}^H(t)\right\}$$

$$+(N^2-N)\mathrm{E}\left\{r_{k_1}(t)r_{l_1}^H(t)\right\}\mathrm{E}\left\{r_{m_1}^H(t)r_{n_1}(t)r_{k_2}^H(t)r_{l_2}(t)r_{m_2}(t)r_{n_2}^H(t)\right\}$$

$$+(N^2-N)\mathrm{E}\left\{r_{m_1}^H(t)r_{n_1}(t)\right\}\mathrm{E}\left\{r_{k_1}(t)r_{l_1}^H(t)r_{k_2}^H(t)r_{l_2}(t)r_{m_2}(t)r_{n_2}^H(t)\right\}$$

$$+(N^2-N)\mathrm{E}\left\{r_{k_2}^H(t)r_{l_2}(t)\right\}\mathrm{E}\left\{r_{k_1}(t)r_{l_1}^H(t)r_{m_1}^H(t)r_{n_1}(t)r_{m_2}(t)r_{n_2}^H(t)\right\}$$

$$+(N^2-N)\mathrm{E}\left\{r_{m_2}(t)r_{n_2}^H(t)\right\}\mathrm{E}\left\{r_{k_1}(t)r_{l_1}^H(t)r_{m_1}^H(t)r_{n_1}(t)r_{k_2}^H(t)r_{l_2}(t)\right\}$$

$$+(N^2-N)\mathrm{E}\left\{r_{k_1}(t)r_{l_1}^H(t)r_{m_1}^H(t)r_{n_1}(t)\right\}\mathrm{E}\left\{r_{k_2}^H(t)r_{l_2}(t)r_{m_2}(t)r_{n_2}^H(t)\right\}$$

$$+(N^2-N)\mathrm{E}\left\{r_{k_1}(t)r_{l_1}^H(t)r_{k_2}^H(t)r_{l_2}(t)\right\}\mathrm{E}\left\{r_{m_1}^H(t)r_{n_1}(t)r_{m_2}(t)r_{n_2}^H(t)\right\}$$

$$+(N^2-N)\mathrm{E}\left\{r_{k_1}(t)r_{l_1}^H(t)r_{m_2}(t)r_{n_2}^H(t)\right\}\mathrm{E}\left\{r_{m_1}^H(t)r_{n_1}(t)r_{k_2}^H(t)r_{l_2}(t)\right\} \tag{3.5}$$

$$+(N^3-3N^2+2N)\mathrm{E}\left\{r_{k_1}(t)r_{l_1}^H(t)r_{m_1}^H(t)r_{n_1}(t)\right\}\mathrm{E}\left\{r_{k_2}^H(t)r_{l_2}(t)\right\}\mathrm{E}\left\{r_{m_2}(t)r_{n_2}^H(t)\right\}$$

$$+(N^3-3N^2+2N)\mathrm{E}\left\{r_{k_1}(t)r_{l_1}^H(t)r_{k_2}^H(t)r_{l_2}(t)\right\}\mathrm{E}\left\{r_{m_1}^H(t)r_{n_1}(t)\right\}\mathrm{E}\left\{r_{m_2}(t)r_{n_2}^H(t)\right\}$$

$$+(N^3-3N^2+2N)\mathrm{E}\left\{r_{k_1}(t)r_{l_1}^H(t)r_{m_2}(t)r_{n_2}^H(t)\right\}\mathrm{E}\left\{r_{m_1}^H(t)r_{n_1}(t)\right\}\mathrm{E}\left\{r_{k_2}^H(t)r_{l_2}(t)\right\}$$

$$+(N^3-3N^2+2N)\mathrm{E}\left\{r_{m_1}^H(t)r_{n_1}(t)r_{k_2}^H(t)r_{l_2}(t)\right\}\mathrm{E}\left\{r_{k_1}(t)r_{l_1}^H(t)\right\}\mathrm{E}\left\{r_{m_2}(t)r_{n_2}^H(t)\right\}$$

$$+(N^3-3N^2+2N)\mathrm{E}\left\{r_{m_1}^H(t)r_{n_1}(t)r_{m_2}(t)r_{n_2}^H(t)\right\}\mathrm{E}\left\{r_{k_1}(t)r_{l_1}^H(t)\right\}\mathrm{E}\left\{r_{k_2}^H(t)r_{l_2}(t)\right\}$$

$$+(N^3-3N^2+2N)\mathrm{E}\left\{r_{k_2}^H(t)r_{l_2}(t)r_{m_2}(t)r_{n_2}^H(t)\right\}\mathrm{E}\left\{r_{k_1}(t)r_{l_1}^H(t)\right\}\mathrm{E}\left\{r_{m_1}^H(t)r_{n_1}(t)\right\}$$

$$+(N^4-6N^3+11N^2-6N)\mathrm{E}\left\{r_{k_1}(t)r_{l_1}^H(t)\right\}\mathrm{E}\left\{r_{m_1}^H(t)r_{n_1}(t)\right\}$$

$$\mathrm{E}\left\{r_{k_2}^H(t)r_{l_2}(t)\right\}\mathrm{E}\left\{r_{m_2}(t)r_{n_2}^H(t)\right\}.$$

Again, many terms in **a**, **b** and **c** can be obtained by changing indices. The key terms are:

I $\quad \mathrm{E}\left\{r_{k_1} r_{l_1}^H r_{m_1}^H r_{n_1} r_{k_2}^H r_{l_2} r_{m_2} r_{n_2}^H\right\}$

II $\quad \mathrm{E}\left\{r_{k_1} r_{l_1}^H r_{m_1}^H r_{n_1} r_{k_2}^H r_{l_2}\right\}$

III $\quad \mathrm{E}\left\{r_{k_1} r_{l_1}^H r_{m_1}^H r_{n_1}\right\}$

IV $\quad \mathrm{E}\left\{r_{k_1} r_{l_1}^H\right\}.$

Closed-form formulas for **II, III, IV** were derived in the previous sections, so only term **I** must be calculated for the model defined in (1.1).

3.1 Eighth-order moment

For the eighth-order moment, we obtain the following 2^8 terms:

$$
\begin{aligned}
&\mathrm{E}\left\{r_{k_1} r_{l_1}^H r_{m_1}^H r_{n_1} r_{k_2}^H r_{l_2} r_{m_2} r_{n_2}^H\right\} \\
&\quad = \mathrm{E}\Big\{(b_{k_1}+n_{k_1})(b_{l_1}^H+n_{l_1}^H)(n_{m_1}^H+n_{m_1}^H)(b_{n_1}+n_{n_1}) \\
&\qquad\qquad (b_{k_2}^H+n_{k_2}^H)(b_{l_2}+n_{l_2})(b_{m_2}+n_{m_2})(b_{n_2}^H+n_{n_2}^H)\Big\} \\
&\quad = \mathrm{E}\left\{b_{k_1} b_{l_1}^H b_{m_1}^H b_{n_1} b_{k_2}^H b_{l_2} b_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{b_{k_1} b_{l_1}^H b_{m_1}^H b_{n_1} b_{k_2}^H b_{l_2} b_{m_2} n_{n_2}^H\right\} + \\
&\qquad \mathrm{E}\left\{b_{k_1} b_{l_1}^H b_{m_1}^H b_{n_1} b_{k_2}^H b_{l_2} n_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{b_{k_1} b_{l_1}^H b_{m_1}^H b_{n_1} b_{k_2}^H b_{l_2} n_{m_2} n_{n_2}^H\right\} + \\
&\qquad \mathrm{E}\left\{b_{k_1} b_{l_1}^H b_{m_1}^H b_{n_1} b_{k_2}^H n_{l_2} b_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{b_{k_1} b_{l_1}^H b_{m_1}^H b_{n_1} b_{k_2}^H n_{l_2} b_{m_2} n_{n_2}^H\right\} + \\
&\qquad \mathrm{E}\left\{b_{k_1} b_{l_1}^H b_{m_1}^H b_{n_1} b_{k_2}^H n_{l_2} n_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{b_{k_1} b_{l_1}^H b_{m_1}^H b_{n_1} b_{k_2}^H n_{l_2} n_{m_2} n_{n_2}^H\right\} + \\
&\qquad \mathrm{E}\left\{b_{k_1} b_{l_1}^H b_{m_1}^H b_{n_1} n_{k_2}^H b_{l_2} b_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{b_{k_1} b_{l_1}^H b_{m_1}^H b_{n_1} n_{k_2}^H b_{l_2} b_{m_2} n_{n_2}^H\right\} + \\
&\qquad \mathrm{E}\left\{b_{k_1} b_{l_1}^H b_{m_1}^H b_{n_1} n_{k_2}^H b_{l_2} n_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{b_{k_1} b_{l_1}^H b_{m_1}^H b_{n_1} n_{k_2}^H b_{l_2} n_{m_2} n_{n_2}^H\right\} + \\
&\qquad \mathrm{E}\left\{b_{k_1} b_{l_1}^H b_{m_1}^H b_{n_1} n_{k_2}^H n_{l_2} b_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{b_{k_1} b_{l_1}^H b_{m_1}^H b_{n_1} n_{k_2}^H n_{l_2} b_{m_2} n_{n_2}^H\right\} + \\
&\qquad \mathrm{E}\left\{b_{k_1} b_{l_1}^H b_{m_1}^H b_{n_1} n_{k_2}^H n_{l_2} n_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{b_{k_1} b_{l_1}^H b_{m_1}^H b_{n_1} n_{k_2}^H n_{l_2} n_{m_2} n_{n_2}^H\right\} + \\
&\qquad \mathrm{E}\left\{b_{k_1} b_{l_1}^H b_{m_1}^H n_{n_1} b_{k_2}^H b_{l_2} b_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{b_{k_1} b_{l_1}^H b_{m_1}^H n_{n_1} b_{k_2}^H b_{l_2} b_{m_2} n_{n_2}^H\right\} +
\end{aligned}
$$

$$\mathrm{E}\left\{b_{k_1}b_{l_1}^H b_{m_1}^H n_{n_1} b_{k_2}^H b_{l_2} n_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{b_{k_1}b_{l_1}^H b_{m_1}^H n_{n_1} b_{k_2}^H b_{l_2} n_{m_2} n_{n_2}^H\right\} +$$
$$\mathrm{E}\left\{b_{k_1}b_{l_1}^H b_{m_1}^H n_{n_1} b_{k_2}^H n_{l_2} b_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{b_{k_1}b_{l_1}^H b_{m_1}^H n_{n_1} b_{k_2}^H n_{l_2} b_{m_2} n_{n_2}^H\right\} +$$
$$\mathrm{E}\left\{b_{k_1}b_{l_1}^H b_{m_1}^H n_{n_1} b_{k_2}^H n_{l_2} n_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{b_{k_1}b_{l_1}^H b_{m_1}^H n_{n_1} b_{k_2}^H n_{l_2} n_{m_2} n_{n_2}^H\right\} +$$
$$\mathrm{E}\left\{b_{k_1}b_{l_1}^H b_{m_1}^H n_{n_1} n_{k_2}^H b_{l_2} b_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{b_{k_1}b_{l_1}^H b_{m_1}^H n_{n_1} n_{k_2}^H b_{l_2} b_{m_2} n_{n_2}^H\right\} +$$
$$\mathrm{E}\left\{b_{k_1}b_{l_1}^H b_{m_1}^H n_{n_1} n_{k_2}^H b_{l_2} n_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{b_{k_1}b_{l_1}^H b_{m_1}^H n_{n_1} n_{k_2}^H b_{l_2} n_{m_2} n_{n_2}^H\right\} +$$
$$\mathrm{E}\left\{b_{k_1}b_{l_1}^H b_{m_1}^H n_{n_1} n_{k_2}^H n_{l_2} b_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{b_{k_1}b_{l_1}^H b_{m_1}^H n_{n_1} n_{k_2}^H n_{l_2} b_{m_2} n_{n_2}^H\right\} +$$
$$\mathrm{E}\left\{b_{k_1}b_{l_1}^H b_{m_1}^H n_{n_1} n_{k_2}^H n_{l_2} n_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{b_{k_1}b_{l_1}^H b_{m_1}^H n_{n_1} n_{k_2}^H n_{l_2} n_{m_2} n_{n_2}^H\right\} +$$
$$\mathrm{E}\left\{b_{k_1}b_{l_1}^H n_{m_1}^H b_{n_1} b_{k_2}^H b_{l_2} b_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{b_{k_1}b_{l_1}^H n_{m_1}^H b_{n_1} b_{k_2}^H b_{l_2} b_{m_2} n_{n_2}^H\right\} +$$
$$\mathrm{E}\left\{b_{k_1}b_{l_1}^H n_{m_1}^H b_{n_1} b_{k_2}^H b_{l_2} n_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{b_{k_1}b_{l_1}^H n_{m_1}^H b_{n_1} b_{k_2}^H b_{l_2} n_{m_2} n_{n_2}^H\right\} +$$
$$\mathrm{E}\left\{b_{k_1}b_{l_1}^H n_{m_1}^H b_{n_1} b_{k_2}^H n_{l_2} b_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{b_{k_1}b_{l_1}^H n_{m_1}^H b_{n_1} b_{k_2}^H n_{l_2} b_{m_2} n_{n_2}^H\right\} +$$
$$\mathrm{E}\left\{b_{k_1}b_{l_1}^H n_{m_1}^H b_{n_1} b_{k_2}^H n_{l_2} n_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{b_{k_1}b_{l_1}^H n_{m_1}^H b_{n_1} b_{k_2}^H n_{l_2} n_{m_2} n_{n_2}^H\right\} +$$
$$\mathrm{E}\left\{b_{k_1}b_{l_1}^H n_{m_1}^H b_{n_1} n_{k_2}^H b_{l_2} b_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{b_{k_1}b_{l_1}^H n_{m_1}^H b_{n_1} n_{k_2}^H b_{l_2} b_{m_2} n_{n_2}^H\right\} +$$
$$\mathrm{E}\left\{b_{k_1}b_{l_1}^H n_{m_1}^H b_{n_1} n_{k_2}^H b_{l_2} n_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{b_{k_1}b_{l_1}^H n_{m_1}^H b_{n_1} n_{k_2}^H b_{l_2} n_{m_2} n_{n_2}^H\right\} +$$
$$\mathrm{E}\left\{b_{k_1}b_{l_1}^H n_{m_1}^H b_{n_1} n_{k_2}^H n_{l_2} b_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{b_{k_1}b_{l_1}^H n_{m_1}^H b_{n_1} n_{k_2}^H n_{l_2} b_{m_2} n_{n_2}^H\right\} +$$
$$\mathrm{E}\left\{b_{k_1}b_{l_1}^H n_{m_1}^H b_{n_1} n_{k_2}^H n_{l_2} n_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{b_{k_1}b_{l_1}^H n_{m_1}^H b_{n_1} n_{k_2}^H n_{l_2} n_{m_2} n_{n_2}^H\right\} +$$
$$\mathrm{E}\left\{b_{k_1}b_{l_1}^H n_{m_1}^H n_{n_1} b_{k_2}^H b_{l_2} b_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{b_{k_1}b_{l_1}^H n_{m_1}^H n_{n_1} b_{k_2}^H b_{l_2} b_{m_2} n_{n_2}^H\right\} +$$
$$\mathrm{E}\left\{b_{k_1}b_{l_1}^H n_{m_1}^H n_{n_1} b_{k_2}^H b_{l_2} n_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{b_{k_1}b_{l_1}^H n_{m_1}^H n_{n_1} b_{k_2}^H b_{l_2} n_{m_2} n_{n_2}^H\right\} +$$
$$\mathrm{E}\left\{b_{k_1}b_{l_1}^H n_{m_1}^H n_{n_1} b_{k_2}^H n_{l_2} b_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{b_{k_1}b_{l_1}^H n_{m_1}^H n_{n_1} b_{k_2}^H n_{l_2} b_{m_2} n_{n_2}^H\right\} +$$
$$\mathrm{E}\left\{b_{k_1}b_{l_1}^H n_{m_1}^H n_{n_1} b_{k_2}^H n_{l_2} n_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{b_{k_1}b_{l_1}^H n_{m_1}^H n_{n_1} b_{k_2}^H n_{l_2} n_{m_2} n_{n_2}^H\right\} +$$
$$\mathrm{E}\left\{b_{k_1}b_{l_1}^H n_{m_1}^H n_{n_1} n_{k_2}^H b_{l_2} b_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{b_{k_1}b_{l_1}^H n_{m_1}^H n_{n_1} n_{k_2}^H b_{l_2} b_{m_2} n_{n_2}^H\right\} +$$
$$\mathrm{E}\left\{b_{k_1}b_{l_1}^H n_{m_1}^H n_{n_1} n_{k_2}^H b_{l_2} n_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{b_{k_1}b_{l_1}^H n_{m_1}^H n_{n_1} n_{k_2}^H b_{l_2} n_{m_2} n_{n_2}^H\right\} +$$
$$\mathrm{E}\left\{b_{k_1}b_{l_1}^H n_{m_1}^H n_{n_1} n_{k_2}^H n_{l_2} b_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{b_{k_1}b_{l_1}^H n_{m_1}^H n_{n_1} n_{k_2}^H n_{l_2} b_{m_2} n_{n_2}^H\right\} +$$
$$\mathrm{E}\left\{b_{k_1}b_{l_1}^H n_{m_1}^H n_{n_1} n_{k_2}^H n_{l_2} n_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{b_{k_1}b_{l_1}^H n_{m_1}^H n_{n_1} n_{k_2}^H n_{l_2} n_{m_2} n_{n_2}^H\right\} +$$
$$\mathrm{E}\left\{b_{k_1}n_{l_1}^H b_{m_1}^H b_{n_1} b_{k_2}^H b_{l_2} b_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{b_{k_1}n_{l_1}^H b_{m_1}^H b_{n_1} b_{k_2}^H b_{l_2} b_{m_2} n_{n_2}^H\right\} +$$
$$\mathrm{E}\left\{b_{k_1}n_{l_1}^H b_{m_1}^H b_{n_1} b_{k_2}^H b_{l_2} n_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{b_{k_1}n_{l_1}^H b_{m_1}^H b_{n_1} b_{k_2}^H b_{l_2} n_{m_2} n_{n_2}^H\right\} +$$

$$\mathrm{E}\left\{b_{k_1}n_{l_1}^H b_{m_1}^H b_{n_1} b_{k_2}^H n_{l_2} b_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{b_{k_1}n_{l_1}^H b_{m_1}^H b_{n_1} b_{k_2}^H n_{l_2} b_{m_2} n_{n_2}^H\right\} +$$
$$\mathrm{E}\left\{b_{k_1}n_{l_1}^H b_{m_1}^H b_{n_1} b_{k_2}^H n_{l_2} n_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{b_{k_1}n_{l_1}^H b_{m_1}^H b_{n_1} b_{k_2}^H n_{l_2} n_{m_2} n_{n_2}^H\right\} +$$
$$\mathrm{E}\left\{b_{k_1}n_{l_1}^H b_{m_1}^H b_{n_1} n_{k_2}^H b_{l_2} b_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{b_{k_1}n_{l_1}^H b_{m_1}^H b_{n_1} n_{k_2}^H b_{l_2} b_{m_2} n_{n_2}^H\right\} +$$
$$\mathrm{E}\left\{b_{k_1}n_{l_1}^H b_{m_1}^H b_{n_1} n_{k_2}^H b_{l_2} n_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{b_{k_1}n_{l_1}^H b_{m_1}^H b_{n_1} n_{k_2}^H b_{l_2} n_{m_2} n_{n_2}^H\right\} +$$
$$\mathrm{E}\left\{b_{k_1}n_{l_1}^H b_{m_1}^H b_{n_1} n_{k_2}^H n_{l_2} b_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{b_{k_1}n_{l_1}^H b_{m_1}^H b_{n_1} n_{k_2}^H n_{l_2} b_{m_2} n_{n_2}^H\right\} +$$
$$\mathrm{E}\left\{b_{k_1}n_{l_1}^H b_{m_1}^H b_{n_1} n_{k_2}^H n_{l_2} n_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{b_{k_1}n_{l_1}^H b_{m_1}^H b_{n_1} n_{k_2}^H n_{l_2} n_{m_2} n_{n_2}^H\right\} +$$
$$\mathrm{E}\left\{b_{k_1}n_{l_1}^H b_{m_1}^H n_{n_1} b_{k_2}^H b_{l_2} b_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{b_{k_1}n_{l_1}^H b_{m_1}^H n_{n_1} b_{k_2}^H b_{l_2} b_{m_2} n_{n_2}^H\right\} +$$
$$\mathrm{E}\left\{b_{k_1}n_{l_1}^H b_{m_1}^H n_{n_1} b_{k_2}^H b_{l_2} n_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{b_{k_1}n_{l_1}^H b_{m_1}^H n_{n_1} b_{k_2}^H b_{l_2} n_{m_2} n_{n_2}^H\right\} +$$
$$\mathrm{E}\left\{b_{k_1}n_{l_1}^H b_{m_1}^H n_{n_1} b_{k_2}^H n_{l_2} b_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{b_{k_1}n_{l_1}^H b_{m_1}^H n_{n_1} b_{k_2}^H n_{l_2} b_{m_2} n_{n_2}^H\right\} +$$
$$\mathrm{E}\left\{b_{k_1}n_{l_1}^H b_{m_1}^H n_{n_1} b_{k_2}^H n_{l_2} n_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{b_{k_1}n_{l_1}^H b_{m_1}^H n_{n_1} b_{k_2}^H n_{l_2} n_{m_2} n_{n_2}^H\right\} +$$
$$\mathrm{E}\left\{b_{k_1}n_{l_1}^H b_{m_1}^H n_{n_1} n_{k_2}^H b_{l_2} b_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{b_{k_1}n_{l_1}^H b_{m_1}^H n_{n_1} n_{k_2}^H b_{l_2} b_{m_2} n_{n_2}^H\right\} +$$
$$\mathrm{E}\left\{b_{k_1}n_{l_1}^H b_{m_1}^H n_{n_1} n_{k_2}^H b_{l_2} n_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{b_{k_1}n_{l_1}^H b_{m_1}^H n_{n_1} n_{k_2}^H b_{l_2} n_{m_2} n_{n_2}^H\right\} +$$
$$\mathrm{E}\left\{b_{k_1}n_{l_1}^H b_{m_1}^H n_{n_1} n_{k_2}^H n_{l_2} b_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{b_{k_1}n_{l_1}^H b_{m_1}^H n_{n_1} n_{k_2}^H n_{l_2} b_{m_2} n_{n_2}^H\right\} +$$
$$\mathrm{E}\left\{b_{k_1}n_{l_1}^H b_{m_1}^H n_{n_1} n_{k_2}^H n_{l_2} n_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{b_{k_1}n_{l_1}^H b_{m_1}^H n_{n_1} n_{k_2}^H n_{l_2} n_{m_2} n_{n_2}^H\right\} +$$
$$\mathrm{E}\left\{b_{k_1}n_{l_1}^H n_{m_1}^H b_{n_1} b_{k_2}^H b_{l_2} b_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{b_{k_1}n_{l_1}^H n_{m_1}^H b_{n_1} b_{k_2}^H b_{l_2} b_{m_2} n_{n_2}^H\right\} +$$
$$\mathrm{E}\left\{b_{k_1}n_{l_1}^H n_{m_1}^H b_{n_1} b_{k_2}^H b_{l_2} n_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{b_{k_1}n_{l_1}^H n_{m_1}^H b_{n_1} b_{k_2}^H b_{l_2} n_{m_2} n_{n_2}^H\right\} +$$
$$\mathrm{E}\left\{b_{k_1}n_{l_1}^H n_{m_1}^H b_{n_1} b_{k_2}^H n_{l_2} b_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{b_{k_1}n_{l_1}^H n_{m_1}^H b_{n_1} b_{k_2}^H n_{l_2} b_{m_2} n_{n_2}^H\right\} +$$
$$\mathrm{E}\left\{b_{k_1}n_{l_1}^H n_{m_1}^H b_{n_1} b_{k_2}^H n_{l_2} n_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{b_{k_1}n_{l_1}^H n_{m_1}^H b_{n_1} b_{k_2}^H n_{l_2} n_{m_2} n_{n_2}^H\right\} +$$
$$\mathrm{E}\left\{b_{k_1}n_{l_1}^H n_{m_1}^H b_{n_1} n_{k_2}^H b_{l_2} b_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{b_{k_1}n_{l_1}^H n_{m_1}^H b_{n_1} n_{k_2}^H b_{l_2} b_{m_2} n_{n_2}^H\right\} +$$
$$\mathrm{E}\left\{b_{k_1}n_{l_1}^H n_{m_1}^H b_{n_1} n_{k_2}^H b_{l_2} n_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{b_{k_1}n_{l_1}^H n_{m_1}^H b_{n_1} n_{k_2}^H b_{l_2} n_{m_2} n_{n_2}^H\right\} +$$
$$\mathrm{E}\left\{b_{k_1}n_{l_1}^H n_{m_1}^H b_{n_1} n_{k_2}^H n_{l_2} b_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{b_{k_1}n_{l_1}^H n_{m_1}^H b_{n_1} n_{k_2}^H n_{l_2} b_{m_2} n_{n_2}^H\right\} +$$
$$\mathrm{E}\left\{b_{k_1}n_{l_1}^H n_{m_1}^H b_{n_1} n_{k_2}^H n_{l_2} n_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{b_{k_1}n_{l_1}^H n_{m_1}^H b_{n_1} n_{k_2}^H n_{l_2} n_{m_2} n_{n_2}^H\right\} +$$
$$\mathrm{E}\left\{b_{k_1}n_{l_1}^H n_{m_1}^H n_{n_1} b_{k_2}^H b_{l_2} b_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{b_{k_1}n_{l_1}^H n_{m_1}^H n_{n_1} b_{k_2}^H b_{l_2} b_{m_2} n_{n_2}^H\right\} +$$
$$\mathrm{E}\left\{b_{k_1}n_{l_1}^H n_{m_1}^H n_{n_1} b_{k_2}^H b_{l_2} n_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{b_{k_1}n_{l_1}^H n_{m_1}^H n_{n_1} b_{k_2}^H b_{l_2} n_{m_2} n_{n_2}^H\right\} +$$
$$\mathrm{E}\left\{b_{k_1}n_{l_1}^H n_{m_1}^H n_{n_1} b_{k_2}^H n_{l_2} b_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{b_{k_1}n_{l_1}^H n_{m_1}^H n_{n_1} b_{k_2}^H n_{l_2} b_{m_2} n_{n_2}^H\right\} +$$

$$
\begin{aligned}
&\mathrm{E}\left\{b_{k_1} n_{l_1}^H n_{m_1}^H n_{n_1} b_{k_2}^H n_{l_2} n_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{b_{k_1} n_{l_1}^H n_{m_1}^H n_{n_1} b_{k_2}^H n_{l_2} n_{m_2} n_{n_2}^H\right\} + \\
&\mathrm{E}\left\{b_{k_1} n_{l_1}^H n_{m_1}^H n_{n_1} n_{k_2}^H b_{l_2} b_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{b_{k_1} n_{l_1}^H n_{m_1}^H n_{n_1} n_{k_2}^H b_{l_2} b_{m_2} n_{n_2}^H\right\} + \\
&\mathrm{E}\left\{b_{k_1} n_{l_1}^H n_{m_1}^H n_{n_1} n_{k_2}^H b_{l_2} n_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{b_{k_1} n_{l_1}^H n_{m_1}^H n_{n_1} n_{k_2}^H b_{l_2} n_{m_2} n_{n_2}^H\right\} + \\
&\mathrm{E}\left\{b_{k_1} n_{l_1}^H n_{m_1}^H n_{n_1} n_{k_2}^H n_{l_2} b_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{b_{k_1} n_{l_1}^H n_{m_1}^H n_{n_1} n_{k_2}^H n_{l_2} b_{m_2} n_{n_2}^H\right\} + \\
&\mathrm{E}\left\{b_{k_1} n_{l_1}^H n_{m_1}^H n_{n_1} n_{k_2}^H n_{l_2} n_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{b_{k_1} n_{l_1}^H n_{m_1}^H n_{n_1} n_{k_2}^H n_{l_2} n_{m_2} n_{n_2}^H\right\} + \\
&\mathrm{E}\left\{n_{k_1} b_{l_1}^H b_{m_1}^H b_{n_1} b_{k_2}^H b_{l_2} b_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{n_{k_1} b_{l_1}^H b_{m_1}^H b_{n_1} b_{k_2}^H b_{l_2} b_{m_2} n_{n_2}^H\right\} + \\
&\mathrm{E}\left\{n_{k_1} b_{l_1}^H b_{m_1}^H b_{n_1} b_{k_2}^H b_{l_2} n_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{n_{k_1} b_{l_1}^H b_{m_1}^H b_{n_1} b_{k_2}^H b_{l_2} n_{m_2} n_{n_2}^H\right\} + \\
&\mathrm{E}\left\{n_{k_1} b_{l_1}^H b_{m_1}^H b_{n_1} b_{k_2}^H n_{l_2} b_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{n_{k_1} b_{l_1}^H b_{m_1}^H b_{n_1} b_{k_2}^H n_{l_2} b_{m_2} n_{n_2}^H\right\} + \\
&\mathrm{E}\left\{n_{k_1} b_{l_1}^H b_{m_1}^H b_{n_1} b_{k_2}^H n_{l_2} n_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{n_{k_1} b_{l_1}^H b_{m_1}^H b_{n_1} b_{k_2}^H n_{l_2} n_{m_2} n_{n_2}^H\right\} + \\
&\mathrm{E}\left\{n_{k_1} b_{l_1}^H b_{m_1}^H b_{n_1} n_{k_2}^H b_{l_2} b_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{n_{k_1} b_{l_1}^H b_{m_1}^H b_{n_1} n_{k_2}^H b_{l_2} b_{m_2} n_{n_2}^H\right\} + \\
&\mathrm{E}\left\{n_{k_1} b_{l_1}^H b_{m_1}^H b_{n_1} n_{k_2}^H b_{l_2} n_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{n_{k_1} b_{l_1}^H b_{m_1}^H b_{n_1} n_{k_2}^H b_{l_2} n_{m_2} n_{n_2}^H\right\} + \\
&\mathrm{E}\left\{n_{k_1} b_{l_1}^H b_{m_1}^H b_{n_1} n_{k_2}^H n_{l_2} b_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{n_{k_1} b_{l_1}^H b_{m_1}^H b_{n_1} n_{k_2}^H n_{l_2} b_{m_2} n_{n_2}^H\right\} + \\
&\mathrm{E}\left\{n_{k_1} b_{l_1}^H b_{m_1}^H b_{n_1} n_{k_2}^H n_{l_2} n_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{n_{k_1} b_{l_1}^H b_{m_1}^H b_{n_1} n_{k_2}^H n_{l_2} n_{m_2} n_{n_2}^H\right\} + \\
&\mathrm{E}\left\{n_{k_1} b_{l_1}^H b_{m_1}^H n_{n_1} b_{k_2}^H b_{l_2} b_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{n_{k_1} b_{l_1}^H b_{m_1}^H n_{n_1} b_{k_2}^H b_{l_2} b_{m_2} n_{n_2}^H\right\} + \\
&\mathrm{E}\left\{n_{k_1} b_{l_1}^H b_{m_1}^H n_{n_1} b_{k_2}^H b_{l_2} n_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{n_{k_1} b_{l_1}^H b_{m_1}^H n_{n_1} b_{k_2}^H b_{l_2} n_{m_2} n_{n_2}^H\right\} + \\
&\mathrm{E}\left\{n_{k_1} b_{l_1}^H b_{m_1}^H n_{n_1} b_{k_2}^H n_{l_2} b_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{n_{k_1} b_{l_1}^H b_{m_1}^H n_{n_1} b_{k_2}^H n_{l_2} b_{m_2} n_{n_2}^H\right\} + \\
&\mathrm{E}\left\{n_{k_1} b_{l_1}^H b_{m_1}^H n_{n_1} b_{k_2}^H n_{l_2} n_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{n_{k_1} b_{l_1}^H b_{m_1}^H n_{n_1} b_{k_2}^H n_{l_2} n_{m_2} n_{n_2}^H\right\} + \\
&\mathrm{E}\left\{n_{k_1} b_{l_1}^H b_{m_1}^H n_{n_1} n_{k_2}^H b_{l_2} b_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{n_{k_1} b_{l_1}^H b_{m_1}^H n_{n_1} n_{k_2}^H b_{l_2} b_{m_2} n_{n_2}^H\right\} + \\
&\mathrm{E}\left\{n_{k_1} b_{l_1}^H b_{m_1}^H n_{n_1} n_{k_2}^H b_{l_2} n_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{n_{k_1} b_{l_1}^H b_{m_1}^H n_{n_1} n_{k_2}^H b_{l_2} n_{m_2} n_{n_2}^H\right\} + \\
&\mathrm{E}\left\{n_{k_1} b_{l_1}^H b_{m_1}^H n_{n_1} n_{k_2}^H n_{l_2} b_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{n_{k_1} b_{l_1}^H b_{m_1}^H n_{n_1} n_{k_2}^H n_{l_2} b_{m_2} n_{n_2}^H\right\} + \\
&\mathrm{E}\left\{n_{k_1} b_{l_1}^H b_{m_1}^H n_{n_1} n_{k_2}^H n_{l_2} n_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{n_{k_1} b_{l_1}^H b_{m_1}^H n_{n_1} n_{k_2}^H n_{l_2} n_{m_2} n_{n_2}^H\right\} + \\
&\mathrm{E}\left\{n_{k_1} b_{l_1}^H n_{m_1}^H b_{n_1} b_{k_2}^H b_{l_2} b_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{n_{k_1} b_{l_1}^H n_{m_1}^H b_{n_1} b_{k_2}^H b_{l_2} b_{m_2} n_{n_2}^H\right\} + \\
&\mathrm{E}\left\{n_{k_1} b_{l_1}^H n_{m_1}^H b_{n_1} b_{k_2}^H b_{l_2} n_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{n_{k_1} b_{l_1}^H n_{m_1}^H b_{n_1} b_{k_2}^H b_{l_2} n_{m_2} n_{n_2}^H\right\} + \\
&\mathrm{E}\left\{n_{k_1} b_{l_1}^H n_{m_1}^H b_{n_1} b_{k_2}^H n_{l_2} b_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{n_{k_1} b_{l_1}^H n_{m_1}^H b_{n_1} b_{k_2}^H n_{l_2} b_{m_2} n_{n_2}^H\right\} + \\
&\mathrm{E}\left\{n_{k_1} b_{l_1}^H n_{m_1}^H b_{n_1} b_{k_2}^H n_{l_2} n_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{n_{k_1} b_{l_1}^H n_{m_1}^H b_{n_1} b_{k_2}^H n_{l_2} n_{m_2} n_{n_2}^H\right\} +
\end{aligned}
$$

