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**Subspace-Based Direction Finding Methods**

**by**

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# Summary

The direction finding problem is to estimate the bearings [i.e., directions of arrival (DOA)] of a collection of sources from an array of measurements made at a collection of time points. In applications, the Rayleigh criterion sets a bound on the resolution power of classical direction finding methods. In this report we summarize many of the so-called super-resolution subspace-based direction finding methods which may overcome the Rayleigh bound. We divide these methods into two classes, those that use second-order statistics, and those that use second- and higher-order statistics.

Not only do we describe all of the popular subspace-based algorithms, but we also provide flowcharts and figures that guide a reader to understand when to use a second-order or a higher-order statistics based method. Using these flowcharts and figures, it is possible for a potential user of a subspace-based direction finding method to decide which method(s) is (are) most likely to give best results for his/her application.

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# Subspace-Based Direction Finding Methods

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## 1 Introduction

Estimating bearings of multiple narrowband signals from measurements collected by an array of sensors has been a very active research problem for the last two decades. Typical applications of this problem are radar, communication, and underwater acoustics. Many algorithms have been proposed to solve the bearing estimation problem. One of the first techniques that appeared was beamforming which has a resolution limited by the array structure. Spectral estimation techniques were also applied to the problem. However, these techniques fail to resolve closely spaced arrival angles for low signal-to-noise ratios. Another approach is the maximum-likelihood (ML) solution. This approach has been well documented in the literature. In the stochastic ML method [Schmidt, 1981], the signals are assumed to be Gaussian whereas they are regarded arbitrary and deterministic in the deterministic ML method [Wax, 1985]. The sensor noise is modeled as Gaussian in both methods, which is a reasonable assumption due to the central limit theorem. The stochastic ML

estimates of the bearings achieve the Cramer-Rao bound (CRB). On the other hand this does not hold for deterministic ML estimates [Stoica and Nehorai, 1990]. The common problem with the ML methods in general is the necessity of solving a nonlinear multidimensional optimization problem which has a high computational cost and for which there is no guarantee of global convergence. *Subspace-based* (or, super-resolution) approaches have attracted much attention, after the work of [Schmidt, 1981], due to their computational simplicity as compared to the ML approach, and their possibility of overcoming the Rayleigh bound on the resolution power of classical direction finding methods. Subspace-based direction finding methods are summarized in this section.

## 2 Formulation of the Problem

Consider an array of $M$ antenna elements receiving a set of plane waves emitted by $P$ ($P < M$) sources in the far field of the array. We assume a narrow-band propagation model, i.e., the signal envelopes do not change during the time it takes for the wavefronts to travel from one sensor to another. Suppose that the signals have a common frequency of $f_0$; then, the wavelength $\lambda = c/f_0$ where $c$ is the speed of propagation. The received $M$-vector $\mathbf{r}(t)$ at time $t$ is

$$\mathbf{r}(t) = \mathbf{A}\mathbf{s}(t) + \mathbf{n}(t) \tag{1}$$

where $\mathbf{s}(t) = [s_1(t), \cdots, s_P(t)]^T$ is the $P$-vector of sources; $\mathbf{A} = [\mathbf{a}(\theta_1), \cdots, \mathbf{a}(\theta_P)]$ is the $M \times P$ steering matrix in which $\mathbf{a}(\theta_i)$, the $i$th steering vector, is the response of the array to the $i$th source arriving from $\theta_i$; and, $\mathbf{n}(t) = [n_1(t), \cdots, n_M(t)]^T$ is an additive noise process.

We assume: (1) The source signals may be statistically independent, partially correlated, or completely correlated (i.e., coherent); the distributions are unknown; (2) The array may have an

arbitrary shape and response; and, (3) The noise process is independent of the sources, zero-mean, and it may be either partially white or colored; its distribution is unknown. These assumptions will be relaxed, as required by specific methods, as we proceed.

The direction finding problem is to estimate the bearings [i.e., directions of arrival (DOA)] $\{\theta_i\}_{i=1}^{P}$ of the sources from the snapshots $\mathbf{r}(t)$, $t = 1, \cdots, N$.

In applications, the Rayleigh criterion sets a bound on the resolution power of classical direction finding methods. In the next sections we summarize some of the so-called super-resolution direction finding methods which may overcome the Rayleigh bound. We divide these methods into two classes, those that use second-order and those that use second- and higher-order statistics.

## 3 Second-Order Statistics-Based Methods

The second-order methods use the sample estimate of the array spatial covariance matrix $\mathbf{R} = E\{\mathbf{r}(t)\mathbf{r}(t)^H\} = \mathbf{A}\mathbf{R}_s\mathbf{A}^H + \mathbf{R}_n$, where $\mathbf{R}_s = E\{\mathbf{s}(t)\mathbf{s}(t)^H\}$ is the $P \times P$ signal covariance matrix and $\mathbf{R}_n = E\{\mathbf{n}(t)\mathbf{n}(t)^H\}$ is the $M \times M$ noise covariance matrix. For the time being, let us assume that the noise is spatially white, i.e., $\mathbf{R}_n = \sigma^2\mathbf{I}$. If the noise is colored and its covariance matrix is known or can be estimated, the measurements can be "whitened" by multiplying the measurements from the left by the matrix $\mathbf{\Lambda}^{-1/2}\mathbf{E}_n^H$ obtained by the orthogonal eigendecomposition $\mathbf{R}_n = \mathbf{E}_n\mathbf{\Lambda}\mathbf{E}_n^H$. The array spatial covariance matrix is estimated as $\hat{\mathbf{R}} = \sum_{t=1}^{N} \mathbf{r}(t)\mathbf{r}(t)^H/N$.

Some spectral estimation approaches to the direction finding problem are based on optimization. Consider the *minimum variance* algorithm, for example. The received signal is processed by a beamforming vector $\mathbf{w}_o$ which is designed such that the output power is minimized subject to the constraint that a signal from a desired direction is passed to the output with unit gain. Solving

this optimization problem, we obtain the array output power as a function of the arrival angle $\theta$ as $P_{mv}(\theta) = \frac{1}{\mathbf{a}^H(\theta)\hat{\mathbf{R}}^{-1}\mathbf{a}(\theta)}$. The arrival angles are obtained by scanning the range $[-90^o, 90^o]$ of $\theta$ and locating the peaks of $P_{mv}(\theta)$. At low signal-to-noise ratios the conventional methods, like minimum variance, fail to resolve closely spaced arrival angles. The resolution of conventional methods are limited by signal-to-noise ratio even if exact $\mathbf{R}$ is used, whereas in subspace methods, there is no resolution limit; hence, the latter are also referred to as *super-resolution* methods. The limit comes from the sample estimate of $\mathbf{R}$.

The subspace-based methods exploit the eigendecomposition of the estimated array covariance matrix $\hat{\mathbf{R}}$. To see the implications of the eigendecomposition of $\hat{\mathbf{R}}$, let us first state the properties of $\mathbf{R}$: (1) If the source signals are independent or partially correlated, $rank(\mathbf{R}_s) = P$. If there are coherent sources, $rank(\mathbf{R}_s) < P$. In the methods explained in Sections 3.1 and 3.2, except for the WSF method (see Section 3.1.1), it will be assumed that there are no coherent sources. The coherent signals case is described in Section 3.3. (2) If the columns of $\mathbf{A}$ are independent, which is generally true when the source bearings are different, then $\mathbf{A}$ is of full-rank $P$. (3) Properties 1 and 2 imply $rank(\mathbf{A}\mathbf{R}_s\mathbf{A}^H) = P$; therefore, $\mathbf{A}\mathbf{R}_s\mathbf{A}^H$ must have $P$ nonzero eigenvalues and $M - P$ zero eigenvalues. Let the eigendecomposition of $\mathbf{A}\mathbf{R}_s\mathbf{A}^H$ be $\mathbf{A}\mathbf{R}_s\mathbf{A}^H = \sum_{i=1}^{M} \alpha_i \mathbf{e}_i \mathbf{e}_i^H$; then $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_P \geq \alpha_{P+1} = \cdots = \alpha_M = 0$ are the rank-ordered eigenvalues, and $\{\mathbf{e}_i\}_{i=1}^{M}$ are the corresponding eigenvectors. (4) Because $\mathbf{R}_n = \sigma^2\mathbf{I}$, the eigenvectors of $\mathbf{R}$ are the same as those of $\mathbf{A}\mathbf{R}_s\mathbf{A}^H$, and its eigenvalues are $\lambda_i = \alpha_i + \sigma^2$, if $1 \leq i \leq P$, or $\lambda_i = \sigma^2$, if $P + 1 \leq i \leq M$. The eigenvectors can be partitioned into two sets: $\mathbf{E}_s \triangleq [\mathbf{e}_1, \cdots, \mathbf{e}_P]$ forms the *signal subspace*, whereas $\mathbf{E}_n \triangleq [\mathbf{e}_{P+1}, \cdots, \mathbf{e}_M]$ forms the *noise subspace*. These subspaces are orthogonal. The signal eigenvalues $\mathbf{\Lambda}_s \triangleq diag\{\lambda_1, \cdots, \lambda_P\}$, and the noise eigenvalues $\mathbf{\Lambda}_n \triangleq diag\{\lambda_{P+1}, \cdots, \lambda_M\}$. (5) The eigenvectors corresponding to zero eigenvalues satisfy $\mathbf{A}\mathbf{R}_s\mathbf{A}^H\mathbf{e}_i = \mathbf{0}$, $i = P + 1, \cdots, M$;

hence, $\mathbf{A}^H\mathbf{e}_i = \mathbf{0}$, $i = P+1, \cdots, M$, because $\mathbf{A}$ and $\mathbf{R}_s$ are full rank. This last equation means that *steering vectors are orthogonal to noise subspace eigenvectors.* It further implies that, because of the orthogonality of signal and noise subspaces, *spans of signal eigenvectors and steering vectors are equal.* Consequently there exists a nonsingular $P \times P$ matrix $\mathbf{T}$ such that $\mathbf{E}_s = \mathbf{AT}$.

Alternatively, the signal and noise subspaces can also be obtained by performing a singular value decomposition directly on the received data without having to calculate the array covariance matrix. Li and Vaccaro [1991] state that the properties of the bearing estimates do not depend on which method is used; however, singular value decomposition must then deal with a data matrix that increases in size as the new snapshots are received. In the sequel, we assume that the array covariance matrix is estimated from the data and an eigendecomposition is performed on the estimated covariance matrix.

The eigenvalue decomposition of the spatial array covariance matrix, and the eigenvector partitionment into signal and noise subspaces, leads to a number of subspace-based direction finding methods. The signal subspace contains information about where the signals are whereas the noise subspace informs us where they are not. Use of either subspace results in better resolution performance than conventional methods. In practice, the performance of the subspace-based methods is limited fundamentally by the accuracy of separating the two subspaces when the measurements are noisy [Marple, 1987]. These methods can be broadly classified into signal subspace and noise subspace methods. A summary of direction-finding methods based on both approaches follows next.

## 3.1 Signal Subspace Methods

In these methods, only the signal subspace information is retained. Their rationale is that by discarding the noise subspace we effectively enhance the SNR, because the contribution of the noise power to the covariance matrix is eliminated. Signal subspace methods are divided into search-based and algebraic methods, which are explained next.

### 3.1.1 Search-based methods

In search-based methods, it is assumed that the response of the array to a single source, *the array manifold* $\mathbf{a}(\theta)$, is either known analytically as a function of arrival angle, or is obtained through the calibration of the array. For example, for an M-element uniform linear array, the array response to a signal from angle $\theta$ is analytically known and is given by $\mathbf{a}(\theta) = \left[1, e^{-j2\pi\frac{d}{\lambda}\sin(\theta)}, \cdots, e^{-j2\pi(M-1)\frac{d}{\lambda}\sin(\theta)}\right]^T$, where $d$ is the separation between the elements, and $\lambda$ is the wavelength.