$$\mathrm{E}\left\{n_{k_1} b_{l_1}^H n_{m_1}^H b_{n_1} n_{k_2}^H b_{l_2} b_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{n_{k_1} b_{l_1}^H n_{m_1}^H b_{n_1} n_{k_2}^H b_{l_2} b_{m_2} n_{n_2}^H\right\} +$$

$$\mathrm{E}\left\{n_{k_1} b_{l_1}^H n_{m_1}^H b_{n_1} n_{k_2}^H b_{l_2} n_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{n_{k_1} b_{l_1}^H n_{m_1}^H b_{n_1} n_{k_2}^H b_{l_2} n_{m_2} n_{n_2}^H\right\} +$$

$$\mathrm{E}\left\{n_{k_1} b_{l_1}^H n_{m_1}^H b_{n_1} n_{k_2}^H n_{l_2} b_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{n_{k_1} b_{l_1}^H n_{m_1}^H b_{n_1} n_{k_2}^H n_{l_2} b_{m_2} n_{n_2}^H\right\} +$$

$$\mathrm{E}\left\{n_{k_1} b_{l_1}^H n_{m_1}^H b_{n_1} n_{k_2}^H n_{l_2} n_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{n_{k_1} b_{l_1}^H n_{m_1}^H b_{n_1} n_{k_2}^H n_{l_2} n_{m_2} n_{n_2}^H\right\} +$$

$$\mathrm{E}\left\{n_{k_1} b_{l_1}^H n_{m_1}^H n_{n_1} b_{k_2}^H b_{l_2} b_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{n_{k_1} b_{l_1}^H n_{m_1}^H n_{n_1} b_{k_2}^H b_{l_2} b_{m_2} n_{n_2}^H\right\} +$$

$$\mathrm{E}\left\{n_{k_1} b_{l_1}^H n_{m_1}^H n_{n_1} b_{k_2}^H b_{l_2} n_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{n_{k_1} b_{l_1}^H n_{m_1}^H n_{n_1} b_{k_2}^H b_{l_2} n_{m_2} n_{n_2}^H\right\} +$$

$$\mathrm{E}\left\{n_{k_1} b_{l_1}^H n_{m_1}^H n_{n_1} b_{k_2}^H n_{l_2} b_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{n_{k_1} b_{l_1}^H n_{m_1}^H n_{n_1} b_{k_2}^H n_{l_2} b_{m_2} n_{n_2}^H\right\} +$$

$$\mathrm{E}\left\{n_{k_1} b_{l_1}^H n_{m_1}^H n_{n_1} b_{k_2}^H n_{l_2} n_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{n_{k_1} b_{l_1}^H n_{m_1}^H n_{n_1} b_{k_2}^H n_{l_2} n_{m_2} n_{n_2}^H\right\} +$$

$$\mathrm{E}\left\{n_{k_1} b_{l_1}^H n_{m_1}^H n_{n_1} n_{k_2}^H b_{l_2} b_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{n_{k_1} b_{l_1}^H n_{m_1}^H n_{n_1} n_{k_2}^H b_{l_2} b_{m_2} n_{n_2}^H\right\} +$$

$$\mathrm{E}\left\{n_{k_1} b_{l_1}^H n_{m_1}^H n_{n_1} n_{k_2}^H b_{l_2} n_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{n_{k_1} b_{l_1}^H n_{m_1}^H n_{n_1} n_{k_2}^H b_{l_2} n_{m_2} n_{n_2}^H\right\} +$$

$$\mathrm{E}\left\{n_{k_1} b_{l_1}^H n_{m_1}^H n_{n_1} n_{k_2}^H n_{l_2} b_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{n_{k_1} b_{l_1}^H n_{m_1}^H n_{n_1} n_{k_2}^H n_{l_2} b_{m_2} n_{n_2}^H\right\} +$$

$$\mathrm{E}\left\{n_{k_1} b_{l_1}^H n_{m_1}^H n_{n_1} n_{k_2}^H n_{l_2} n_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{n_{k_1} b_{l_1}^H n_{m_1}^H n_{n_1} n_{k_2}^H n_{l_2} n_{m_2} n_{n_2}^H\right\} +$$

$$\mathrm{E}\left\{n_{k_1} n_{l_1}^H b_{m_1}^H b_{n_1} b_{k_2}^H b_{l_2} b_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{n_{k_1} n_{l_1}^H b_{m_1}^H b_{n_1} b_{k_2}^H b_{l_2} b_{m_2} n_{n_2}^H\right\} +$$

$$\mathrm{E}\left\{n_{k_1} n_{l_1}^H b_{m_1}^H b_{n_1} b_{k_2}^H b_{l_2} n_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{n_{k_1} n_{l_1}^H b_{m_1}^H b_{n_1} b_{k_2}^H b_{l_2} n_{m_2} n_{n_2}^H\right\} +$$

$$\mathrm{E}\left\{n_{k_1} n_{l_1}^H b_{m_1}^H b_{n_1} b_{k_2}^H n_{l_2} b_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{n_{k_1} n_{l_1}^H b_{m_1}^H b_{n_1} b_{k_2}^H n_{l_2} b_{m_2} n_{n_2}^H\right\} +$$

$$\mathrm{E}\left\{n_{k_1} n_{l_1}^H b_{m_1}^H b_{n_1} b_{k_2}^H n_{l_2} n_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{n_{k_1} n_{l_1}^H b_{m_1}^H b_{n_1} b_{k_2}^H n_{l_2} n_{m_2} n_{n_2}^H\right\} +$$

$$\mathrm{E}\left\{n_{k_1} n_{l_1}^H b_{m_1}^H b_{n_1} n_{k_2}^H b_{l_2} b_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{n_{k_1} n_{l_1}^H b_{m_1}^H b_{n_1} n_{k_2}^H b_{l_2} b_{m_2} n_{n_2}^H\right\} +$$

$$\mathrm{E}\left\{n_{k_1} n_{l_1}^H b_{m_1}^H b_{n_1} n_{k_2}^H b_{l_2} n_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{n_{k_1} n_{l_1}^H b_{m_1}^H b_{n_1} n_{k_2}^H b_{l_2} n_{m_2} n_{n_2}^H\right\} +$$

$$\mathrm{E}\left\{n_{k_1} n_{l_1}^H b_{m_1}^H b_{n_1} n_{k_2}^H n_{l_2} b_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{n_{k_1} n_{l_1}^H b_{m_1}^H b_{n_1} n_{k_2}^H n_{l_2} b_{m_2} n_{n_2}^H\right\} +$$

$$\mathrm{E}\left\{n_{k_1} n_{l_1}^H b_{m_1}^H b_{n_1} n_{k_2}^H n_{l_2} n_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{n_{k_1} n_{l_1}^H b_{m_1}^H b_{n_1} n_{k_2}^H n_{l_2} n_{m_2} n_{n_2}^H\right\} +$$

$$\mathrm{E}\left\{n_{k_1} n_{l_1}^H b_{m_1}^H n_{n_1} b_{k_2}^H b_{l_2} b_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{n_{k_1} n_{l_1}^H b_{m_1}^H n_{n_1} b_{k_2}^H b_{l_2} b_{m_2} n_{n_2}^H\right\} +$$

$$\mathrm{E}\left\{n_{k_1} n_{l_1}^H b_{m_1}^H n_{n_1} b_{k_2}^H b_{l_2} n_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{n_{k_1} n_{l_1}^H b_{m_1}^H n_{n_1} b_{k_2}^H b_{l_2} n_{m_2} n_{n_2}^H\right\} +$$

$$\mathrm{E}\left\{n_{k_1} n_{l_1}^H b_{m_1}^H n_{n_1} b_{k_2}^H n_{l_2} b_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{n_{k_1} n_{l_1}^H b_{m_1}^H n_{n_1} b_{k_2}^H n_{l_2} b_{m_2} n_{n_2}^H\right\} +$$

$$\mathrm{E}\left\{n_{k_1} n_{l_1}^H b_{m_1}^H n_{n_1} b_{k_2}^H n_{l_2} n_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{n_{k_1} n_{l_1}^H b_{m_1}^H n_{n_1} b_{k_2}^H n_{l_2} n_{m_2} n_{n_2}^H\right\} +$$

$$\mathrm{E}\left\{n_{k_1} n_{l_1}^H b_{m_1}^H n_{n_1} n_{k_2}^H b_{l_2} b_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{n_{k_1} n_{l_1}^H b_{m_1}^H n_{n_1} n_{k_2}^H b_{l_2} b_{m_2} n_{n_2}^H\right\} +$$

$$
\begin{aligned}
&\mathrm{E}\left\{n_{k_1}n_{l_1}^H b_{m_1}^H n_{n_1} n_{k_2}^H b_{l_2} n_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{n_{k_1}n_{l_1}^H b_{m_1}^H n_{n_1} n_{k_2}^H b_{l_2} n_{m_2} n_{n_2}^H\right\} + \\
&\mathrm{E}\left\{n_{k_1}n_{l_1}^H b_{m_1}^H n_{n_1} n_{k_2}^H n_{l_2} b_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{n_{k_1}n_{l_1}^H b_{m_1}^H n_{n_1} n_{k_2}^H n_{l_2} b_{m_2} n_{n_2}^H\right\} + \\
&\mathrm{E}\left\{n_{k_1}n_{l_1}^H b_{m_1}^H n_{n_1} n_{k_2}^H n_{l_2} n_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{n_{k_1}n_{l_1}^H b_{m_1}^H n_{n_1} n_{k_2}^H n_{l_2} n_{m_2} n_{n_2}^H\right\} + \\
&\mathrm{E}\left\{n_{k_1}n_{l_1}^H n_{m_1}^H b_{n_1} b_{k_2}^H b_{l_2} b_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{n_{k_1}n_{l_1}^H n_{m_1}^H b_{n_1} b_{k_2}^H b_{l_2} b_{m_2} n_{n_2}^H\right\} + \\
&\mathrm{E}\left\{n_{k_1}n_{l_1}^H n_{m_1}^H b_{n_1} b_{k_2}^H b_{l_2} n_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{n_{k_1}n_{l_1}^H n_{m_1}^H b_{n_1} b_{k_2}^H b_{l_2} n_{m_2} n_{n_2}^H\right\} + \\
&\mathrm{E}\left\{n_{k_1}n_{l_1}^H n_{m_1}^H b_{n_1} b_{k_2}^H n_{l_2} b_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{n_{k_1}n_{l_1}^H n_{m_1}^H b_{n_1} b_{k_2}^H n_{l_2} b_{m_2} n_{n_2}^H\right\} + \\
&\mathrm{E}\left\{n_{k_1}n_{l_1}^H n_{m_1}^H b_{n_1} b_{k_2}^H n_{l_2} n_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{n_{k_1}n_{l_1}^H n_{m_1}^H b_{n_1} b_{k_2}^H n_{l_2} n_{m_2} n_{n_2}^H\right\} + \\
&\mathrm{E}\left\{n_{k_1}n_{l_1}^H n_{m_1}^H b_{n_1} n_{k_2}^H b_{l_2} b_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{n_{k_1}n_{l_1}^H n_{m_1}^H b_{n_1} n_{k_2}^H b_{l_2} b_{m_2} n_{n_2}^H\right\} + \\
&\mathrm{E}\left\{n_{k_1}n_{l_1}^H n_{m_1}^H b_{n_1} n_{k_2}^H b_{l_2} n_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{n_{k_1}n_{l_1}^H n_{m_1}^H b_{n_1} n_{k_2}^H b_{l_2} n_{m_2} n_{n_2}^H\right\} + \\
&\mathrm{E}\left\{n_{k_1}n_{l_1}^H n_{m_1}^H b_{n_1} n_{k_2}^H n_{l_2} b_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{n_{k_1}n_{l_1}^H n_{m_1}^H b_{n_1} n_{k_2}^H n_{l_2} b_{m_2} n_{n_2}^H\right\} + \\
&\mathrm{E}\left\{n_{k_1}n_{l_1}^H n_{m_1}^H b_{n_1} n_{k_2}^H n_{l_2} n_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{n_{k_1}n_{l_1}^H n_{m_1}^H b_{n_1} n_{k_2}^H n_{l_2} n_{m_2} n_{n_2}^H\right\} + \\
&\mathrm{E}\left\{n_{k_1}n_{l_1}^H n_{m_1}^H n_{n_1} b_{k_2}^H b_{l_2} b_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{n_{k_1}n_{l_1}^H n_{m_1}^H n_{n_1} b_{k_2}^H b_{l_2} b_{m_2} n_{n_2}^H\right\} + \\
&\mathrm{E}\left\{n_{k_1}n_{l_1}^H n_{m_1}^H n_{n_1} b_{k_2}^H b_{l_2} n_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{n_{k_1}n_{l_1}^H n_{m_1}^H n_{n_1} b_{k_2}^H b_{l_2} n_{m_2} n_{n_2}^H\right\} + \\
&\mathrm{E}\left\{n_{k_1}n_{l_1}^H n_{m_1}^H n_{n_1} b_{k_2}^H n_{l_2} b_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{n_{k_1}n_{l_1}^H n_{m_1}^H n_{n_1} b_{k_2}^H n_{l_2} b_{m_2} n_{n_2}^H\right\} + \\
&\mathrm{E}\left\{n_{k_1}n_{l_1}^H n_{m_1}^H n_{n_1} b_{k_2}^H n_{l_2} n_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{n_{k_1}n_{l_1}^H n_{m_1}^H n_{n_1} b_{k_2}^H n_{l_2} n_{m_2} n_{n_2}^H\right\} + \\
&\mathrm{E}\left\{n_{k_1}n_{l_1}^H n_{m_1}^H n_{n_1} n_{k_2}^H b_{l_2} b_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{n_{k_1}n_{l_1}^H n_{m_1}^H n_{n_1} n_{k_2}^H b_{l_2} b_{m_2} n_{n_2}^H\right\} + \\
&\mathrm{E}\left\{n_{k_1}n_{l_1}^H n_{m_1}^H n_{n_1} n_{k_2}^H b_{l_2} n_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{n_{k_1}n_{l_1}^H n_{m_1}^H n_{n_1} n_{k_2}^H b_{l_2} n_{m_2} n_{n_2}^H\right\} + \\
&\mathrm{E}\left\{n_{k_1}n_{l_1}^H n_{m_1}^H n_{n_1} n_{k_2}^H n_{l_2} b_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{n_{k_1}n_{l_1}^H n_{m_1}^H n_{n_1} n_{k_2}^H n_{l_2} b_{m_2} n_{n_2}^H\right\} + \\
&\mathrm{E}\left\{n_{k_1}n_{l_1}^H n_{m_1}^H n_{n_1} n_{k_2}^H n_{l_2} n_{m_2} b_{n_2}^H\right\} + \mathrm{E}\left\{n_{k_1}n_{l_1}^H n_{m_1}^H n_{n_1} n_{k_2}^H n_{l_2} n_{m_2} n_{n_2}^H\right\}.
\end{aligned}
\tag{3.6}
$$

Following the same idea of changing indices, only 7 Terms in (3.6) need further examination. They are:

1) $\mathrm{E}\left\{b_{k_1} b_{l_1}^H b_{m_1}^H b_{n_1} b_{k_2}^H b_{l_2} b_{m_2} b_{n_2}^H\right\} =$
$\left(\mathbf{a}_{k_1}^T \otimes \mathbf{a}_{l_1}^H \otimes \mathbf{a}_{m_1}^H \otimes \mathbf{a}_{n_1}^T \otimes \mathbf{a}_{k_2}^H \otimes \mathbf{a}_{l_2}^T \otimes \mathbf{a}_{m_2}^T \otimes \mathbf{a}_{n_2}^H\right) \mathrm{E}\left\{\mathbf{s} \otimes \mathbf{s}^* \otimes \mathbf{s}^* \otimes \mathbf{s} \otimes \mathbf{s}^* \otimes \mathbf{s} \otimes \mathbf{s} \otimes \mathbf{s}^*\right\}$

2) $\mathrm{E}\left\{b_{k_1} b_{l_1}^H b_{m_1}^H b_{n_1} b_{k_2}^H b_{l_2} n_{m_2} n_{n_2}^H\right\} =$
$\left(\mathbf{a}_{k_1}^T \otimes \mathbf{a}_{l_1}^H \otimes \mathbf{a}_{m_1}^H \otimes \mathbf{a}_{n_1}^T \otimes \mathbf{a}_{k_2}^H \otimes \mathbf{a}_{l_2}^T\right) \mathrm{E}\left\{\mathbf{s} \otimes \mathbf{s}^* \otimes \mathbf{s}^* \otimes \mathbf{s} \otimes \mathbf{s}^* \otimes \mathbf{s}\right\} \mathrm{E}\left\{n_{m_2} n_{n_2}^H\right\}$

3) $\mathrm{E}\left\{b_{k_1} b_{l_1}^H b_{m_1}^H b_{n_1} b_{k_2}^H n_{l_2} n_{m_2} n_{n_2}^H\right\} =$
$\left(\mathbf{a}_{k_1}^T \otimes \mathbf{a}_{l_1}^H \otimes \mathbf{a}_{m_1}^H \otimes \mathbf{a}_{n_1}^T \otimes \mathbf{a}_{k_2}^H\right) \mathrm{E}\left\{\mathbf{s} \otimes \mathbf{s}^* \otimes \mathbf{s}^* \otimes \mathbf{s} \otimes \mathbf{s}^*\right\} \mathrm{E}\left\{n_{l_2} n_{m_2} n_{n_2}^H\right\}$

4) $\mathrm{E}\left\{b_{k_1} b_{l_1}^H b_{m_1}^H b_{n_1} n_{k_2}^H n_{l_2} n_{m_2} n_{n_2}^H\right\} =$
$\left(\mathbf{a}_{k_1}^T \otimes \mathbf{a}_{l_1}^H \otimes \mathbf{a}_{m_1}^H \otimes \mathbf{a}_{n_1}^T\right) \mathrm{E}\left\{\mathbf{s} \otimes \mathbf{s}^* \otimes \mathbf{s}^* \otimes \mathbf{s}\right\} \mathrm{E}\left\{n_{k_2}^H n_{l_2} n_{m_2} n_{n_2}^H\right\}$

5) $\mathrm{E}\left\{b_{k_1} b_{l_1}^H b_{m_1}^H n_{n_1} n_{k_2}^H n_{l_2} n_{m_2} n_{n_2}^H\right\} =$
$\left(\mathbf{a}_{k_1}^T \otimes \mathbf{a}_{l_1}^H \otimes \mathbf{a}_{m_1}^H\right) \mathrm{E}\left\{\mathbf{s} \otimes \mathbf{s}^* \otimes \mathbf{s}^*\right\} \mathrm{E}\left\{n_{n_1} n_{k_2}^H n_{l_2} n_{m_2} n_{n_2}^H\right\}$

6) $\mathrm{E}\left\{b_{k_1} b_{l_1}^H n_{m_1}^H n_{n_1} n_{k_2}^H n_{l_2} n_{m_2} n_{n_2}^H\right\} =$
$\left(\mathbf{a}_{k_1}^T \otimes \mathbf{a}_{l_1}^H\right) \mathrm{E}\left\{\mathbf{s} \otimes \mathbf{s}^*\right\} \mathrm{E}\left\{n_{m_1}^H n_{n_1} n_{k_2}^H n_{l_2} n_{m_2} n_{n_2}^H\right\}$

7) $\mathrm{E}\left\{n_{k_1} n_{l_1}^H n_{m_1}^H n_{n_1} n_{k_2}^H n_{l_2} n_{m_2} n_{n_2}^H\right\}$

where **7)** can be directly obtained from the given noise moments. To complete the work, we must compute the expected value of the fourth-order sample cumulant.

$$\begin{aligned} \mathrm{E}\left\{\hat{c}_{k_1,l_1,m_1,n_1}^{(4)}\right\} &= \frac{1}{\alpha} \sum_{t=1}^{N} \mathrm{E}\left\{r_{k_1}(t) r_{l_1}^H(t) r_{m_1}^H(t) r_{n_1}(t)\right\} \\ &\quad -\frac{1}{\beta} \sum_{t=1}^{N} \sum_{p=1}^{N} \mathrm{E}\left\{r_{k_1}(t) r_{l_1}^H(t) r_{m_1}^H(p) r_{n_1}(p)\right\} \end{aligned}$$

$$
\begin{aligned}
& -\frac{1}{\beta}\sum_{t=1}^{N}\sum_{p=1}^{N}\mathrm{E}\left\{r_{k_1}(t)r_{l_1}^H(p)r_{m_1}^H(t)r_{n_1}(p)\right\} \\
& -\frac{1}{\beta}\sum_{t=1}^{N}\sum_{p=1}^{N}\mathrm{E}\left\{r_{k_1}(t)r_{l_1}^H(p)r_{m_1}^H(p)r_{n_1}(t)\right\} \qquad (3.7)
\end{aligned}
$$

$$
\begin{aligned}
= \; & \frac{N}{\alpha}\mathrm{E}\left\{r_{k_1}(t)r_{l_1}^H(t)r_{m_1}^H(t)r_{n_1}(t)\right\} \\
& -\frac{N}{\beta}\mathrm{E}\left\{r_{k_1}(t)r_{l_1}^H(t)r_{m_1}^H(t)r_{n_1}(t)\right\} - \frac{N^2-N}{\beta}\mathrm{E}\left\{r_{k_1}(t)r_{l_1}^H(t)\right\}\mathrm{E}\left\{r_{m_1}^H(t)r_{n_1}(t)\right\} \\
& -\frac{N}{\beta}\mathrm{E}\left\{r_{k_1}(t)r_{l_1}^H(t)r_{m_1}^H(t)r_{n_1}(t)\right\} - \frac{N^2-N}{\beta}\mathrm{E}\left\{r_{k_1}(t)r_{m_1}^H(t)\right\}\mathrm{E}\left\{r_{l_1}^H(t)r_{n_1}(t)\right\} \\
& -\frac{N}{\beta}\mathrm{E}\left\{r_{k_1}(t)r_{l_1}^H(t)r_{m_1}^H(t)r_{n_1}(t)\right\} - \frac{N^2-N}{\beta}\mathrm{E}\left\{r_{k_1}(t)r_{n_1}(t)\right\}\mathrm{E}\left\{r_{l_1}^H(t)r_{m_1}^H(t)\right\} \\
= \; & \left(\frac{N}{\alpha}-\frac{3N}{\beta}\right)\mathrm{E}\left\{r_{k_1}(t)r_{l_1}^H(t)r_{m_1}^H(t)r_{n_1}(t)\right\} \\
& -\left(\frac{N^2-N}{\beta}\right)\Big[\mathrm{E}\left\{r_{k_1}(t)r_{l_1}^H(t)\right\}\mathrm{E}\left\{r_{m_1}^H(t)r_{n_1}(t)\right\}+ \\
& \qquad \mathrm{E}\left\{r_{k_1}(t)r_{m_1}^H(t)\right\}\mathrm{E}\left\{r_{l_1}^H(t)r_{n_1}(t)\right\}+ \\
& \qquad \mathrm{E}\left\{r_{k_1}(t)r_{n_1}(t)\right\}\mathrm{E}\left\{r_{l_1}^H(t)r_{m_1}^H(t)\right\}\Big].
\end{aligned}
$$

Note, that if

$$
\alpha = \frac{N^2-N}{N+2} \qquad \text{and} \qquad \beta = N^2 - N
$$

the fourth-order sample cumulant is an *unbiased* estimator.

3.2 Finite-sample covariance of the fourth-order sample cumulant as a function of N

Since we deal with the finite-sample case, the principal behaviour of the finite-sample covariance of the fourth-order sample cumulant as a function of N is interesting. Using eqs. (3.2), (3.3), (3.4), (3.5), it can be shown that

$$\begin{aligned}
\mathrm{E}\left\{\hat{c}^{(4)}_{k_1,l_1,m_1,n_1}\hat{c}^{(4),H}_{k_2,l_2,m_2,n_2}\right\} &= \frac{1}{\alpha^2}\left(N^2c_1+Nc_2\right)\\
&+\frac{1}{\alpha\beta}\left(N^3c_3+N^2c_4+Nc_5\right)\\
&+\frac{1}{\beta^2}\left(N^4c_6+N^3c_7+N^2c_8+Nc_9\right)
\end{aligned}$$

and (3.7) can be written as

$$\mathrm{E}\left\{\hat{c}^{(4)}_{k_1,l_1,m_1,n_1}\right\} = \frac{1}{\alpha}Nc_{10}+\frac{1}{\beta}\left(N^2c_{11}+Nc_{12}\right), \tag{3.8}$$

where $c_1,...,c_{12}$ are independent of N. This leads to

$$\begin{aligned}
&\mathrm{Cov}\left(\hat{c}^{(4)}_{k_1,l_1,m_1,n_1},\hat{c}^{(4),H}_{k_2,l_2,m_2,n_2}\right) =\\
&\quad\mathrm{E}\left\{\hat{c}^{(4)}_{k_1,l_1,m_1,n_1}\hat{c}^{(4),H}_{k_2,l_2,m_2,n_2}\right\}-\mathrm{E}\left\{\hat{c}^{(4)}_{k_1,l_1,m_1,n_1}\right\}\mathrm{E}\left\{\hat{c}^{(4),H}_{k_2,l_2,m_2,n_2}\right\}\\
&\quad=\frac{1}{\alpha^2}\left(N^2b_1+Nb_2\right)\\
&\quad\quad+\frac{1}{\alpha\beta}\left(N^3b_3+N^2b_4+Nb_5\right)\\
&\quad\quad+\frac{1}{\beta^2}\left(N^4b_6+N^3b_7+N^2b_8+Nb_9\right),
\end{aligned}\tag{3.9}$$

where $b_1,...,b_9$ are independent of N. By some tedious calculation it can be shown, that

$$b_1=b_3=b_6=0,$$

which confirms that $\mathrm{Cov}\left(\hat{c}^{(4)}_{k_1,l_1,m_1,n_1}, \hat{c}^{(4),H}_{k_2,l_2,m_2,n_2}\right)$ is $O(N^{-1})$ if $\alpha \sim N$ and $\beta \sim N^2$. Hence, the functional behaviour of the finite-sample covariance for the biased estimator ($\alpha = N$, $\beta = N^2$) is given by

$$\mathrm{Cov}\left(\hat{c}^{(4)}_{k_1,l_1,m_1,n_1}, \hat{c}^{(4),H}_{k_2,l_2,m_2,n_2}\right) = \frac{a_1}{N} + \frac{a_2}{N^2} + \frac{a_3}{N^3}$$

where a_1, a_2, a_3 are independent of N.

Chapter 4

M-Files

In this section we show how the finite-sample covariance can be implemented under MATLAB$^{\text{TM}}$ in so-called *M-Files.* We provide the code for:

1. "`cov2all.m`": Calculation of the finite-sample covariance of the second-order sample cumulant (moment).
2. "`cov3all.m`": Calculation of the finite-sample covariance of the third-order sample cumulant (moment).
3. "`cov4all.m`": Calculation of the finite-sample covariance of the fourth-order sample cumulant.
4. "`xmaln.m`": Calculation of the expected value

   $$\mathrm{E}\left\{\prod_{j=1}^{J} X_j\right\}$$

   of the product of J complex random variables X_j. The moments of these random variables must be given up to J. The only purpose of this M-File is to simplify the notation in the other M-Files.

For reasons of clearness, the important equations in the M-Files are cross-referenced with the former sections and chapters.

4.1 cov2all.m

```
function [cov2,E2all1,E2all2]=cov2all(N,index,A,gs,gn)
%COV2ALL Calculation of the covariance of the correlation estimator
%
%         c^ = 1/N *sum_t=1^N r_k(t) * r_l^H(t)
%
%         by an analytic formula in case of the usual model in narrowband
%         array processing:
%
%                 r_vec(t) = A_mat s_vec(t) + n_vec(t)
%
%         where s_vec(t) includes the independent source signals, matrix A_mat
%         consists in the steering vectors and n_vec(t) is measurement noise
%         of the M sensors with arbitrary distribution. This M-File can be
%         also used if some or all of the signals are coherent by a proper
%         modification of the matrix A_mat (see "Applications of Cumulants
%         to Array Processing: Direction Finding in Coherent Signal
%         Environment", E. Goenen, M. C. Dogan, J. M. Mendel, 28th. Asilomar
%         Conference, October 31-November 2, 1994, Pacific Grove, CA.).
%         Furthermore the near-field case can be also investigated by matrix
%         modification (see "Higher-Order Subspace Algorithms for Passive
%         Localization of Near-Field Sources", R. N. Challa and S. Shamsunder,
%         29th Annual Asilomar Conference on Signals, Systems and Computers,
%         Pacific Grove, CA, October 30 - November 1, 1995.
%
%         Syntax:
%                [cov2,E2all1,E2all2]=cov2all(N,index,A,gs,gn)
%
%         where
%
%            cov2 = Covariance of the correlation estimator (as a function of N).
%         E2all1 = Expected value of the correlation estimator for the first
%                  two indices [k1 l1]  (as a function of N).
%         E2all2 = Expected value of the correlation estimator for the second
%                  two indices [k2 l2]  (as a function of N).
%               N = vector with data lengths.
%           index = signal indices, index=[k1 l1 k2 l2], if k2=k1,l2=l1, then
```

```
%                   calculation of the variance otherwise of the covariance
%               A = Steering vector matrix
%              gs = [gRs1,2 gRs1,4
%                    gIs1,2 gIs1,4
%                      :       :
%                    gRsP,2 gRsP,4
%                    gIsP,2 gIsP,4 ]
%
%                    Matrix that contains the second and the fourth of the
%                    source signals. The following example explains the
%                    abbrevations:
%
%                        gRs1,4 = 4th moment of the realpart of signal 1
%
%              gn = [gRn1,2 gRn1,4
%                    gIn1,2 gIn1,4
%                      :       :
%                    gRnM,2 gRnM,4
%                    gInM,2 gInM,4]
%
%                    Matrix that contains the second and the fourth moment of the
%                    noise signals. The abbrevations are analogous to "gs".

%Inserting the first- and third order moment only for a more systematic
%M-file regarding "cov3all.m" and "cov4all.m".
[H,gs_cols]=size(gs);
gs=[zeros(H,1) gs(:,1) zeros(H,1) gs(:,2)]; gs_cols=gs_cols+2;

%Inserting the third order moment only for a more systematic
%M-file regarding "cov3all.m" and "cov4all.m" (The first order moment
%is setting to zero by the M-file "xmaln.m".
[H1,gn_cols]=size(gn);   gn_cols=gn_cols+1;
gn=[gn(:,1) zeros(H1,1) gn(:,2)];

%Checking for correct parameter dimensions
P=ceil(H/2);                % P=Number of sources
if (gs_cols~=4)
  disp('gs-Matrix not valid')
  return
end
M=ceil(H1/2);                % M=Number of Sensors
if (gn_cols~=3)
  disp('gn-Matrix not valid')
```

```
  return
end
[A_rows,A_cols]=size(A);
if (A_cols~=P)|(A_rows~=M)
  disp('A-Matrix not valid')
  return
end

%Index-abbrevations
k1=index(1); l1=index(2); k2=index(3); l2=index(4);

%Noise-abbrevations: Ek1l1H means E{n_k1 n_l1H} and so on ...
Ek1l1H=xmaln([k1 j*l1],gn);
Ek1k2H=xmaln([k1 j*k2],gn);
Ek1l2=xmaln([k1 l2],gn);
El1Hk2H=xmaln([j*l1 j*k2],gn);
El1Hl2=xmaln([j*l1 l2],gn);
Ek2Hl2=xmaln([j*k2 l2],gn);
Ek1l1Hk2Hl2=xmaln([k1 j*l1 j*k2 l2],gn);

%Calculation of modified steering vectors
A=[A sqrt(-1)*A];       %a_k(1:H)= A(k,:)

%Calculation of the source signal based Kronecker-products:
%E2malS means E{s \kron s*}, E4malS means E{s \kron s* \kron s* \kron s} ...
z2=1;z4=1;
E2malS=ones(1,H^2);E4malS=ones(1,H^4);

%Steering vector abbrevations
ak1=A(k1,:);
al1H=conj(A(l1,:));
ak2H=conj(A(k2,:));
al2=A(l2,:);

for a=1:H
 for b=1:H
  %Begin 2nd signal moment vector
  sig_ind=[sort([a b]) -1];
  momz2=1;
  for i=2:2+1
   if sig_ind(i)==sig_ind(i-1)
    momz2=momz2+1;
   else
```

```
    E2malS(z2)=E2malS(z2)*gs(sig_ind(i-1),momz2);
    momz2=1;
   end
  end
  z2=z2+1;
  %End 2nd signal moment vector
  for c=1:H
   for d=1:H
    %Begin 4th signal moment vector
    sig_ind=[sort([a b c d]) -1];
    momz4=1;
    for i=2:4+1
     if sig_ind(i)==sig_ind(i-1)
      momz4=momz4+1;
     else
      E4malS(z4)=E4malS(z4)*gs(sig_ind(i-1),momz4);
      momz4=1;
     end
    end
    z4=z4+1;
    %End 4th signal moment vector
   end
  end
 end
end

%Kronecker-product abbrevations: Kk1l1h means (a_k1 \kron a_l1H) and so on...
Kk1l1H=kron(ak1,al1H);
Kk1k2H=kron(ak1,ak2H);
Kk1l2=kron(ak1,al2);
Kl1Hk2H=kron(al1H,ak2H);
Kl1Hl2=kron(al1H,al2);
Kk2Hl2=kron(ak2H,al2);
Kk1l1Hk2Hl2=kron(ak1,kron(al1H,kron(ak2H,al2)));

%CALCULATION OF a)  See Section "Fourth-order Moment"
%
%      E{rk1 rl1H rk2H rl2}
%
Rk1l1Hk2Hl2=Kk1l1Hk2Hl2*E4malS'+Kk1l1H*E2malS'*Ek2Hl2+...
            Kk1k2H*E2malS'*El1Hl2+Kk1l2*E2malS'*El1Hk2H+...
            Kl1Hk2H*E2malS'*Ek1l2+Kl1Hl2*E2malS'*Ek1k2H+...
            Kk2Hl2*E2malS'*Ek1l1H+Ek1l1Hk2Hl2;
```

```
%CALCULATION OF b1) See Section "Second-order Moment"
%
%       E{rk1 rl1H}
%
Rk1l1H=Kk1l1H*E2malS'+Ek1l1H;

%CALCULATION OF b1) See Section "Second-order Moment"
%
%       E{rk2H rl2}
%
Rk2Hl2=Kk2Hl2*E2malS'+Ek2Hl2;

E2all1 = Rk1l1H;
E2all2 = Rk2Hl2;

cov2=(Rk1l1Hk2Hl2-Rk1l1H*Rk2Hl2)./N;
```

4.2 cov3all.m

```
function [cov3,E3all1,E3all2]=cov3all(N,index,A,gs,gn)
%COV3ALL Calculation of the covariance of the cumulant estimator
%
%         c^ = 1/alpha*sum_t=1^N r_k(t) * r_l^H(t) * r_m^H(t)
%
%         by an analytic formula in case of the usual model in narrowband
%         array processing:
%
%                 r_vec(t) = A_mat s_vec(t) + n_vec(t)
%
%         where s_vec(t) includes the independent source signals, matrix A_mat
%         consists in the steering vectors and n_vec(t) is measurement noise
%         of the M sensors with arbitrary distribution. This M-File can be
%         also used if some or all of the signals are coherent by a proper
%         modification of the matrix A_mat (see "Applications of Cumulants
%         to Array Processing: Direction Finding in Coherent Signal
%         Environment", E. Goenen, M. C. Dogan, J. M. Mendel, 28th. Asilomar
%         Conference, October 31-November 2, 1994, Pacific Grove, CA.).
%         Furthermore the near-field case can be also investigated by matrix
%         modification (see "Higher-Order Subspace Algorithms for Passive
%         Localization of Near-Field Sources", R. N. Challa and S. Shamsunder,
%         29th Annual Asilomar Conference on Signals, Systems and Computers,
```

```
%         Pacific Grove, CA, October 30 - November 1, 1995.
%
%         Syntax:
%                 [cov3,E3all1,E3all2]=cov3all(N,index,A,gs,gn)
%         where
%
%           cov3 = Covariance of the cumulant estimator (as a function of N).
%         E3all1 = Expected value of the cumulant estimator for the first
%                  three indices [k1 l1 m1]  (as a function of N).
%         E3all2 = Expected value of the cumulant estimator for the second
%                  three indices [k2 l2 m2]  (as a function of N).
%              N = vector with data lengths.
%          index = signal indices, index=[k1 l1 m1 k2 l2 m2], if k2=k1,l2=l1
%                  and m1=m2, then calculation of the variance otherwise of
%                  the covariance.
%              A = Steering vector matrix
%             gs = [gRs1,2 gRs1,3 gRs1,4 gRs1,6
%                   gIs1,2 gIs1,3 gIs1,4 gIs1,6
%                     :       :       :      :
%                   gRsP,2 gRsP,3 gRsP,4 gRsP,6
%                   gIsP,2 gIsP,3 gIsP,4 gIsP,6 ]
%
%                   Matrix that contains the second, third, fourth and sixth
%                   moment of the source signals. The following example
%                   explains the abbrevations:
%
%                       gRs1,4 = 4th moment of the realpart of signal 1
%
%             gn = [gRn1,2 gRn1,3 gRn1,4 gRn1,6
%                   gIn1,2 gIn1,3 gIn1,4 gIn1,6
%                     :       :       :      :
%                   gRnM,2 gRnM,3 gRnM,4 gRnM,6
%                   gInM,2 gInM,3 gInM,4 gInM,6]
%
%                   Matrix that contains the second, third, fourth and sixth-
%                   moment of the noise signals. The abbrevations are
%                   analogous to "gs".

%Inserting the first- and fifth-order moment only for a more systematic
%M-file regarding "cov2all.m" and "cov4all.m".
[H,gs_cols]=size(gs);
gs=[zeros(H,1) gs(:,1:3) zeros(H,1) gs(:,4)]; gs_cols=gs_cols+2;
```

```
%Inserting the fifth order moment only for a more systematic
%M-file regarding "cov3all.m" and "cov4all.m" (The first order moment
%is setting to zero by the M-file "xmaln.m".
[H1,gn_cols]=size(gn);
gn=[gn(:,1:3) zeros(H1,1) gn(:,4)]; gn_cols=gn_cols+1;


%Checking for correct parameter dimensions
P=ceil(H/2);                % P=Number of sources
if (gs_cols~=6)
  disp('gs-Matrix not valid')
  return
end
M=ceil(H1/2);               % M=Number of sensors
if (gn_cols~=5)
  disp('gn-Matrix not valid')
  return
end
[A_rows,A_cols]=size(A);
if (A_cols~=P)|(A_rows~=M)
  disp('A-Matrix not valid')
  return
end

%Index-abbrevations
k1=index(1); l1=index(2); m1=index(3);
k2=index(4); l2=index(5); m2=index(6);

%Noise-abbrevations: Ek1l1H means E{n_k1 n_l1H} and so on ...
%Sixth moment
Ek1l1Hm1k2Hl2m2H=xmaln([k1 j*l1 m1 j*k2 l2 j*m2],gn);

%Fourth moments
Ek1l1Hm1k2H=xmaln([k1 j*l1 m1 j*k2],gn);
Ek1l1Hm1l2=xmaln([k1 j*l1 m1 l2],gn);
Ek1l1Hm1m2H=xmaln([k1 j*l1 m1 j*m2],gn);
Ek1l1Hk2Hl2=xmaln([k1 j*l1 j*k2 l2],gn);
Ek1l1Hk2Hm2H=xmaln([k1 j*l1 j*k2 j*m2],gn);
Ek1l1Hl2m2H=xmaln([k1 j*l1 l2 j*m2],gn);
Ek1m1k2Hl2=xmaln([k1 m1 j*k2 l2],gn);
Ek1m1k2Hm2H=xmaln([k1 m1 j*k2 j*m2],gn);
Ek1m1l2m2H=xmaln([k1 m1 l2 j*m2],gn);
Ek1k2Hl2m2H=xmaln([k1 j*k2 l2 j*m2],gn);
```

```
El1Hm1k2Hl2=xmaln([j*l1 m1 j*k2 l2],gn);
El1Hm1k2Hm2H=xmaln([j*l1 m1 j*k2 j*m2],gn);
El1Hm1l2m2H=xmaln([j*l1 m1 l2 j*m2],gn);
El1Hk2Hl2m2H=xmaln([j*l1 j*k2 l2 j*m2],gn);
Em1k2Hl2m2H=xmaln([m1 j*k2 l2 j*m2],gn);

%Third moments
Ek1l1Hm1=xmaln([k1 j*l1 m1],gn);
Ek1l1Hk2H=xmaln([k1 j*l1 j*k2],gn);
Ek1l1Hl2=xmaln([k1 j*l1 l2],gn);
Ek1l1Hm2H=xmaln([k1 j*l1 j*m2],gn);
Ek1m1k2H=xmaln([k1 m1 j*k2],gn);
Ek1m1l2=xmaln([k1 m1 l2],gn);
Ek1m1m2H=xmaln([k1 m1 j*m2],gn);
Ek1k2Hl2=xmaln([k1 j*k2 l2],gn);
Ek1k2Hm2H=xmaln([k1 j*k2 j*m2],gn);
Ek1l2m2H=xmaln([k1 l2 j*m2],gn);
El1Hm1k2H=xmaln([j*l1 m1 j*k2],gn);
El1Hm1l2=xmaln([j*l1 m1 l2],gn);
El1Hm1m2H=xmaln([j*l1 m1 j*m2],gn);
El1Hk2Hl2=xmaln([j*l1 j*k2 l2],gn);
El1Hk2Hm2H=xmaln([j*l1 j*k2 j*m2],gn);
El1Hl2m2H=xmaln([j*l1 l2 j*m2],gn);
Em1k2Hl2=xmaln([m1 j*k2 l2],gn);
Em1k2Hm2H=xmaln([m1 j*k2 j*m2],gn);
Em1l2m2H=xmaln([m1 l2 j*m2],gn);
Ek2Hl2m2H=xmaln([j*k2 l2 j*m2],gn);

%Second moments
Ek1l1H=xmaln([k1 j*l1],gn);
Ek1m1=xmaln([k1 m1],gn);
Ek1k2H=xmaln([k1 j*k2],gn);
Ek1l2=xmaln([k1 l2],gn);
Ek1m2H=xmaln([k1 j*m2],gn);
El1Hm1=xmaln([j*l1 m1],gn);
El1Hk2H=xmaln([j*l1 j*k2],gn);
El1Hl2=xmaln([j*l1 l2],gn);
El1Hm2H=xmaln([j*l1 j*m2],gn);
Em1k2H=xmaln([m1 j*k2],gn);
Em1l2=xmaln([m1 l2],gn);
Em1m2H=xmaln([m1 j*m2],gn);
Ek2Hl2=xmaln([j*k2 l2],gn);
Ek2Hm2H=xmaln([j*k2 j*m2],gn);
```

```
El2m2H=xmaln([l2 j*m2],gn);


%Calculation of the modified steering matrix
A=[A sqrt(-1)*A];        %a_k(1:H)= A(k,:)


%Calculation of the source signal based Kronecker-products:
%E2malS means E{s \kron s*}, E4malS means E{s \kron s* \kron s* \kron s} ...
z2=1;z3=1;z4=1;z6=1;
E2malS=ones(1,H^2);E3malS=ones(1,H^3);E4malS=ones(1,H^4);E6malS=ones(1,H^6);

%Steering vector abbrevations
ak1=A(k1,:);
al1H=conj(A(l1,:));
am1=A(m1,:);
ak2H=conj(A(k2,:));
al2=A(l2,:);
am2H=conj(A(m2,:));

for a=1:H
 for b=1:H
  %Begin 2nd signal moment vector
  sig_ind=[sort([a b]) -1];
  momz2=1;
  for i=2:2+1
   if sig_ind(i)==sig_ind(i-1)
    momz2=momz2+1;
   else
    E2malS(z2)=E2malS(z2)*gs(sig_ind(i-1),momz2);
    momz2=1;
   end
  end
  z2=z2+1;
  %End 2nd signal moment vector
  for c=1:H
   %Begin 3rd signal moment vector
   sig_ind=[sort([a b c]) -1];
   momz3=1;
   for i=2:3+1
    if sig_ind(i)==sig_ind(i-1)
     momz3=momz3+1;
    else
```

```
      E3malS(z3)=E3malS(z3)*gs(sig_ind(i-1),momz3);
      momz3=1;
     end
    end
    z3=z3+1;
    %End 3rd signal moment vector

    for d=1:H
     %Begin 4th signal moment vector
     sig_ind=[sort([a b c d]) -1];
     momz4=1;
     for i=2:4+1
      if sig_ind(i)==sig_ind(i-1)
       momz4=momz4+1;
      else
       E4malS(z4)=E4malS(z4)*gs(sig_ind(i-1),momz4);
       momz4=1;
      end
     end
     z4=z4+1;
     %End 4th signal moment vector
     for e=1:H
      for f=1:H
       %Begin 6th signal moment vector
       sig_ind=[sort([a b c d e f]) -1];
       momz6=1;
       for i=2:6+1
        if sig_ind(i)==sig_ind(i-1)
         momz6=momz6+1;
        else
         E6malS(z6)=E6malS(z6)*gs(sig_ind(i-1),momz6);
         momz6=1;
        end
       end
       z6=z6+1;
       %End 6th signal moment vector
      end
     end
    end
   end
  end
end
```

```
%Kronecker product abbrevations: Kk1l1h means (a_k1 \kron a_l1H) and so on...
%Six
Kk1l1Hm1k2Hl2m2H=kron(ak1,kron(al1H,kron(am1,kron(ak2H,kron(al2,am2H)))));

%Four
Kk1l1Hm1k2H=kron(ak1,kron(al1H,kron(am1,ak2H)));
Kk1l1Hm1l2=kron(ak1,kron(al1H,kron(am1,al2)));
Kk1l1Hm1m2H=kron(ak1,kron(al1H,kron(am1,am2H)));
Kk1l1Hk2Hl2=kron(ak1,kron(al1H,kron(ak2H,al2)));
Kk1l1Hk2Hm2H=kron(ak1,kron(al1H,kron(ak2H,am2H)));
Kk1l1Hl2m2H=kron(ak1,kron(al1H,kron(al2,am2H)));
Kk1m1k2Hl2=kron(ak1,kron(am1,kron(ak2H,al2)));
Kk1m1k2Hm2H=kron(ak1,kron(am1,kron(ak2H,am2H)));
Kk1m1l2m2H=kron(ak1,kron(am1,kron(al2,am2H)));
Kk1k2Hl2m2H=kron(ak1,kron(ak2H,kron(al2,am2H)));
Kl1Hm1k2Hl2=kron(al1H,kron(am1,kron(ak2H,al2)));
Kl1Hm1k2Hm2H=kron(al1H,kron(am1,kron(ak2H,am2H)));
Kl1Hm1l2m2H=kron(al1H,kron(am1,kron(al2,am2H)));
Kl1Hk2Hl2m2H=kron(al1H,kron(ak2H,kron(al2,am2H)));
Km1k2Hl2m2H=kron(am1,kron(ak2H,kron(al2,am2H)));

%Three
Kk1l1Hm1=kron(ak1,kron(al1H,am1));
Kk1l1Hk2H=kron(ak1,kron(al1H,ak2H));
Kk1l1Hl2=kron(ak1,kron(al1H,al2));
Kk1l1Hm2H=kron(ak1,kron(al1H,am2H));
Kk1m1k2H=kron(ak1,kron(am1,ak2H));
Kk1m1l2=kron(ak1,kron(am1,al2));
Kk1m1m2H=kron(ak1,kron(am1,am2H));
Kk1k2Hl2=kron(ak1,kron(ak2H,al2));
Kk1k2Hm2H=kron(ak1,kron(ak2H,am2H));
Kk1l2m2H=kron(ak1,kron(al2,am2H));
Kl1Hm1k2H=kron(al1H,kron(am1,ak2H));
Kl1Hm1l2=kron(al1H,kron(am1,al2));
Kl1Hm1m2H=kron(al1H,kron(am1,am2H));
Kl1Hk2Hl2=kron(al1H,kron(ak2H,al2));
Kl1Hk2Hm2H=kron(al1H,kron(ak2H,am2H));
Kl1Hl2m2H=kron(al1H,kron(al2,am2H));
Km1k2Hl2=kron(am1,kron(ak2H,al2));
Km1k2Hm2H=kron(am1,kron(ak2H,am2H));
Km1l2m2H=kron(am1,kron(al2,am2H));
Kk2Hl2m2H=kron(ak2H,kron(al2,am2H));
```

```
%Two
Kk1l1H=kron(ak1,al1H);
Kk1m1=kron(ak1,am1);
Kk1k2H=kron(ak1,ak2H);
Kk1l2=kron(ak1,al2);
Kk1m2H=kron(ak1,am2H);
Kl1Hm1=kron(al1H,am1);
Kl1Hk2H=kron(al1H,ak2H);
Kl1Hl2=kron(al1H,al2);
Kl1Hm2H=kron(al1H,am2H);
Km1k2H=kron(am1,ak2H);
Km1l2=kron(am1,al2);
Km1m2H=kron(am1,am2H);
Kk2Hl2=kron(ak2H,al2);
Kk2Hm2H=kron(ak2H,am2H);
Kl2m2H=kron(al2,am2H);


%CALCULATION OF a) See Section "Sixth-order Moment"
%
%      E{rk1 rl1H rm1 rk2H rl2 rm2H}
%
TERM1=....
Kk1l1Hm1k2Hl2m2H*E6malS'+Kk1l1Hm1k2H*E4malS'*El2m2H+...
Kk1l1Hm1l2*E4malS'*Ek2Hm2H+Kk1l1Hm1m2H*E4malS'*Ek2Hl2+...
Kk1l1Hk2Hl2*E4malS'*Em1m2H+Kk1l1Hk2Hm2H*E4malS'*Em1l2+...
Kk1l1Hl2m2H*E4malS'*Em1k2H+Kk1m1k2Hl2*E4malS'*El1Hm2H+...
Kk1m1k2Hm2H*E4malS'*El1Hl2+Kk1m1l2m2H*E4malS'*El1Hk2H+...
Kk1k2Hl2m2H*E4malS'*El1Hm1+Kl1Hm1k2Hl2*E4malS'*Ek1m2H+...
Kl1Hm1k2Hm2H*E4malS'*Ek1l2+Kl1Hm1l2m2H*E4malS'*Ek1k2H+...
Kl1Hk2Hl2m2H*E4malS'*Ek1m1+Km1k2Hl2m2H*E4malS'*Ek1l1H+...
Kk1l1Hm1*E3malS'*Ek2Hl2m2H+Kk1l1Hk2H*E3malS'*Em1l2m2H+...
Kk1l1Hl2*E3malS'*Em1k2Hm2H+Kk1l1Hm2H*E3malS'*Em1k2Hl2+...
Kk1m1k2H*E3malS'*El1Hl2m2H+Kk1m1l2*E3malS'*El1Hk2Hm2H+...
Kk1m1m2H*E3malS'*El1Hk2Hl2+Kk1k2Hl2*E3malS'*El1Hm1m2H+...
Kk1k2Hm2H*E3malS'*El1Hm1l2+Kk1l2m2H*E3malS'*El1Hm1k2H+...
Kl1Hm1k2H*E3malS'*Ek1l2m2H+Kl1Hm1l2*E3malS'*Ek1k2Hm2H;

TERM2=...
Kl1Hm1m2H*E3malS'*Ek1k2Hl2+Kl1Hk2Hl2*E3malS'*Ek1m1m2H+...
Kl1Hk2Hm2H*E3malS'*Ek1m1l2+Kl1Hl2m2H*E3malS'*Ek1m1k2H+...
Km1k2Hl2*E3malS'*Ek1l1Hm2H+Km1k2Hm2H*E3malS'*Ek1l1Hl2+...
```

```
Km1l2m2H*E3malS'*Ek1l1Hk2H+Kk2Hl2m2H*E3malS'*Ek1l1Hm1+...
Kk1l1H*E2malS'*Em1k2Hl2m2H+Kk1m1*E2malS'*El1Hk2Hl2m2H+...
Kk1k2H*E2malS'*El1Hm1l2m2H+Kk1l2*E2malS'*El1Hm1k2Hm2H+...
Kk1m2H*E2malS'*El1Hm1k2Hl2+Kl1Hm1*E2malS'*Ek1k2Hl2m2H+...
Kl1Hk2H*E2malS'*Ek1m1l2m2H+Kl1Hl2*E2malS'*Ek1m1k2Hm2H+...
Kl1Hm2H*E2malS'*Ek1m1k2Hl2+Km1k2H*E2malS'*Ek1l1Hl2m2H+...
Km1l2*E2malS'*Ek1l1Hk2Hm2H+Km1m2H*E2malS'*Ek1l1Hk2Hl2+...
Kk2Hl2*E2malS'*Ek1l1Hm1m2H+Kk2Hm2H*E2malS'*Ek1l1Hm1l2+...
Kl2m2H*E2malS'*Ek1l1Hm1k2H+Ek1l1Hm1k2Hl2m2H;

Rk1l1Hm1k2Hl2m2H=TERM1+TERM2;
%CALCULATION OF b1) See Section "Third-order Moment"
%
%       E{rk1 rl1H rm1}
%
Rk1l1Hm1=Kk1l1Hm1*E3malS'+Ek1l1Hm1;

%CALCULATION OF b2) See Section "Third-order Moment"
%
%       E{rk2H rl2 rm2H}
%
Rk2Hl2m2H=Kk2Hl2m2H*E3malS'+Ek2Hl2m2H;

E3all1 = Rk1l1Hm1;
E3all2 = Rk2Hl2m2H;

cov3 = (Rk1l1Hm1k2Hl2m2H-Rk1l1Hm1*Rk2Hl2m2H)./N;
```