In search-based methods to follow (except for the subspace fitting methods), which are spatial versions of widely-known power spectral density estimators, the estimated array covariance matrix is approximated by its signal subspace eigenvectors, or its *principal components*, as $\hat{\mathbf{R}} \approx \sum_{i=1}^{P} \lambda_i \mathbf{e}_i \mathbf{e}_i^H$. Then the arrival angles are estimated by locating the peaks of a function, $S(\theta)$ $(-90^o \leq \theta \leq 90^o)$ which depends on the particular method. Some of these methods and the associated function $S(\theta)$ are summarized in the following [Johnson and Dudgeon, 1993, Marple, 1987, Nikias and Petropulu, 1993]:

**Correlogram method:** In this method, $S(\theta) = \mathbf{a}(\theta)^H \hat{\mathbf{R}} \mathbf{a}(\theta)$. The resolution obtained from the Correlogram method is lower than that obtained from the MV and AR methods.

**Minimum Variance (MV) [Capon, 1969] method:** In this method, $S(\theta) = 1/\mathbf{a}(\theta)^H \hat{\mathbf{R}}^{-1} \mathbf{a}(\theta)$.

The MV method is known to have a higher resolution than the correlogram method, but lower resolution and variance than the AR method.

**Autoregressive (AR) method:** In this method, $S(\theta) = 1/|\mathbf{u}^T\hat{\mathbf{R}}^{-1}\mathbf{a}(\theta)|^2$ where $\mathbf{u} = [1, 0, \cdots, 0]^T$. This method is known to have a better resolution than the previous ones.

**Subspace Fitting (SSF) and Weighted Subspace Fitting (WSF) Methods:** In Section 2 we saw that the spans of signal eigenvectors and steering vectors are equal; therefore, bearings can be solved from the best least-squares fit of the two spanning sets when the array is calibrated [Viberg and Ottersten, 1991]. In the Subspace Fitting Method the criterion $[\hat{\theta}, \hat{\mathbf{T}}] = argmin\ ||\mathbf{E}_s\mathbf{W}^{1/2} - \mathbf{A}(\theta)\mathbf{T}||^2$ is used, where $||.||$ denotes the Frobenius norm, $\mathbf{W}$ is a positive definite weighting matrix, $\mathbf{E}_s$ is the matrix of signal subspace eigenvectors, and the notation for the steering matrix is changed to show its dependence on the bearing vector $\theta$. This criterion can be minimized directly with respect to $\mathbf{T}$, and the result for $\mathbf{T}$ can then be substituted back into it, so that $\hat{\theta} = argmin\ Tr\{(\mathbf{I} - \mathbf{A}(\theta)\mathbf{A}(\theta)^{\#})\mathbf{E}_s\mathbf{W}\mathbf{E}_s^H\}$, where $\mathbf{A}^{\#} = (\mathbf{A}^H\mathbf{A})^{-1}\mathbf{A}^H$.

Viberg and Ottersten have shown that a class of direction finding algorithms can be approximated by this subspace fitting formulation for appropriate choices of the weighting matrix $\mathbf{W}$. For example, for the deterministic ML method $\mathbf{W} = \mathbf{\Lambda}_s - \sigma^2\mathbf{I}$, which is implemented using the empirical values of the signal eigenvalues, $\mathbf{\Lambda}_s$, and the noise eigenvalue $\sigma^2$. TLS-ESPRIT, which is explained in the next subsection, can also be formulated in a similar but more involved way. Viberg and Ottersten have also derived an optimal Weighted Subspace Fitting (WSF) Method, which yields the smallest estimation error variance among the class of subspace fitting methods. In WSF, $\mathbf{W} = (\mathbf{\Lambda}_s - \sigma^2\mathbf{I})^2\mathbf{\Lambda}_s^{-1}$. The WSF method works regardless of the source covariance (including coherence) and has been shown to have the same asymptotic properties as the stochastic ML method; hence, it is asymptotically efficient for Gaussian signals (i.e., it achieves the stochastic

CRB). Its behavior in the finite sample case may be different from the asymptotic case [Viberg, Ottersten and Kailath, 1991]. Viberg and Ottersten have also shown that the asymptotic properties of the WSF estimates are identical for both cases of Gaussian and non-Gaussian sources. They have also developed a consistent detection method for arbitrary signal correlation, and an algorithm for minimizing the WSF criterion. They do point out several practical implementation problems of their method, such as the need for accurate calibrations of the array manifold and knowledge of the derivative of the steering vectors w.r.t $\theta$. For nonlinear and nonuniform arrays, multidimensional search methods are required for SSF, hence it is computationally expensive.

### 3.1.2 Algebraic methods

Algebraic methods do not require a search procedure and yield DOA estimates directly.

**ESPRIT (Estimation of Signal Parameters via Rotational Invariance Techniques) [Paulraj, Roy and Kailath, 1985]:** The ESPRIT Algorithm requires "translationally invariant"arrays, i.e., an array with its *identical copy* displaced in space. The geometry and response of the arrays do not have to be known; only the measurements from these arrays, and the displacement between the identical arrays are required. The computational complexity of ESPRIT is less than that of the search-based methods.

Let $\mathbf{r}^1(t)$ and $\mathbf{r}^2(t)$ be the measurements from these arrays. Due to the displacement of the arrays the following holds: $\mathbf{r}^1(t) = \mathbf{A}\mathbf{s}(t) + \mathbf{n}_1(t)$ and $\mathbf{r}^2(t) = \mathbf{A}\mathbf{\Phi}\mathbf{s}(t) + \mathbf{n}_2(t)$, where $\mathbf{\Phi} = diag\{e^{-j2\pi\frac{d}{\lambda}\sin\theta_1}, \cdots, e^{-j2\pi\frac{d}{\lambda}\sin\theta_P}\}$ in which $d$ is the separation between the identical arrays, and the angles $\{\theta_i\}_{i=1}^P$ are measured with respect to the normal to the displacement vector between the identical arrays. Note that the auto covariance of $\mathbf{r}^1(t)$, $\mathbf{R}^{11}$, and the cross covariance between $\mathbf{r}^1(t)$ and $\mathbf{r}^2(t)$, $\mathbf{R}^{21}$, are given by $\mathbf{R}^{11} = \mathbf{A}\mathbf{D}\mathbf{A}^H + \mathbf{R}_{n_1}$ and $\mathbf{R}^{21} = \mathbf{A}\mathbf{\Phi}\mathbf{D}\mathbf{A}^H + \mathbf{R}_{n_2 n_1}$, where $\mathbf{D}$ is

the covariance matrix of the sources, and $\mathbf{R}_{n_1}$ and $\mathbf{R}_{n_2 n_1}$ are the noise auto- and cross-covariance matrices.

The ESPRIT algorithm solves for $\Phi$, which then gives the bearing estimates. Although the subspace separation concept is not used in ESPRIT, its LS and TLS versions are based on a signal subspace formulation. The LS and TLS versions are more complicated, but are more accurate than the original ESPRIT, and are summarized in the next subsection. Here we summarize the original ESPRIT:

(1) Estimate the autocovariance of $\mathbf{r}^1(t)$ and cross covariance between $\mathbf{r}^1(t)$ and $\mathbf{r}^2(t)$, as $\mathbf{R}^{11} = \frac{1}{N}\sum_{t=1}^{N}\mathbf{r}^1(t)\mathbf{r}^1(t)^H$ and $\mathbf{R}^{21} = \frac{1}{N}\sum_{t=1}^{N}\mathbf{r}^2(t)\mathbf{r}^1(t)^H$. (2) Calculate $\hat{\mathbf{R}}^{11} = \mathbf{R}^{11} - \mathbf{R}_{n_1}$ and $\hat{\mathbf{R}}^{21} = \mathbf{R}^{21} - \mathbf{R}_{n_2 n_1}$ where $\mathbf{R}_{n_1}$ and $\mathbf{R}_{n_2 n_1}$ are the estimated noise covariance matrices. (3) Find the singular values $\lambda_i$ of the matrix pencil $\hat{\mathbf{R}}^{11} - \lambda_i \hat{\mathbf{R}}^{21}$, $i = 1, \cdots, P$. (4) The bearings, $\theta_i$ $(i = 1, \cdots, P)$, are readily obtained by solving the equation $\lambda_i = e^{j2\pi \frac{d}{\lambda}\sin\theta_i}$ for $\theta_i$. In the above steps, it is assumed that the noise is spatially and temporally white or the covariance matrices $\mathbf{R}_{n_1}$ and $\mathbf{R}_{n_2 n_1}$ are known.

**LS and TLS ESPRIT [Roy, 1987]:**

(1) Follow Steps 1 and 2 of ESPRIT; (2) stack $\hat{\mathbf{R}}^{11}$ and $\hat{\mathbf{R}}^{21}$ into a $2M \times M$ matrix $\mathbf{R}$, as $\mathbf{R} \triangleq \left[\hat{\mathbf{R}}^{11T} \;\; \hat{\mathbf{R}}^{21T}\right]^T$, and, perform an SVD of $\mathbf{R}$, keeping the first $2M \times P$ submatrix of the left singular vectors of $\mathbf{R}$. Let this submatrix be $\mathbf{E}_s$; (3) partition $\mathbf{E}_s$ into two $M \times P$ matrices $\mathbf{E}_{s1}$ and $\mathbf{E}_{s2}$ such that $\mathbf{E}_s = \left[\mathbf{E}_{s1}{}^T \;\; \mathbf{E}_{s2}{}^T\right]^T$. (4) For LS-ESPRIT, calculate the eigendecomposition of $(\mathbf{E}_{s1}^H \mathbf{E}_{s1})^{-1}\mathbf{E}_{s1}^H \mathbf{E}_{s2}$. The eigenvalue matrix gives $\Phi = diag\{e^{-j2\pi \frac{d}{\lambda}\sin\theta_1}, \cdots, e^{-j2\pi \frac{d}{\lambda}\sin\theta_P}\}$ from which the arrival angles are readily obtained. For TLS-ESPRIT, proceed as follows: (5) Perform an SVD of the $M \times 2P$ matrix $[\mathbf{E}_{s1}, \mathbf{E}_{s2}]$, and stack the last $P$ right singular vectors of $[\mathbf{E}_{s1}, \mathbf{E}_{s2}]$ into a $2P \times P$ matrix denoted $\mathbf{F}$; (6) Partition $\mathbf{F}$ as $\mathbf{F} \triangleq \left[\mathbf{F}_x{}^T \;\; \mathbf{F}_y{}^T\right]^T$ where $\mathbf{F}_x$

and $\mathbf{F}_y$ are $P \times P$; (7) Perform the eigendecomposition of $-\mathbf{F}_x\mathbf{F}_y^{-1}$. The eigenvalue matrix gives $\Phi = diag\{e^{-j2\pi\frac{d}{\lambda}\sin\theta_1}, \cdots, e^{-j2\pi\frac{d}{\lambda}\sin\theta_P}\}$ from which the arrival angles are readily obtained.

Different versions of ESPRIT have different statistical properties. The Toeplitz Approximation Method (TAM) [Kung, Lo and Foka, 1986], in which the array measurement model is represented as a state-variable model, although different in implementation from LS-ESPRIT, is equivalent to LS-ESPRIT; hence, it has the same error variance as LS-ESPRIT.

**Generalized Eigenvalues Utilizing Signal Subspace Eigenvectors (GEESE) [Kwon and Pillai, 1989]:** (1) Follow Steps 1-3 of TLS ESPRIT. (2) Find the singular values $\lambda_i$ of the pencil $\mathbf{E}_{s1} - \lambda_i\mathbf{E}_{s2}$, $i = 1, \cdots, P$; (3) The bearings, $\theta_i$ $(i = 1, \cdots, P)$, are readily obtained from $\lambda_i = e^{j2\pi\frac{d}{\lambda}\sin\theta_i}$. The GEESE method is claimed to be better than ESPRIT [Kwon and Pillai, 1989].

## 3.2 Noise Subspace Methods

These methods, in which only the noise subspace information is retained, are based on the property that the steering vectors are orthogonal to any linear combination of the noise subspace eigenvectors. Noise subspace methods are also divided into search-based and algebraic methods, which are explained next.

### 3.2.1 Search-based methods

In search-based methods, the array manifold is assumed to be known, and the arrival angles are estimated by locating the peaks of the function $S(\theta) = 1/\mathbf{a}(\theta)^H\mathbf{N}\mathbf{a}(\theta)$ where N is a matrix formed using the noise space eigenvectors.