4.3 cov4all.m

```
function [cov4,E4all1,E4all2,cov4biased,relerror,relerrorbiased]=
          cov4all(N,index,A,gs,gn,alpha,beta)
%COV4ALL Calculation of the covariance of the cumulant estimator
%
%         c^ = 1/alpha*sum_t=1^N r_k(t) * r_l^H(t) * r_m^H(t) * r_n(t)
%             -1/beta*(sum_t=1^N r_k(t)*r_l^H(t))*(sum_t=1^N r_m^H(t)*r_n(t))
%             -1/beta*(sum_t=1^N r_k(t)*r_m^H(t))*(sum_t=1^N r_l^H(t)*r_n(t))
%             -1/beta*(sum_t=1^N r_k(t)*r_n(t))*(sum_t=1^N r_l^H(t)*r_m^H(t))
%
%         by an analytic formula in case of the usual model in narrowband
%         array processing:
%
```

```
%               r_vec(t) = A_mat s_vec(t) + n_vec(t)
%
%         where s_vec(t) includes the independent source signals, matrix A_mat
%         consists in the steering vectors and n_vec(t) is measurement noise
%         of the M sensors with arbitrary distribution. This M-File can be
%         also used if some or all of the signals are coherent by a proper
%         modification of the matrix A_mat (see "Applications of Cumulants
%         to Array Processing: Direction Finding in Coherent Signal
%         Environment", E. Goenen, M. C. Dogan, J. M. Mendel, 28th. Asilomar
%         Conference, October 31-November 2, 1994, Pacific Grove, CA.).
%         Furthermore the near-field case can be also investigated by matrix
%         modification (see "Higher-Order Subspace Algorithms for Passive
%         Localization of Near-Field Sources", R. N. Challa and S. Shamsunder,
%         29th Annual Asilomar Conference on Signals, Systems and Computers,
%         Pacific Grove, CA, October 30 - November 1, 1995.
%
%         Syntax:
%             [cov4,E4all1,E4all2,cov4biased,relerror,relerrorbiased] =
%                             cov4all(N,index,A,gs,gn,alpha,beta,r)
%         where
%
%           cov4 = Covariance of the cumulant estimator (as a function of N).
%         E4all1 = Expected value of the cumulant estimator for the first
%                  four indices [k1 l1 m1 n1]  (as a function of N).
%         E4all2 = Expected value of the cumulant estimator for the second
%                  four indices [k2 l2 m2 n2]  (as a function of N).
%     cov4biased = If more than three output arguments exists, then the
%                  covariance of the "Biased Estimator" is also calculated.
%       relerror = Calculation of the relativ error between N*Cov(c^,N) and
%                  infty*Cov(c^,infty) as a function of N.
%                  (relerror=[N*Cov(c^,N)-infty*Cov(c^,infty)]/(N*Cov(c^,N)))
% relerrorbiased = If more than five output arguments exists, then relative
%                  error in case of the "Biased Estimator" is also calculated.
%              N = vector with data lengths.
%          index = signal indices, index=[k1 l1 m1 n1 k2 l2 m2 n2], if k2=k1,
%                  l2=l1, m1=m2 and n1=n2, then calculation of the variance
%                  otherwise of the covariance.
%              A = Steering vector matrix
%             gs = [gRs1,2 gRs1,3 gRs1,4 gRs1,5 gRs1,6 gRs1,7 gRs1,8
%                   gIs1,2 gIs1,3 gIs1,4 gIs1,5 gIs1,6 gIs1,7 gIs1,8
%                     :      :      :      :      :      :      :      :
%                   gRsP,2 gRsP,3 gRsP,4 gRsP,5 gRsP,6 gRsP,7 gRsP,8
%                   gIsP,2 gIsP,3 gIsP,4 gIsP,5 gIsP,6 gIsP,7 gIsP,8]
```

```
%
%                   Matrix that contains the second, third, fourth, fifth,
%                   sixth, seventh and eighth-moment of the source signals.
%                   The following example explains the abbrevations:
%
%                       gRs1,4 = 4th moment of the realpart of signal 1
%
%             gn = [gRn1,2 gRn1,3 gRn1,4 gRn1,5 gRn1,6 gRn1,7 gRn1,8
%                   gIn1,2 gIn1,3 gIn1,4 gIn1,5 gIn1,6 gIn1,7 gIn1,8
%                     :      :      :      :      :      :      :      :
%                   gRnM,2 gRnM,3 gRnM,4 gRnM,5 gRnM,6 gRnM,7 gRnM,8
%                   gInM,2 gInM,3 gInM,4 gInM,5 gInM,6 gInM,7 gInM,8]
%
%                   Matrix that contains the second, third, fourth, fifth,
%                   sixth, seventh and eighth-moment of the noise signals.
%                   The abbrevations are analogous to "gs".
%
%          alpha = If alpha=(N^2-N)/(N+2) and beta=N^2-N, than the "Unbiased
%          beta  = Estimator is choosen and otherwise a "Biased Estimator"
%                  (Usual alpha=N and beta=N^2).
%
% Please note, that P>2 needs a huge amount of memory since vectors of the
% size (2*P)^8 are used.

%Inserting the first-order moment only for a more systematic
%M-file regarding "cov3all.m" and "cov4all.m".
[H,gs_cols]=size(gs);
gs=[zeros(H,1) gs]; gs_cols=gs_cols+1;

[H1,gn_cols]=size(gn);

%Checking for correct parameter dimensions
P=ceil(H/2);              % P=Number of sources
if (gs_cols~=8)
  disp('gs-Matrix not valid')
  return
end
M=ceil(H1/2);             % M=Number of sensors
if (gn_cols~=7)
  disp('gn-Matrix not valid')
  return
end
[A_rows,A_cols]=size(A);
```

```
if (A_cols~=P)|(A_rows~=M)
  disp('A-Matrix not valid')
  return
end

if nargout>4   %Calculation of the asymptotic solution
  Nasymp=10^6*N(length(N));
  N=[N Nasymp];
  if nargin>5, alpha=[alpha Nasymp]; end;
  if nargin>6, beta=[beta Nasymp^2]; end;
end

%Index-abbrevations
k1=index(1); l1=index(2); m1=index(3); n1=index(4);
k2=index(5); l2=index(6); m2=index(7); n2=index(8);

%Noise-abbrevations: Em2n2H means E{n_m2 n_n2H} and so on ...
%T11111
Em2n2H=xmaln([m2 j*n2],gn);
El2n2H=xmaln([l2 j*n2],gn);
El2m2=xmaln([l2 m2],gn);
El2m2n2H=xmaln([l2 m2 j*n2],gn);

%T11112
Ek2Hn2H=xmaln([j*k2 j*n2],gn);
Ek2Hm2=xmaln([j*k2 m2],gn);
Ek2Hm2n2H=xmaln([j*k2 m2 j*n2],gn);
Ek2Hl2=xmaln([j*k2 l2],gn);
Ek2Hl2n2H=xmaln([j*k2 l2 j*n2],gn);
Ek2Hl2m2=xmaln([j*k2 l2 m2],gn);
Ek2Hl2m2n2H=xmaln([j*k2 l2 m2 j*n2],gn);

%T11121
En1n2H=xmaln([n1 j*n2],gn);
En1m2=xmaln([n1 m2],gn);
En1m2n2H=xmaln([n1 m2 j*n2],gn);
En1l2=xmaln([n1 l2],gn);
En1l2n2H=xmaln([n1 l2 j*n2],gn);
En1l2m2=xmaln([n1 l2 m2],gn);
En1l2m2n2H=xmaln([n1 l2 m2 j*n2],gn);

%T11122
En1k2H=xmaln([n1 j*k2],gn);
```

```
En1k2Hn2H=xmaln([n1 j*k2 j*n2],gn);
En1k2Hm2=xmaln([n1 j*k2 m2],gn);
En1k2Hm2n2H=xmaln([n1 j*k2 m2 j*n2],gn);
En1k2Hl2=xmaln([n1 j*k2 l2],gn);
En1k2Hl2n2H=xmaln([n1 j*k2 l2 j*n2],gn);
En1k2Hl2m2=xmaln([n1 j*k2 l2 m2],gn);
En1k2Hl2m2n2H=xmaln([n1 j*k2 l2 m2 j*n2],gn);

%T11211
Em1Hn2H=xmaln([j*m1 j*n2],gn);
Em1Hm2=xmaln([j*m1 m2],gn);
Em1Hm2n2H=xmaln([j*m1 m2 j*n2],gn);
Em1Hl2=xmaln([j*m1 l2],gn);
Em1Hl2n2H=xmaln([j*m1 l2 j*n2],gn);
Em1Hl2m2=xmaln([j*m1 l2 m2],gn);
Em1Hl2m2n2H=xmaln([j*m1 l2 m2 j*n2],gn);

%T11212
Em1Hk2H=xmaln([j*m1 j*k2],gn);
Em1Hk2Hn2H=xmaln([j*m1 j*k2 j*n2],gn);
Em1Hk2Hm2=xmaln([j*m1 j*k2 m2],gn);
Em1Hk2Hm2n2H=xmaln([j*m1 j*k2 m2 j*n2],gn);
Em1Hk2Hl2=xmaln([j*m1 j*k2 l2],gn);
Em1Hk2Hl2n2H=xmaln([j*m1 j*k2 l2 j*n2],gn);
Em1Hk2Hl2m2=xmaln([j*m1 j*k2 l2 m2],gn);
Em1Hk2Hl2m2n2H=xmaln([j*m1 j*k2 l2 m2 j*n2],gn);

%T11221
Em1Hn1=xmaln([j*m1 n1],gn);
Em1Hn1n2H=xmaln([j*m1 n1 j*n2],gn);
Em1Hn1m2=xmaln([j*m1 n1 m2],gn);
Em1Hn1m2n2H=xmaln([j*m1 n1 m2 j*n2],gn);
Em1Hn1l2=xmaln([j*m1 n1 l2],gn);
Em1Hn1l2n2H=xmaln([j*m1 n1 l2 j*n2],gn);
Em1Hn1l2m2=xmaln([j*m1 n1 l2 m2],gn);
Em1Hn1l2m2n2H=xmaln([j*m1 n1 l2 m2 j*n2],gn);

%T11222
Em1Hn1k2H=xmaln([j*m1 n1 j*k2],gn);
Em1Hn1k2Hn2H=xmaln([j*m1 n1 j*k2 j*n2],gn);
Em1Hn1k2Hm2=xmaln([j*m1 n1 j*k2 m2],gn);
Em1Hn1k2Hm2n2H=xmaln([j*m1 n1 j*k2 m2 j*n2],gn);
Em1Hn1k2Hl2=xmaln([j*m1 n1 j*k2 l2],gn);
```

```
Em1Hn1k2Hl2n2H=xmaln([j*m1 n1 j*k2 l2 j*n2],gn);
Em1Hn1k2Hl2m2=xmaln([j*m1 n1 j*k2 l2 m2],gn);
Em1Hn1k2Hl2m2n2H=xmaln([j*m1 n1 j*k2 l2 m2 j*n2],gn);

%T12111
El1Hn2H=xmaln([j*l1 j*n2],gn);
El1Hm2=xmaln([j*l1 m2],gn);
El1Hm2n2H=xmaln([j*l1 m2 j*n2],gn);
El1Hl2=xmaln([j*l1 l2],gn);
El1Hl2n2H=xmaln([j*l1 l2 j*n2],gn);
El1Hl2m2=xmaln([j*l1 l2 m2],gn);
El1Hl2m2n2H=xmaln([j*l1 l2 m2 j*n2],gn);

%T12112
El1Hk2H=xmaln([j*l1 j*k2],gn);
El1Hk2Hn2H=xmaln([j*l1 j*k2 j*n2],gn);
El1Hk2Hm2=xmaln([j*l1 j*k2 m2],gn);
El1Hk2Hm2n2H=xmaln([j*l1 j*k2 m2 j*n2],gn);
El1Hk2Hl2=xmaln([j*l1 j*k2 l2],gn);
El1Hk2Hl2n2H=xmaln([j*l1 j*k2 l2 j*n2],gn);
El1Hk2Hl2m2=xmaln([j*l1 j*k2 l2 m2],gn);
El1Hk2Hl2m2n2H=xmaln([j*l1 j*k2 l2 m2 j*n2],gn);



%T12121
El1Hn1=xmaln([j*l1 n1],gn);
El1Hn1n2H=xmaln([j*l1 n1 j*n2],gn);
El1Hn1m2=xmaln([j*l1 n1 m2],gn);
El1Hn1m2n2H=xmaln([j*l1 n1 m2 j*n2],gn);
El1Hn1l2=xmaln([j*l1 n1 l2],gn);
El1Hn1l2n2H=xmaln([j*l1 n1 l2 j*n2],gn);
El1Hn1l2m2=xmaln([j*l1 n1 l2 m2],gn);
El1Hn1l2m2n2H=xmaln([j*l1 n1 l2 m2 j*n2],gn);

%T12122
El1Hn1k2H=xmaln([j*l1 n1 j*k2],gn);
El1Hn1k2Hn2H=xmaln([j*l1 n1 j*k2 j*n2],gn);
El1Hn1k2Hm2=xmaln([j*l1 n1 j*k2 m2],gn);
El1Hn1k2Hm2n2H=xmaln([j*l1 n1 j*k2 m2 j*n2],gn);
El1Hn1k2Hl2=xmaln([j*l1 n1 j*k2 l2],gn);
El1Hn1k2Hl2n2H=xmaln([j*l1 n1 j*k2 l2 j*n2],gn);
El1Hn1k2Hl2m2=xmaln([j*l1 n1 j*k2 l2 m2],gn);
```

```
El1Hn1k2Hl2m2n2H=xmaln([j*l1 n1 j*k2 l2 m2 j*n2],gn);

%T12211
El1Hm1H=xmaln([j*l1 j*m1],gn);
El1Hm1Hn2H=xmaln([j*l1 j*m1 j*n2],gn);
El1Hm1Hm2=xmaln([j*l1 j*m1 m2],gn);
El1Hm1Hm2n2H=xmaln([j*l1 j*m1 m2 j*n2],gn);
El1Hm1Hl2=xmaln([j*l1 j*m1 l2],gn);
El1Hm1Hl2n2H=xmaln([j*l1 j*m1 l2 j*n2],gn);
El1Hm1Hl2m2=xmaln([j*l1 j*m1 l2 m2],gn);
El1Hm1Hl2m2n2H=xmaln([j*l1 j*m1 l2 m2 j*n2],gn);

%T12212
El1Hm1Hk2H=xmaln([j*l1 j*m1 j*k2],gn);
El1Hm1Hk2Hn2H=xmaln([j*l1 j*m1 j*k2 j*n2],gn);
El1Hm1Hk2Hm2=xmaln([j*l1 j*m1 j*k2 m2],gn);
El1Hm1Hk2Hm2n2H=xmaln([j*l1 j*m1 j*k2 m2 j*n2],gn);
El1Hm1Hk2Hl2=xmaln([j*l1 j*m1 j*k2 l2],gn);
El1Hm1Hk2Hl2n2H=xmaln([j*l1 j*m1 j*k2 l2 j*n2],gn);
El1Hm1Hk2Hl2m2=xmaln([j*l1 j*m1 j*k2 l2 m2],gn);
El1Hm1Hk2Hl2m2n2H=xmaln([j*l1 j*m1 j*k2 l2 m2 j*n2],gn);

%T12221
El1Hm1Hn1=xmaln([j*l1 j*m1 n1],gn);
El1Hm1Hn1n2H=xmaln([j*l1 j*m1 n1 j*n2],gn);
El1Hm1Hn1m2=xmaln([j*l1 j*m1 n1 m2],gn);
El1Hm1Hn1m2n2H=xmaln([j*l1 j*m1 n1 m2 j*n2],gn);
El1Hm1Hn1l2=xmaln([j*l1 j*m1 n1 l2],gn);
El1Hm1Hn1l2n2H=xmaln([j*l1 j*m1 n1 l2 j*n2],gn);
El1Hm1Hn1l2m2=xmaln([j*l1 j*m1 n1 l2 m2],gn);
El1Hm1Hn1l2m2n2H=xmaln([j*l1 j*m1 n1 l2 m2 j*n2],gn);

%T12222
El1Hm1Hn1k2H=xmaln([j*l1 j*m1 n1 j*k2],gn);
El1Hm1Hn1k2Hn2H=xmaln([j*l1 j*m1 n1 j*k2 j*n2],gn);
El1Hm1Hn1k2Hm2=xmaln([j*l1 j*m1 n1 j*k2 m2],gn);
El1Hm1Hn1k2Hm2n2H=xmaln([j*l1 j*m1 n1 j*k2 m2 j*n2],gn);
El1Hm1Hn1k2Hl2=xmaln([j*l1 j*m1 n1 j*k2 l2],gn);
El1Hm1Hn1k2Hl2n2H=xmaln([j*l1 j*m1 n1 j*k2 l2 j*n2],gn);
El1Hm1Hn1k2Hl2m2=xmaln([j*l1 j*m1 n1 j*k2 l2 m2],gn);

%T21111
Ek1n2H=xmaln([k1 j*n2],gn);
```

```
Ek1m2=xmaln([k1 m2],gn);
Ek1m2n2H=xmaln([k1 m2 j*n2],gn);
Ek1l2=xmaln([k1 l2],gn);
Ek1l2n2H=xmaln([k1 l2 j*n2],gn);
Ek1l2m2=xmaln([k1 l2 m2],gn);
Ek1l2m2n2H=xmaln([k1 l2 m2 j*n2],gn);

%T21112
Ek1k2H=xmaln([k1 j*k2],gn);
Ek1k2Hn2H=xmaln([k1 j*k2 j*n2],gn);
Ek1k2Hm2=xmaln([k1 j*k2 m2],gn);
Ek1k2Hm2n2H=xmaln([k1 j*k2 m2 j*n2],gn);
Ek1k2Hl2=xmaln([k1 j*k2 l2],gn);
Ek1k2Hl2n2H=xmaln([k1 j*k2 l2 j*n2],gn);
Ek1k2Hl2m2=xmaln([k1 j*k2 l2 m2],gn);
Ek1k2Hl2m2n2H=xmaln([k1 j*k2 l2 m2 j*n2],gn);

%T21121
Ek1n1=xmaln([k1 n1],gn);
Ek1n1n2H=xmaln([k1 n1 j*n2],gn);
Ek1n1m2=xmaln([k1 n1 m2],gn);
Ek1n1m2n2H=xmaln([k1 n1 m2 j*n2],gn);
Ek1n1l2=xmaln([k1 n1 l2],gn);
Ek1n1l2n2H=xmaln([k1 n1 l2 j*n2],gn);
Ek1n1l2m2=xmaln([k1 n1 l2 m2],gn);
Ek1n1l2m2n2H=xmaln([k1 n1 l2 m2 j*n2],gn);

%T21122
Ek1n1k2H=xmaln([k1 n1 j*k2],gn);
Ek1n1k2Hn2H=xmaln([k1 n1 j*k2 j*n2],gn);
Ek1n1k2Hm2=xmaln([k1 n1 j*k2 m2],gn);
Ek1n1k2Hm2n2H=xmaln([k1 n1 j*k2 m2 j*n2],gn);
Ek1n1k2Hl2=xmaln([k1 n1 j*k2 l2],gn);
Ek1n1k2Hl2n2H=xmaln([k1 n1 j*k2 l2 j*n2],gn);
Ek1n1k2Hl2m2=xmaln([k1 n1 j*k2 l2 m2],gn);
Ek1n1k2Hl2m2n2H=xmaln([k1 n1 j*k2 l2 m2 j*n2],gn);

%T21211
Ek1m1H=xmaln([k1 j*m1],gn);
Ek1m1Hn2H=xmaln([k1 j*m1 j*n2],gn);
Ek1m1Hm2=xmaln([k1 j*m1 m2],gn);
Ek1m1Hm2n2H=xmaln([k1 j*m1 m2 j*n2],gn);
Ek1m1Hl2=xmaln([k1 j*m1 l2],gn);
```

```
Ek1m1Hl2n2H=xmaln([k1 j*m1 l2 j*n2],gn);
Ek1m1Hl2m2=xmaln([k1 j*m1 l2 m2],gn);
Ek1m1Hl2m2n2H=xmaln([k1 j*m1 l2 m2 j*n2],gn);

%T21212
Ek1m1Hk2H=xmaln([k1 j*m1 j*k2],gn);
Ek1m1Hk2Hn2H=xmaln([k1 j*m1 j*k2 j*n2],gn);
Ek1m1Hk2Hm2=xmaln([k1 j*m1 j*k2 m2],gn);
Ek1m1Hk2Hm2n2H=xmaln([k1 j*m1 j*k2 m2 j*n2],gn);
Ek1m1Hk2Hl2=xmaln([k1 j*m1 j*k2 l2],gn);
Ek1m1Hk2Hl2n2H=xmaln([k1 j*m1 j*k2 l2 j*n2],gn);
Ek1m1Hk2Hl2m2=xmaln([k1 j*m1 j*k2 l2 m2],gn);
Ek1m1Hk2Hl2m2n2H=xmaln([k1 j*m1 j*k2 l2 m2 j*n2],gn);

%T21221
Ek1m1Hn1=xmaln([k1 j*m1 n1],gn);
Ek1m1Hn1n2H=xmaln([k1 j*m1 n1 j*n2],gn);
Ek1m1Hn1m2=xmaln([k1 j*m1 n1 m2],gn);
Ek1m1Hn1m2n2H=xmaln([k1 j*m1 n1 m2 j*n2],gn);
Ek1m1Hn1l2=xmaln([k1 j*m1 n1 l2],gn);
Ek1m1Hn1l2n2H=xmaln([k1 j*m1 n1 l2 j*n2],gn);
Ek1m1Hn1l2m2=xmaln([k1 j*m1 n1 l2 m2],gn);
Ek1m1Hn1l2m2n2H=xmaln([k1 j*m1 n1 l2 m2 j*n2],gn);

%T21222
Ek1m1Hn1k2H=xmaln([k1 j*m1 n1 j*k2],gn);
Ek1m1Hn1k2Hn2H=xmaln([k1 j*m1 n1 j*k2 j*n2],gn);
Ek1m1Hn1k2Hm2=xmaln([k1 j*m1 n1 j*k2 m2],gn);
Ek1m1Hn1k2Hm2n2H=xmaln([k1 j*m1 n1 j*k2 m2 j*n2],gn);
```

```
Ek1m1Hn1k2Hl2=xmaln([k1 j*m1 n1 j*k2 l2],gn);
Ek1m1Hn1k2Hl2n2H=xmaln([k1 j*m1 n1 j*k2 l2 j*n2],gn);
Ek1m1Hn1k2Hl2m2=xmaln([k1 j*m1 n1 j*k2 l2 m2],gn);

%T22111
Ek1l1H=xmaln([k1 j*l1],gn);
Ek1l1Hn2H=xmaln([k1 j*l1 j*n2],gn);
Ek1l1Hm2=xmaln([k1 j*l1 m2],gn);
Ek1l1Hm2n2H=xmaln([k1 j*l1 m2 j*n2],gn);
Ek1l1Hl2=xmaln([k1 j*l1 l2],gn);
Ek1l1Hl2n2H=xmaln([k1 j*l1 l2 j*n2],gn);
Ek1l1Hl2m2=xmaln([k1 j*l1 l2 m2],gn);
Ek1l1Hl2m2n2H=xmaln([k1 j*l1 l2 m2 j*n2],gn);

%%T22112
Ek1l1Hk2H=xmaln([k1 j*l1 j*k2],gn);
Ek1l1Hk2Hn2H=xmaln([k1 j*l1 j*k2 j*n2],gn);
Ek1l1Hk2Hm2=xmaln([k1 j*l1 j*k2 m2],gn);
Ek1l1Hk2Hm2n2H=xmaln([k1 j*l1 j*k2 m2 j*n2],gn);
Ek1l1Hk2Hl2=xmaln([k1 j*l1 j*k2 l2],gn);
Ek1l1Hk2Hl2n2H=xmaln([k1 j*l1 j*k2 l2 j*n2],gn);
Ek1l1Hk2Hl2m2=xmaln([k1 j*l1 j*k2 l2 m2],gn);
Ek1l1Hk2Hl2m2n2H=xmaln([k1 j*l1 j*k2 l2 m2 j*n2],gn);

%T22121
Ek1l1Hn1=xmaln([k1 j*l1 n1],gn);
Ek1l1Hn1n2H=xmaln([k1 j*l1 n1 j*n2],gn);
Ek1l1Hn1m2=xmaln([k1 j*l1 n1 m2],gn);
Ek1l1Hn1m2n2H=xmaln([k1 j*l1 n1 m2 j*n2],gn);
Ek1l1Hn1l2=xmaln([k1 j*l1 n1 l2],gn);
Ek1l1Hn1l2n2H=xmaln([k1 j*l1 n1 l2 j*n2],gn);
Ek1l1Hn1l2m2=xmaln([k1 j*l1 n1 l2 m2],gn);
Ek1l1Hn1l2m2n2H=xmaln([k1 j*l1 n1 l2 m2 j*n2],gn);

%T22122
Ek1l1Hn1k2H=xmaln([k1 j*l1 n1 j*k2],gn);
Ek1l1Hn1k2Hn2H=xmaln([k1 j*l1 n1 j*k2 j*n2],gn);
Ek1l1Hn1k2Hm2=xmaln([k1 j*l1 n1 j*k2 m2],gn);
Ek1l1Hn1k2Hm2n2H=xmaln([k1 j*l1 n1 j*k2 m2 j*n2],gn);
Ek1l1Hn1k2Hl2=xmaln([k1 j*l1 n1 j*k2 l2],gn);
Ek1l1Hn1k2Hl2n2H=xmaln([k1 j*l1 n1 j*k2 l2 j*n2],gn);
Ek1l1Hn1k2Hl2m2=xmaln([k1 j*l1 n1 j*k2 l2 m2],gn);
```

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%T22211
Ek1l1Hm1H=xmaln([k1 j*l1 j*m1],gn);
Ek1l1Hm1Hn2H=xmaln([k1 j*l1 j*m1 j*n2],gn);
Ek1l1Hm1Hm2=xmaln([k1 j*l1 j*m1 m2],gn);
Ek1l1Hm1Hm2n2H=xmaln([k1 j*l1 j*m1 m2 j*n2],gn);
Ek1l1Hm1Hl2=xmaln([k1 j*l1 j*m1 l2],gn);
Ek1l1Hm1Hl2n2H=xmaln([k1 j*l1 j*m1 l2 j*n2],gn);
Ek1l1Hm1Hl2m2=xmaln([k1 j*l1 j*m1 l2 m2],gn);
Ek1l1Hm1Hl2m2n2H=xmaln([k1 j*l1 j*m1 l2 m2 j*n2],gn);

%T22212
Ek1l1Hm1Hk2H=xmaln([k1 j*l1 j*m1 j*k2],gn);
Ek1l1Hm1Hk2Hn2H=xmaln([k1 j*l1 j*m1 j*k2 j*n2],gn);
Ek1l1Hm1Hk2Hm2=xmaln([k1 j*l1 j*m1 j*k2 m2],gn);
Ek1l1Hm1Hk2Hm2n2H=xmaln([k1 j*l1 j*m1 j*k2 m2 j*n2],gn);
Ek1l1Hm1Hk2Hl2=xmaln([k1 j*l1 j*m1 j*k2 l2],gn);
Ek1l1Hm1Hk2Hl2n2H=xmaln([k1 j*l1 j*m1 j*k2 l2 j*n2],gn);
Ek1l1Hm1Hk2Hl2m2=xmaln([k1 j*l1 j*m1 j*k2 l2 m2],gn);

%T22221
Ek1l1Hm1Hn1=xmaln([k1 j*l1 j*m1 n1],gn);
Ek1l1Hm1Hn1n2H=xmaln([k1 j*l1 j*m1 n1 j*n2],gn);
Ek1l1Hm1Hn1m2=xmaln([k1 j*l1 j*m1 n1 m2],gn);
Ek1l1Hm1Hn1m2n2H=xmaln([k1 j*l1 j*m1 n1 m2 j*n2],gn);
Ek1l1Hm1Hn1l2=xmaln([k1 j*l1 j*m1 n1 l2],gn);
Ek1l1Hm1Hn1l2n2H=xmaln([k1 j*l1 j*m1 n1 l2 j*n2],gn);
Ek1l1Hm1Hn1l2m2=xmaln([k1 j*l1 j*m1 n1 l2 m2],gn);

%T22222
Ek1l1Hm1Hn1k2H=xmaln([k1 j*l1 j*m1 n1 j*k2],gn);
Ek1l1Hm1Hn1k2Hn2H=xmaln([k1 j*l1 j*m1 n1 j*k2 j*n2],gn);
Ek1l1Hm1Hn1k2Hm2=xmaln([k1 j*l1 j*m1 n1 j*k2 m2],gn);
Ek1l1Hm1Hn1k2Hl2=xmaln([k1 j*l1 j*m1 n1 j*k2 l2],gn);
Ek1l1Hm1Hn1k2Hl2m2n2H=xmaln([k1 j*l1 j*m1 n1 j*k2 l2 m2 j*n2],gn);

%Calculation of the modified steering matrix
A=[A sqrt(-1)*A];       %a_k(1:H)= A(k,:)

%Calculation of the source signal based Kronecker-products
%E2malS means E{s \kron s*}, E4malS means E{s \kron s* \kron s* \kron s} ...
z2=1;z3=1;z4=1;z5=1;z6=1;z8=1;
E2malS=ones(1,H^2);E3malS=ones(1,H^3);E4malS=ones(1,H^4);
E5malS=ones(1,H^5);E6malS=ones(1,H^6);E8malaS=zeros(1,H^8);
```

```
%Steering vector abbrevations
ak1=A(k1,:);
al1H=conj(A(l1,:));
am1H=conj(A(m1,:));
an1=A(n1,:);
ak2H=conj(A(k2,:));
al2=A(l2,:);
am2=A(m2,:);
an2H=conj(A(n2,:));

for a=1:H
 for b=1:H
  %Begin 2nd signal moment vector
  sig_ind=[sort([a b]) -1];
  momz2=1;
  for i=2:2+1
   if sig_ind(i)==sig_ind(i-1)
    momz2=momz2+1;
   else
    E2malS(z2)=E2malS(z2)*gs(sig_ind(i-1),momz2);
    momz2=1;
   end
  end
  z2=z2+1;
  %End 2nd signal moment vector
  for c=1:H
   [a b c]
   %Begin 3rd signal moment vector
   sig_ind=[sort([a b c]) -1];
   momz3=1;
   for i=2:3+1
    if sig_ind(i)==sig_ind(i-1)
     momz3=momz3+1;
    else
     E3malS(z3)=E3malS(z3)*gs(sig_ind(i-1),momz3);
     momz3=1;
    end
   end
   z3=z3+1;
   %End 3rd signal moment vector
   for d=1:H
    %Begin 4th signal moment vector
```

```
sig_ind=[sort([a b c d]) -1];
momz4=1;
for i=2:4+1
 if sig_ind(i)==sig_ind(i-1)
  momz4=momz4+1;
 else
  E4malS(z4)=E4malS(z4)*gs(sig_ind(i-1),momz4);
  momz4=1;
 end
end
z4=z4+1;
%End 4th signal moment vector
for e=1:H
 %Begin 5th signal moment vector
 sig_ind=[sort([a b c d e]) -1];
 momz5=1;
 for i=2:5+1
  if sig_ind(i)==sig_ind(i-1)
   momz5=momz5+1;
  else
   E5malS(z5)=E5malS(z5)*gs(sig_ind(i-1),momz5);
   momz5=1;
  end
 end
 z5=z5+1;
 %End 5th signal moment vector
 for f=1:H
  %Begin 6th signal moment vector
  sig_ind=[sort([a b c d e f]) -1];
  momz6=1;
  for i=2:6+1
   if sig_ind(i)==sig_ind(i-1)
    momz6=momz6+1;
   else
    E6malS(z6)=E6malS(z6)*gs(sig_ind(i-1),momz6);
    momz6=1;
  ]end
  end
  z6=z6+1;
  NEnd 6th signal moment vector
  for g=1:H
   for h=1:H
    %Begin 8th signal moment vector
```

```
        sig_ind=[sort([a b c d e f g h]) -1];
        momz8=1;
        E8malS=1;
        for i=2:8+1
         if sig_ind(i)==sig_ind(i-1)
          momz8=momz8+1;
         else
          E8malS=E8malS*gs(sig_ind(i-1),momz8);
          momz8=1;
         end
        end
        E8malaS(z8)=E8malS*ak1(a)*al1H(b)*am1H(c)*an1(d)*ak2H(e)*al2(f)...
                   *am2(g)*an2H(h);
        z8=z8+1;
        %End 8th signal moment vector
       end
      end
     end
    end
   end
  end
 end
end
E8malb=sum(E8malaS);

%Kronecker product abbrevations: Kk1l1h means (a_k1 \kron a_l1H) and so on...
%T11111
Kk1l1Hm1Hn1k2Hl2=kron(ak1,kron(al1H,kron(am1H,kron(an1,kron(ak2H,al2)))));
Kk1l1Hm1Hn1k2Hm2=kron(ak1,kron(al1H,kron(am1H,kron(an1,kron(ak2H,am2)))));
Kk1l1Hm1Hn1k2Hn2H=kron(ak1,kron(al1H,kron(am1H,kron(an1,kron(ak2H,an2H)))));
Kk1l1Hm1Hn1k2H=kron(ak1,kron(al1H,kron(am1H,kron(an1,ak2H))));

%T11112
Kk1l1Hm1Hn1l2m2=kron(ak1,kron(al1H,kron(am1H,kron(an1,kron(al2,am2)))));
Kk1l1Hm1Hn1l2n2H=kron(ak1,kron(al1H,kron(am1H,kron(an1,kron(al2,an2H)))));
Kk1l1Hm1Hn1l2=kron(ak1,kron(al1H,kron(am1H,kron(an1,al2))));
Kk1l1Hm1Hn1m2n2H=kron(ak1,kron(al1H,kron(am1H,kron(an1,kron(am2,an2H)))));
Kk1l1Hm1Hn1m2=kron(ak1,kron(al1H,kron(am1H,kron(an1,am2))));
Kk1l1Hm1Hn1n2H=kron(ak1,kron(al1H,kron(am1H,kron(an1,an2H))));
Kk1l1Hm1Hn1=kron(ak1,kron(al1H,kron(am1H,an1)));

%T11121
Kk1l1Hm1Hk2Hl2m2=kron(ak1,kron(al1H,kron(am1H,kron(ak2H,kron(al2,am2)))));
```

```
Kk1l1Hm1Hk2Hl2n2H=kron(ak1,kron(al1H,kron(am1H,kron(ak2H,kron(al2,an2H)))));
Kk1l1Hm1Hk2Hl2=kron(ak1,kron(al1H,kron(am1H,kron(ak2H,al2))));
Kk1l1Hm1Hk2Hm2n2H=kron(ak1,kron(al1H,kron(am1H,kron(an1,kron(am2,an2H)))));
Kk1l1Hm1Hk2Hm2=kron(ak1,kron(al1H,kron(am1H,kron(ak2H,am2))));
Kk1l1Hm1Hk2Hn2H=kron(ak1,kron(al1H,kron(am1H,kron(ak2H,an2H))));
Kk1l1Hm1Hk2H=kron(ak1,kron(al1H,kron(am1H,ak2H)));

%T11122
Kk1l1Hm1Hl2m2n2H=kron(ak1,kron(al1H,kron(am1H,kron(al2,kron(am2,an2H)))));
Kk1l1Hm1Hl2m2=kron(ak1,kron(al1H,kron(am1H,kron(al2,am2))));
Kk1l1Hm1Hl2n2H=kron(ak1,kron(al1H,kron(am1H,kron(al2,an2H))));
Kk1l1Hm1Hl2=kron(ak1,kron(al1H,kron(am1H,al2)));
Kk1l1Hm1Hm2n2H=kron(ak1,kron(al1H,kron(am1H,kron(am2,an2H))));
Kk1l1Hm1Hm2=kron(ak1,kron(al1H,kron(am1H,am2)));
Kk1l1Hm1Hn2H=kron(ak1,kron(al1H,kron(am1H,an2H)));
Kk1l1Hm1H=kron(ak1,kron(al1H,am1H));

%T11211
Kk1l1Hn1k2Hl2m2=kron(ak1,kron(al1H,kron(an1,kron(ak2H,kron(al2,am2)))));
Kk1l1Hn1k2Hl2n2H=kron(ak1,kron(al1H,kron(an1,kron(ak2H,kron(al2,an2H)))));
Kk1l1Hn1k2Hl2=kron(ak1,kron(al1H,kron(an1,kron(ak2H,al2))));
Kk1l1Hn1k2Hm2n2H=kron(ak1,kron(al1H,kron(an1,kron(ak2H,kron(am2,an2H)))));
Kk1l1Hn1k2Hm2=kron(ak1,kron(al1H,kron(an1,kron(ak2H,am2))));
Kk1l1Hn1k2Hn2H=kron(ak1,kron(al1H,kron(an1,kron(ak2H,an2H))));
Kk1l1Hn1k2H=kron(ak1,kron(al1H,kron(an1,ak2H)));

%T11212
Kk1l1Hn1l2m2n2H=kron(ak1,kron(al1H,kron(an1,kron(al2,kron(am2,an2H)))));
Kk1l1Hn1l2m2=kron(ak1,kron(al1H,kron(an1,kron(al2,am2))));
Kk1l1Hn1l2n2H=kron(ak1,kron(al1H,kron(an1,kron(al2,an2H))));
Kk1l1Hn1l2=kron(ak1,kron(al1H,kron(an1,al2)));
Kk1l1Hn1m2n2H=kron(ak1,kron(al1H,kron(an1,kron(am2,an2H))));
Kk1l1Hn1m2=kron(ak1,kron(al1H,kron(an1,am2)));
Kk1l1Hn1n2H=kron(ak1,kron(al1H,kron(an1,an2H)));
Kk1l1Hn1=kron(ak1,kron(al1H,an1));

%T11221
Kk1l1Hk2Hl2m2n2H=kron(ak1,kron(al1H,kron(ak2H,kron(al2,kron(am2,an2H)))));
Kk1l1Hk2Hl2m2=kron(ak1,kron(al1H,kron(ak2H,kron(al2,am2))));
Kk1l1Hk2Hl2n2H=kron(ak1,kron(al1H,kron(ak2H,kron(al2,an2H))));
Kk1l1Hk2Hl2=kron(ak1,kron(al1H,kron(ak2H,al2)));
Kk1l1Hk2Hm2n2H=kron(ak1,kron(al1H,kron(ak2H,kron(am2,an2H))));
Kk1l1Hk2Hm2=kron(ak1,kron(al1H,kron(ak2H,am2)));
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Kk1l1Hk2Hn2H=kron(ak1,kron(al1H,kron(ak2H,an2H)));
Kk1l1Hk2H=kron(ak1,kron(al1H,ak2H));

%T11222
Kk1l1Hl2m2n2H=kron(ak1,kron(al1H,kron(al2,kron(am2,an2H))));
Kk1l1Hl2m2=kron(ak1,kron(al1H,kron(al2,am2)));
Kk1l1Hl2n2H=kron(ak1,kron(al1H,kron(al2,an2H)));
Kk1l1Hl2=kron(ak1,kron(al1H,al2));
Kk1l1Hm2n2H=kron(ak1,kron(al1H,kron(am2,an2H)));
Kk1l1Hm2=kron(ak1,kron(al1H,am2));
Kk1l1Hn2H=kron(ak1,kron(al1H,an2H));
Kk1l1H=kron(ak1,al1H);

%T12111
Kk1m1Hn1k2Hl2m2=kron(ak1,kron(am1H,kron(an1,kron(ak2H,kron(al2,am2)))));
Kk1m1Hn1k2Hl2n2H=kron(ak1,kron(am1H,kron(an1,kron(ak2H,kron(al2,an2H)))));
Kk1m1Hn1k2Hl2=kron(ak1,kron(am1H,kron(an1,kron(ak2H,al2))));
Kk1m1Hn1k2Hm2n2H=kron(ak1,kron(am1H,kron(an1,kron(ak2H,kron(am2,an2H)))));
Kk1m1Hn1k2Hm2=kron(ak1,kron(am1H,kron(an1,kron(ak2H,am2))));
Kk1m1Hn1k2Hn2H=kron(ak1,kron(am1H,kron(an1,kron(ak2H,an2H))));
Kk1m1Hn1k2H=kron(ak1,kron(am1H,kron(an1,ak2H)));

%T12112
Kk1m1Hn1l2m2n2H=kron(ak1,kron(am1H,kron(an1,kron(al2,kron(am2,an2H)))));
Kk1m1Hn1l2m2=kron(ak1,kron(am1H,kron(an1,kron(al2,am2))));
Kk1m1Hn1l2n2H=kron(ak1,kron(am1H,kron(an1,kron(al2,an2H))));
Kk1m1Hn1l2=kron(ak1,kron(am1H,kron(an1,al2)));
Kk1m1Hn1m2n2H=kron(ak1,kron(am1H,kron(an1,kron(am2,an2H))));
Kk1m1Hn1m2=kron(ak1,kron(am1H,kron(an1,am2)));
Kk1m1Hn1n2H=kron(ak1,kron(am1H,kron(an1,an2H)));
Kk1m1Hn1=kron(ak1,kron(am1H,an1));

%T12121
Kk1m1Hk2Hl2m2n2H=kpon(ak1,kron(am1H,kron(ak2H,kron(al2,kron(am2,an2H)))));
Kk1m1Hk2Hl2m2=kron(ak1,kron(am1H,kron(ak2H,kron(al2,am2))));
Kk1m1Hk2Hl2n2H=kron(ak1,kron(am1H,kron(ak2H,kron(al2,an2H))));
Kk1m1Hk2Hl2=kron(ak1,kron(am1H,kron(ak2H,al2)));
Kk1m1Hk2Hm2n2H=kron(ak1,kron(am1H,kron(ak2H,kron(am2,an2H))));
Kk1m1Hk2Hm2=kron(ak1,kron(am1H,kron(ak2H,am2)));
Kk1m1Hk2Hn2H=kron(ak1,kron(am1H,kron(ak2H,an2H)));
Kk1m1Hk2H=kron(ak1,kron(am1H,ak2H));

%T12122
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Kk1m1Hl2m2n2H=kron(ak1,kron(am1H,kron(al2,kron(am2,an2H))));
Kk1m1Hl2m2=kron(ak1,kron(am1H,kron(al2,am2)));
Kk1m1Hl2n2H=kron(ak1,kron(am1H,kron(al2,an2H)));
Kk1m1Hl2=kron(ak1,kron(am1H,al2));
Kk1m1Hm2n2H=kron(ak1,kron(am1H,kron(am2,an2H)));
Kk1m1Hm2=kron(ak1,kron(am1H,am2));
Kk1m1Hn2H=kron(ak1,kron(am1H,an2H));
Kk1m1H=kron(ak1,am1H);

%T12211
 k1n1k2Hl2m2n2H=kron(ak1,kron(an1,kron(ak2H,kron(al2,kron(am2,an2H)))));
Kk1n1k2Hl2m2=kron(ak1,kron(an1,kron(ak2H,kron(al2,am2))));
Kk1n1k2Hl2n2H=kron(ak1,kron(an1,kron(ak2H,kron(al2,an2H))));
Kk1n1k2Hl2=kron(ak1,kron(an1,kron(ak2H,al2)));
Kk1n1k2Hm2n2H=kron(ak1,kron(an1,kron(ak2H,kron(am2,an2H))));
Kk1n1k2Hm2=kron(ak1,kron(an1,kron(ak2H,am2)));
Kk1n1k2Hn2H=kron(ak1,kron(an1,kron(ak2H,an2H)));
Kk1n1k2H=kron(ak1,kron(an1,ak2H));

%T12212
Kk1n1l2m2n2H=kron(ak1,kron(an1,kron(al2,kron(am2,an2H))));
Kk1n1l2m2=kron(ak1,kron(an1,kron(al2,am2)));
Kk1n1l2n2H=kron(ak1,kron(an1,kron(al2,an2H)));
Kk1n1l2=kron(ak1,kron(an1,al2));
Kk1n1m2n2H=kron(ak1,kron(an1,kron(am2,an2H)));
Kk1n1m2=kron(ak1,kron(an1,am2));
Kk1n1n2H=kron(ak1,kron(an1,an2H));
Kk1n1=kron(ak1,an1);

%T12221
Kk1k2Hl2m2n2H=kron(ak1,kron(ak2H,kron(al2,kron(am2,an2H))));
Kk1k2Hl2m2=kron(ak1,kron(ak2H,kron(al2,am2)));
Kk1k2Hl2n2H=kron(ak1,kron(ak2H,kron(al2,an2H)));
Kk1k2Hl2=kron(ak1,kron(ak2H,al2));
Kk1k2Hm2n2H=kron(ak1,kron(ak2H,kron(am2,an2H)));
Kk1k2Hm2=kron(ak1,kron(ak2H,am2));
Kk1k2Hn2H=kron(ak1,kron(ak2H,an2H));
Kk1k2H=kron(ak1,ak2H);

%T12222
Kk1l2m2n2H=kron(ak1,kron(al2,kron(am2,an2H)));
Kk1l2m2=kron(ak1,kron(al2,am2));
Kk1l2n2H=kron(ak1,kron(al2,an2H));
```

```
Kk1l2=kron(ak1,al2);
Kk1m2n2H=kron(ak1,kron(am2,an2H));
Kk1m2=kron(ak1,am2);
Kk1n2H=kron(ak1,an2H);

%T21111
Kl1Hm1Hn1k2Hl2m2=kron(al1H,kron(am1H,kron(an1,kron(ak2H,kron(al2,am2)))));
Kl1Hm1Hn1k2Hl2n2H=kron(al1H,kron(am1H,kron(an1,kron(ak2H,kron(al2,an2H)))));
Kl1Hm1Hn1k2Hl2=kron(al1H,kron(am1H,kron(an1,kron(ak2H,al2))));
Kl1Hm1Hn1k2Hm2n2H=kron(al1H,kron(am1H,kron(an1,kron(ak2H,kron(am2,an2H)))));
Kl1Hm1Hn1k2Hm2=kron(al1H,kron(am1H,kron(an1,kron(ak2H,am2))));
Kl1Hm1Hn1k2Hn2H=kron(al1H,kron(am1H,kron(an1,kron(ak2H,an2H))));
Kl1Hm1Hn1k2H=kron(al1H,kron(am1H,kron(an1,ak2H)));

%T21112
Kl1Hm1Hn1l2m2n2H=kron(al1H,kron(am1H,kron(an1,kron(al2,kron(am2,an2H)))));
Kl1Hm1Hn1l2m2=kron(al1H,kron(am1H,kron(an1,kron(al2,am2))));
Kl1Hm1Hn1l2n2H=2ron(al1H,kron(am1H,kron(an1,kron(al2,an2H))));
Kl1Hm1Hn1l2=kron(al1H,kron(am1H,kron(an1,al2)));
Kl1Hm1Hn1m2n2H=kron(al1H,kron(am1H,kron(an1,kron(am2,an2H))));
Kl1Hm1Hn1m2=kron(al1H,kron(am1H,kron(an1,am2)));
Kl1Hm1Hn1n2H=kron(al1H,kron(am1H,kron(an1,an2H)));
Kl1Hm1Hn1=kron(al1H,kron(am1H,an1));

%T21121
Kl1Hm1Hk2Hl2m2n2H=kron(al1H,kron(am1H,kron(ak2H,kron(al2,kron(am2,an2H)))));
Kl1Hm1Hk2Hl2m2=kron(al1H,kron(am1H,kron(ak2H,kron(al2,am2))));
Kl1Hm1Hk2Hl2n2H=kron(al1H,kron(am1H,kron(ak2H,kron(al2,an2H))));
Kl1Hm1Hk2Hl2=kron(al1H,kron(am1H,kron(ak2H,al2)));
Kl1Hm1Hk2Hm2n2H=kron(al1H,kron(am1H,kron(ak2H,kron(am2,an2H))));
Kl1Hm1Hk2Hm2=kron(al1H,kron(am1H,kron(ak2H,am2)));
Kl1Hm1Hk2Hn2H=kron(al1H,kron(am1H,kron(ak2H,an2H)));
Kl1Hm1Hk2H=kron(al1H,kron(am1H,ak2H));

%T21122
Kl1Hm1Hl2m2n2H=kron(al1H,kron(am1H,kron(al2,kron(am2,an2H))));
Kl1Hm1Hl2m2=kron(al1H,kron(am1H,kron(al2,am2)));
Kl1Hm1Hl2n2H=kron(al1H,kron(am1H,kron(al2,an2H)));
Kl1Hm1Hl2=kron(al1H,kron(am1H,al2));
Kl1Hm1Hm2n2H=kron(al1H,kron(am1H,kron(am2,an2H)));
Kl1Hm1Hm2=kron(al1H,kron(am1H,am2));
Kl1Hm1Hn2H=kron(al1H,kron(am1H,an2H));
Kl1Hm1H=kron(al1H,am1H);
```

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%T21211
Kl1Hn1k2Hl2m2n2H=kron(al1H,kron(an1,kron(ak2H,kron(al2,kron(am2,an2H)))));
Kl1Hn1k2Hl2m2=kron(al1H,kron(an1,kron(ak2H,kron(al2,am2))));
Kl1Hn1k2Hl2n2H=kron(al1H,kron(an1,kron(ak2H,kron(al2,an2H))));
Kl1Hn1k2Hl2=kron(al1H,kron(an1,kron(ak2H,al2)));
Kl1Hn1k2Hm2n2H=kron(al1H,kron(an1,kron(ak2H,kron(am2,an2H))));
Kl1Hn1k2Hm2=kron(al1H,kron(an1,kron(ak2H,am2)));
Kl1Hn1k2Hn2H=kron(al1H,kron(an1,kron(ak2H,an2H)));
Kl1Hn1k2H=kron(al1H,kron(an1,ak2H));

%T21212
Kl1Hn1l2m2n2H=kron(al1H,kron(an1,kron(al2,kron(am2,an2H))));
Kl1Hn1l2m2=kron(al1H,kron(an1,kron(al2,am2)));
Kl1Hn1l2n2H=kron(al1H,kron(an1,kron(al2,an2H)));
Kl1Hn1l2=kron(al1H,kron(an1,al2));
Kl1Hn1m2n2H=kron(al1H,kron(an1,kron(am2,an2H)));
Kl1Hn1m2=kron(al1H,kron(an1,am2));
Kl1Hn1n2H=kron(al1H,kron(an1,an2H));
Kl1Hn1=kron(al1H,an1);

%T21221
Kl1Hk2Hl2m2n2H=kron(al1H,kron(ak2H,kron(al2,kron(am2,an2H))));
Kl1Hk2Hl2m2=kron(al1H,kron(ak2H,kron(al2,am2)));
Kl1Hk2Hl2n2H=kron(al1H,kron(ak2H,kron(al2,an2H)));
Kl1Hk2Hl2=kron(al1H,kron(ak2H,al2));
Kl1Hk2Hm2n2H=kron(al1H,kron(ak2H,kron(am2,an2H)));
Kl1Hk2Hm2=kron(al1H,kron(ak2H,am2));
Kl1Hk2Hn2H=kron(al1H,kron(ak2H,an2H));
Kl1Hk2H=kron(al1H,ak2H);

%T21222
Kl1Hl2m2n2H=kron(al1H,kron(al2,kron(am2,an2H)));
Kl1Hl2m2=kron(al1H,kron(al2,am2));
Kl1Hl2n2H=kron(al1H,kron(al2,an2H));
Kl1Hl2=kron(al1H,al2);
Kl1Hm2n2H=kron(al1H,kron(am2,an2H));
Kl1Hm2=kron(al1H,am2);
Kl1Hn2H=kron(al1H,an2H);

%T22111
Km1Hn1k2Hl2m2n2H=kron(am1H,kron(an1,kron(ak2H,kron(al2,kron(am2,an2H)))));
Km1Hn1k2Hl2m2=kron(am1H,kron(an1,kron(ak2H,kron(al2,am2))));
```

```
Km1Hn1k2Hl2n2H=kron(am1H,kron(an1,kron(ak2H,kron(al2,an2H))));
Km1Hn1k2Hl2=kron(am1H,kron(an1,kron(ak2H,al2)));
Km1Hn1k2Hm2n2H=kron(am1H,kron(an1,kron(ak2H,kron(am2,an2H))));
Km1Hn1k2Hm2=kron(am1H,kron(an1,kron(ak2H,am2)));
Km1Hn1k2Hn2H=kron(am1H,kron(an1,kron(ak2H,an2H)));
Km1Hn1k2H=kron(am1H,kron(an1,ak2H));

%T22112
Km1Hn1l2m2n2H=kron(am1H,kron(an1,kron(al2,kron(am2,an2H))));
Km1Hn1l2m2=kron(am1H,kron(an1,kron(al2,am2)));
Km1Hn1l2n2H=kron(am1H,kron(an1,kron(al2,an2H)));
Km1Hn1l2=kron(am1H,kron(an1,al2));
Km1Hn1m2n2H=kron(am1H,kron(an1,kron(am2,an2H)));
Km1Hn1m2=kron(am1H,kron(an1,am2));
Km1Hn1n2H=kron(am1H,kron(an1,an2H));
Km1Hn1=kron(am1H,an1);

%T22121
Km1Hk2Hl2m2n2H=kron(am1H,kron(ak2H,kron(al2,kron(am2,an2H))));
Km1Hk2Hl2m2=kron(am1H,kron(ak2H,kron(al2,am2)));
Km1Hk2Hl2n2H=kron(am1H,kron(ak2H,kron(al2,an2H)));
Km1Hk2Hl2=kron(am1H,kron(ak2H,al2));
Km1Hk2Hm2n2H=kron(am1H,kron(ak2H,kron(am2,an2H)));
Km1Hk2Hm2=kron(am1H,kron(ak2H,am2));
Km1Hk2Hn2H=kron(am1H,kron(ak2H,an2H));
Km1Hk2H=kron(am1H,ak2H);

%T22122
Km1Hl2m2n2H=kron(am1H,kron(al2,kron(am2,an2H)));
Km1Hl2m2=kron(am1H,kron(al2,am2));
Km1Hl2n2H=kron(am1H,kron(al2,an2H));
Km1Hl2=kron(am1H,al2);
Km1Hm2n2H=kron(am1H,kron(am2,an2H));
Km1Hm2=kron(am1H,am2);
Km1Hn2H=kron(am1H,an2H);

%T22211
Kn1k2Hl2m2n2H=kron(an1,kron(ak2H,kron(al2,kron(am2,an2H))));
Kn1k2Hl2m2=kron(an1,kron(ak2H,kron(al2,am2)));
Kn1k2Hl2n2H=kron(an1,kron(ak2H,kron(al2,an2H)));
Kn1k2Hl2=kron(an1,kron(ak2H,al2));
Kn1k2Hm2n2H=kron(an1,kron(ak2H,kron(am2,an2H)));
Kn1k2Hm2=kron(an1,kron(ak2H,am2));
```

```
Kn1k2Hn2H=kron(an1,kron(ak2H,an2H));
Kn1k2H=kron(an1,ak2H);

%T22212
Kn1l2m2n2H=kron(an1,kron(al2,kron(am2,an2H)));
Kn1l2m2=kron(an1,kron(al2,am2));
Kn1l2n2H=kron(an1,kron(al2,an2H));
Kn1l2=kron(an1,al2);
Kn1m2n2H=kron(an1,kron(am2,an2H));
Kn1m2=kron(an1,am2);
Kn1n2H=kron(an1,an2H);

%T22221
Kk2Hl2m2n2H=kron(ak2H,kron(al2,kron(am2,an2H)));
Kk2Hl2m2=kron(ak2H,kron(al2,am2));
Kk2Hl2n2H=kron(ak2H,kron(al2,an2H));
Kk2Hl2=kron(ak2H,al2);
Kk2Hm2n2H=kron(ak2H,kron(am2,an2H));
Kk2Hm2=kron(ak2H,am2);
Kk2Hn2H=kron(ak2H,an2H);

%T22222
Kl2m2n2H=kron(al2,kron(am2,an2H));
Kl2m2=kron(al2,am2);
Kl2n2H=kron(al2,an2H);
Km2n2H=kron(am2,an2H);

%CALCULATION OF a)  See Section "Eigth-order Moment"
%
%       E{rk1 rl1H rm1H rn1 rk2H rl2 rm2 rn2H}
%
T11111=...
  E8malb+Kk1l1Hm1Hn1k2Hl2*E6malS'*Em2n2H+...
  Kk1l1Hm1Hn1k2Hm2*E6malS'*El2n2H+Kk1l1Hm1Hn1k2Hn2H*E6malS'*El2m2+...
  Kk1l1Hm1Hn1k2H*E5malS'*El2m2n2H;

T11112=...
  Kk1l1Hm1Hn1l2m2*E6malS'*Ek2Hn2H+Kk1l1Hm1Hn1l2n2H*E6malS'*Ek2Hm2+...
  Kk1l1Hm1Hn1l2*E5malS'*Ek2Hm2n2H+Kk1l1Hm1Hn1m2n2H*E6malS'*Ek2Hl2+...
  Kk1l1Hm1Hn1m2*E5malS'*Ek2Hl2n2H+Kk1l1Hm1Hn1n2H*E5malS'*Ek2Hl2m2+...
  Kk1l1Hm1Hn1*E4malS'*Ek2Hl2m2n2H;
```

```
T11121=...
  Kk1l1Hm1Hk2Hl2m2*E6malS'*En1n2H+Kk1l1Hm1Hk2Hl2n2H*E6malS'*En1m2+...
  Kk1l1Hm1Hk2Hl2*E5malS'*En1m2n2H+Kk1l1Hm1Hk2Hm2n2H*E6malS'*En1l2+...
  Kk1l1Hm1Hk2Hm2*E5malS'*En1l2n2H+Kk1l1Hm1Hk2Hn2H*E5malS'*En1l2m2+...
  Kk1l1Hm1Hk2H*E4malS'*En1l2m2n2H;

T11122=...
  Kk1l1Hm1Hl2m2n2H*E6malS'*En1k2H+Kk1l1Hm1Hl2m2*E5malS'*En1k2Hn2H+...
  Kk1l1Hm1Hl2n2H*E5malS'*En1k2Hm2+Kk1l1Hm1Hl2*E4malS'*En1k2Hm2n2H+...
  Kk1l1Hm1Hm2n2H*E5malS'*En1k2Hl2+Kk1l1Hm1Hm2*E4malS'*En1k2Hl2n2H+...
  Kk1l1Hm1Hn2H*E4malS'*En1k2Hl2m2+Kk1l1Hm1H*E3malS'*En1k2Hl2m2n2H;

T11211=...
  Kk1l1Hn1k2Hl2m2*E6malS'*Em1Hn2H+Kk1l1Hn1k2Hl2n2H*E6malS'*Em1Hm2+...
  Kk1l1Hn1k2Hl2*E5malS'*Em1Hm2n2H+Kk1l1Hn1k2Hm2n2H*E6malS'*Em1Hl2+...
  Kk1l1Hn1k2Hm2*E5malS'*Em1Hl2n2H+Kk1l1Hn1k2Hn2H*E5malS'*Em1Hl2m2+...
  Kk1l1Hn1k2H*E4malS'*Em1Hl2m2n2H;

T11212=...
  Kk1l1Hn1l2m2n2H*E6malS'*Em1Hk2H+Kk1l1Hn1l2m2*E5malS'*Em1Hk2Hn2H+...
  Kk1l1Hn1l2n2H*E5malS'*Em1Hk2Hm2+Kk1l1Hn1l2*E4malS'*Em1Hk2Hm2n2H+...
  Kk1l1Hn1m2n2H*E5malS'*Em1Hk2Hl2+Kk1l1Hn1m2*E4malS'*Em1Hk2Hl2n2H+...
  Kk1l1Hn1n2H*E4malS'*Em1Hk2Hl2m2+Kk1l1Hn1*E3malS'*Em1Hk2Hl2m2n2H;

T11221=...
  Kk1l1Hk2Hl2m2n2H*E6malS'*Em1Hn1+Kk1l1Hk2Hl2m2*E5malS'*Em1Hn1n2H+...
  Kk1l1Hk2Hl2n2H*E5malS'*Em1Hn1m2+Kk1l1Hk2Hl2*E4malS'*Em1Hn1m2n2H+...
  Kk1l1Hk2Hm2n2H*E5malS'*Em1Hn1l2+Kk1l1Hk2Hm2*E4malS'*Em1Hn1l2n2H+...
  Kk1l1Hk2Hn2H*E4malS'*Em1Hn1l2m2+Kk1l1Hk2H*E3malS'*Em1Hn1l2m2n2H;

T11222=...
  Kk1l1Hl2m2n2H*E5malS'*Em1Hn1k2H+Kk1l1Hl2m2*E4malS'*Em1Hn1k2Hn2H+...
  Kk1l1Hl2n2H*E4malS'*Em1Hn1k2Hm2+Kk1l1Hl2*E3malS'*Em1Hn1k2Hm2n2H+...
  Kk1l1Hm2n2H*E4malS'*Em1Hn1k2Hl2+Kk1l1Hm2*E3malS'*Em1Hn1k2Hl2n2H+...
  Kk1l1Hn2H*E3malS'*Em1Hn1k2Hl2m2+Kk1l1H*E2malS'*Em1Hn1k2Hl2m2n2H;

T12111=...
  Kk1m1Hn1k2Hl2m2*E6malS'*El1Hn2H+Kk1m1Hn1k2Hl2n2H*E6malS'*El1Hm2+...
  Kk1m1Hn1k2Hl2*E5malS'*El1Hm2n2H+Kk1m1Hn1k2Hm2n2H*E6malS'*El1Hl2+...
  Kk1m1Hn1k2Hm2*E5malS'*El1Hl2n2H+Kk1m1Hn1k2Hn2H*E5malS'*El1Hl2m2+...
  Kk1m1Hn1k2H*E4malS'*El1Hl2m2n2H;

T12112=...
```

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  Kk1m1Hn1l2m2n2H*E6malS'*El1Hk2H+Kk1m1Hn1l2m2*E5malS'*El1Hk2Hn2H+...
  Kk1m1Hn1l2n2H*E5malS'*El1Hk2Hm2+Kk1m1Hn1l2*E4malS'*El1Hk2Hm2n2H+...
  Kk1m1Hn1m2n2H*E5malS'*El1Hk2Hl2+Kk1m1Hn1m2*E4malS'*El1Hk2Hl2n2H+...
  Kk1m1Hn1n2H*E4malS'*El1Hk2Hl2m2+Kk1m1Hn1*E3malS'*El1Hk2Hl2m2n2H;

T12121=...
  Kk1m1Hk2Hl2m2n2H*E6malS'*El1Hn1+Kk1m1Hk2Hl2m2*E5malS'*El1Hn1n2H+...
  Kk1m1Hk2Hl2n2H*E5malS'*El1Hn1m2+Kk1m1Hk2Hl2*E4malS'*El1Hn1m2n2H+...
  Kk1m1Hk2Hm2n2H*E5malS'*El1Hn1l2+Kk1m1Hk2Hm2*E4malS'*El1Hn1l2n2H+...
  Kk1m1Hk2Hn2H*E4malS'*El1Hn1l2m2+Kk1m1Hk2H*E3malS'*El1Hn1l2m2n2H;

T12122=...
  Kk1m1Hl2m2n2H*E5malS'*El1Hn1k2H+Kk1m1Hl2m2*E4malS'*El1Hn1k2Hn2H+...
  Kk1m1Hl2n2H*E4malS'*El1Hn1k2Hm2+Kk1m1Hl2*E3malS'*El1Hn1k2Hm2n2H+...
  Kk1m1Hm2n2H*E4malS'*El1Hn1k2Hl2+Kk1m1Hm2*E3malS'*El1Hn1k2Hl2n2H+...
  Kk1m1Hn2H*E3malS'*El1Hn1k2Hl2m2+Kk1m1H*E2malS'*El1Hn1k2Hl2m2n2H;

T12211=...
  Kk1n1k2Hl2m2n2H*E6malS'*El1Hm1H+Kk1n1k2Hl2m2*E5malS'*El1Hm1Hn2H+...
  Kk1n1k2Hl2n2H*E5malS'*El1Hm1Hm2+Kk1n1k2Hl2*E4malS'*El1Hm1Hm2n2H+...
  Kk1n1k2Hm2n2H*E5malS'*El1Hm1Hl2+Kk1n1k2Hm2*E4malS'*El1Hm1Hl2n2H+...
  Kk1n1k2Hn2H*E4malS'*El1Hm1Hl2m2+Kk1n1k2H*E3malS'*El1Hm1Hl2m2n2H;

T12212=...
  Kk1n1l2m2n2H*E5malS'*El1Hm1Hk2H+Kk1n1l2m2*E4malS'*El1Hm1Hk2Hn2H+...
  Kk1n1l2n2H*E4malS'*El1Hm1Hk2Hm2+Kk1n1l2*E3malS'*El1Hm1Hk2Hm2n2H+...
  Kk1n1m2n2H*E4malS'*El1Hm1Hk2Hl2+Kk1n1m2*E3malS'*El1Hm1Hk2Hl2n2H+...
  Kk1n1n2H*E3malS'*El1Hm1Hk2Hl2m2+Kk1n1*E2malS'*El1Hm1Hk2Hl2m2n2H;

T12221=...
  Kk1k2Hl2m2n2H*E5malS'*El1Hm1Hn1+Kk1k2Hl2m2*E4malS'*El1Hm1Hn1n2H+...
  Kk1k2Hl2n2H*E4malS'*El1Hm1Hn1m2+Kk1k2Hl2*E3malS'*El1Hm1Hn1m2n2H+...
  Kk1k2Hm2n2H*E4malS'*El1Hm1Hn1l2+Kk1k2Hm2*E3malS'*El1Hm1Hn1l2n2H+...
  Kk1k2Hn2H*E3malS'*El1Hm1Hn1l2m2+Kk1k2H*E2malS'*El1Hm1Hn1l2m2n2H;

T12222=...
  Kk1l2m2n2H*E4malS'*El1Hm1Hn1k2H+Kk1l2m2*E3malS'*El1Hm1Hn1k2Hn2H+...
  Kk1l2n2H*E3malS'*El1Hm1Hn1k2Hm2+Kk1l2*E2malS'*El1Hm1Hn1k2Hm2n2H+...
  Kk1m2n2H*E3malS'*El1Hm1Hn1k2Hl2+Kk1m2*E2malS'*El1Hm1Hn1k2Hl2n2H+...
  Kk1n2H*E2malS'*El1Hm1Hn1k2Hl2m2;