**Pisarenko method:** In this method, $\mathbf{N} = \mathbf{e}_M\mathbf{e}_M{}^H$, where $\mathbf{e}_M$ is the eigenvector corresponding to the minimum eigenvalue of R. If the minimum eigenvalue is repeated, any unit-norm vector

which is a linear combination of the eigenvectors corresponding to the minimum eigenvalue can be used as $\mathbf{e}_M$. The basis of this method is that when the search angle $\theta$ corresponds to an actual arrival angle, the denominator of $S(\theta)$ in the Pisarenko method, $|\mathbf{a}(\theta)^H \mathbf{e}_M|^2$, becomes small due to orthogonality of steering vectors and noise subspace eigenvectors; hence, $S(\theta)$ will peak at an arrival angle.

**MUSIC (Multiple Signal Classification) [Schmidt, 1981] method:** In this method, $\mathbf{N} = \sum_{i=P+1}^{M} \mathbf{e}_i \mathbf{e}_i^H$. The idea is similar to that of the Pisarenko method; the inner product $|\mathbf{a}(\theta)^H \sum_{i=P+1}^{M} \mathbf{e}_i|^2$ is small when $\theta$ is an actual arrival angle. An obvious signal-subspace formulation of MUSIC is also possible. The MUSIC spectrum is equivalent to the MV method using the exact covariance matrix when SNR is infinite, and therefore performs better than the MV method.

Asymptotic properties of MUSIC are well established [e.g., Stoica and Nehorai 1989, Swindlehurst and Kailath, 1992], e.g., MUSIC is known to have the same asymptotic variance as the deterministic ML method for uncorrelated sources. It is shown by Xu and Buckley [1994] that although, asymptotically, bias is insignificant compared to standard deviation, it is an important factor limiting the performance for resolving closely spaced sources when they are correlated.

In order to overcome the problems due to finite sample effects and source correlation, a multidimensional (MD) version of MUSIC has been proposed [Schmidt, 1981, Roy, 1987]; however, this approach involves a computationally involved search, as in the ML method. MD MUSIC can be interpreted as a norm minimization problem, as shown in [Ephraim, Merhav and Van Trees, 1995]; using this interpretation, strong consistency of MD MUSIC has been demonstrated. An optimally weighted version of MD MUSIC, which outperforms the deterministic ML method, has also been proposed in [Viberg and Ottersten, 1991].

**Eigenvector (EV) method:** In this method, $\mathbf{N} = \sum_{i=P+1}^{M} \frac{1}{\lambda_i} \mathbf{e}_i \mathbf{e}_i^H$. The only difference between

the EV method and MUSIC is the use of inverse eigenvalue (the $\lambda_i$ are the noise subspace eigenvalues of $\mathbf{R}$) weighting in EV and unity weighting in MUSIC, which causes EV to yield fewer spurious peaks than MUSIC [Johnson and Dudgeon, 1993]. The EV Method is also claimed to shape the noise spectrum better than MUSIC.

**Method of Direction Estimation (MODE):** MODE is equivalent to WSF when there are no coherent sources. Viberg and Ottersten [1991] claim that, for coherent sources, only WSF is asymptotically efficient. A minimum norm interpretation and proof of strong consistency of MODE for ergodic and stationary signals, has also been reported [Ephraim, Merhav and Van Trees, 1995]. The norm measure used in that work involves the source covariance matrix. By contrasting this norm with the Frobenius norm that is used in MD MUSIC, Ephraim *et al* relate MODE and MD MUSIC.

**Minimum-Norm [Kumaresan and Tufts, 1983] method:** In this method, the matrix $\mathbf{N}$ is obtained as follows [Haykin, 1991]: (1) Form $\mathbf{E}_n = [\mathbf{e}_{P+1}, \cdots, \mathbf{e}_M]$; (2) partition $\mathbf{E}_n$ as $\mathbf{E}_n = \begin{bmatrix} \mathbf{c} & \mathbf{C}^T \end{bmatrix}^T$, to establish $\mathbf{c}$ and $\mathbf{C}$; (3) compute $\mathbf{d} = \begin{bmatrix} 1 & ((\mathbf{c}^H\mathbf{c})^{-1}\mathbf{C}^*\mathbf{c})^T \end{bmatrix}^T$, and, finally, $\mathbf{N} = \mathbf{d}\mathbf{d}^H$. For two closely spaced, equal power signals, the Minimum Norm method has been shown to have a lower SNR threshold (i.e., the minimum SNR required to separate the two sources) than MUSIC [Kaveh and Barabell, 1986]. [Li and Vaccaro, 1991] derive and compare the mean-squared errors of the DOA estimates from Minimum Norm and MUSIC algorithms due to finite sample effects, calibration errors, and, noise modeling errors for the case of finite samples and high SNR. They show that mean-squared errors for DOA estimates produced by the MUSIC algorithm are always lower than the corresponding mean-squared errors for the Minimum Norm algorithm.

### 3.2.2 Algebraic methods

When the array is uniform linear, so that $\mathbf{a}(\theta) = \left[1, e^{-j2\pi\frac{d}{\lambda}\sin(\theta)}, \cdots, e^{-j2\pi(M-1)\frac{d}{\lambda}\sin(\theta)}\right]^T$, the search in $S(\theta) = 1/\mathbf{a}(\theta)^H\mathbf{N}\mathbf{a}(\theta)$ for the peaks can be replaced by a root-finding procedure which yields the arrival angles. So doing, results in better resolution than the search-based alternative, because the root-finding procedure can give distinct roots corresponding to each source whereas the search function may not have distinct maxima for closely spaced sources. In addition, the computational complexity of algebraic methods is lower than that of the search-based ones. In the algebraic methods, the roots $\{z_i\}_{i=1}^M$ of the $M$-th order polynomial $p(z) = s_1 + s_2z^{-1} + \cdots + s_Mz^{-M}$ are calculated, where $\mathbf{s} \triangleq [s_1, \cdots, s_M]$ is $\mathbf{e}_M$ for Pisarenko, and $\mathbf{d}$ for Minimum-Norm methods. The DOAs are obtained by solving the equation $z_i = 2\pi\frac{d}{\lambda}\sin\theta_i$ for $\theta_i$. The algebraic version of MUSIC (Root-MUSIC) is given next; for EV method, $\mathbf{N} = \sum_{i=P+1}^M \frac{1}{\lambda_i}\mathbf{e}_i\mathbf{e}_i^H$ in the first step of Root-MUSIC.

**Root-MUSIC:**

In root-MUSIC, the array is required to be uniform linear, and the search procedure in MUSIC is converted into the following root-finding approach: (1) Form the $M \times M$ matrix $\mathbf{N} = \sum_{i=P+1}^M \mathbf{e}_i\mathbf{e}_i^H$; (2) form a polynomial $p(z)$ of degree $2M-1$ which has for its $i$th coefficient $c_i = tr_i[\mathbf{N}]$, where $tr_i$ denotes the trace of the $i$th diagonal, and $i = -(M-1), \cdots, 0, \cdots, M-1$. Note that $tr_0$ denotes the main diagonal, $tr_1$ denotes the first super-diagonal, and $tr_{-1}$denotes the first sub-diagonal. (3) The roots of $p(z)$ exhibit inverse symmetry with respect to the unit circle in the $z$-plane. Express $p(z)$ as the product of two polynomials $p(z) = h(z)h^*(z^{-1})$. (4) Find the roots $z_i$ $(i = 1, \cdots, M)$ of $h(z)$. The angles $\theta_i$ of the roots that are very close to (or, ideally on) the unit circle yield the direction of arrival estimates, as $\theta_i = \sin^{-1}(\frac{\lambda}{2\pi d}z_i)$, where $i = 1, \cdots, P$. The root-MUSIC algorithm has been shown to have better resolution power than MUSIC [Rao and Hari, 1989]; however, as

mentioned previously, root-MUSIC is restricted to uniform linear arrays. Steps (2)-(4) make use of this knowledge. Li and Vaccaro show that algebraic versions of the MUSIC and Minimum Norm algorithms have the same mean-squared errors as their search-based versions for finite samples and high SNR case. The advantages of root-MUSIC over search-based MUSIC is increased resolution of closely spaced sources and reduced computations.

### 3.3 Spatial Smoothing [Evans, Johnson and Sun, 1981, Shan and Kailath, 1985]

When there are coherent (completely correlated) sources, $rank(\mathbf{R}_s)$, and consequently $rank(\mathbf{R})$ is less than $P$, and hence the above described subspace methods fail. If the array is uniform linear, then by applying the spatial smoothing method, described below, a new rank-$P$ matrix is obtained which can be used in place of $\mathbf{R}$ in any of the subspace methods described earlier.

Spatial smoothing starts by dividing the $M$-vector $\mathbf{r}(t)$ of the ULA into $K = M - S + 1$ overlapping subvectors of size $S$, $\mathbf{r}^f_{S,k}$ $(k = 1, \cdots, K)$, with elements $\{r_k, \cdots, r_{k+S-1}\}$, and $\mathbf{r}^b_{S,k}$ $(k = 1, \cdots, K)$, with elements $\{r^*_{M-k+1}, \cdots, r^*_{M-S-k+2}\}$. Then, a forward and backward spatially smoothed matrix $\mathbf{R}^{fb}$ is calculated as $\mathbf{R}^{fb} = \sum_{t=1}^{N} \sum_{k=1}^{K} (\mathbf{r}^f_{S,k}(t) \mathbf{r}^{f\,H}_{S,k}(t) + \mathbf{r}^b_{S,k}(t) \mathbf{r}^{b\,H}_{S,k}(t))/KN$. The rank of $\mathbf{R}^{fb}$ is $P$ if there are at most $2M/3$ coherent sources. $S$ must be selected such that $P_c + 1 \leq S \leq M - P_c/2 + 1$ in which $P_c$ is the number of coherent sources. Then, any subspace-based method can be applied to $\mathbf{R}^{fb}$ to determine the directions of arrival. It is also possible to do spatial smoothing based only on $\mathbf{r}^f_{S,k}$ or $\mathbf{r}^b_{S,k}$, but in this case at most $M/2$ coherent sources can be handled.

## 3.4 Discussion

The application of all the subspace-based methods requires exact knowledge of the number of signals, in order to separate the signal and noise subspaces. The number of signals can be estimated from the data using either the Akaike Information Criterion (AIC) [Wax and Kailath, 1985] or Minimum Descriptive Length (MDL) [Wax, 1985] methods. The effect of underestimating the number of sources is analysed by [Radich and Buckley, 1994], whereas the case of overestimating the number of signals can be treated as a special case of the analysis in [Stoica and Nehorai, 1990].

The second-order methods described above have the following disadvantages: (1) Except for ESPRIT (which requires a special array structure), all of the above methods require calibration of the array which means that the response of the array for every possible combination of the source parameters should be measured and stored; or, analytical knowledge of the array response is required. However, at any time, the antenna response can be different from when it was last calibrated due to environmental effects such as weather conditions for radar, or water waves for sonar. Even if the analytical response of the array elements are known, it may be impossible to know or track the precise locations of the elements in some applications (e.g., towed array). Consequently, these methods are sensitive to errors and perturbations in the array response. In addition, physically identical sensors may not respond identically in practice due to lack of synchronization or imbalances in the associated electronic circuitry. (2) In deriving the above methods it was assumed that the noise covariance structure is known; however, it is often unrealistic to assume that the noise statistics are known due to several reasons. In practice, the noise is not isolated; it is often observed along with the signals. Moreover, as [Swindlehurst and Kailath, 1992] state, there are noise phonemena effects which can not be modeled accurately, e.g., channel crosstalk, reverberation, near-field, wide-

band and distributed sources. (3) None of the methods in Sections 3.1 and 3.2, except for the WSF method and other multidimensional search-based approaches, which are computationally very expensive, work when there are coherent (completely correlated) sources. Only if the array is uniform linear, can the spatial smoothing method in Section 3.3 be used. On the other hand, higher-order statistics of the received signals can be exploited to develop direction finding methods which have less restrictive requirements.