T21111=...
  Kl1Hm1Hn1k2Hl2m2*E6malS'*Ek1n2H+Kl1Hm1Hn1k2Hl2n2H*E6malS'*Ek1m2+...
```

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  Kl1Hm1Hn1k2Hl2*E5malS'*Ek1m2n2H+Kl1Hm1Hn1k2Hm2n2H*E6malS'*Ek1l2+...
  Kl1Hm1Hn1k2Hm2*E5malS'*Ek1l2n2H+Kl1Hm1Hn1k2Hn2H*E5malS'*Ek1l2m2+...
  Kl1Hm1Hn1k2H*E4malS'*Ek1l2m2n2H;

T21112=...
  Kl1Hm1Hn1l2m2n2H*E6malS'*Ek1k2H+Kl1Hm1Hn1l2m2*e5malS'*Ek1k2Hn2H+...
  Kl1Hm1Hn1l2n2H*E5malS'*Ek1k2Hm2+Kl1Hm1Hn1l2*E4malS'*Ek1k2Hm2n2H+...
  Kl1Hm1Hn1m2n2H*E5malS'*Ek1k2Hl2+Kl1Hm1Hn1m2*E4malS'*Ek1k2Hl2n2H+...
  Kl1Hm1Hn1n2H*E4malS'*Ek1k2Hl2m2+Kl1Hm1Hn1*E3malS'*Ek1k2Hl2m2n2H;

T21121=...
  Kl1Hm1Hk2Hl2m2n2H*E6malS'*Ek1n1+Kl1Hm1Hk2Hl2m2*E5malS'*Ek1n1n2H+...
  Kl1Hm1Hk2Hl2n2H*E5malS'*Ek1n1m2+Kl1Hm1Hk2Hl2*E4malS'*Ek1n1m2n2H+...
  Kl1Hm1Hk2Hm2n2H*E5malS'*Ek1n1l2+Kl1Hm1Hk2Hm2*E4malS'*Ek1n1l2n2H+...
  Kl1Hm1Hk2Hn2H*E4malS'*Ek1n1l2m2+Kl1Hm1Hk2H*E3malS'*Ek1n1l2m2n2H;

T21122=...
  Kl1Hm1Hl2m2n2H*E5malS'*Ek1n1k2H+Kl1Hm1Hl2m2*E4malS'*Ek1n1k2Hn2H+...
  Kl1Hm1Hl2n2H*E4malS'*Ek1n1k2Hm2+Kl1Hm1Hl2*E3malS'*Ek1n1k2Hm2n2H+...
  Kl1Hm1Hm2n2H*E4malS'*Ek1n1k2Hl2+Kl1Hm1Hm2*E3malS'*Ek1n1k2Hl2n2H+...
  Kl1Hm1Hn2H*E3malS'*Ek1n1k2Hl2m2+Kl1Hm1H*E2malS'*Ek1n1k2Hl2m2n2H;

T21211=...
  Kl1Hn1k2Hl2m2n2H*E6malS'*Ek1m1H+Kl1Hn1k2Hl2m2*E5malS'*Ek1m1Hn2H+...
  Kl1Hn1k2Hl2n2H*E5malS'*Ek1m1Hm2+Kl1Hn1k2Hl2*E4malS'*Ek1O1Hm2n2H+...
  Kl1Hn1k2Hm2n2H*E5malS'*Ek1m1Hl2+Kl1Hn1k2Hm2*E4malS'*Ek1m1Hl2n2H+...
  Kl1Hn1k2Hn2H*E4malS'*Ek1m1Hl2m2+Kl1Hn1k2H*E3malS'*Ek1m1Hl2m2n2H;

T21212=...
  Kl1Hn1l2m2n2H*E5malS'*Ek1m1Hk2H+Kl1Hn1l2m2*E4malS'*Ek1m1Hk2Hn2H+...
  Kl1Hn1l2n2H*E4malS'*Ek1m1Hk2Hm2+Kl1Hn1l2*E3malS'*Ek1m1Hk2Hm2n2H+...
  Kl1Hn1m2n2H*E4malS'*Ek1m1Hk2Hl2+Kl1Hn1m2*E3malS'*Ek1m1Hk2Hl2n2H+...
  Kl1Hn1n2H*E3malS'*Ek1m1Hk2Hl2m2+Kl1Hn1*E2malS'*Ek1m1Hk2Hl2m2n2H;

T21221=...
  Kl1Hk2Hl2m2n2H*E5malS'*Ek1m1Hn1+Kl1Hk2Hl2m2*E4malS'*Ek1m1Hn1n2H+...
  Kl1Hk2Hl2n2H*E4malS'*Ek1m1Hn1m2+Kl1Hk2Hl2*E3malS'*Ek1m1Hn1m2n2H+...
  Kl1Hk2Hm2n2H*E4malS'*Ek1m1Hn1l2+Kl1Hk2Hm2*E3malS'*Ek1m1Hn1l2n2H+...
  Kl1Hk2Hn2H*E3malS'*Ek1m1Hn1l2m2+Kl1Hk2H*E2malS'*Ek1m1Hn1l2m2n2H;

T21222=...
  Kl1Hl2m2n2H*E4malS'*Ek1m1Hn1k2H+Kl1Hl2m2*E3malS'*Ek1m1Hn1k2Hn2H+...
  Kl1Hl2n2H*E3malS'*Ek1m1Hn1k2Hm2+Kl1Hl2*E2malS'*Ek1m1Hn1k2Hm2n2H+...
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  Kl1Hm2n2H*E3malS'*Ek1m1Hn1k2Hl2+Kl1Hm2*E2malS'*Ek1m1Hn1k2Hl2n2H+...
  Kl1Hn2H*E2malS'*Ek1m1Hn1k2Hl2m2;

T22111=...
  Km1Hn1k2Hl2m2n2H*E6malS'*Ek1l1H+Km1Hn1k2Hl2m2*E5malS'*Ek1l1Hn2H+...
  Km1Hn1k2Hl2n2H*E5malS'*Ek1l1Hm2+Km1Hn1k2Hl2*E4malS'*Ek1l1Hm2n2H+...
  Km1Hn1k2Hm2n2H*E5malS'*Ek1l1Hl2+Km1Hn1k2Hm2*E4malS'*Ek1l1Hl2n2H+...
  Km1Hn1k2Hn2H*E4malS'*Ek1l1Hl2m2+Km1Hn1k2H*E3malS'*Ek1l1Hl2m2n2H;

T22112=...
  Km1Hn1l2m2n2H*E5malS'*Ek1l1Hk2H+Km1Hn1l2m2*E4malS'*Ek1l1Hk2Hn2H+...
  Km1Hn1l2n2H*E4malS'*Ek1l1Hk2Hm2+Km1Hn1l2*E3malS'*Ek1l1Hk2Hm2n2H+...
  Km1Hn1m2n2H*E4malS'*Ek1l1Hk2Hl2+Km1Hn1m2*E3malS'*Ek1l1Hk2Hl2n2H+...
  Km1Hn1n2H*E3malS'*Ek1l1Hk2Hl2m2+Km1Hn1*E2malS'*Ek1l1Hk2Hl2m2n2H;

T22121=...
  Km1Hk2Hl2m2n2H*E5malS'*Ek1l1Hn1+Km1Hk2Hl2m2*E4malS'*Ek1l1Hn1n2H+...
  Km1Hk2Hl2n2H*E4malS'*Ek1l1Hn1m2+Km1Hk2Hl2*E3malS'*Ek1l1Hn1m2n2H+...
  Km1Hk2Hm2n2H*E4malS'*Ek1l1Hn1l2+Km1Hk2Hm2*E3malS'*Ek1l1Hn1l2n2H+...
  Km1Hk2Hn2H*E3malS'*Ek1l1Hn1l2m2+Km1Hk2H*E2malS'*Ek1l1Hn1l2m2n2H;

T22122=...
  Km1Hl2m2n2H*E4malS'*Ek1l1Hn1k2H+Km1Hl2m2*E3malS'*Ek1l1Hn1k2Hn2H+...
  Km1Hl2n2H*E3malS'*Ek1l1Hn1k2Hm2+Km1Hl2*E2malS'*Ek1l1Hn1k2Hm2n2H+...
  Km1Hm2n2H*E3malS'*Ek1l1Hn1k2Hl2+Km1Hm2*E2malS'*Ek1l1Hn1k2Hl2n2H+...
  Km1Hn2H*E2malS'*Ek1l1Hn1k2Hl2m2;

T22211=...
  Kn1k2Hl2m2n2H*E5malS'*Ek1l1Hm1H+Kn1k2Hl2m2*E4malS'*Ek1l1Hm1Hn2H+...
  Kn1k2Hl2n2H*E4malS'*Ek1l1Hm1Hm2+Kn1k2Hl2*E3malS'*Ek1l1Hm1Hm2n2H+...
  Kn1k2Hm2n2H*E4malS'*Ek1l1Hm1Hl2+Kn1k2Hm2*E3malS'*Ek1l1Hm1Hl2n2H+...
  Kn1k2Hn2H*E3malS'*Ek1l1Hm1Hl2m2+Kn1k2H*E2malS'*Ek1l1Hm1Hl2m2n2H;

T222_2=...
  Kn2l2m2n2H*E4malS'* k1m1H21Hk2H+Kn1l2m2*E1malS'*Ek1l1Hm1Hk2Hn2H+...
  Kn1l2n2H*E3malS'*Ek1l1Hm1Hk2Hm2+Kn1l2*E2malS'*Ek1l1Hm1Hk2Hm2n2H+...
  Kn1m2n2H*E3malS'*Ek1l1Hm1Hk2Hl2+Kn1m2*E2malS'*Ek1l1Hm1Hk2Hl2n2H+...
  Kn1n2H*E2malS'*Ek1l1Hm1Hk2Hl2m2;

T22221=...
  Kk2Hl2m2n2H*E4malS'*Ek1l1Hm1Hn1+Kk2Hl2m2*E3malS'*Ek1l1Hm1Hn1n2H+...
  Kk2Hl2n2H*E3malS'*Ek1l1Hm1Hn1m2+Kk2Hl2*E2malS'*Ek1l1Hm1Hn1m2n2H+...
  Kk2Hm2n2H*E3malS'*Ek1l1Hm1Hn1l2+Kk2Hm2*E2malS'*Ek1l1Hm1Hn1l2n2H+...
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  Kk2Hn2H*E2malS'*Ek1l1Hm1Hn1l2m2;

T22222=...
  Kl2m2n2H*E3malS'*Ek1l1Hm1Hn1k2H+Kl2m2*E2malS'*Ek1l1Hm1Hn1k2Hn2H+...
  Kl2n2H*E2malS'*Ek1l1Hm1Hn1k2Hm2+Km2n2H*E2malS'*Ek1l1Hm1Hn1k2Hl2+...
  Ek1l1Hm1Hn1k2Hl2m2n2H;

Rk1l1Hm1Hn1k2Hl2m2n2H= T11111+T11112+T11121+T11122+...
                       T11211+T11212+T11221+T11222+...
                       T12111+T12112+T12121+T12122+...
                       T12211+T12212+T12221+T12222+...
                       T21111+T21112+T21121+T21122+...
                       T21211+T21212+T21221+T21222+...
                       T22111+T22112+T22121+T22122+...
                       T22211+T22212+T22221+T22222;

%CALCULATION OF b) See Section "Sixth-order Moment"
%
%      E{rk1 rl1H rm1H rn1 rk2H rl2}
%
T111=...
Kk1l1Hm1Hn1k2Hl2*E6malS'+Kk1l1Hm1Hn1*E4malS'*Ek2Hl2 +...
Kk1l1Hm1Hk2H*E4malS'*En1l2+Kk1l1Hm1Hl2*E4malS'*En1k2H +...
Kk1l1Hm1H*E3malS'*En1k2Hl2;

T112=...
Kk1l1Hn1k2H*E4malS'*Em1Hl2+Kk1l1Hn1l2*E4malS'*Em1Hk2H +...
Kk1l1Hn1*E3malS'*Em1Hk2Hl2+Kk1l1Hk2Hl2*E4malS'*Em1Hn1 +...
Kk1l1Hk2H*E3malS'*Em1Hn1l2+Kk1l1Hl2*E3malS'*Em1Hn1k2H +...
Kk1l1H*E2malS'*Em1Hn1k2Hl2;

T121=...
Kk1m1Hn1k2H*E4malS'*El1Hl2+Kk1m1Hn1l2*E4malS'*El1Hk2H +...
Kk1m1Hn1*E3malS'*El1Hk2Hl2+Kk1m1Hk2Hl2*E4malS'*El1Hn1 +...
Kk1m1Hk2H*E3malS'*El1Hn1l2+Kk1m1Hl2*E3malS'*El1Hn1k2H +...
Kk1m1H*E2malS'*El1Hn1k2Hl2;

T122=...
Kk1n1k2Hl2*E4malS'*El1Hm1H+Kk1n1k2H*E3malS'*El1Hm1Hl2 +...
Kk1n1l2*E3malS'*El1Hm1Hk2H+Kk1n1wE2malS'*El1Hm1H 2Hl2 +...
Kk1k2_l2HE3 alS'*El1Hm1Hn1+{k1k2H*E2malS'*El1Hm1Hn1l2 +...
Kk1l2*E2malS'*El1Hm1Hn1k2H;
```

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T211=...
Kl1Hm1Hn1k2H*E4malS'*Ek1l2+Kl1Hm1Hn1l2*E4malS'*Ek1k2H +...
Kl1Hm1Hn1*E3malS'*Ek1k2Hl2+Kl1Hm1Hk2Hl2*E4malS'*Ek1n1 +...
Kl1Hm1Hk2H*E3malS'*Ek1n1l2+Kl1Hm1Hl2*E3malS'*Ek1n1k2H +...
Kl1Hm1H*E2malS'*Ek1n1k2Hl2;

T212=...
Kl1Hn1k2Hl2*E4malS'*Ek1m1H+Kl1Hn1k2H*E3malS'*Ek1m1Hl2 +...
Kl1Hn1l2*E3malS'*Ek1m1Hk2H+Kl1Hn1*E2malS'*Ek1m1Hk2Hl2 +...
Kl1Hk2Hl2*E3malS'*Ek1m1Hn1+Kl1Hk2H*E2malS'*Ek1m1Hn1l2 +...
Kl1Hl2*E2malS'*Ek1m1Hn1k2H;

T221=...
Km1Hn1k2Hl2*E4malS'*Ek1l1H+Km1Hn1k2H*E3malS'*Ek1l1Hl2 +...
Km1Hn1l2*E3malS'*Ek1l1Hk2H+Km1Hn1*E2malS'*Ek1l1Hk2Hl2 +...
Km1Hk2Hl2*E3malS'*Ek1l1Hn1+Km1Hk2H*E2malS'*Ek1l1Hn1l2 +...
Km1Hl2*E2malS'*Ek1l1Hn1k2H;

T222=...
Kn1k2Hl2*E3malS'*Ek1l1Hm1H+Kn1k2H*E2malS'*Ek1l1Hm1Hl2 +...
Kn1l2*E2malS'*Ek1l1Hm1Hk2H+Kk2Hl2*E2malS'*Ek1l1Hm1Hn1 +...
Ek1l1Hm1Hn1k2Hl2;

Rk1l1Hm1Hn1k2Hl2=T111+T112mT121+T122+T211+T212+T221+T222;

%CALCULATION OF b2) See Section "Sixth-order Moment"
%
%       E{rk1 rl1H rm1H rn1 rm2 rn2H}
%
T111=...
Kk1l1Hm1Hn1m2n2H*E6malS'+Kk1l1Hm1Hn1*E4malS'*Em2n2H +...
Kk1l1Hm1Hm2*E4malS'*En1n2H+Kk1l1Hm1Hn2H*E4malS'*En1m2 +...
Kk1l1Hm1H*E3malS'*En1m2n2H;

T112=...
Kk1l1Hn1m2*E4malS'*Em1Hn2H+Kk1l1Hn1n2H*E4malS'*Em1Hm2 +...
Kk1l1Hn1*E3malS'*Em1Hm2n2H+Kk1l1Hm2n2H*E4malS'*Em1Hn1 +...
Kk1l1Hm2*E3malS'*Em1Hn1n2H+Kk1l1Hn2H*E3malS'*Em1Hn1m2 +...
Kk1l1H*E2malS'*Em1Hn1m2n2H;

T121=...
Kk1m1Hn1m2*E4malS'*El1Hn2H+Kk1m1Hn1n2H*E4malS'*El1Hm2 +...
Kk1m1Hn1*E3malS'*El1Hm2n2H+Kk1m1Hm2n2H*E4malS'*El1Hn1 +...
```

```
Kk1m1Hm2*E3malS'*El1Hn1n2H+Kk1m1Hn2H*E3malS'*El1Hn1m2 +...
Kk1m1H*E2malS'*El1Hn1m2n2H;

T122=...
Kk1n1m2n2H*E4malS'*El1Hm1H+Kk1n1m2*E3malS'*El1Hm1Hn2H +...
Kk1n1n2H*E3malS'*El1Hm1Hm2+Kk1n1*E2malS'*El1Hm1Hm2n2H +...
Kk1m2n2H*E3malS'*El1Hm1Hn1+Kk1m2*E2malS'*El1Hm1Hn1n2H +...
Kk1n2H*E2malS'*El1Hm1Hn1m2;

T211=...
Kl1Hm1Hn1m2*E4malS'*Ek1n2H+Kl1Hm1Hn1n2H*E4malS'*Ek1m2 +...
Kl1Hm1Hn1*E3malS'*Ek1m2n2H+Kl1Hm1Hm2n2H*E4malS'*Ek1n1 +...
Kl1Hm1Hm2*E3malS'*Ek1n1n2H+Kl1Hm1Hn2H*E3malS'*Ek1n1m2 +...
Kl1Hm1H*E2malS'*Ek1n1m2n2H;

T212=...
Kl1Hn1m2n2H*E4malS'*Ek1m1H+Kl1Hn1m2*E3malS'*Ek1m1Hn2H +...
Kl1Hn1n2H*E3malS'*Ek1m1Hm2+Kl1Hn1*E2malS'*Ek1m1Hm2n2H +...
Kl1Hm2n2H*E3malS'*Ek1m1Hn1+Kl1Hm2*E2malS'*Ek1m1Hn1n2H +...
Kl1Hn2H*E2malS'*Ek1m1Hn1m2;

T221=...
Km1Hn1m2n2H*E4malS'*Ek1l1H+Km1Hn1m2*E3malS'*Ek1l1Hn2H +...
Km1Hn1n2H*E3malS'*Ek1l1Hm2+Km1Hn1*E2malS'*Ek1l1Hm2n2H +...
Km1Hm2n2H*E3malS'*Ek1l1Hn1+Km1Hm2*E2malS'*Ek1l1Hn1n2H +...
Km1Hn2H*E2malS'*Ek1l1Hn1m2;

T222=...
Kn1m2n2H*E3malS'*Ek1l1Hm1H+Kn1m2*E2malS'*Ek1l1Hm1Hn2H +...
Kn1n2H*E2malS'*Ek1l1Hm1Hm2+Km2n2H*E2malS'*Ek1l1Hm1Hn1 +...
Ek1l1Hm1Hn1m2n2H;

Rk1l1Hm1Hn1m2n2H=T111+T112+T121+T122+T211+T212+T221+T222;

%CALCULATION OF b3) See Section "Sixth-order Moment"
%
%      E{rk1 rl1H lm1H rn1 rk2H rm2}
%
T111=...
Kk1l1Hm1Hn1k2Hm2*E6malS'+Kk1l1Hm1Hn1*E4malS'*Ek2Hm2 +...
Kk1l1Hm1Hk2H*E4malS'*En1m2+Kk1l1Hm1Hm2*E4malS'*En1k2H +...
Kk1l1Hm1H*E3malS'*En1k2Hm2;
```

```
T112=...
Kk1l1Hn1k2H*E4malS'*Em1Hm2+Kk1l1Hn1m2*E4malS'*Em1Hk2H +...
Kk1l1Hn1*E3malS'*Em1Hk2Hm2+Kk1l1Hk2Hm2*E4malS'*Em1Hn1 +...
Kk1l1Hk2H*E3malS'*Em1Hn1m2+Kk1l1Hm2*E3malS'*Em1Hn1k2H +...
Kk1l1H*E2malS'*Em1Hn1k2Hm2;

T121=...
Kk1m1Hn1k2H*E4malS'*El1Hm2+Kk1m1Hn1m2*E4malS'*El1Hk2H +...
Kk1m1Hn1*E3malS'*El1Hk2Hm2+Kk1m1Hk2Hm2*E4malS'*El1Hn1 +...
Kk1m1Hk2H*E3malS'*El1Hn1m2+Kk1m1Hm2*E3malS'*El1Hn1k2H +...
Kk1m1H*E2malS'*El1Hn1k2Hm2;

T122=...
Kk1n1k2Hm2*E4malS'*El1Hm1H+Kk1n1k2H*E3malS'*El1Hm1Hm2 +...
Kk1n1m2*E3malS'*El1Hm1Hk2H+Kk1n1*E2malS'*El1Hm1Hk2Hm2 +...
Kk1k2Hm2*E3malS'*El1Hm1Hn1+Kk1k2H*E2malS'*El1Hm1Hn1m2 +...
Kk1m2*E2malS'*El1Hm1Hn1k2H;

T211=...
Kl1Hm1Hn1k2H*E4malS'*Ek1m2+Kl1Hm1Hn1m2*E4malS'*Ek1k2H +...
Kl1Hm1Hn1*E3malS'*Ek1k2Hm2+Kl1Hm1Hk2Hm2*E4malS'*Ek1n1 +...
Kl1Hm1Hk2H*E3malS'*Ek1n1m2+Kl1Hm1Hm2*E3malS'*Ek1n1k2H +...
Kl1Hm1H*E2malS'*Ek1n1k2Hm2;

T212=...
Kl1Hn1k2Hm2*E4malS'*Ek1m1H+Kl1Hn1k2H*E3malS'*Ek1m1Hm2 +...
Kl1Hn1m2*E3malS'*Ek1m1Hk2H+Kl1Hn1*E2malS'*Ek1m1Hk2Hm2 +...
Kl1Hk2Hm2*E3malS'*Ek1m1Hn1+Kl1Hk2H*E2malS'*Ek1m1Hn1m2 +...
Kl1Hm2*E2malS'*Ek1m1Hn1k2H;

T221=...
Km1Hn1k2Hm2*E4malS'*Ek1l1H+Km1Hn1k2H*E3malS'*Ek1l1Hm2 +...
Km1Hn1m2*E3malS'*Ek1l1Hk2H+Km1Hn1*E2malS'*Ek1l1Hk2Hm2 +...
Km1Hk2Hm2*E3malS'*Ek1l1Hn1+Km1Hk2H*E2malS'*Ek1l1Hn1m2 +...
Km1Hm2*E2malS'*Ek1l1Hn1k2H;

T222=...
Kn1k2Hm2*E3malS'*Ek1l1Hm1H+Kn1k2H*E2malS'*Ek1l1Hm1Hm2 +...
Kn1m2*E2malS'*Ek1l1Hm1Hk2H+Kk2Hm2*E2malS'*Ek1l1Hm1Hn1 +...
Ek1l1Hm1Hn1k2Hm2;

Rk1l1Hm1Hn1k2Hm2=T111+T112+T121+T122+T211+T212+T221+T222;
```

```
%CALCULATION OF b4) See Section "Sixth-order Moment"
%
%       E{rk1 rl1H rm1H rn1 rl2 rn2H}
%
T111=...
Kk1l1Hm1Hn1l2n2H*E6malS'+Kk1l1Hm1Hn1*E4malS'*El2n2H +...
Kk1l1Hm1Hl2*E4malS'*En1n2H+Kk1l1Hm1Hn2H*E4malS'*En1l2 +...
Kk1l1Hm1H*E3malS'*En1l2n2H;

T112=...
Kk1l1Hn1l2*E4malS'*Em1Hn2H+Kk1l1Hn1n2H*E4malS'*Em1H22 +...
Kk1l1Hn1*E3malS'*Em1Hl2n2H+Kk1l1Hl2n2H*E4malS'*Em1Hn1 +...
Kk1l1Hl2*E3malS'*Em1Hn1n2H+Kk1l1Hn2H*E3malS'*Em1Hn1l2 +...
Kk1l1H*E2malS'*Em1Hn1l2n2H;

T121=...
Kk1m1Hn1l2*E4malS'*El1Hn2H+Kk1m1Hn1n2H*E4malS'*El1Hl2 +...
Kk1m1Hn1*E3malS'*El1Hl2n2H+Kk1m1Hl2n2H*E4malS'*El1Hn1 +...
Kk1m1Hl2*E3malS'*El1Hn1n2H+Kk1m1Hn2H*E3malS'*El1Hn1l2 +...
Kk1m1H*E2malS'*El1Hn1l2n2H;

T122=...
Kk1n1l2n2H*E4malS'*El1Hm1H+Kk1n1l2*E3malS'*El1Hm1Hn2H +...
Kk1n1n2H*E3malS'*El1Hm1Hl2+Kk1n1*E2malS'*El1Hm1Hl2n2H +...
Kk1l2n2H*E3malS'*El1Hm1Hn1+Kk1l2*E2malS'*El1Hm1Hn1n2H +...
Kk1n2H*E2malS'*El1Hm1Hn1l2;

T211=...
Kl1Hm1Hn1l2*E4malS'*Ek1n2H+Kl1Hm1Hn1n2H*E4malS'*Ek1l2 +...
Kl1Hm1Hn1*E3malS'*Ek1l2n2H+Kl1Hm1Hl2n2H*E4malS'*Ek1n1 +...
Kl1Hm1Hl2*E3malS'*Ek1n1n2H+Kl1Hm1Hn2H*E3malS'*Ek1n1l2 +...
Kl1Hm1H*E2malS'*Ek1n1l2n2H;

T212=...
Kl1Hn1l2n2H*E4malS'*Ek1m1H+Kl1Hn1l2*E3malS'*Ek1m1Hn2H +...
Kl1Hn1n2H*E3malS'*Ek1m1Hl2+Kl1Hn1*E2malS'*Ek1m1Hl2n2H +...
Kl1Hl2n2H*E3malS'*Ek1m1Hn1+Kl1Hl2*E2malS'*Ek1m1Hn1n2H +...
Kl1Hn2H*E2malS'*Ek1m1Hn1l2;

T221=...
Km1Hn1l2n2H*E4malS'*Ek1l1H+Km1Hn1l2*E3malS'*Ek1l1Hn2H +...
Km1Hn1n2H*E3malS'*Ek1l1Hl2+Km1Hn1*E2malS'*Ek1l1Hl2n2H +...
Km1Hl2n2H*E3malS'*Ek1l1Hn1+Km1Hl2*E2malS'*Ek1l1Hn1n2H +...
```

```
Km1Hn2H*E2malS'*Ek1l1Hn1l2;

T222=...
Kn1l2n2H*E3malS'*Ek1l1Hm1H+Kn1l2*E2malS'*Ek1l1Hm1Hn2H +...
Kn1n2H*E2malS'*Ek1l1Hm1Hl2+Kl2n2H*E2malS'*Ek1l1Hm1Hn1 +...
Ek1l1Hm1Hn1l2n2H;

Rk1l1Hm1Hn1l2n2H=T111+T112+T121+T122+T211+T212+T221+T222;

%CALCULATION OF b5) See Section "Sixth-order Moment"
%
%       E{rk1 rl1H rm1H rn1 rk2H rn2H}
%
T111=...
Kk1l1Hm1Hn1k2Hn2H*E6malS'+Kk1l1Hm1Hn1*E4malS'*Ek2Hn2H +...
Kk1l1Hm1Hk2H*E4malS'*En1n2H+Kk1l1Hm1Hn2H*E4malS'*En1k2H +...
Kk1l1Hm1H*E3malS'*En1k2Hn2H;

T112=...
Kk1l1Hn1k2H*E4malS'*Em1Hn2H+Kk1l1Hn1n2H*E4malS'*Em1Hk2H +...
Kk1l1Hn1*E3malS'*Em1Hk2Hn2H+Kk1l1Hk2Hn2H*E4malS'*Em1Hn1 +...
Kk1l1Hk2H*E3malS'*Em1Hn1n2H+Kk1l1Hn2H*E3malS'*Em1Hn1k2H +...
Kk1l1H*E2malS'*Em1Hn1k2Hn2H;

T121=...
Kk1m1Hn1k2H*E4malS'*El1Hn2H+Kk1m1Hn1n2H*E4malS'*El1Hk2H +...
Kk1m1Hn1*E3malS'*El1Hk2Hn2H+Kk1m1Hk2Hn2H*E4malS'*El1Hn1 +...
Kk1m1Hk2H*E3malS'*El1Hn1n2H+Kk1m1Hn2H*E3malS'*El1Hn1k2H +...
Kk1m1H*E2malS'*El1Hn1k2Hn2H;

T122=...
Kk1n1k2Hn2H*E4malS'*El1Hm1H+Kk1n1k2H*E3malS'*El1Hm1Hn2H +...
Kk1n1n2H*E3malS'*El1Hm1Hk2H+Kk1n1*E2malS'*El1Hm1Hk2Hn2H +...
Kk1k2Hn2H*E3malS'*El1Hm1Hn1+Kk1k2H*E2malS'*El1Hm1Hn1n2H +...
Kk1n2H*E2malS'*El1Hm1Hn1k2H;

T211=...
Kl1Hm1Hn1k2H*E4malS'*Ek1n2H+Kl1Hm1Hn1n2H*E4malS'*Ek1k2H +...
Kl1Hm1Hn1*E3malS'*Ek1k2Hn2H+Kl1Hm1Hk2Hn2H*E4malS'*Ek1n1 +...
Kl1Hm1Hk2H*E3malS'*Ek1n1n2H+Kl1Hm1Hn2H*E3malS'*Ek1n1k2H +...
Kl1Hm1H*E2malS'*Ek1n1k2Hn2H;

T212=...
```

```
Kl1Hn1k2Hn2H*E4malS'*Ek1m1H+Kl1Hn1k2H*E3malS'*Ek1m1Hn2H +...
Kl1Hn1n2H*E3malS'*Ek1m1Hk2H+Kl1Hn1*E2malS'*Ek1m1Hk2Hn2H +...
Kl1Hk2Hn2H*E3malS'*Ek1m1Hn1+Kl1Hk2H*E2malS'*Ek1m1Hn1n2H +...
Kl1Hn2H*E2malS'*Ek1m1Hn1k2H;

T221=...
Km1Hn1k2Hn2H*E4malS'*Ek1l1H+Km1Hn1k2H*E3malS'*Ek1l1Hn2H +...
Km1Hn1n2H*E3malS'*Ek1l1Hk2H+Km1Hn1*E2malS'*Ek1l1Hk2Hn2H +...
Km1Hk2Hn2H*E3malS'*Ek1l1Hn1+Km1Hk2H*E2malS'*Ek1l1Hn1n2H +...
Km1Hn2H*E2malS'*Ek1l1Hn1k2H;

T222=...
Kn1k2Hn2H*E3malS'*Ek1l1Hm1H+Kn1k2H*E2malS'*Ek1l1Hm1Hn2H +...
Kn1n2H*E2malS'*Ek1l1Hm1Hk2H+Kk2Hn2H*E2malS'*Ek1l1Hm1Hn1 +...
Ek1l1Hm1Hn1k2Hn2H;

Rk1l1Hm1Hn1k2Hn2H=T111+T112+T121+T122+T211+T212+T221+T222;

%CALCULATION OF b6) See Section "Sixth-order Moment"
%
%       E{rk1 rl1H rm1H rn1 rl2 rm2}
%
T111=...
Kk1l1Hm1Hn1l2m2*E6malS'+Kk1l1Hm1Hn1*E4malS'*El2m2 +...
Kk1l1Hm1Hl2*E4malS'*En1m2+Kk1l1Hm1Hm2*E4malS'*En1l2 +...
Kk1l1Hm1H*E3malS'*En1l2m2;

T112=...
Kk1l1Hn1l2*E4malS'*Em1Hm2+Kk1l1Hn1m2*E4malS'*Em1Hl2 +...
Kk1l1Hn1*E3malS'*Em1Hl2m2+Kk1l1Hl2m2*E4malS'*Em1Hn1 +...
Kk1l1Hl2*E3malS'*Em1Hn1m2+Kk1l1Hm2*E3malS'*Em1Hn1l2 +...
Kk1l1H*E2malS'*Em1Hn1l2m2;

T121=...
Kk1m1Hn1l2*E4malS'*El1Hm2+Kk1m1Hn1m2*E4malS'*El1Hl2 +...
Kk1m1Hn1*E3malS'*El1Hl2m2+Kk1m1Hl2m2*E4malS'*El1Hn1 +...
Kk1m1Hl2*E3malS'*El1Hn1m2+Kk1m1Hm2*E3malS'*El1Hn1l2 +...
Kk1m1H*E2malS'*El1Hn1l2m2;

T122=...
Kk1n1l2m2*E4malS'*El1Hm1H+Kk1n1l2*E3malS'*El1Hm1Hm2 +...
Kk1n1m2*E3malS'*El1Hm1Hl2+Kk1n1*E2malS'*El1Hm1Hl2m2 +...
Kk1l2m2*E3malS'*El1Hm1Hn1+Kk1l2*E2malS'*El1Hm1Hn1m2 +...
```

```
Kk1m2*E2malS'*El1Hm1Hn1l2;

T211=...
Kl1Hm1Hn1l2*E4malS'*Ek1m2+Kl1Hm1Hn1m2*E4malS'*Ek1l2 +...
Kl1Hm1Hn1*E3malS'*Ek1l2m2+Kl1Hm1Hl2m2*E4malS'*Ek1n1 +...
Kl1Hm1Hl2*E3malS'*Ek1n1m2+Kl1Hm1Hm2*E3malS'*Ek1n1l2 +...
Kl1Hm1H*E2malS'*Ek1n1l2m2;