# 4 Higher-Order Statistics-Based Methods

The higher-order statistical direction finding methods use the spatial cumulant matrices of the array. They require that the source signals be non-Gaussian so that their higher- than second order statistics convey extra information. Most communication signals (e.g., QAM) are *complex circular* (a signal is complex circular if its real and imaginary parts are independent and symmetrically distributed with equal variances) and hence their third-order cumulants vanish; therefore, even-order cumulants are used, and, usually fourth-order cumulants are employed. The fourth-order cumulant of the source signals must be nonzero in order to use these methods. One important feature of cumulant-based methods is that they can suppress Gaussian noise regardless of its coloring. Consequently, the requirement of having to estimate the noise covariance, as in second-order statistical processing methods, is avoided in cumulant-based methods. It is also possible to suppress non-Gaussian noise, [Dogan and Mendel, 1995], and, when properly applied, cumulants extend the aperture of an array [Dogan and Mendel, 1995, Shamsunder and Giannakis, 1994] which means that more sources than sensors can be detected. As in the second-order statistics-based methods, it is assumed that the number of sources is known or is estimated from the data.

The fourth-order moments of the signal $\mathbf{s}(t)$ are $E\{s_i s_j^* s_k s_l^*\}$ $1 \leq i,j,k,l \leq P$ and the fourth-order cumulants are defined as $c_{4,s}(i,j,k,l) \triangleq cum(s_i, s_j^*, s_k, s_l^*) = E\{s_i s_j^* s_k s_l^*\} - E\{s_i s_j^*\}E\{s_k s_l^*\} - E\{s_i s_l^*\}E\{s_k s_j^*\} - E\{s_i s_j\}E\{s_k^* s_l^*\}$, where $1 \leq i,j,k,l \leq P$. Note that two arguments in the above fourth-order moments and cumulants are conjugated and the other two are unconjugated. For circularly symmetric signals, which is often the case in communication applications, the last term in $c_{4,s}(i,j,k,l)$ is zero.

In practice, sample estimates of the cumulants are used in place of the theoretical cumulants, and these sample estimates are obtained from the received signal vector $\mathbf{r}(t)$ $(t = 1, \cdots, N)$, as: $\hat{c}_{4,r}(i,j,k,l) = \sum_{t=1}^{N} r_i(t) r_j^*(t) r_k(t) r_l^*(t)/N - \sum_{t=1}^{N} r_i(t) r_j^*(t) \sum_{t=1}^{N} r_k(t) r_l^*(t)/N^2 - \sum_{t=1}^{N} r_i(t) r_l^*(t) \sum_{t=1}^{N} r_k(t) r_j^*(t)/N^2$, where $1 \leq i,j,k,l \leq M$. Note that the last term in $c_{4,r}(i,j,k,l)$ is zero, and therefore, it is omitted.

Higher-order statistical subspace methods use fourth-order spatial cumulant matrices of the array output, which can be obtained in a number of ways by suitably selecting the arguments $i,j,k,l$ of $c_{4,r}(i,j,k,l)$. Existing methods for the selection of the cumulant matrix, and their associated processing schemes are summarized next.

**Pan-Nikias [1988] and Cardoso-Moulines [1995] method:** In this method, the array needs to be calibrated, or its response must be known in analytical form. The source signals are assumed to be independent or partially correlated [i.e, there are no coherent signals]. The method is as follows: (1) An estimate of an $M \times M$ fourth-order cumulant matrix $\mathbf{C}$ is obtained from the data. The following two selections for $\mathbf{C}$ are possible [Pan and Nikias, 1988, Cardoso and Moulines, 1995]: $c_{ij} = c_{4,r}(i,j,j,j)$ $1 \leq i,j \leq M$, or, $c_{ij} = \sum_{m=1}^{M} c_{4,r}(i,j,m,m)$ $1 \leq i,j \leq M$. Using cumulant properties [Mendel, 1991], and (1), and $a_{ij}$ for the $ij$th element of $\mathbf{A}$, it is easy to verify that $c_{4,r}(i,j,j,j) = \sum_{p=1}^{P} a_{ip} \sum_{q,r,s=1}^{P} a_{jq}^* a_{jr} a_{js}^* c_{4,s}(p,q,r,s)$, which, in matrix format, is $\mathbf{C} = \mathbf{AB}$ where $\mathbf{A}$ is the

steering matrix and $\mathbf{B}$ is a $P \times M$ matrix with elements $b_{ij} = \sum_{q,r,s=1}^{P} a_{jq}{}^* a_{jr} a_{js}{}^* c_{4,s}(i,q,r,s)$. Similarly, $\sum_{m=1}^{M} c_{4,r}(i,j,m,m) = \sum_{p,q=1}^{P} a_{ip} \left( \sum_{r,s=1}^{P} \sum_{m=1}^{M} a_{mr} a_{ms}{}^* c_{4,s}(p,q,r,s) \right) a_{jq}{}^*$, $1 \leq i,j \leq M$, which, in matrix form can be expressed as $\mathbf{C} = \mathbf{ADA}^H$, where $\mathbf{D}$ is a $P \times P$ matrix with elements $d_{ij} = \sum_{r,s=1}^{P} \sum_{m=1}^{M} a_{mr} a_{ms}{}^* c_{4,s}(i,j,r,s)$. Note that additive Gaussian noise is suppressed in both $C$ matrices, because higher than second-order statistics of a Gaussian process are zero. (2) The $P$ left singular vectors of $\mathbf{C} = \mathbf{AB}$, corresponding to nonzero singular values, or, the $P$ eigenvectors of $\mathbf{C} = \mathbf{ADA}^H$ corresponding to nonzero eigenvalues, form the signal subspace. The orthogonal complement of the signal subspace gives the noise subspace. Any of the Section 3 covariance-based search and algebraic DF methods (except for the EV method and ESPRIT) can now be applied (in exactly the same way as described in Section 3) either by replacing the signal and noise subspace eigenvectors and eigenvalues of the array covariance matrix by the corresponding subspace eigenvectors and eigenvalues of $\mathbf{ADA}^H$, or by the corresponding subspace singular vectors and singular values of $\mathbf{AB}$. A cumulant-based analog of the EV method does not exist, because the eigenvalues and singular values of $\mathbf{ADA}^H$ and $\mathbf{AB}$ corresponding to the noise subspace are theoretically zero. The cumulant-based analog of ESPRIT is explained later.

The same assumptions and restrictions for the covariance-based methods apply to their analogs in the cumulant domain. The advantage of using the cumulant-based analogs of these methods is that there is no need to know or estimate the noise-covariance matrix.

The asymptotic covariance of the DOA estimates obtained by MUSIC based on the above fourth-order cumulant matrices are derived in [Cardoso and Moulines, 1995] for the case of Gaussian measurement noise with arbitrary spatial covariance, and are compared to the asymptotic covariance of the DOA estimates from the covariance-based MUSIC algorithm. Cardoso and Moulines show that covariance- and fourth-order cumulant-based MUSIC have similar performance for the

high SNR case, and that, as SNR decreases below a certain SNR threshold, the variances of the fourth-order cumulant-based MUSIC DOA estimates increase with the fourth power of the reciprocal of the SNR, whereas the variances of covariance-based MUSIC DOA estimates increase with the square of the reciprocal of the SNR. They also observe that for high SNR and uncorrelated sources, the covariance-based MUSIC DOA estimates are uncorrelated, and the asymptotic variance of any particular source depends only on the power of that source (i.e., it is independent of the powers of the other sources). They observe, on the other hand, that DOA estimates from cumulant-based MUSIC, for the same case, are correlated, and the variance of the DOA estimate of a weak source increases in the presence of strong sources. This observation limits the use of cumulant-based MUSIC when the sources have a high dynamic range, even for the case of high SNR. Cardoso and Moulines state that this problem may be alleviated when the source of interest has a large fourth-order cumulant.

**Porat and Friedlander [1991] method:** In this method, the array also needs to be calibrated, or its response is required in analytical form. The model used in this method divides the sources into groups that are partially correlated (but, not coherent) within each group, but are statistically independent across the groups, i.e., $\mathbf{r}(t) = \sum_{g=1}^{G} \mathbf{A}_g \mathbf{s}_g + \mathbf{n}(t)$, where $G$ is the number of groups each having $p_g$ sources ($\sum_{g=1}^{G} p_g = P$). In this model, the $p_g$ sources in the $g$th group are partially correlated, and they are received from different directions. The method is as follows: (1) Estimate the fourth-order cumulant matrix, $\mathbf{C}_r$, of $\mathbf{r}(t) \otimes \mathbf{r}(t)^*$ where $\otimes$ denotes the Kronecker product. It can be verified that $\mathbf{C}_r = \sum_{g=1}^{G} (\mathbf{A}_g \otimes \mathbf{A}^*_g) \mathbf{C}_{s_g} (\mathbf{A}_g \otimes \mathbf{A}^*_g)^H$ where $\mathbf{C}_{s_g}$ is the fourth-order cumulant matrix of $\mathbf{s}_g$. The rank of $\mathbf{C}_r$ is $\sum_{g=1}^{G} {p_g}^2$, and since $\mathbf{C}_r$ is $M^2 \times M^2$, it has $M^2 - \sum_{g=1}^{G} {p_g}^2$ zero eigenvalues which correspond to the noise subspace. The other eigenvalues corrrespond to the signal subspace. (2) Compute the SVD of $\mathbf{C}_r$ and identify the signal and noise subspace singular

vectors. Now, second-order subspace-based search methods can be applied, using the signal or noise subspaces, by replacing the array response vector $\mathbf{a}(\theta)$ by $\mathbf{a}(\theta) \otimes \mathbf{a}^*(\theta)$.

The eigendecomposition in this method has computational complexity $O(M^6)$ due to the Kronecker product, whereas the second-order statistics-based methods (e.g., MUSIC) have complexity $O(M^3)$.

**Chiang-Nikias [1989] method:** This method uses the ESPRIT algorithm and requires an array with its entire identical copy displaced in space by distance $d$; however, no calibration of the array is required. The signals $\mathbf{r}^1(t) = \mathbf{A}\mathbf{s}(t) + \mathbf{n}_1(t)$ and $\mathbf{r}^2(t) = \mathbf{A}\mathbf{\Phi}\mathbf{s}(t) + \mathbf{n}_2(t)$. Two $M \times M$ matrices $\mathbf{C}^1$ and $\mathbf{C}^2$ are generated as follows: $c^1{}_{ij} = cum(r^1{}_i, r^1{}_j{}^*, r^1{}_k, r^1{}_k{}^*)$, $1 \leq i, j, k \leq M$, and $c^2{}_{ij} = cum(r^2{}_i, r^1{}_j{}^*, r^1{}_k, r^1{}_k{}^*)$ $1 \leq i, j, k \leq M$. It can be shown that $\mathbf{C}^1 = \mathbf{A}\mathbf{E}\mathbf{A}^H$ and $\mathbf{C}^2 = \mathbf{A}\mathbf{\Phi}\mathbf{E}\mathbf{A}^H$, where $\Phi = diag\{e^{-j2\pi\frac{d}{\lambda}\sin\theta_1}, \cdots, e^{-j2\pi\frac{d}{\lambda}\sin\theta_P}\}$ in which $d$ is the separation between the identical arrays, and $\mathbf{E}$ is a $P \times P$ matrix with elements $e_{ij} = \sum_{q,r=1}^{P} a_{kq} a_{kr}{}^* c_{4,s}(i, q, r, j)$. Note that these equations are in the same form as those for covariance-based ESPRIT [the noise cumulants do not appear in $\mathbf{C}^1$ and $\mathbf{C}^2$ because the fourth-order cumulants of Gaussian noises are zero]; therefore, any version of ESPRIT or GEESE can be used to solve for $\Phi$ by replacing $\mathbf{R}^{11}$ and $\mathbf{R}^{21}$ by $\mathbf{C}^1$ and $\mathbf{C}^2$, respectively.

**Virtual Cross Correlation Computer ($VC^3$) [Dogan and Mendel, 1995]:** In $VC^3$, the source signals are assumed to be statistically independent. The idea of $VC^3$ can be demonstrated as follows: Suppose we have 3 identical sensors as in Fig. 1, where $r_1(t)$, $r_2(t)$ and $r_3(t)$ are measurements, and, $\vec{d_1}$, $\vec{d_2}$ and $\vec{d_3}$ ($\vec{d_3} = \vec{d_1} + \vec{d_2}$) are the vectors joining these sensors. Let the response of each sensor to a signal from $\theta$ be $a(\theta)$. A *virtual* sensor is one at which no measurement is actually made. Suppose that we wish to compute the correlation between the virtual sensor $v_1(t)$ and $r_2(t)$, which (using the plane wave assumption) is $E\{r_2^*(t)v_1(t)\} = \sum_{p=1}^{P} |a(\theta_p)|^2 {\sigma_p}^2 e^{-j\vec{k_p}.\vec{d_3}}$.

Consider the following cumulant $cum(r_2^*(t), r_1(t), r_2^*(t), r_3(t)) = \sum_{p=1}^{P} |a(\theta_p)|^4 \gamma_p e^{-j\vec{k_p}.\vec{d_1}} e^{-j\vec{k_p}.\vec{d_2}} = \sum_{p=1}^{P} |a(\theta_p)|^4 \gamma_p e^{-j\vec{k_p}.\vec{d_3}}$. This cumulant carries the same angular information as the cross correlation $E\{r_2^*(t)v_1(t)\}$, but for sources having different powers.

The fact that we are interested only in the directional information carried by correlations between the sensors lets us therefore interpret a cross correlation as a vector (e.g., $\vec{d_3}$), and a fourth-order cumulant as the addition of two vectors (e.g., $\vec{d_1} + \vec{d_2}$). This interpretation leads to the idea of decomposing the computation of a cross correlation into that of computing a cumulant. Doing this means that the directional information that would be obtained from the cross correlation between nonexisting sensors (or between an actual sensor and a nonexisting sensor) at certain virtual locations in the space can be obtained from a suitably defined cumulant that uses the real sensor measurements.

One advantage of virtual cross correlation computation is that it is possible to obtain a larger aperture than would be obtained by using only second-order statistics. This means that more sources than sensors can be detected using cumulants. For example, given an $M$ element uniform linear array, $VC^3$ lets its aperture be extended from $M$ to $2M-1$ sensors, so that $2M-2$ targets can be detected (rather than $M-1$) just by using the array covariance matrix obtained by $VC^3$ in any of the subspace-based search methods explained earlier. This use of $VC^3$ requires the array to be calibrated. Another advantage of $VC^3$ is a fault tolerance capability. If sensors at certain locations in a given array fail to operate properly, these sensors can be replaced using $VC^3$.

**Virtual ESPRIT (VESPA) [Dogan and Mendel, 1995]:** For VESPA, the array only needs two identical sensors; the rest of the array may have arbitrary and unknown geometry and response. The sources are assumed to be statistically independent. VESPA uses the ESPRIT solution applied to cumulant matrices. By choosing a suitable pair of cumulants in VESPA, the need for a copy of

the entire array, as required in ESPRIT, is totally eliminated. VESPA preserves the computational advantage of ESPRIT over search-based algorithms. An example array configuration is given in Fig. 2.

Without loss of generality, let the signals received by the identical sensor pair be $r_1$ and $r_2$. The sensors $r_1$ and $r_2$ are collectively referred to as the *guiding sensor pair*. The VESPA algorithm is: (1) Two $M \times M$ matrices, $\mathbf{C}^1$ and $\mathbf{C}^2$, are generated as follows: $c^1{}_{ij} = cum(r_1, r_1^*, r_i, r_j^*)$, $1 \leq i,j \leq M$, and $c^2{}_{ij} = cum(r_2, r_1^*, r_i, r_j^*)$ $1 \leq i,j \leq M$. It can be shown that these relations can be expressed, as $\mathbf{C}^1 = \mathbf{AFA}^H$ and $\mathbf{C}^2 = \mathbf{A\Phi FA}^H$, where the $P \times P$ matrix $\mathbf{F} \triangleq diag\{\gamma_{4,s_1}|a_{11}|^2, \cdots, \gamma_{4,s_P}|a_{1P}|^2\}$, $\{\gamma_{4,s_P}\}_{p=1}^P$, and $\Phi$ has been defined before. (2) Note that these equations are in the same form as ESPRIT and Chiang and Nikias's ESPRIT-like method; however, as opposed to these methods, there is no need for an identical copy of the array; only an identical response sensor pair is necessary for VESPA. Consequently, any version of ESPRIT or GEESE can be used to solve for $\Phi$ by replacing $\mathbf{R}^{11}$ and $\mathbf{R}^{21}$ by $\mathbf{C}^1$ and $\mathbf{C}^2$, respectively.

Note, also, that there exists a very close link between $VC^3$ and VESPA. Although the way we chose $\mathbf{C}^1$ and $\mathbf{C}^2$ above seems to be not very obvious, there is a unique geometric interpretation to it. According to $VC^3$, as far as the bearing information is concerned, $\mathbf{C}^1$ is equivalent to the autocorrelation matrix of the array, and $\mathbf{C}^2$ is equivalent to the cross-correlation matrix between the array and its virtual copy (which is created by displacing the array by the vector that connects the second and the first sensors).

If the noise component of the signal received by one of the guiding sensor pair elements is independent of the noises at the other sensors, VESPA supresses the noise regardless of its distribution [Dogan and Mendel, 1995]. In practice, the noise does affect the standard deviations of results obtained from VESPA.

An iterative version of VESPA has also been developed for cases where the source powers have a high dynamic range [Gonen, 1996]. Iterative VESPA has the same hardware requirements and assumptions as in VESPA.

**Extended VESPA [Gonen, Mendel and Dogan, 1994]:** When there are coherent (or, completely correlated) sources, all of the above second- and higher-order statistics methods, except for the WSF method and other multidimensional search-based approaches, fail. For the WSF and other multidimensional methods, however, the array must be calibrated accurately and the computational load is expensive. The coherent signals case arises in practice when there are multipaths. Porat and Friedlander present a modified version of their algorithm to handle the case of coherent signals; however, their method is not practical, because it requires selection of a highly redundant subset of fourth-order cumulants that contains $O(N^4)$ elements, and no guidelines exist for its selection; and, 2nd-, 4th-, 6th- and 8th-order moments of the data are required. If the array is *uniform linear*, coherence can be handled using spatial smoothing as a preprocessor to the usual second- or higher-order [Chen and Lin, 1994, Yuen and Friedlander, 1995] methods; however, the array aperture is reduced. Extended VESPA can handle coherence and provides increased aperture. Additionally, the array does not have to be completely uniform linear or calibrated; however, a uniform linear subarray is still needed. An example array configuration is shown in Figure 3.

Consider a scenario in which there are $G$ statistically independent narrowband sources, $\{u_g(t)\}_{i=1}^{G}$. These source signals undergo multipath propagation, and each produces $p_i$ coherent wavefronts $\{s_{1,1}, \cdots, s_{1,p_1}, \cdots, s_{G,1}, \cdots, s_{G,p_G}\}$ $(\sum_{i=1}^{G} p_i = P)$ that impinge on an $M$ element sensor array from directions $\{\theta_{1,1}, \cdots, \theta_{1,p_1}, \cdots, \theta_{G,1}, \cdots, \theta_{G,p_G}\}$, where $\theta_{m,p}$ represents the angle-of-arrival of the wavefront $s_{g,p}$ that is the $p$th coherent signal in the $g$th group. The collection of $p_i$ coherent wavefronts, which are scaled and delayed replicas of the $i$th source, are referred to as

the $i$th *group*. The wavefronts are represented by the $P$-vector $\mathbf{s}(t)$. The problem is to estimate the DOAs $\{\theta_{1,1}, \cdots, \theta_{1,p_1}, \cdots, \theta_{G,1}, \cdots, \theta_{G,p_G}\}$.

When the multipath delays are insignificant compared to the bit durations of signals, then the signals received from different paths differ by only amplitude and phase shifts, thus the coherence among the received wavefronts can be expressed by the following equation:

$$\mathbf{s}(t) = \begin{bmatrix} \mathbf{s}_1(t) \\ \mathbf{s}_2(t) \\ \vdots \\ \mathbf{s}_G(t) \end{bmatrix} = \begin{bmatrix} \mathbf{c}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{c}_2 & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{c}_G \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_G(t) \end{bmatrix} \triangleq \mathbf{Q}\mathbf{u}(t) \tag{2}$$

where $\mathbf{s}_i(t)$ is a $p_i \times 1$ signal vector representing the coherent wavefronts from the $i$th independent source $u_i(t)$, $\mathbf{c}_i$ is a $p_i \times 1$ complex attenuation vector for the $i$th source ($1 \leq i \leq G$), and $\mathbf{Q}$ is $P \times G$. The elements of $\mathbf{c}_i$ account for the attenuation and phase differences among the multipaths due to different arrival times. The received signal can then be written in terms of the independent sources as follows:

$$\mathbf{r}(t) = \mathbf{A}\mathbf{s}(t) + \mathbf{n}(t) = \mathbf{A}\mathbf{Q}\mathbf{u}(t) + \mathbf{n}(t) = \mathbf{B}\mathbf{u}(t) + \mathbf{n}(t) \tag{3}$$

where $\mathbf{B} \triangleq \mathbf{A}\mathbf{Q}$. The columns of $M \times G$ matrix $\mathbf{B}$ are known as the *generalized steering vectors.*

Extended VESPA has three major steps:

**Step 1**: Use step (1) of VESPA by choosing $r_1(t)$ and $r_2(t)$ as any two sensor measurements. In this case $\mathbf{C}^1 = \mathbf{B}\mathbf{G}\mathbf{B}^H$ and $\mathbf{C}^2 = \mathbf{B}\mathbf{C}\mathbf{G}\mathbf{B}^H$. where $\mathbf{G} \triangleq diag(\gamma_{4,u_1}|b_{11}|^2, \cdots, \gamma_{4,u_G}|b_{1G}|^2)$, $\{\gamma_{4,u_g}\}_{g=1}^G$ and $\mathbf{C} \triangleq diag(\frac{b_{21}}{b_{11}}, \cdots, \frac{b_{2G}}{b_{1G}})$. Because of the coherence, the DOAs can not be obtained at this step

from just $\mathbf{C}^1$ and $\mathbf{C}^2$, since the columns of $\mathbf{B}$ depend on a vector of DOAs (all those within a group). In the independent sources case, the columns of $\mathbf{A}$ depend only on a single DOA. Fortunately, the columns of $\mathbf{B}$ can be solved for, as follows: (1.1) Follow Steps 2-5 of TLS ESPRIT by replacing $\mathbf{R}^{11}$ and $\mathbf{R}^{21}$ by $\mathbf{C}^1$ and $\mathbf{C}^2$, respectively, and using appropriate matrix dimensions; (1.2) determine the eigenvectors and eigenvalues of $-\mathbf{F}_x\mathbf{F}_y^{-1}$; Let the eigenvector and eigenvalue matrices of $-\mathbf{F}_x\mathbf{F}_y^{-1}$ be $\mathbf{E}$ and $\mathbf{D}$, respectively; and, (1.3) obtain an estimate of $\mathbf{B}$ to within a diagonal matrix, as $\mathbf{B} = (\mathbf{U}_{11}\mathbf{E} + \mathbf{U}_{12}\mathbf{E}\mathbf{D}^{-1})/2$, for use in Step 2.