T212=...
Kl1Hn1l2m2*E4malS'*Ek1m1H+Kl1Hn1l2*E3malS'*Ek1m1Hm2 +...
Kl1Hn1m2*E3malS'*Ek1m1Hl2+Kl1Hn1*E2malS'*Ek1m1Hl2m2 +...
Kl1Hl2m2*E3malS'*Ek1m1Hn1+Kl1Hl2*E2malS'*Ek1m1Hn1m2 +...
Kl1Hm2*E2malS'*Ek1m1Hn1l2;

T221=...
Km1Hn1l2m2*E4malS'*Ek1l1H+Km1Hn1l2*E3malS'*Ek1l1Hm2 +...
Km1Hn1m2*E3malS'*Ek1l1Hl2+Km1Hn1*E2malS'*Ek1l1Hl2m2 +...
Km1Hl2m2*E3malS'*Ek1l1Hn1+Km1Hl2*E2malS'*Ek1l1Hn1m2 +...
Km1Hm2*E2malS'*Ek1l1Hn1l2;

T222=...
Kn1l2m2*E3malS'*Ek1l1Hm1H+Kn1l2*E2malS'*Ek1l1Hm1Hm2 +...
Kn1m2*E2malS'*Ek1l1Hm1Hl2+Kl2m2*E2malS'*Ek1l1Hm1Hn1 +...
Ek1l1Hm1Hn1l2m2;

Rk1l1Hm1Hn1l2m2=T111+T112+T121+T122+T211+T212+T221+T222;

%CALCULATION OF b7) See Section "Sixth-order Moment"
%
%       E{rm1H rn1 rk2H rl2 rm2 rn2H}
%
T111=...
Km1Hn1k2Hl2m2n2H*E6malS'+Km1Hn1k2Hl2*E4malS'*Em2n2H +...
Km1Hn1k2Hm2*E4malS'*El2n2H+Km1Hn1k2Hn2H*E4malS'*El2m2 +...
Km1Hn1k2H*E3malS'*El2m2n2H;

T112=...
Km1Hn1l2m2*E4malS'*Ek2Hn2H+Km1Hn1l2n2H*E4malS'*Ek2Hm2 +...
Km1Hn1l2*E3malS'*Ek2Hm2n2H+Km1Hn1m2n2H*E4malS'*Ek2Hl2 +...
Km1Hn1m2*E3malS'*Ek2Hl2n2H+Km1Hn1n2H*E3malS'*Ek2Hl2m2 +...
Km1Hn1*E2malS'*Ek2Hl2m2n2H;

T121=...
```

```
Km1Hk2Hl2m2*E4malS'*En1n2H+Km1Hk2Hl2n2H*E4malS'*En1m2 +...
Km1Hk2Hl2*E3malS'*En1m2n2H+Km1Hk2Hm2n2H*E4malS'*En1l2 +...
Km1Hk2Hm2*E3malS'*En1l2n2H+Km1Hk2Hn2H*E3malS'*En1l2m2 +...
Km1Hk2H*E2malS'*En1l2m2n2H;

T122=...
Km1Hl2m2n2H*E4malS'*En1k2H+KmoHl2m2*E3malS'*En1k2Hn2H +...
Km1Hl2n2H*E3ma^S`*En1k2Hm2+Km1Hl2*E2malS'*En1k2Hm2"2H + ..
Km_Hm2n2H*E3malS'*En1k2Hl2+Km1Hm2*E2malS'*En1k2Hl2n2H +...
Km1Hn2H*E2malS'*En1k2Hl2m2;

T211=...
Kn1k2Hl2m2*E4malS'*Em1Hn2H+Kn1k2Hl2n2H*E4malS'*Em1Hm2 +...
Kn1k2Hl2*E3malS'*Em1Hm2n2H+Kn1k2Hm2n2H*E4malS'*Em1Hl2 +...
Kn1k2Hm2*E3malS'*Em1Hl2n2H+Kn1k2Hn2H*E3malS'*Em1Hl2m2 +...
Kn1k2H*E2malS'*Em1Hl2m2n2H;

T212=...
Kn1l2m2n2H*E4malS'*Em1Hk2H+Kn1l2m2*E3malS'*Em1Hk2Hn2H +...
Kn1l2n2H*E3malS'*Em1Hk2Hm2+Kn1l2*E2malS'*Em1Hk2Hm2n2H +...
Kn1m2n2H*E3malS'*Em1Hk2Hl2+Kn1m2*E2malS'*Em1Hk2Hl2n2H +...
Kn1n2H*E2malS'*Em1Hk2Hl2m2;

T221=...
Kk2Hl2m2n2H*E4malS'*Em1Hn1+Kk2Hl2m2*E3malS'*Em1Hn1n2H +...
Kk2Hl2n2H*E3malS'*Em1Hn1m2+Kk2Hl2*E2malS'*Em1Hn1m2n2H +...
Kk2Hm2n2H*E3malS'*Em1Hn1l2+Kk2Hm2*E2malS'*Em1Hn1l2n2H +...
Kk2Hn2H*E2malS'*Em1Hn1l2m2;

T222=...
Kl2m2n2H*E3malS'*Em1Hn1k2H+Kl2mr*E2malS'*Em1Hn1k2Hn2H +...
Kl2n2H*E2malS'*Em1Hn1k2Hm2+Km2n2H*E2malS'*Em1Hn1k2Hl2 +...
Em1Hn1k2Hl2m2n2H;

Rm1Hn1k2Hl2m2n2H=T111+T112+T121+T122+T211+T212+T221+T222;

%CALCULATION OF b8) See Section "Sixth-order Moment"
%
%       E{rk1 rl1H rk2H rl2 rm2 rn2H}
%
```

```
T111=...
Kk1l1Hk2Hl2m2n2H*E6malS'+Kk1l1Hk2Hl2*E4malS'*Em2n2H +...
Kk1l1Hk2Hm2*E4malS'*El2n2H+Kk1l1Hk2Hn2H*E4malS'*El2m2 +...
Kk1l1Hk2H*E3malS'*El2m2n2H;

T112=...
Kk1l1Hl2m2*E4malS'*Ek2Hn2H+Kk1l1Hl2n2H*E4malS'*Ek2Hm2 +...
Kk1l1Hl2*E3malS'*Ek2Hm2n2H+Kk1l1Hm2n2H*E4malS'*Ek2Hl2 +...
Kk1l1Hm2*E3malS'*Ek2Hl2n2H+Kk1l1Hn2H*E3malS'*Ek2Hl2m2 +...
Kk1l1H*E2malS'*Ek2Hl2m2n2H;

T121=...
Kk1k2Hl2m2*E4malS'*El1Hn2H+Kk1k2Hl2n2H*E4malS'*El1Hm2 +...
Kk1k2Hl2*E3malS'*El1Hm2n2H+Kk1k2Hm2n2H*E4malS'*El1Hl2 +...
Kk1k2Hm2*E3malS'*El1Hl2n2H+Kk1k2Hn2H*E3malS'*El1Hl2m2 +...
Kk1k2H*E2malS'*El1Hl2m2n2H;

T122=...
Kk1l2m2n2H*E4malS'*El1Hk2H+Kk1l2m2*E3malS'*El1Hk2Hn2H +...
Kk1l2n2H*E3malS'*El1Hk2Hm2+Kk1l2*E2malS'*El1Hk2Hm2n2H +...
Kk1m2n2H*E3malS'*El1Hk2Hl2+Kk1m2*E2malS'*El1Hk2Hl2n2H +...
Kk1n2H*E2malS'*El1Hk2Hl2m2;

T211=...
Kl1Hk2Hl2m2*E4malS'*Ek1n2H+Kl1Hk2Hl2n2H*E4malS'*Ek1m2 +...
Kl1Hk2Hl2*E3malS'*Ek1m2n2H+Kl1Hk2Hm2n2H*E4malS'*Ek1l2 +...
Kl1Hk2Hm2*E3malS'*Ek1l2n2H+Kl1Hk2Hn2H*E3malS'*Ek1l2m2 +...
Kl1Hk2H*E2malS'*Ek1l2m2n2H;

T212=...
Kl1Hl2m2n2H*E4malS'*Ek1k2H+Kl1Hl2m2*E3malS'*Ek1k2Hn2H +...
Kl1Hl2n2H*E3malS'*Ek1k2Hm2+Kl1Hl2*E2malS'*Ek1k2Hm2n2H +...
Kl1Hm2n2H*E3malS'*Ek1k2Hl2+Kl1Hm2*E2malS'*Ek1k2Hl2n2H +...
Kl1Hn2H*E2malS'*Ek1k2Hl2m2;

T221=...
Kk2Hl2m2n2H*E4malS'*Ek1l1H+Kk2Hl2m2*E3malS'*Ek1l1Hn2H +...
Kk2Hl2n2H*E3malS'*Ek1l1Hm2+Kk2Hl2*E2malS'*Ek1l1Hm2n2H +...
Kk2Hm2n2H*E3malS'*Ek1l1Hl2+Kk2Hm2*E2malS'*Ek1l1Hl2n2H +...
Kk2Hn2H*E2malS'*Ek1l1Hl2m2;

T222=...
Kl2m2n2H*E3malS'*Ek1l1Hk2H+Kl2m2*E2malS'*Ek1l1Hk2Hn2H +...
```

```
Kl2n2H*E2malS'*Ek1l1Hk2Hm2+Km2n2H*E2malS'*Ekol1Hk2Hl2 +...
Ek1l1Hk2Hl2m2n2H;

Rk1l1Hk2Hl2m2n2H=T111+T112+T121+T122+T211+T212+T22m+T222;

%CALCULATION OF b9) See Section "Sixth-order Moment"
%
%       E{ro1H rn1 rk2d rl2 rm2 rn2H}
%

T111=...
Kl1Hn1k2Hl2m2n2H*E6malS'+Kl1Hn1k2Hl2*E4malS'*Em2n2H +...
Kl1Hn1k2Hm2*E4malS'*El2n2H+Kl1Hn1k2Hn2H*E4malS'*El2m2 +...
Kl1Hn1k2H*E3malS'*El2m2n2H;

T112=...
Kl1Hn1l2m2*E4malS'*Ek2Hn2H+Kl1Hn1l2n2H*E4malS'*Ek2Hm2 +...
Kl1Hn1l2*E3malS'*Ek2Hm2n2H+Kl1Hn1m2n2H*E4malS'*Ek2Hl2 +...
Kl1Hn1m2*E3malS'*Ek2Hl2n2H+Kl1Hn1n2H*E3malS'*Ek2Hl2m2 +...
Kl1Hn1*E2malS'*Ek2Hl2m2n2H;

T121=...
Kl1Hk2Hl2m2*E4malS'*En1n2H+Kl1Hk2Hl2n2H*E4malS'*En1m2 +...
Kl1Hk2Hl2*E3malS'*En1m2n2H+Kl1Hk2Hm2n2H*E4malS'*En1l2 +...
Kl1Hk2Hm2*E3malS'*En1l2n2H+Kl1Hk2Hn2H*E3malS'*En1l2m2 +...
Kl1Hk2H*E2malS'*En1l2m2n2H;

T122=...
Kl1Hl2m2n2H*E4malS'*En1k2H+Kl1Hl2m2*E3malS'*En1k2Hn2H +...
Kl1Hl2n2H*E3malS'*En1k2Hm2+Kl1Hl2*E2malS'*En1k2Hm2n2H +...
Kl1Hm2n2H*E3malS'*En1k2Hl2+Kl1Hm2*E2malS'*En1k2Hl2n2H +...
Kl1Hn2H*E2malS'*En1k2Hl2m2;

T211=...
Kn1k2Hl2m2*E4malS'*El1Hn2H+Kn1k2Hl2n2H*E4malS'*El1Hm2 +...
Kn1k2Hl2*E3malS'*El1Hm2n2H+Kn1k2Hm2n2H*E4malS'*El1Hl2 +...
Kn1k2Hm2*E3malS'*El1Hl2n2H+Kn1k2Hn2H*E3malS'*El1Hl2m2 +...
Kn1k2H*E2malS'*El1Hl2m2n2H;

T212=...
Kn1l2m2n2H*E4malS'*El1Hk2H+Kn1l2m2*E3malS'*El1Hk2Hn2H +...
Kn1l2n2H*E3malS'*El1Hk2Hm2+Kn1l2*E2malS'*El1Hk2Hm2n2H +...
Kn1m2n2H*E3malS'*El1Hk2Hl2+Kn1m2*E2malS'*El1Hk2Hl2n2H +...
```

```
Kn1n2H*E2malS'*El1Hk2Hl2m2;

T221=...
Kk2Hl2m2n2H*E4malS'*El1Hn1+Kk2Hl2m2*E3malS'*El1Hn1n2H +...
Kk2Hl2n2H*E3malS'*El1Hn1m2+Kk2Hl2*E2malS'*El1Hn1m2n2H +...
Kk2Hm2n2H*E3malS'*El1Hn1l2+Kk2Hm2*E2malS'*El1Hn1l2n2H +...
Kk2Hn2H*E2malS'*El1Hn1l2m2;

T222=...
Kl2m2n2H*E3malS'*El1Hn1k2H+Kl2m2*E2malS'*El1Hn1k2Hn2H +...
Kl2n2H*E2malS'*El1Hn1k2Hm2+Km2n2H*E2malS'*El1Hn1k2Hl2 +...
El1Hn1k2Hl2m2n2H;

Rl1Hn1k2Hl2m2n2H=T111+T112+T121+T122+T211+T212+T221+T222;

%CALCULATION OF b10) See Section "Sixth-order Moment"
%
%       E{rk1 rm1H rk2H rl2 rm2 rn2H}
%

T111=...
Kk1m1Hk2Hl2m2n2H*E6malS'+Kk1m1Hk2Hl2*E4malS'*Em2n2H +...
Kk1m1Hk2Hm2*E4malS'*El2n2H+Kk1m1Hk2Hn2H*E4malS'*El2m2 +...
Kk1m1Hk2H*E3malS'*El2m2n2H;

T112=...
Kk1m1Hl2m2*E4malS'*Ek2Hn2H+Kk1m1Hl2n2H*E4malS'*Ek2Hm2 +...
Kk1m1Hl2*E3malS'*Ek2Hm2n2H+Kk1m1Hm2n2H*E4malS'*Ek2Hl2 +...
Kk1m1Hm2*E3malS'*Ek2Hl2n2H+Kk1m1Hn2H*E3malS'*Ek2Hl2m2 +...
Kk1m1H*E2malS'*Ek2Hl2m2n2H;

T121=...
Kk1k2Hl2m2*E4malS'*Em1Hn2H+Kk1k2Hl2n2H*E4malS'*Em1Hm2 +...
Kk1k2Hl2*E3malS'*Em1Hm2n2H+Kk1k2Hm2n2H*E4malS'*Em1Hl2 +...
Kk1k2Hm2*E3malS'*Em1Hl2n2H+Kk1k2Hn2H*E3malS'*Em1Hl2m2 +...
Kk1k2H*E2malS'*Em1Hl2m2n2H;

T122=...
Kk1l2m2n2H*E4malS'*Em1Hk2H+Kk1l2m2*E3malS'*Em1Hk2Hn2H +...
Kk1l2n2H*E3malS'*Em1Hk2Hm2+Kk1l2*E2malS'*Em1Hk2Hm2n2H +...
Kk1m2n2H*E3malS'*Em1Hk2Hl2+Kk1m2*E2malS'*Em1Hk2Hl2n2H +...
Kk1n2H*E2malS'*Em1Hk2Hl2m2;
```

```
T211=...
Km1Hk2Hl2m2*E4malS'*Ek1n2H+Km1Hk2Hl2n2H*E4malS'*Ek1m2 +...
Km1Hk2Hl2*E3malS'*Ek1m2n2H+Km1Hk2Hm2n2H*E4malS'*Ek1l2 +...
Km1Hk2Hm2*E3malS'*Ek1l2n2H+Km1Hk2Hn2H*E3malS'*Ek1l2m2 +...
Km1Hk2H*E2malS'*Ek1l2m2n2H;

T212=...
Km1Hl2m2n2H*E4malS'*Ek1k2H+Km1Hl2m2*E3malS'*Ek1k2Hn2H +...
Km1Hl2n2H*E3malS'*Ek1k2Hm2+Km1Hl2*E2malS'*Ek1k2Hm2n2H +...
Km1Hm2n2H*E3malS'*Ek1k2Hl2+Km1Hm2*E2malS'*Ek1k2Hl2n2H +...
Km1Hn2H*E2malS'*Ek1k2Hl2m2;

T221=...
Kk2Hl2m2n2H*E4malS'*Ek1m1H+Kk2Hl2m2*E3malS'*Ek1m1Hn2H +...
Kk2Hl2n2H*E3malS'*Ek1m1Hm2+Kk2Hl2*E2malS'*Ek1m1Hm2n2H +...
Kk2Hm2n2H*E3malS'*Ek1m1Hl2+Kk2Hm2*E2malS'*Ek1m1Hl2n2H +...
Kk2Hn2H*E2malS'*Ek1m1Hl2m2;

T222=...
Kl2m2n2H*E3malS'*Ek1m1Hk2H+Kl2m2*E2malS'*Ek1m1Hk2Hn2H +...
Kl2n2H*E2malS'*Ek1m1Hk2Hm2+Km2n2H*E2malS'*Ek1m1Hk2Hl2 +...
Ek1m1Hk2Hl2m2n2H;

Rk1m1Hk2Hl2m2n2H=T111+T112+T121+T122+T211+T212+T221+T222;

%CALCULATION OF b11) See Section "Sixth-order Moment"
%
%       E{rl1H rm1H rk2H rl2 rm2 rn2H}
%

T111=...
Kl1Hm1Hk2Hl2m2n2H*E6malS'+Kl1Hm1Hk2Hl2*E4malS'*Em2n2H +...
Kl1Hm1Hk2Hm2*E4malS'*El2n2H+Kl1Hm1Hk2Hn2H*E4malS'*El2m2 +...
Kl1Hm1Hk2H*E3malS'*El2m2n2H;

T112=...
Kl1Hm1Hl2m2*E4malS'*Ek2Hn2H+Kl1Hm1Hl2n2H*E4malS'*Ek2Hm2 +...
Kl1Hm1Hl2*E3malS'*Ek2Hm2n2H+Kl1Hm1Hm2n2H*E4malS'*Ek2Hl2 +...
Kl1Hm1Hm2*E3malS'*Ek2Hl2n2H+Kl1Hm1Hn2H*E3malS'*Ek2Hl2m2 +...
Kl1Hm1H*E2malS'*Ek2Hl2m2n2H;

T121=...
Kl1Hk2Hl2m2*E4malS'*Em1Hn2H+Kl1Hk2Hl2n2H*E4malS'*Em1Hm2 +...
```

```
Kl1Hk2Hl2*E3malS'*Em1Hm2n2H+Kl1Hk2Hm2n2H*E4malS'*Em1Hl2 +...
Kl1Hk2Hm2*E3malS'*Em1Hl2n2H+Kl1Hk2Hn2H*E3malS'*Em1Hl2m2 +...
Kl1Hk2H*E2malS'*Em1Hl2m2n2H;

T122=...
Kl1Hl2m2n2H*E4malS'*Em1Hk2H+Kl1Hl2m2*E3malS'*Em1Hk2Hn2H +...
Kl1Hl2n2H*E3malS'*Em1Hk2Hm2+Kl1Hl2*E2malS'*Em1Hk2Hm2n2H +...
Kl1Hm2n2H*E3malS'*Em1Hk2Hl2+Kl1Hm2*E2malS'*Em1Hk2Hl2n2H +...
Kl1Hn2H*E2malS'*Em1Hk2Hl2m2;

T211=...
Km1Hk2Hl2m2*E4malS'*El1Hn2H+Km1Hk2Hl2n2H*E4malS'*El1Hm2 +...
Km1Hk2Hl2*E3malS'*El1Hm2n2H+Km1Hk2Hm2n2H*E4malS'*El1Hl2 +...
Km1Hk2Hm2*E3malS'*El1Hl2n2H+Km1Hk2Hn2H*E3malS'*El1Hl2m2 +...
Km1Hk2H*E2malS'*El1Hl2m2n2H;

T212=...
Km1Hl2m2n2H*E4malS'*El1Hk2H+Km1Hl2m2*E3malS'*El1Hk2Hn2H +...
Km1Hl2n2H*E3malS'*El1Hk2Hm2+Km1Hl2*E2malS'*El1Hk2Hm2n2H +...
Km1Hm2n2H*E3malS'*El1Hk2Hl2+Km1Hm2*E2malS'*El1Hk2Hl2n2H +...
Km1Hn2H*E2malS'*El1Hk2Hl2m2;

T221=...
Kk2Hl2m2n2H*E4malS'*El1Hm1H+Kk2Hl2m2*E3malS'*El1Hm1Hn2H +...
Kk2Hl2n2H*E3malS'*El1Hm1Hm2+Kk2Hl2*E2malS'*El1Hm1Hm2n2H +...
Kk2Hm2n2H*E3malS'*El1Hm1Hl2+Kk2Hm2*E2malS'*El1Hm1Hl2n2H +...
Kk2Hn2H*E2malS'*El1Hm1Hl2m2;

T222=...
Kl2m2n2H*E3malS'*El1Hm1Hk2H+Kl2m2*E2malS'*El1Hm1Hk2Hn2H +...
Kl2n2H*E2malS'*El1Hm1Hk2Hm2+Km2n2H*E2malS'*El1Hm1Hk2Hl2 +...
El1Hm1Hk2Hl2m2n2H;

Rl1Hm1Hk2Hl2m2n2H=T111+T112+T121+T122+T211+T212+T221+T222;

%CALCULATION OF b12) See Section "Sixth-order Moment"
%
%      E{rk1 rn1 rk2H rl2 rm2 rn2H}
%

T111=...
Kk1n1k2Hl2m2n2H*E6malS'+Kk1n1k2Hl2*E4malS'*Em2n2H +...
Kk1n1k2Hm2*E4malS'*El2n2H+Kk1n1k2Hn2H*E4malS'*El2m2 +...
```

```
Kk1n1k2H*E3malS'*El2m2n2H;

T112=...
Kk1n1l2m2*E4malS'*Ek2Hn2H+Kk1n1l2n2H*E4malS'*Ek2Hm2 +...
Kk1n1l2*E3malS'*Ek2Hm2n2H+Kk1n1m2n2H*E4malS'*Ek2Hl2 +...
Kk1n1m2*E3malS'*Ek2Hl2n2H+Kk1n1n2H*E3malS'*Ek2Hl2m2 +...
Kk1n1*E2malS'*Ek2Hl2m2n2H;

T121=...
Kk1k2Hl2m2*E4malS'*En1n2H+Kk1k2Hl2n2H*E4malS'*En1m2 +...
Kk1k2Hl2*E3malS'*En1m2n2H+Kk1k2Hm2n2H*E4malS'*En1l2 +...
Kk1k2Hm2*E3malS'*En1l2n2H+Kk1k2Hn2H*E3malS'*En1l2m2 +...
Kk1k2H*E2malS'*En1l2m2n2H;

T122=...
Kk1l2m2n2H*E4malS'*En1k2H+Kk1l2m2*E3malS'*En1k2Hn2H +...
Kk1l2n2H*E3malS'*En1k2Hm2+Kk1l2*E2malS'*En1k2Hm2n2H +...
Kk1m2n2H*E3malS'*En1k2Hl2+Kk1m2*E2malS'*En1k2Hl2n2H +...
Kk1n2H*E2malS'*En1k2Hl2m2;

T211=...
Kn1k2Hl2m2*E4malS'*Ek1n2H+Kn1k2Hl2n2H*E4malS'*Ek1m2 +...
Kn1k2Hl2*E3malS'*Ek1m2n2H+Kn1k2Hm2n2H*E4malS'*Ek1l2 +...
Kn1k2Hm2*E3malS'*Ek1l2n2H+Kn1k2Hn2H*E3malS'*Ek1l2m2 +...
Kn1k2H*E2malS'*Ek1l2m2n2H;

T212=...
Kn1l2m2n2H*E4malS'*Ek1k2H+Kn1l2m2*E3malS'*Ek1k2Hn2H +...
Kn1l2n2H*E3malS'*Ek1k2Hm2+Kn1l2*E2malS'*Ek1k2Hm2n2H +...
Kn1m2n2H*E3malS'*Ek1k2Hl2+Kn1m2*E2malS'*Ek1k2Hl2n2H +...
Kn1n2H*E2malS'*Ek1k2Hl2m2;

T221=...
Kk2Hl2m2n2H*E4malS'*Ek1n1+Kk2Hl2m2*E3malS'*Ek1n1n2H +...
Kk2Hl2n2H*E3malS'*Ek1n1m2+Kk2Hl2*E2malS'*Ek1n1m2n2H +...
Kk2Hm2n2H*E3malS'*Ek1n1l2+Kk2Hm2*E2malS'*Ek1n1l2n2H +...
Kk2Hn2H*E2malS'*Ek1n1l2m2;

T222=...
Kl2m2n2H*E3malS'*Ek1n1k2H+Kl2m2*E2malS'*Ek1n1k2Hn2H +...
Kl2n2H*E2malS'*Ek1n1k2Hm2+Km2n2H*E2malS'*Ek1n1k2Hl2 +...
Ek1n1k2Hl2m2n2H;
```

```
Rk1n1k2Hl2m2n2H=T111+T112+T121+T122+T211+T212+T221+T222;

%CALCULATION OF c) See Section "Fourth-order Moment"
%
%       E{rk1 rl1H rm1H rn1}
%

Rk1l1Hm1Hn1=...
Kk1l1Hm1Hn1*E4malS'+Kk1l1H*E2malS'*Em1Hn1 +...
Kk1m1H*E2malS'*El1Hn1+Kk1n1*E2malS'*El1Hm1H +...
Kl1Hm1H*E2malS'*Ek1n1+Kl1Hn1*E2malS'*Ek1m1H +...
Km1Hn1*E2malS'*Ek1l1H+Ek1l1Hm1Hn1;

%CALCULATION OF c2) See Section "Fourth-order Moment"
%
%       E{rk2H rl2 rm2 rn2H}
%

Rk2Hl2m2n2H=...
Kk2Hl2m2n2H*E4malS'+Kk2Hl2*E2malS'*Em2n2H +...
Kk2Hm2*E2malS'*El2n2H+Kk2Hn2H*E2malS'*El2m2 +...
Kl2m2*E2malS'*Ek2Hn2H+Kl2n2H*E2malS'*Ek2Hm2 +...
Km2n2H*E2malS'*Ek2Hl2+Ek2Hl2m2n2H;

%CALCULATION OF c3) See Section "Fourth-order Moment"
%
%       E{rk1 rl1H rk2H rl2}
%
Rk1l1Hk2Hl2=...
Kk1l1Hk2Hl2*E4malS'+Kk1l1H*E2malS'*Ek2Hl2 +...
Kk1k2H*E2malS'*El1Hl2+Kk1l2*E2malS'*El1Hk2H +...
Kl1Hk2H*E2malS'*Ek1l2+Kl1Hl2*E2malS'*Ek1k2H +...
Kk2Hl2*E2malS'*Ek1l1H+Ek1l1Hk2Hl2;

%CALCULATION OF c4) See Section "Fourth-order Moment"
%
%       E{rm1H rn1 rm2 rn2H}
%
Rm1Hn1m2n2H=...
Km1Hn1m2n2H*E4malS'+Km1Hn1*E2malS'*Em2n2H +...
Km1Hm2*E2malS'*En1n2H+Km1Hn2H*E2malS'*En1m2 +...
Kn1m2*E2malS'*Em1Hn2H+Kn1n2H*E2malS'*Em1Hm2 +...
Km2n2H*E2malS'*Em1Hn1+Em1Hn1m2n2H;
```

```
%CALCULATION OF c5) See Section "Fourth-order Moment"
%
%       E{rk1 rl1H rm2 rn2H}
%
Rk1l1Hm2n2H=...
Kk1l1Hm2n2H*E4malS'+Kk1l1H*E2malS'*Em2n2H +...
Kk1m2*E2malS'*El1Hn2H+Kk1n2H*E2malS'*El1Hm2 +...
Kl1Hm2*E2malS'*Ek1n2H+Kl1Hn2H*E2malS'*Ek1m2 +...
Km2n2H*E2malS'*Ek1l1H+Ek1l1Hm2n2H;

%CALCULATION OF c6) See Section "Fourth-order Moment"
%
%       E{rm1H rn1 rk2H rl2}
%
Rm1Hn1k2Hl2=...
Km1Hn1k2Hl2*E4malS'+Km1Hn1*E2malS'*Ek2Hl2 +...
Km1Hk2H*E2malS'*En1l2+Km1Hl2*E2malS'*En1k2H +...
Kn1k2H*E2malS'*Em1Hl2+Kn1l2*E2malS'*Em1Hk2H +...
Kk2Hl2*E2malS'*Em1Hn1+Em1Hn1k2Hl2;

%CALCULATION OF c7) See Section "Fourth-order Moment"
%
%       E{rk1 rl1H rk2H rm2}
%

Rk1l1Hk2Hm2=...
Kk1l1Hk2Hm2*E4malS'+Kk1l1H*E2malS'*Ek2Hm2 +...
Kk1k2H*E2malS'*El1Hm2+Kk1m2*E2malS'*El1Hk2H +...
Kl1Hk2H*E2malS'*Ek1m2+Kl1Hm2*E2malS'*Ek1k2H +...
Kk2Hm2*E2malS'*Ek1l1H+Ek1l1Hk2Hm2;

%CALCULATION OF c8) See Section "Fourth-order Moment"
%
%       E{rm1H rn1 rl2 rn2H}
%
Rm1Hn1l2n2H=...
Km1Hn1l2n2H*E4malS'+Km1Hn1*E2malS'*El2n2H +...
Km1Hl2*E2malS'*En1n2H+Km1Hn2H*E2malS'*En1l2 +...
Kn1l2*E2malS'*Em1Hn2H+Kn1n2H*E2malS'*Em1Hl2 +...
Kl2n2H*E2malS'*Em1Hn1+Em1Hn1l2n2H;

%CALCULATION OF c9) See Section "Fourth-order Moment"
```

```
%
%       E{rk1 rl1H rl2 rn2H}
%
Rk1l1Hl2n2H=...
Kk1l1Hl2n2H*E4malS'+Kk1l1H*E2malS'*El2n2H +...
Kk1l2*E2malS'*El1Hn2H+Kk1n2H*E2malS'*El1Hl2 +...
Kl1Hl2*E2malS'*Ek1n2H+Kl1Hn2H*E2malS'*Ek1l2 +...
Kl2n2H*E2malS'*Ek1l1H+Ek1l1Hl2n2H;

%CALCULATION OF c10) See Section "Fourth-order Moment"
%
%       E{rm1H rn1 rk2H rm2}
%
Rm1Hn1k2Hm2=...
Km1Hn1k2Hm2*E4malS'+Km1Hn1*E2malS'*Ek2Hm2 +...
Km1Hk2H*E2malS'*En1m2+Km1Hm2*E2malS'*En1k2H +...
Kn1k2H*E2malS'*Em1Hm2+Kn1m2*E2malS'*Em1Hk2H +...
Kk2Hm2*E2malS'*Em1Hn1+Em1Hn1k2Hm2;



%CALCULATION OF c11)     See Section "Fourth-order Moment"
%
%       E{rk1 rl1H rk2H rn2H}
%
Rk1l1Hk2Hn2H=...
Kk1l1Hk2Hn2H*E4malS'+Kk1l1H*E2malS'*Ek2Hn2H +...
Kk1k2H*E2malS'*El1Hn2H+Kk1n2H*E2malS'*El1Hk2H +...
Kl1Hk2H*E2malS'*Ek1n2H+Kl1Hn2H*E2malS'*Ek1k2H +...
Kk2Hn2H*E2malS'*Ek1l1H+Ek1l1Hk2Hn2H;