**Step 2:** Partition the matrices $\mathbf{B}$ and $\mathbf{A}$ as $\mathbf{B} = [\mathbf{b}_1, \cdots, \mathbf{b}_G]$ and $\mathbf{A} = [\mathbf{A}_1, \cdots, \mathbf{A}_G]$, where the steering vector for the $i$th group $\mathbf{b}_i$ is $M \times 1$, $\mathbf{A}_i \triangleq [\mathbf{a}(\theta_{i,1}), \cdots, \mathbf{a}(\theta_{i,p_i})]$ is $M \times p_i$, and, $\theta_{i,m}$ is the angle-of-arrival of the $m$th source in the $i$th coherent group ($1 \leq m \leq p_i$). Using the fact that the $i$th column of $\mathbf{Q}$ has $p_i$ nonzero elements, express $\mathbf{B}$ as $\mathbf{B} = \mathbf{A}\mathbf{Q} = [\mathbf{A}_1\mathbf{c}_1, \cdots, \mathbf{A}_G\mathbf{c}_G]$; therefore, the $i$th column of $\mathbf{B}$, $\mathbf{b}_i$, is $\mathbf{b}_i = \mathbf{A}_i\mathbf{c}_i$ where $i = 1, \cdots, G$. Now, the problem of solving for the steering vectors is transformed into the problem of solving for the steering vectors from *each* coherent group *separately.* To solve this new problem, each generalized steering vector $\mathbf{b}_i$ can be interpreted as a received signal for an array illuminated by $p_i$ coherent signals having a steering matrix $\mathbf{A}_i$, and covariance matrix $\mathbf{c}_i\mathbf{c}_i^H$. The DOAs could then be solved for by using a second-order-statistics-based high-resolution method such as MUSIC, if the array was calibrated, and the rank of $\mathbf{c}_i\mathbf{c}_i^H$ was $p_i$; however, the array is not calibrated and $rank(\mathbf{c}_i\mathbf{c}_i^H) = 1$. The solution is to keep the portion of each $\mathbf{b}_i$ that corresponds to the uniform linear part of the array, $\mathbf{b}_{L,i}$, and to then apply the Section 3.3 spatial smoothing technique to a pseudocovariance matrix $\mathbf{b}_{L,i}\mathbf{b}_{L,i}^H$ for $i = 1, \cdots, G$. Doing this *restores* the rank of $\mathbf{c}_i\mathbf{c}_i^H$ to $p_i$. In the Section 3.3 spatial smoothing technique, we must replace $\mathbf{r}(t)$ by $\mathbf{b}_{L,i}$ and set $N = 1$.

The conditions on the length of the linear subarray and the parameter $S$ under which the

rank of $\mathbf{b}_{S,i}\mathbf{b}_{S,i}{}^H$ is restored to $p_i$ are [Gonen, 1996]: (a) $L \geq 3p_i/2$, which means that the linear subarray must have at least $3p_{max}/2$ elements, where $p_{max}$ is the maximum number of multipaths in anyone of the $G$ groups; and, (b) given $L$ and $p_{max}$, the parameter $S$ must be selected such that $p_{max} + 1 \leq S \leq L - p_{max}/2 + 1$.

**Step 3:** Apply any second-order-statistics-based subspace technique (e.g., root-MUSIC, etc.) to $\mathbf{R}_i^{fb}$ $(i = 1, \cdots, G)$ to estimate DOAs of up to $2L/3$ coherent signals in each group.

Note that the matrices $\mathbf{C}$ and $\mathbf{G}$ in $\mathbf{C}^1$ and $\mathbf{C}^2$ are not used; however, if the received signals are independent, choosing $r_1(t)$ and $r_2(t)$ from the linear subarray lets DOA estimates be obtained from $\mathbf{C}$ in Step 1, because, in that case, $\mathbf{C} = diag\{e^{-j2\pi\frac{d}{\lambda}\sin\theta_1}, \cdots, e^{-j2\pi\frac{d}{\lambda}\sin\theta_P}\}$; hence, extended VESPA can also be applied to the case of independent sources.

## 4.1 Discussion

One advantage of using higher-order statistics-based methods over second-order methods is that the covariance matrix of the noise is not needed when the noise is Gaussian. The fact that higher-order statistics have more arguments than covariances leads to more practical algorithms which have less restrictions on the array structure (for instance, the requirement of maintaining identical arrays for ESPRIT is reduced to only maintaining two identical sensors for VESPA). Another advantage is more sources than sensors can be detected, i.e., the array aperture is increased when higher-order statistics are properly applied; or, depending on the array geometry, unreliable sensor measurements can be replaced by using the $VC^3$ idea. One disadvantage of using higher-order statistics-based methods is that sample estimates of higher-order statistics require longer data lengths than covariances; hence, computational complexity is increased. In their recent study, [Cardoso and Moulines, 1995] present a comparative performance analysis of second- and fourth-order statistics based MU-

SIC methods. Their results indicate that dynamic range of the sources may be a factor limiting the performance of the fourth-order statistics based MUSIC. A comprehensive performance analysis of the above higher-order statistical methods is still lacking; therefore, a detailed comparison of these methods remains as a very important research topic.

# 5 Flowchart Comparison of Subspace-based Methods

Clearly, there are many subspace-based direction finding methods. In order to see the forest from the trees, to know when to use a second-order or a higher-order statistics-based method, we present Figs. 4-9. These figures provide a comprehensive summary of the existing subspace-based methods for direction finding and constitute guidelines to selection of a proper direction finding method for a given application.

Note that:

Figure 4: Independent sources and ULA

Figure 5: Independent sources and NL/Mixed array

Figure 6: Coherent and correlated sources and ULA

Figure 7: Coherent and correlated sources and NL/Mixed array

All four figures show two paths: SOS (second-order statistics) and HOS (higher-order statistics). Each path terminates in one or more method boxes, each of which may contain a multitude of methods. Figures 8 and 9 summarize the pros and cons of all the methods that we have considered in this chapter.

Using Figures 4-9, it is possible for a potential user of a subspace-based direction finding method to decide which method(s) is (are) most likely to give best results for his/her application.

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## References

[1] J. Capon, "High-resolution frequency-wavenumber spectral anlysis," *Proceedings of IEEE*, vol. 57, no. 8, pp. 1408–1418, Aug. 1969.

[2] J.-F. Cardoso and E. Moulines, "Asymptotic performance analysis of direction-finding algorithms based on fourth-order cumulants," *IEEE Trans. on Signal Processing*, vol. 43, no. 1, pp. 214–224, Jan. 1995.

[3] Y. H. Chen and Y. S. Lin, "A modified cumulant matrix for DOA estimation," *IEEE Trans. on Signal Processing*, vol. 42, pp. 3287–3291, Nov. 1994.

[4] H. H. Chiang and C. L. Nikias, "The ESPRIT algorithm with higher-order statistics," *Proceedings of Workshop on Higher-Order Spectral Analysis*, Vail, CO, pp. 163–168, June 28-30, 1989.

[5] M. C. Dogan and J. M. Mendel, "Applications of cumulants to array processing, Part I: Aperture extension and array calibration," *IEEE Trans. on Signal Processing*, vol. 43, no. 5, pp. 1200–1216, May 1995.

[6] M. C. Dogan and J. M. Mendel, "Applications of cumulants to array processing, Part II: Non-Gaussian noise suppression," *IEEE Trans. on Signal Processing*, vol. 43, no. 7, pp. 1661–1676, Jul. 1995.

[7] M. C. Dogan and J. M. Mendel, Oct. 17, 1995, "Method and apparatus for signal analysis employing a virtual cross-correlation computer," U. S. Patent No. 5,459,668.

[8] T. Ephraim, N. Merhav and H. L. Van Trees, "Min-norm interpretations and consistency of MUSIC, MODE and ML," *IEEE Trans. on Signal Processing*, vol. 43, no. 12, pp. 2937–2941, Dec. 1995.

[9] J. E. Evans, J. R. Johnson, and D. F. Sun, "High resolution angular spectrum estimation techniques for terrain scattering analysis and angle of arrival estimation," in *Proc. First ASSP Workshop Spectral Estimation*, Communication Research Lab., McMaster University, Aug. 1981.

[10] E. Gönen, M. C. Doğan, J. M. Mendel, "Applications of cumulants to array processing: direction finding in coherent signal environment," *Proc. of 28th Asilomar Conference on Signals, Systems, and Computers*, Asilomar, CA, pp. 633–637,1994.

[11] E. Gönen, "Cumulants and subspace techniques for array signal processing," Ph.D. Dissertation Thesis, University of Southern California, Los Angeles, CA, Dec. 1996.

[12] S. S. Haykin, Adaptive Filter Theory, Prentice Hall, Englewood Cliffs, New Jersey, 1991.

[13] D. H. Johnson and D. E. Dudgeon, Array Signal Processing: Concepts and Techniques, Prentice Hall, Englewood Hills, New Jersey, 1993.

[14] M. Kaveh and A. J. Barabell, "The statistical performance of the MUSIC and the Minimum-Norm algorithms in resolving plane waves in noise," *IEEE Trans. on Acoustics, Speech and Signal Processing*, vol. 34, pp. 331–341, Apr. 1986.

[15] R. Kumaresan and D. W. Tufts, "Estimating the angles of arrival multiple plane waves," *IEEE Trans. on Aerosp. Electron. Syst.*, vol. AES-19, pp. 134-139, Jan. 1983.

[16] S. Y. Kung, C. K. Lo and R. Foka, "A Toeplitz approximation approach to cohernet source direction finding," in *Proceedings ICASSP*, 1986.

[17] F. Li and R. J. Vaccaro, "Unified anlysis for DOA estimation algorithms in array signal processing," *Signal Processing*, vol. 25, no. 2, pp. 147–169, Nov. 1991.

[18] S. L. Marple, Digital Spectral Analysis with Applications, Prentice Hall, Englewood Cliffs, New Jersey, 1987.

[19] J. M. Mendel, "Tutorial on higher-order statistics (spectra) in signal processing and system theory: theoretical results and some applications," *Proc. IEEE*, vol. 79, no. 3, pp. 278–305, March, 1991.

[20] C. L. Nikias and A. P. Petropulu, Higher-order spectra analysis: a nonlinear signal processing framework, Prentice Hall, Englewood Cliffs, New Jersey, 1993.

[21] B. Ottersten, M. Viberg, and T. Kailath, "Performance analysis of total least squares ESPRIT algorithm," *IEEE Trans. on Signal Processing*, vol. 39, no. 5, pp. 1122–1135, May 1991.

[22] R. Pan and C. L. Nikias, "Harmonic decomposition methods in cumulant domains," *Proceedings of ICASSP'88*, pp. 2356–2359, New York, 1988.

[23] A. Paulraj, R. Roy and T. Kailath, "Estimation of signal parameters via rotational invariance techniques-ESPRIT," Proceedings of 19th Asilomar Conference on Signals, Systems, and Computers, Asilomar, CA, Nov. 1985.

[24] S. U. Pillai, Array Signal Processing, Springer-Verlag, New York, 1989.

[25] B. Porat and B. Friedlander, "Direction finding algorithms based on high-order statistics," *IEEE Trans. on Signal Processing*, vol. 39, no. 9, pp. 2016–2023, Sep. 1991.

[26] B. M. Radich and K. Buckley, "The effect of source number underestimation on MUSIC location estimates," vol. 42, no. 1, pp. 233–235, Jan. 1994.

[27] D. V. B. Rao and K. V. S. Hari, "Performance analysis of Root-MUSIC," *IEEE Trans. on Acoustics, Speech, Signal Processing*, vol. ASSP-37, pp. 1939–1949, Dec. 1989.

[28] R. H. Roy, "ESPRIT-Estimation of signal paramaters via rotational invariance techniques," Ph.D. dissertation, Stanford University, Stanford, CA, 1987.

[29] R. O. Schmidt, "A signal subspace approach to multiple emitter location and spectral estimation," Ph.D. dissertation, Stanford Univ., Stanford, CA, Nov. 1981.

[30] S. Shamsunder and G. B. Giannakis, "Detection and parameter estimation of multiple non-Gaussian sources via higher order statistics," *IEEE Trans. on Signal Processing*, vol. 42, pp. 1145–1155, May 1994.

[31] T. J. Shan, M. Wax and T. Kailath, "On spatial smoothing for direction-of-arrival estimation of coherent signals," *IEEE Trans. on Acoustics, Speech, Signal Processing*, vol. ASSP-33, no. 2, pp. 806–811, Aug. 1985.

[32] P. Stoica and A. Nehorai, "MUSIC, maximum likelihood and Cramer-Rao bound: Further results and comparisons, " *IEEE Trans. on Signal Processing*, vol. 38, pp. 2140–2150, Dec. 1990.

[33] A. L. Swindlehurst and T. Kailath, "A performance analysis of subspace-based methods in the presence of model erros, Part1: the MUSIC algorithm," *IEEE Trans. on Signal Processing*, vol. 40, no. 7, pp. 1758–1774, July 1992.

[34] M. Viberg, B. Ottersten and T. Kailath, "Detection and estimation in sensor arrays using weighted subspace fitting," *IEEE Trans. on Signal Processing*, vol. 39, no. 11, pp. 2436–2448, Nov. 1991.

[35] M. Viberg and B. Ottersten, "Sensor array processing based on subspace fitting," *IEEE Trans. on Signal Processing*, vol. 39, no. 5, pp. 1110–1120, May 1991.

[36] M. Wax and T. Kailath, "Detection of signals by information theoretic criteria," *IEEE Trans. on Acoustics, Speech, Signal Processing*, vol. ASSP-33, no. 2, pp. 387–392, Apr. 1985.

[37] M. Wax, "Detection and estimation of superimposed signals," Ph.D. dissertation, Stanford Univ., Stanford, CA, Mar. 1985.

[38] X.-L. Xu and K. Buckley, "Bias and variance of direction-of-arrival estimates from MUSIC, MIN-NORM and FINE," *IEEE Trans. on Signal Processing*, vol. 42, no. 7, pp. 1812–1816, July 1994.

[39] N. Yuen and B. Friedlander, "DOA estimation in multipath based on fourth-order cumulants," in *Proc. IEEE Signal Processing ATHOS Workshop on Higher-Order Statistics*, pp. 71–75, June 1995.

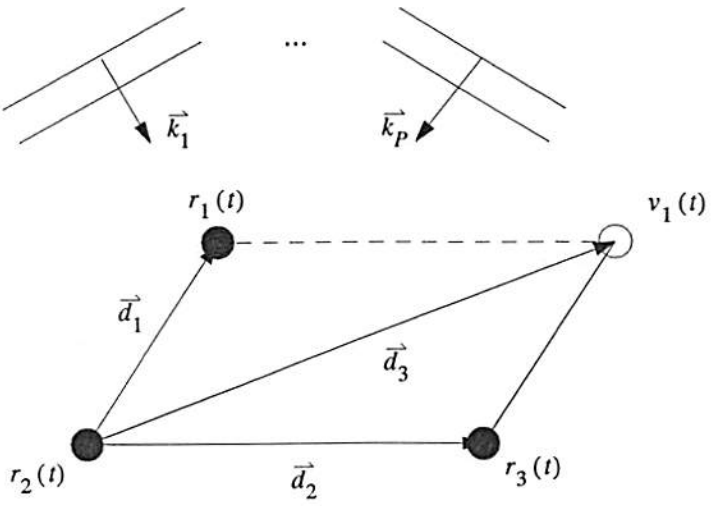


Figure 1: Demonstration of $VC^3$.

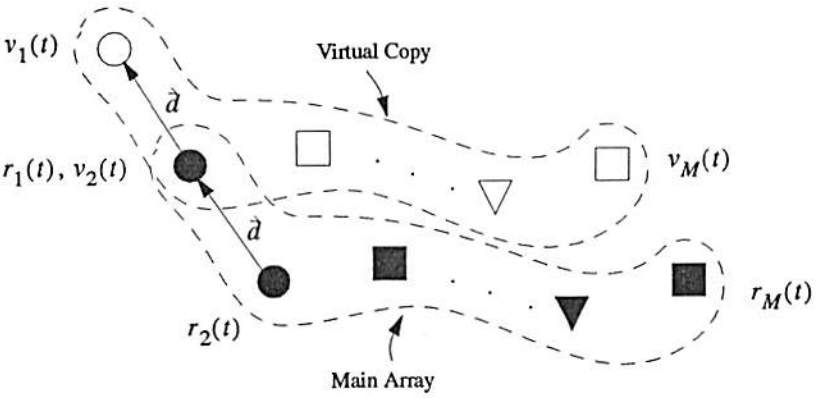


Figure 2: The main array and its virtual copy.

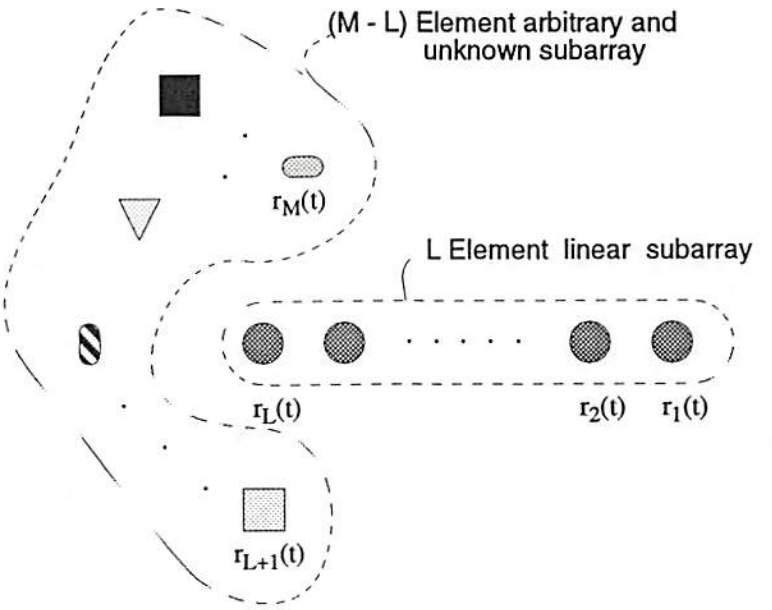


Figure 3: An example array configuration. There are $M$ sensors, $L$ of which are uniform linearly positioned; $r_1(t)$ and $r_2(t)$ are identical guiding sensors. Linear subarray elements are separated by $\Delta$.

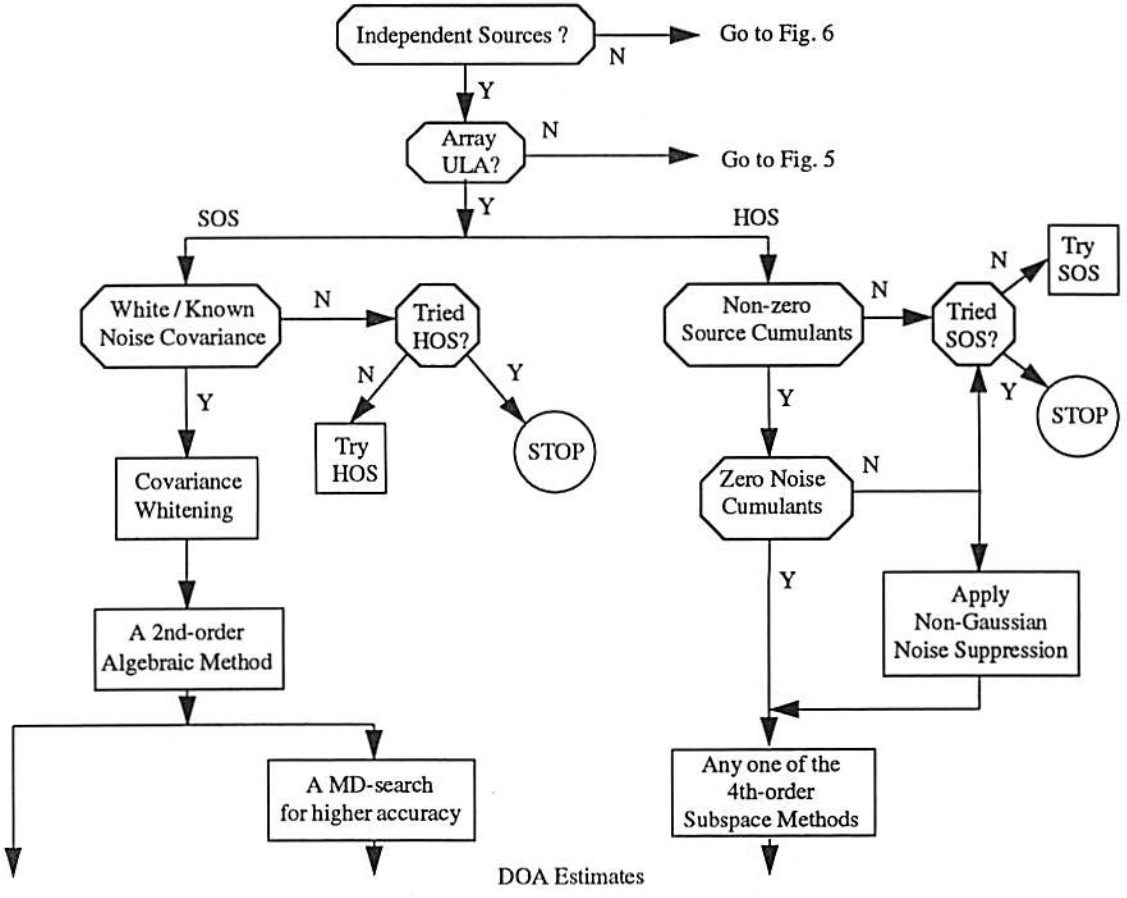


Figure 4: Second- or higher-order statistics based subspace DF algorithms. Independent sources and ULA.

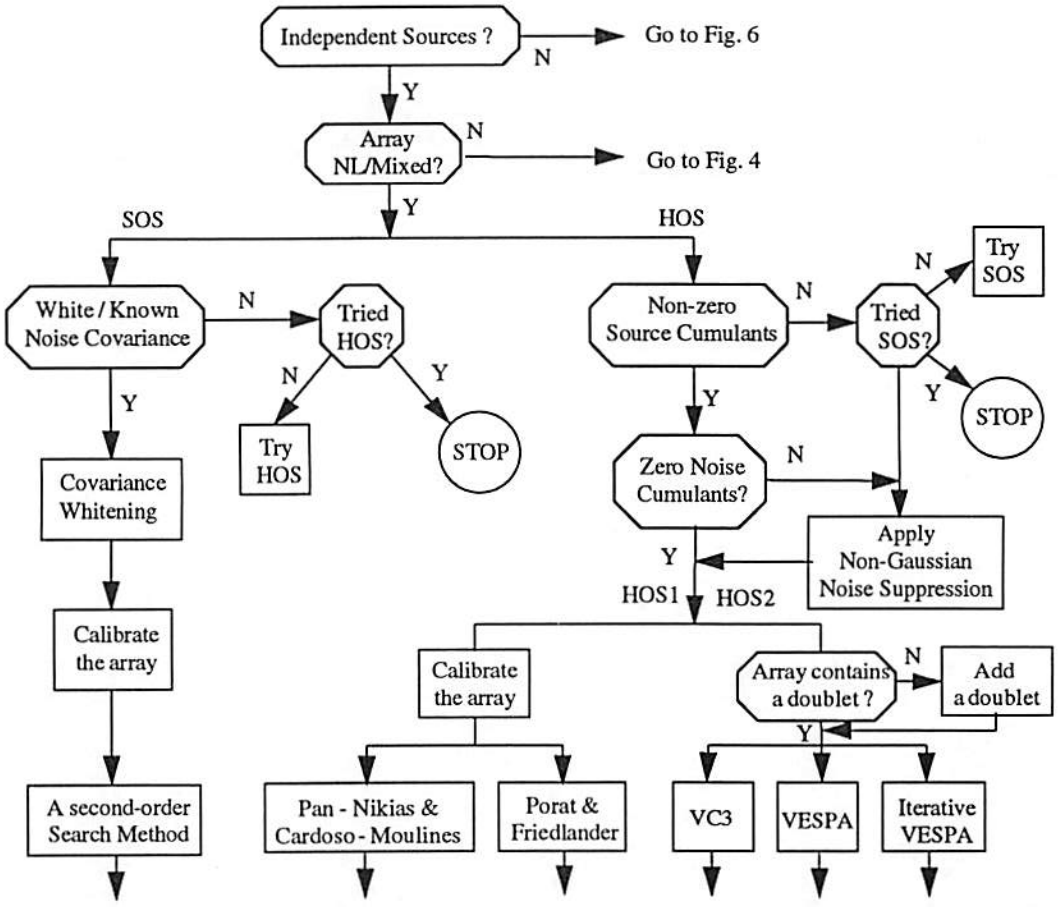


Figure 5: Second- or higher-order statistics based subspace DF algorithms. Independent sources and NL/Mixed Array.