%CALCULATION OF c12) See Section "Fourth-order Moment"
%
%       E{rm1H rn1 rl2 rm2}
%
Rm1Hn1l2m2=...
Km1Hn1l2m2*E4malS'+Km1Hn1*E2malS'*El2m2 +...
Km1Hl2*E2malS'*En1m2+Km1Hm2*E2malS'*En1l2 +...
Kn1l2*E2malS'*Em1Hm2+Kn1m2*E2malS'*Em1Hl2 +...
Kl2m2*E2malS'*Em1Hn1+Em1Hn1l2m2;

%CALCULATION OF c13) See Section "Fourth-order Moment"
%
```

```
%       E{rk1 rl1H rl2 rm2}
%
Rk1l1Hl2m2=...
Kk1l1Hl2m2*E4malS'+Kk1l1H*E2malS'*El2m2 +...
Kk1l2*E2malS'*El1Hm2+Kk1m2*E2malS'*El1Hl2 +...
Kl1Hl2*E2malS'*Ek1m2+Kl1Hm2*E2malS'*Ek1l2 +...
Kl2m2*E2malS'*Ek1l1H+Ek1l1Hl2m2;

%CALCULATION OF c14) See Section "Fourth-order Moment"
%
%       E{rm1H rn1 rk2H rn2H}
%
Rm1Hn1k2Hn2H=...
Km1Hn1k2Hn2H*E4malS'+Km1Hn1*E2malS'*Ek2Hn2H +...
Km1Hk2H*E2malS'*En1n2H+Km1Hn2H*E2malS'*En1k2H +...
Kn1k2H*E2malS'*Em1Hn2H+Kn1n2H*E2malS'*Em1Hk2H +...
Kk2Hn2H*E2malS'*Em1Hn1+Em1Hn1k2Hn2H;

%CALCULATION OF c15) See Section "Fourth-order Moment"
%
%       E{rk1 rm1H rk2H rm2}
%
Rk1m1Hk2Hm2=...
Kk1m1Hk2Hm2*E4malS'+Kk1m1H*E2malS'*Ek2Hm2 +...
Kk1k2H*E2malS'*Em1Hm2+Kk1m2*E2malS'*Em1Hk2H +...
Km1Hk2H*E2malS'*Ek1m2+Km1Hm2*E2malS'*Ek1k2H +...
Kk2Hm2*E2malS'*Ek1m1H+Ek1m1Hk2Hm2;

%CALCULATION OF c16) See Section "Fourth-order Moment"
%
%       E{rl1H rn1 rl2 rn2H}
%
Rl1Hn1l2n2H=...
Kl1Hn1l2n2H*E4malS'+Kl1Hn1*E2malS'*El2n2H +...
Kl1Hl2*E2malS'*En1n2H+Kl1Hn2H*E2malS'*En1l2 +...
Kn1l2*E2malS'*El1Hn2H+Kn1n2H*E2malS'*El1Hl2 +...
Kl2n2H*E2malS'*El1Hn1+El1Hn1l2n2H;
```

```
%CALCULATION OF c17) See Section "Fourth-order Moment"
%
%       E{rk1 rm1H rl2 rn2H}
%
Rk1m1Hl2n2H=...
Kk1m1Hl2n2H*E4malS'+Kk1m1H*E2malS'*El2n2H +...
Kk1l2*E2malS'*Em1Hn2H+Kk1n2H*E2malS'*Em1Hl2 +...
Km1Hl2*E2malS'*Ek1n2H+Km1Hn2H*E2malS'*Ek1l2 +...
Kl2n2H*E2malS'*Ek1m1H+Ek1m1Hl2n2H;

%CALCULATION OF c18) See Section "Fourth-order Moment"
%
%       E{rl1H rn1 rk2H rm2}
%
Rl1Hn1k2Hm2=...
Kl1Hn1k2Hm2*E4malS'+Kl1Hn1*E2malS'*Ek2Hm2 +...
Kl1Hk2H*E2malS'*En1m2+Kl1Hm2*E2malS'*En1k2H +...
Kn1k2H*E2malS'*El1Hm2+Kn1m2*E2malS'*El1Hk2H +...
Kk2Hm2*E2malS'*El1Hn1+El1Hn1k2Hm2;

%CALCULATION OF c19)     See Section "Fourth-order Moment"
%
%       E{rk1 rm1H rk2H rn2H}
%
Rk1m1Hk2Hn2H=...
Kk1m1Hk2Hn2H*E4malS'+Kk1m1H*E2malS'*Ek2Hn2H +...
Kk1k2H*E2malS'*Em1Hn2H+Kk1n2H*E2malS'*Em1Hk2H +...
Km1Hk2H*E2malS'*Ek1n2H+Km1Hn2H*E2malS'*Ek1k2H +...
Kk2Hn2H*E2malS'*Ek1m1H+Ek1m1Hk2Hn2H;

%CALCULATION OF c20) See Section "Fourth-order Moment"
%
%       E{rl1H rn1 rl2 rm2}
%
Rl1Hn1l2m2=...
Kl1Hn1l2m2*E4malS'+Kl1Hn1*E2malS'*El2m2 +...
Kl1Hl2*E2malS'*En1m2+Kl1Hm2*E2malS'*En1l2 +...
Kn1l2*E2malS'*El1Hm2+Kn1m2*E2malS'*El1Hl2 +...
Kl2m2*E2malS'*El1Hn1+El1Hn1l2m2;

%CALCULATION OF c21) See Section "Fourth-order Moment"
%
%       E{rk1 rm1H rl2 rm2}
```

```
%
Rk1m1H12m2=...
Kk1m1H12m2*E4malS'+Kk1m1H*E2malS'*E12m2 +...
Kk112*E2malS'*Em1Hm2+Kk1m2*E2malS'*Em1H12 +...
Km1H12*E2malS'*Ek1m2+Km1Hm2*E2malS'*Ek112 +...
K12m2*E2malS'*Ek1m1H+Ek1m1H12m2;

%CALCULATION OF c22) See Section "Fourth-order Moment"
%
%      E{rl1H rn1 rk2H rn2H}
%
Rl1Hn1k2Hn2H=...
Kl1Hn1k2Hn2H*E4malS'+Kl1Hn1*E2malS'*Ek2Hn2H +...
Kl1Hk2H*E2malS'*En1n2H+Kl1Hn2H*E2malS'*En1k2H +...
Kn1k2H*E2malS'*El1Hn2H+Kn1n2H*E2malS'*El1Hk2H +...
Kk2Hn2H*E2malS'*El1Hn1+El1Hn1k2Hn2H;

%CALCULATION OF c23)     See Section "Fourth-order Moment"
%
%      E{rk1 rn1 rk2H rn2H}
%
Rk1n1k2Hn2H=...
Kk1n1k2Hn2H*E4malS'+Kk1n1*E2malS'*Ek2Hn2H +...
Kk1k2H*E2malS'*En1n2H+Kk1n2H*E2malS'*En1k2H +...
Kn1k2H*E2malS'*Ek1n2H+Kn1n2H*E2malS'*Ek1k2H +...
Kk2Hn2H*E2malS'*Ek1n1+Ek1n1k2Hn2H;

%CALCULATION OF c24) See Section "Fourth-order Moment"
%
%      E{rl1H rm1H rl2 rm2}
%
Rl1Hm1H12m2=...
Kl1Hm1H12m2*E4malS'+Kl1Hm1H*E2malS'*E12m2 +...
Kl1H12*E2malS'*Em1Hm2+Kl1Hm2*E2malS'*Em1H12 +...
Km1H12*E2malS'*El1Hm2+Km1Hm2*E2malS'*El1H12 +...
K12m2*E2malS'*El1Hm1H+El1Hm1H12m2;

%CALCULATION OF c25) See Section "Fourth-order Moment"
%
%      E{rk1 rn1 rl2 rm2}
%
Rk1n112m2=...
Kk1n112m2*E4malS'+Kk1n1*E2malS'*E12m2 +...
```

```
Kk1l2*E2malS'*En1m2+Kk1m2*E2malS'*En1l2 +...
Kn1l2*E2malS'*Ek1m2+Kn1m2*E2malS'*Ek1l2 +...
Kl2m2*E2malS'*Ek1n1+Ek1n1l2m2;

%CALCULATION OF c26) See Section "Fourth-order Moment"
%
%       E{rl1H rm1H rk2H rn2H}
%
Rl1Hm1Hk2Hn2H=...
Kl1Hm1Hk2Hn2H*E4malS'+Kl1Hm1H*E2malS'*Ek2Hn2H +...
Kl1Hk2H*E2malS'*Em1Hn2H+Kl1Hn2H*E2malS'*Em1Hk2H +...
Km1Hk2H*E2malS'*El1Hn2H+Km1Hn2H*E2malS'*El1Hk2H +...
Kk2Hn2H*E2malS'*El1Hm1H+El1Hm1Hk2Hn2H;

%CALCULATION OF d) See Section "Second-order Moment"
%
%       E{rk1 rl1H}
%
Rk1l1H= Kk1l1H*E2malS'+Ek1l1H;

%CALCULATION OF d2) See Section "Second-order Moment"
%
%       E{rm2 rn2H}
%
Rm2n2H= Km2n2H*E2malS'+Em2n2H;

%CALCULATION OF d3) See Section "Fourth-order Moment"
%
%       E{rk2H rl2}
%
Rk2Hl2= Kk2Hl2*E2malS'+Ek2Hl2;

%CALCULATION OF d4) See Section "Fourth-order Moment"
%
%       E{rl2 rn2H}
%
Rl2n2H= Kl2n2H*E2malS'+El2n2H;

%CALCULATION OF d5) See Section "Fourth-order Moment"
%
%       E{rk2H rm2}
%
Rk2Hm2= Kk2Hm2*E2malS'+Ek2Hm2;
```

```
%CALCULATION OF d6) See Section "Fourth-order Moment"
%
%       E{rl2 rm2}
%
Rl2m2= Kl2m2*E2malS'+El2m2;

%CALCULATION OF d7) See Section "Fourth-order Moment"
%
%       E{rk2H rn2H}
%
Rk2Hn2H= Kk2Hn2H*E2malS'+Ek2Hn2H;

%CALCULATION OF d8) See Section "Fourth-order Moment"
%
%       E{rm1H rn1}
%
Rm1Hn1= Km1Hn1*E2malS'+Em1Hn1;

%CALCULATION OF d9) See Section "Fourth-order Moment"
%
%       E{rk1 rm1H}
%
Rk1m1H= Kk1m1H*E2malS'+Ek1m1H;

%CALCULATION OF d10) See Section "Fourth-order Moment"
%
%       E{rl1H rn1}
%
Rl1Hn1= Kl1Hn1*E2malS'+El1Hn1;

%CALCULATION OF d11) See Section "Fourth-order Moment"
%
%       E{rk1 rn1}
%
Rk1n1= Kk1n1*E2malS'+Ek1n1;

%CALCULATION OF d12) See Section "Fourth-order Moment"
%
%       E{rl1H rm1H}
%
Rl1Hm1H= Kl1Hm1H*E2malS'+El1Hm1H;
```

```
%CALCULATION OF  1)
%See Chapter "The finite-sample covariance of the fourth-order sample cumulant"
%
% sum(sum(E{rk1(t) rl1H(t) rm1H(t) rn1(t) rk2H(p) rl2(p) rm2(p) rn2H(p)}))
%  t   p
%
sum1=N*Rk1l1Hm1Hn1k2Hl2m2n2H+(N.^2-N)*Rk1l1Hm1Hn1*Rk2Hl2m2n2H;

%CALCULATION OF  2a)
%See Chapter "The finite-sample covariance of the fourth-order sample cumulant"
%
% sum(sum(sum(E{rk1(t) rl1H(t) rm1H(t) rn1(t) rk2H(p) rl2(p) rm2(q) rn2H(q)})))
%  t   p   q
%
sum2a= N     *Rk1l1Hm1Hn1k2Hl2m2n2H+...
     (N.^2-N)*Rk1l1Hm1Hn1*Rk2Hl2m2n2H+...
     (N.^2-N)*Rk1l1Hm1Hn1k2Hl2*Rm2n2H+...
     (N.^2-N)*Rk1l1Hm1Hn1m2n2H*Rk2Hl2+...
     (N.^3-3*N.^2+2*N)*Rk1l1Hm1Hn1*Rk2Hl2*Rm2n2H;

%CALCULATION OF  2b)
%See Chapter "The finite-sample covariance of the fourth-order sample cumulant"
% sum(sum(sum(E{rk1(t) rl1H(t) rm1H(t) rn1(t) rk2H(p) rm2(p) rl2(q) rn2H(q)})))
%  t   p   q
%
sum2b= N     *Rk1l1Hm1Hn1k2Hl2m2n2H+...
     (N.^2-N)*Rk1l1Hm1Hn1*Rk2Hl2m2n2H+...
     (N.^2-N)*Rk1l1Hm1Hn1k2Hm2*Rl2n2H+...
     (N.^2-N)*Rk1l1Hm1Hn1l2n2H*Rk2Hm2+...
     (N.^3-3*N.^2+2*N)*Rk1l1Hm1Hn1*Rk2Hm2*Rl2n2H;

%CALCULATION OF  2c)
%See Chapter "The finite-sample covariance of the fourth-order sample cumulant"
%
% sum(sum(sum(E{rk1(t) rl1H(t) rm1H(t) rn1(t) rk2H(p) rn2H(p) rl2(q) rm2(q)})))
%  t   p   q
%
sum2c= N     *Rk1l1Hm1Hn1k2Hl2m2n2H+...
     (N.^2-N)*Rk1l1Hm1Hn1*Rk2Hl2m2n2H+...
     (N.^2-N)*Rk1l1Hm1Hn1k2Hn2H*Rl2m2+...
     (N.^2-N)*Rk1l1Hm1Hn1l2m2*Rk2Hn2H+...
     (N.^3-3*N.^2+2*N)*Rk1l1Hm1Hn1*Rk2Hn2H*Rl2m2;
```

```
%CALCULATION OF  3a)
%See Chapter "The finite-sample covariance of the fourth-order sample cumulant"
%
%sum(sum(sum(sum(E{rk1(t) rl1H(t) rm1H(p) rn1(p) rk2H(q) rl2(q) rm2(s) rn2H(s)}))))
% t   p   q   s
%
sum3a=  N     *Rk1l1Hm1Hn1k2Hl2m2n2H+...
      (N.^2-N)*Rk1l1H*Rm1Hn1k2Hl2m2n2H+...
      (N.^2-N)*Rm1Hn1*Rk1l1Hk2Hl2m2n2H+...
      (N.^2-N)*Rk2Hl2*Rk1l1Hm1Hn1m2n2H+...
      (N.^2-N)*Rm2n2H*Rk1l1Hm1Hn1k2Hl2+...
      (N.^2-N)*Rk1l1Hm1Hn1*Rk2Hl2m2n2H+...
      (N.^2-N)*Rk1l1Hk2Hl2*Rm1Hn1m2n2H+...
      (N.^2-N)*Rk1l1Hm2n2H*Rm1Hn1k2Hl2+...
      (N.^3-3*N.^2+2*N)*Rk1l1Hm1Hn1*Rk2Hl2*Rm2n2H +...
      (N.^3-3*N.^2+2*N)*Rk1l1Hk2Hl2*Rm1Hn1*Rm2n2H +...
      (N.^3-3*N.^2+2*N)*Rk1l1Hm2n2H*Rm1Hn1*Rk2Hl2 +...
      (N.^3-3*N.^2+2*N)*Rm1Hn1k2Hl2*Rk1l1H*Rm2n2H +...
      (N.^3-3*N.^2+2*N)*Rm1Hn1m2n2H*Rk1l1H*Rk2Hl2 +...
      (N.^3-3*N.^2+2*N)*Rk2Hl2m2n2H*Rk1l1H*Rm1Hn1 +...
      (N.^4-6*N.^3+11*N.^2-6*N)*Rk1l1H*Rm1Hn1*Rk2Hl2*Rm2n2H;

%CALCULATION OF  3b)
%See Chapter "The finite-sample covariance of the fourth-order sample cumulant"
%
%sum(sum(sum(sum(E{rk1(t) rl1H(t) rm1H(p) rn1(p) rk2H(q) rm2(q) rl2(s) rn2H(s)}))))
% t   p   q   s
%
sum3b=  N     *Rk1l1Hm1Hn1k2Hl2m2n2H+...
      (N.^2-N)*Rk1l1H*Rm1Hn1k2Hl2m2n2H+...
      (N.^2-N)*Rm1Hn1*Rk1l1Hk2Hl2m2n2H+...
      (N.^2-N)*Rk2Hm2*Rk1l1Hm1Hn1l2n2H+...
      (N.^2-N)*Rl2n2H*Rk1l1Hm1Hn1k2Hm2+...
      (N.^2-N)*Rk1l1Hm1Hn1*Rk2Hl2m2n2H+...
      (N.^2-N)*Rk1l1Hk2Hm2*Rm1Hn1l2n2H+...
      (N.^2-N)*Rk1l1Hl2n2H*Rm1Hn1k2Hm2+...
      (N.^3-3*N.^2+2*N)*Rk1l1Hm1Hn1*Rk2Hm2*Rl2n2H +...
      (N.^3-3*N.^2+2*N)*Rk1l1Hk2Hm2*Rm1Hn1*Rl2n2H +...
      (N.^3-3*N.^2+2*N)*Rk1l1Hl2n2H*Rm1Hn1*Rk2Hm2 +...
      (N.^3-3*N.^2+2*N)*Rm1Hn1k2Hm2*Rk1l1H*Rl2n2H +...
      (N.^3-3*N.^2+2*N)*Rm1Hn1l2n2H*Rk1l1H*Rk2Hm2 +...
      (N.^3-3*N.^2+2*N)*Rk2Hl2m2n2H*Rk1l1H*Rm1Hn1 +...
      (N.^4-6*N.^3+11*N.^2-6*N)*Rk1l1H*Rm1Hn1*Rk2Hm2*Rl2n2H;
```

```
%CALCULATION OF  3c)
%See Chapter "The finite-sample covariance of the fourth-order sample cumulant"
%
%sum(sum(sum(sum(E{rk1(t) rl1H(t) rm1H(p) rn1(p) rk2H(q) rn2H(q) rl2(s) rm2(s)}))))
% t   p   q   s
%
sum3c=  N     *Rk1l1Hm1Hn1k2Hl2m2n2H+...
      (N.^2-N)*Rk1l1H*Rm1Hn1k2Hl2m2n2H+...
      (N.^2-N)*Rm1Hn1*Rk1l1Hk2Hl2m2n2H+...
      (N.^2-N)*Rk2Hn2H*Rk1l1Hm1Hn1l2m2+...
      (N.^2-N)*Rl2m2*Rk1l1Hm1Hn1k2Hn2H+...
      (N.^2-N)*Rk1l1Hm1Hn1*Rk2Hl2m2n2H+...
      (N.^2-N)*Rk1l1Hk2Hn2H*Rm1Hn1l2m2+...
      (N.^2-N)*Rk1l1Hl2m2*Rm1Hn1k2Hn2H+...
      (N.^3-3*N.^2+2*N)*Rk1l1Hm1Hn1*Rk2Hn2H*Rl2m2 +...
      (N.^3-3*N.^2+2*N)*Rk1l1Hk2Hn2H*Rm1Hn1*Rl2m2 +...
      (N.^3-3*N.^2+2*N)*Rk1l1Hl2m2*Rm1Hn1*Rk2Hn2H +...
      (N.^3-3*N.^2+2*N)*Rm1Hn1k2Hn2H*Rk1l1H*Rl2m2 +...
      (N.^3-3*N.^2+2*N)*Rm1Hn1l2m2*Rk1l1H*Rk2Hn2H +...
      (N.^3-3*N.^2+2*N)*Rk2Hl2m2n2H*Rk1l1H*Rm1Hn1 +...
      (N.^4-6*N.^3+11*N.^2-6*N)*Rk1l1H*Rm1Hn1*Rk2Hn2H*Rl2m2;

%CALCULATION OF  3d)
%See Chapter "The finite-sample covariance of the fourth-order sample cumulant"
%
%sum(sum(sum(sum(E{rk1(t) rm1H(t) rl1H(p) rn1(p) rk2H(q) rm2(q) rl2(s) rn2H(s)}))))
% t   p   q   s
%
sum3d=  N     *Rk1l1Hm1Hn1k2Hl2m2n2H+...
      (N.^2-N)*Rk1m1H*Rl1Hn1k2Hl2m2n2H+...
      (N.^2-N)*Rl1Hn1*Rk1m1Hk2Hl2m2n2H+...
      (N.^2-N)*Rk2Hm2*Rk1l1Hm1Hn1l2n2H+...
      (N.^2-N)*Rl2n2H*Rk1l1Hm1Hn1k2Hm2+...
      (N.^2-N)*Rk1l1Hm1Hn1*Rk2Hl2m2n2H+...
      (N.^2-N)*Rk1m1Hk2Hm2*Rl1Hn1l2n2H+...
      (N.^2-N)*Rk1m1Hl2n2H*Rl1Hn1k2Hm2+...
      (N.^3-3*N.^2+2*N)*Rk1l1Hm1Hn1*Rk2Hm2*Rl2n2H +...
      (N.^3-3*N.^2+2*N)*Rk1m1Hk2Hm2*Rl1Hn1*Rl2n2H +...
      (N.^3-3*N.^2+2*N)*Rk1m1Hl2n2H*Rl1Hn1*Rk2Hm2 +...
      (N.^3-3*N.^2+2*N)*Rl1Hn1k2Hm2*Rk1m1H*Rl2n2H +...
      (N.^3-3*N.^2+2*N)*Rl1Hn1l2n2H*Rk1m1H*Rk2Hm2 +...
      (N.^3-3*N.^2+2*N)*Rk2Hl2m2n2H*Rk1m1H*Rl1Hn1 +...
```

```
      (N.^4-6*N.^3+11*N.^2-6*N)*Rk1m1H*Rl1Hn1*Rk2Hm2*Rl2n2H;

%CALCULATION OF  3e)
%See Chapter "The finite-sample covariance of the fourth-order sample cumulant"
%
%sum(sum(sum(sum(E{rk1(t) rm1H(t) rl1H(p) rn1(p) rk2H(q) rn2H(q) rl2(s) rm2(s)}))))
% t   p   q   s
%
sum3e=  N     *Rk1l1Hm1Hn1k2Hl2m2n2H+...
      (N.^2-N)*Rk1m1H*Rl1Hn1k2Hl2m2n2H+...
      (N.^2-N)*Rl1Hn1*Rk1m1Hk2Hl2m2n2H+...
      (N.^2-N)*Rk2Hn2H*Rk1l1Hm1Hn1l2m2+...
      (N.^2-N)*Rl2m2*Rk1l1Hm1Hn1k2Hn2H+...
      (N.^2-N)*Rk1l1Hm1Hn1*Rk2Hl2m2n2H+...
      (N.^2-N)*Rk1m1Hk2Hn2H*Rl1Hn1l2m2+...
      (N.^2-N)*Rk1m1Hl2m2*Rl1Hn1k2Hn2H+...
      (N.^3-3*N.^2+2*N)*Rk1l1Hm1Hn1*Rk2Hn2H*Rl2m2 +...
      (N.^3-3*N.^2+2*N)*Rk1m1Hk2Hn2H*Rl1Hn1*Rl2m2 +...
      (N.^3-3*N.^2+2*N)*Rk1m1Hl2m2*Rl1Hn1*Rk2Hn2H +...
      (N.^3-3*N.^2+2*N)*Rl1Hn1k2Hn2H*Rk1m1H*Rl2m2 +...
      (N.^3-3*N.^2+2*N)*Rl1Hn1l2m2*Rk1m1H*Rk2Hn2H +...
      (N.^3-3*N.^2+2*N)*Rk2Hl2m2n2H*Rk1m1H*Rl1Hn1 +...
      (N.^4-6*N.^3+11*N.^2-6*N)*Rk1m1H*Rl1Hn1*Rk2Hn2H*Rl2m2;

%CALCULATION OF  3f)
%See Chapter "The finite-sample covariance of the fourth-order sample cumulant"
%
%sum(sum(sum(sum(E{rk1(t) rn1(t) rl1H(p) rm1H(p) rk2H(q) rn2H(q) rl2(s) rm2(s)}))))
% t   p   q   s
%
sum3f=  N     *Rk1l1Hm1Hn1k2Hl2m2n2H+...
      (N.^2-N)*Rk1n1*Rl1Hm1Hk2Hl2m2n2H+...
      (N.^2-N)*Rl1Hm1H*Rk1n1k2Hl2m2n2H+...
      (N.^2-N)*Rk2Hn2H*Rk1l1Hm1Hn1l2m2+...
      (N.^2-N)*Rl2m2*Rk1l1Hm1Hn1k2Hn2H+...
      (N.^2-N)*Rk1l1Hm1Hn1*Rk2Hl2m2n2H+...
      (N.^2-N)*Rk1n1k2Hn2H*Rl1Hm1Hl2m2+...
      (N.^2-N)*Rk1n1l2m2*Rl1Hm1Hk2Hn2H+...
      (N.^3-3*N.^2+2*N)*Rk1l1Hm1Hn1*Rk2Hn2H*Rl2m2 +...
      (N.^3-3*N.^2+2*N)*Rk1n1k2Hn2H*Rl1Hm1H*Rl2m2 +...
      (N.^3-3*N.^2+2*N)*Rk1n1l2m2*Rl1Hm1H*Rk2Hn2H +...
      (N.^3-3*N.^2+2*N)*Rl1Hm1Hk2Hn2H*Rk1n1*Rl2m2 +...
      (N.^3-3*N.^2+2*N)*Rl1Hm1Hl2m2*Rk1n1*Rk2Hn2H +...
```

```
    (N.^3-3*N.^2+2*N)*Rk2Hl2m2n2H*Rk1n1*Rl1Hm1H +...
    (N.^4-6*N.^3+11*N.^2-6*N)*Rk1n1*Rl1Hm1H*Rk2Hn2H*Rl2m2;

if nargin<6, alpha=(N.^2-N)./(N+2); end;
if nargin<7, beta=N.^2-N; end;

%
% Calculation of the expected value E{c^}
%

N=N(:)';
1N=ones(1,length(N));
Ecall1 = N.*(1N./alpha-3*1N./beta)*Rk1l1Hm1Hn1...
         -(N.^2-N)./beta*(Rk1l1H*Rm1Hn1+Rk1m1H*Rl1Hn1+Rk1n1*Rl1Hm1H);
Ecall2 = N.*(1N./alpha-3*1N./beta)*Rk2Hl2m2n2H...
         -(N.^2-N)./beta*(Rk2Hl2*Rm2n2H+Rk2Hm2*Rl2n2H+Rk2Hn2H*Rl2m2);

%
% ...finally the last step!!!!
%
cov4=sum1./alpha.^2 -2*real(sum2a+sum2b+sum2c)./(alpha.*beta) +...
        (sum3a+sum3d+sum3f)./beta.^2 + ...
        2*real(sum3b+sum3c+sum3e)./beta.^2-...
        Ecall1.*Ecall2;

if nargout>4
  cov4asymp=cov4(length(cov4));
  cov4=cov4(1:length(cov4)-1);
  Ecall1=Ecall1(1:length(Ecall1)-1);
  Ecall2=Ecall2(1:length(Ecall2)-1);
end

%Calculation of the covariance of the "Biased Estimator"
if nargout>3
   Ecall1b = (1N-3*1N./N)*Rk1l1Hm1Hn1...
             -(1N-1N./N)*(Rk1l1H*Rm1Hn1+Rk1m1H*Rl1Hn1+Rk1n1*Rl1Hm1H);
   Ecall2b = (1N-3*1N./N)*Rk2Hl2m2n2H...
           -(1N-1N./N)*(Rk2Hl2*Rm2n2H+Rk2Hm2*Rl2n2H+Rk2Hn2H*Rl2m2);

   cov4biased = sum1./N.^2 -2*real(sum2a+sum2b+sum2c)./(N.^3) +...
                (sum3a+sum3d+sum3f)./N.^4 + ...
                2*real(sum3b+sum3c+sum3e)./N.^4-...
                Ecall1b.*Ecall2b;
```

```
    if nargout>4
      cov4biasedasymp=cov4biased(length(cov4biased));
      cov4biased=cov4biased(1:length(cov4biased)-1);
      Ecall1b=Ecall1b(1:length(Ecall1b)-1);
      Ecall2b=Ecall2b(1:length(Ecall2b)-1);
    end
end

%Calculation of the relative error between the asymptotic and the finite
%data length covariance

if (nargout>4)
  N=N(1:length(N)-1);
  relerror=(N.*cov4-Nasymp*cov4asymp)./(N.*cov4);
  if nargout>5
    relerrorbiased=(N.*cov4biased-Nasymp*cov4biasedasymp)./(N.*cov4biased);
  end
end
```

4.4 xmaln.m

```
function xmn=xmaln(index,gn)
%XMALN  M-File for simplifying the notation in covxall.m (x=2,3,4).
%
%       Syntax:
%                xmn = xmaln(index,gn)
%       where
%             index = contains the indices of the moment, eventually
%                     multiplied with an "j" to indicate that the
%                     correspondent random variable is conjugated.
%                     Example: For calculating E{n1 n2 n5 n2^H n7^H}
%
%                              index = [1 2 5 j*2 j*7]
%
%             gn = [gRn1,2 gRn1,3 .... gRn1,K
%                   gIn1,2 gIn1,3 .... gIn1,K
%                     :      :     :     :
%                   gRnM,2 gRnM,3 .... gRnM,K
%                   gInM,2 gInM,3 .... gInM,K]
%
%                   Matrix that contains all moments of the real- and
%                   imaginary signal part from the second to the Kth-
```

```
%                    moment, where K=length(index).
%
%                    The following example explains the abbrevations:
%
%                        gRn1,3 = 3rd moment of the realpart of signal 1

%Extracting of the important information in the index-vector
index1=abs(index);
index2=-2*sign(abs(index-real(index)))+1; %index2=1 if random variable is real
                                          %       =-1 if  "       "      " imag.
K=length(index);

%Insertion of the first moment
[H,gn_cols]=size(gn);
gn=[zeros(H,1) gn];

[sig_ind,I]=sort(index1);
sig_ind=[sig_ind -1];
index2=index2(I);          %Proper sorting of index2

xmn=1;
momz=1;
for i=2:K+1   %Loop for determining equal random variables
  if sig_ind(i)==sig_ind(i-1)
    momz=momz+1;
  else
    sum=0;
    hindex2=index2(i-momz:i-1);
    for m=1:2^momz  %Loop on all 2^momz summands
      M=gerade(ceil(m*(ones(1,momz)./exp((momz-1:-1:0)*log(2)))))+1;
      [sig_ind2,I2]=sort(M);
      sig_ind2=[sig_ind2 -1];
      hhindex2=hindex2(I2);
      momz2=1;
      hilf=1;
      hj=1;
      for l=2:momz+1
        if sig_ind2(l)==sig_ind2(l-1)
          momz2=momz2+1;
          Ind=2*sig_ind(i-1)+sig_ind2(l-1)-2;
          if gerade(Ind)
            hj=hj*sqrt(-1)*hhindex2(l-1);
          end
```

```
        else
          Ind=2*sig_ind(i-1)+sig_ind2(l-1)-2;
          if gerade(Ind)
            hilf=hilf*gn(Ind,momz2)*sqrt(-1)*hhindex2(l-1)*hj;
          else
            hilf=hilf*gn(Ind,momz2);
          end
          momz2=1;
        end
      end
      sum=sum+hilf;
    end
    xmn=xmn*sum;
    momz=1;
  end
end
```

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