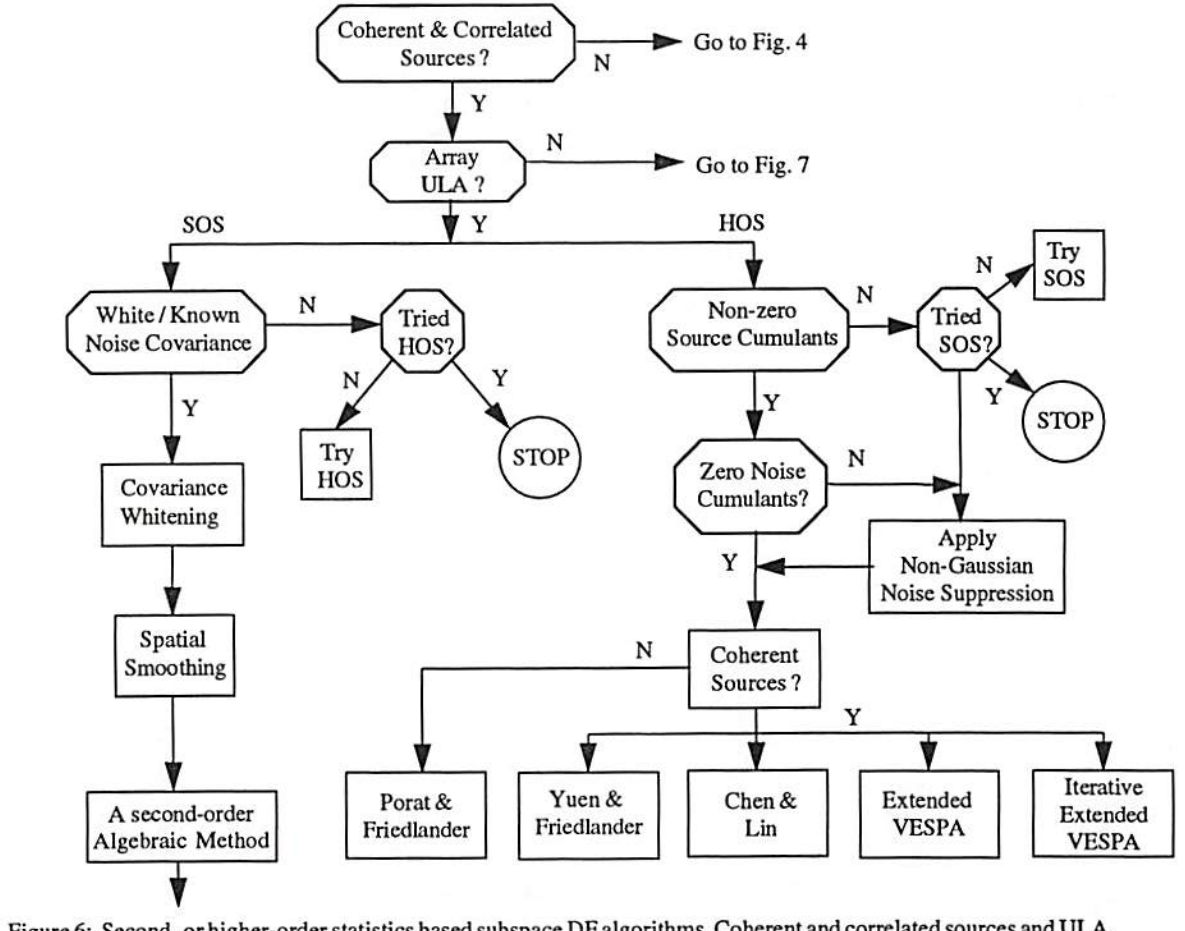


Figure 6: Second- or higher-order statistics based subspace DF algorithms. Coherent and correlated sources and ULA.

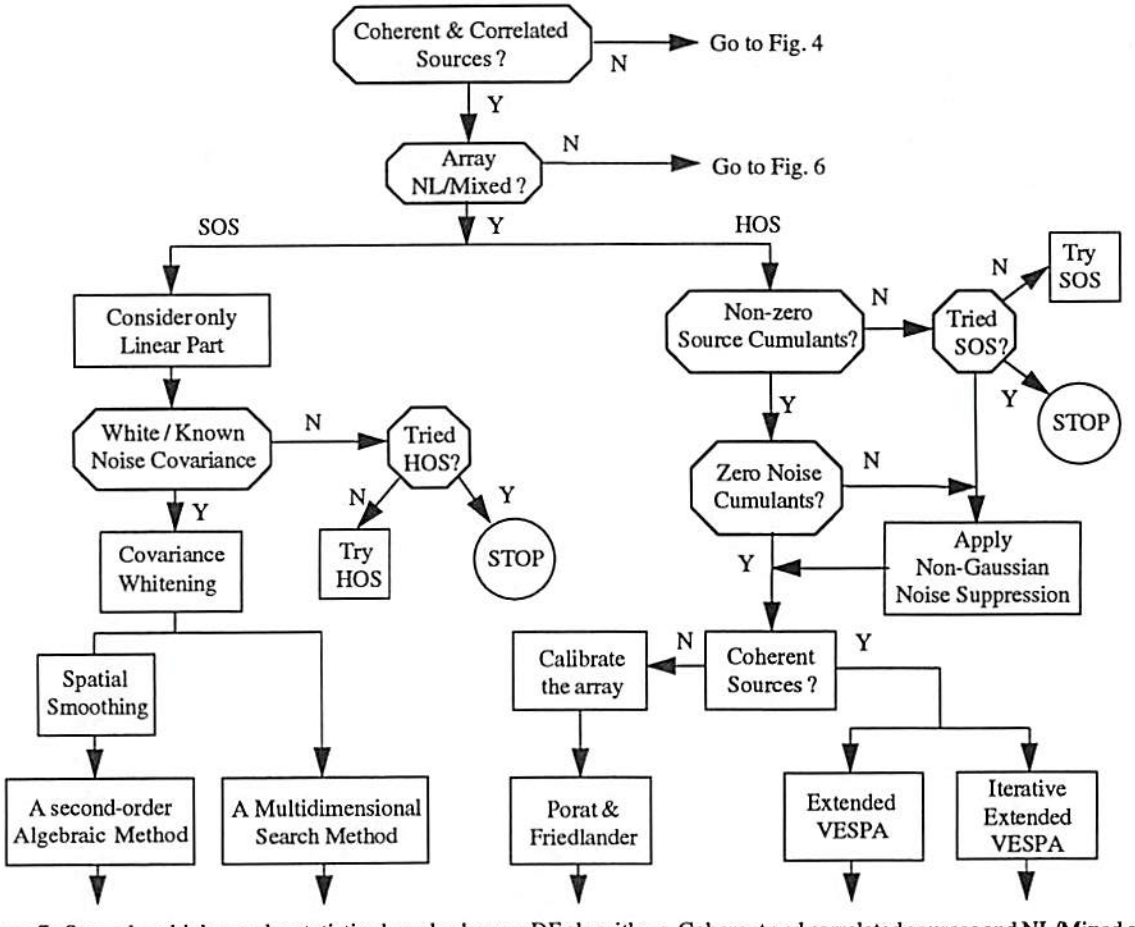


Figure 7: Second- or higher-order statistics based subspace DF algorithms. Coherent and correlated sources and NL/Mixed array.

# Second-Order Statistics based Subspace Methods for Direction Finding

## Signal Subspace Methods

*> SNR is enhanced effectively by retaining the signal subspace only*

### Search Based Methods

*> Select if array is calibrated or response is known analytically*

**Correlogram**

*> Lower resolution than MV and AR*

**Minimum Variance (MV)**

*> Narrower mainlobe and smoother sidelobes than conventional beamformers*
*> Higher resolution than Correlogram*
*> Lower resolution than AR*
*> Lower variance than AR*

***Autoregressive (AR)***

*> Higher resolution than MV and Correlogram*

**Subspace Fitting (SF)**

*> Weighted SF works regardless of source correlation, and has the same asymptotic properties as the stochastic ML method, i.e., it achieves CRB.*
*> Requires accurate calibration of the manifold and its derivative with respect to arrival angle*

### Algebraic Methods

*> Select if the array is ULA or its identical copy exists*
*> Computationally simpler than search-based methods.*

**ESPRIT**

*> Select if the array has an identical copy*
*> Computationally simple as compared to search based methods*
*> Sensitive to perturbations in the sensor response and array geometry*
*> LS and TLS versions are best. They have the same asymptotic performance, but TLS converges faster and is better than LS for low SNR and short data lengths*

**Toeplitz Approximation Method (TAM)**

*> Equivalent to LS-ESPRIT*

**GEESE**

*> Better than ESPRIT*

Figure 8

## Noise Subspace Methods

*> Methods are based on the orthogonality of steering vectors and noise subspace eigenvectors*

### Search Based Methods

*> Select if array is calibrated or response is known analytically*

**Eigenvector (EV)**

*> Produces fewer spurious peaks than MUSIC*
*> Shapes the noise spectrum better than MUSIC*

**Pisarenko**

*> Performance with short data is poor*

**MUSIC**

*> Better than MV*
*> Same asymptotic performance as the deterministic ML for uncorrelated sources*

**Minimum Norm**

*> Select if the array is ULA*
*> Lower SNR threshold than MUSIC for resolution of closely spaced sources*

**Method of Direction Estimation (MODE)**

*> Consistent for ergodic and stationary signals*

### Algebraic Methods

*> Select if the array is ULA*
*> Algebraic versions of EV, Pisarenko, MUSIC and Minimum Norm are possible*
*> Better resolution than search-based versions*

**Root MUSIC**

*> Lower SNR threshold than MUSIC for resolution of closely spaced sources*
*> Simple root-finding procedure*

## Higher-Order Statistics Based Subspace Methods for Direction Finding

> *Advantages over second-order methods: Less restrictions on the array geometry., no need for the noise covariance matrix*
> *Disadvantage: Longer data lengths than second-order methods needed*
> *Detailed analyses and comparisons remain unexplored*

### Pan-Nikias and Cardoso-Moulines Method

> *Calibration or analytical response of the array required*
> *Two cumulant formulations possible*
> *Any of the second-order search-based and algebraic methods (except fpr EV and ESPRIT) can be applied by using cumulant eigen-values and eigenvectors.*
> *Same hardware requirements as the corresponding second-order methods.*
> *Similar performance as second-order MUSIC for high SNR*
> *Limited use when source powers have a high dynamic range*
> *Fails for coherent sources*

### Porat and Friedlander Method

> *Calibration or analytical response of the array required*
> *Second-order search-based methods can be applied*
> *Fails for coherent sources*
> *Computationally expensive*

### Yuen and Friedlander Method

> *Handles coherent sources*
> *Limited to uniform linear arrays*
> *Array aperture is decreased*

### Chen and Lin Method

> *Handles coherent sources*
> *Limited to uniform linear arrays*
> *Array aperture is decreased*

### Chiang and Nikias's ESPRIT

> *An identical copy of the array needed*
> *ESPRIT is used with two cumulant matrices*
> *Fails for coherent sources*

### Dogan and Mendel Methods:

> *Array aperture is increased*

### Virtual Cross Correlation Computer

> *Cross-correlations are replaced (and computed) by their cumulant equivalents.*
> *Any of the second-order search-based and algebaric methods can be applied to virtually-created covariance matrix*
> *More sources than sensors can be detected*
> *Fault tolerance capability*
> *Fails for correlated and coherent sources*

### Virtual ESPRIT (VESPA)

> *Only one pair of identical sensors is required.*
> *Applicable to arbitrary arrays*
> *No calibration required*
> *Similar computational load as ESPRIT*
> *Fails for correlated and coherent sources*

### Gonen and Mendel's Extended VESPA

> *Handles correlated and coherent sources*
> *Applicable to partially linear arrays*
> *More signals than sensors can be detected*
> *Similar computational load as ESPRIT*

### Iterated (Extended) VESPA

> *Handles the case when the source powers have a high dynamic range*
> *Same hardware requirements as VESPA and Extended VESPA*

Figure 9