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**Operations on Type-2 Fuzzy Sets**

**by**

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# Abstract

A type-2 fuzzy set is characterized by a fuzzy membership function, i.e., the membership grade of each element of this set is itself a fuzzy set in $[0, 1]$. Such sets can be used in situations where there is uncertainty about the membership grades of a fuzzy set. Operations on type-2 fuzzy sets are defined by using Zadeh's Extension Principle. In this report, we give some examples of type-2 fuzzy sets; discuss set theoretic operations on type-2 sets and algebraic operations on the membership grades of type-2 sets in great detail; and, introduce the concept of the centroid of a type-2 fuzzy set. We provide easily implementable algorithms for performing these set theoretic and algebraic operations on type-2 sets; and, also provide practical approximations for the cases where actual results are difficult to generalize or implement.

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# List of Commonly Used Symbols

## Set Notation

| Symbol | Meaning |
|---|---|
| $\tilde{A}$ | A type-1 fuzzy set |
| $\mu_{\tilde{A}}(x)$ | The membership grade of $x$ in type-1 fuzzy set $\tilde{A}$ |
| $\int_x \mu_{\tilde{A}}(x)/x$ | A type-1 fuzzy set, $\tilde{A}$, supported on a continuum |
| $\sum_x \mu_{\tilde{A}}(x)/x$ | A type-1 fuzzy set, $\tilde{A}$, having a discrete support |
| $\tilde{\tilde{A}}$ | A type-2 fuzzy set |
| $\tilde{\mu}_{\tilde{\tilde{A}}}(x)$ | The membership grade of $x$ in type-2 fuzzy set $\tilde{\tilde{A}}$ |
| $C_{\tilde{A}}$ | The centroid (a crisp number) of $\tilde{A}$ |
| $\tilde{C}_{\tilde{\tilde{A}}}$ | The centroid (a type-1 fuzzy set) of $\tilde{\tilde{A}}$ |

## Operations on Fuzzy Sets

| Symbol | Meaning |
|---|---|
| $\star, \mathcal{T}$ | $t$-norm |
| $\vee$ | maximum |
| $\wedge$ | minimum |
| $\cup$ | Union |
| $\cap$ | Intersection |
| $\bar{(\cdot)}$ | Complement of $(\cdot)$ |
| $\sqcup$ | Join |
| $\sqcap$ | Meet |
| $\neg$ | Negation |
| $\circ$ | Sup-star composition |
| $\sum, +$ | Algebraic sum or union of discrete quantities (as indicated) |
| $\int$ | Integral or union on a continuum (as indicated) |
| $\times$ | Algebraic or Cartesian product (as indicated) |

# Chapter 1

# Introduction

The concept of a *type-2 fuzzy set* was introduced by Zadeh [9] as an extension of the concept of an ordinary fuzzy set (henceforth called a *type-1 fuzzy set*). A type-2 fuzzy set is characterized by a fuzzy membership function, i.e., the membership value (or membership grade) for each element of this set is a fuzzy set in $[0, 1]$, unlike a type-1 set where the membership grade is a crisp number in $[0, 1]$. Such sets can be used in situations where there is uncertainty about the membership grades themselves, e.g., an uncertainty in the shape of the membership function or in some of its parameters. Consider the transition from ordinary sets to fuzzy sets. When we cannot determine the membership of an element in a set as 0 or 1, we use fuzzy sets of type-1. Similarly, when the circumstances are so fuzzy that we have trouble determining the membership grade even as a crisp number in $[0, 1]$, we use fuzzy sets of type-2.

This does not mean that we need to have extraordinarily fuzzy circumstances to use type-2 sets. We can look at the situation from a different perspective. When something is uncertain (e.g., a measurement), we have trouble determining its exact value, and in this case, using type-1 sets, of course, makes more sense than using crisp sets. But then, even in the type-1 sets, we specify the membership functions exactly, which seems counter-intuitive. If we can not determine the exact value of an uncertain quantity, how can we determine its exact membership grade in a fuzzy set ? Of course, this criticism applies to type-2 sets as well, because even though the membership grade is fuzzy, we specify the membership function of the membership grade exactly, which again seems counter-intuitive. If we continue thinking along these lines, we can say that no finite-type fuzzy set can represent uncertainty "completely". So, ideally, we need to use a type-$\infty$ fuzzy set to "completely" represent uncertainty ! Of course, we can not do this in practice, so we have to use some finite-type sets. So, type-1 fuzzy sets can be thought of as a *first-order approximation* to the uncertainty in real life. Our work with type-2 fuzzy sets tries to get at a *second-order approximation.* One may look at higher types too; but, as we go on to higher types, the complexity of the system increases rapidly. So, in this work we deal just with type-2 sets.

Our aim, in writing this report, is to discuss in detail operations on type-2 fuzzy sets. The operations of interest to us are : (1) set theoretic operations like union, intersection or complement of type-2 sets, and (2) some algebraic operations, which include centroid calculation of a type-2 set, and addition, multiplication and weighted averaging of membership grades of type-2 sets (which themselves are type-1 fuzzy sets). In this chapter, we explain the concept of a type-2 set, and introduce our notation and pictorial representation; and, in Chapter 2, we discuss in detail the aforementioned operations on type-2 sets.

## 1.1 Examples of Type-2 Fuzzy Sets

**Example 1.1** Consider the case of a fuzzy set characterized by a Gaussian membership function with mean $m$ and a standard deviation that can take values in $[\sigma_1, \sigma_2]$, i.e.,

$$\mu(x) = e^{-\frac{1}{2}(\frac{x-m}{\sigma})^2}; \quad \sigma \in [\sigma_1, \sigma_2] \tag{1.1}$$

Corresponding to each value of $\sigma$, we will get a different membership curve (see Fig. 1.1). So, the membership grade of any particular $x$ (except for $x = m$) can take any of a number of possible values depending upon the value of $\sigma$, i.e., *the membership grade is not a crisp number, it is a fuzzy set.* Figure 1.1 shows the domain of the fuzzy set associated with $x = 0.65$; however, the membership function associated with this fuzzy set is not shown in the figure. We return to this point in Section 1.3. □

**Example 1.2** Consider the case of a fuzzy set with a Gaussian membership function having a fixed standard deviation $\sigma$, but an uncertain mean, taking values in $[m_1, m_2]$, i.e.,

$$\mu(x) = e^{-\frac{1}{2}(\frac{x-m}{\sigma})^2}; \quad m \in [m_1, m_2] \tag{1.2}$$

Again, $\mu(x)$ is a fuzzy set. Figure 1.2 shows an example of such a set. As in Fig. 1.1, it is not possible to deduce the membership function associated with the fuzzy membership of any $x$ from Fig. 1.2. □

In Examples 1.1 and 1.2, we considered fuzzy sets with Gaussian membership functions which had their standard deviations or means uncertain. Such sets can be used in situations where we want to use Gaussian fuzzy sets, but are not certain about their center or spread locations. If the situation is such, however, that we are uncertain even about the *shape* of the membership function (Gaussian / triangular / any other arbitrary shape), we can use a *Gaussian type-2 fuzzy set* defined as in Example 1.3.

**Example 1.3** Consider a type-1 fuzzy set characterized by a Gaussian membership function (mean $M$ and standard deviation $\sigma_x$), which gives one crisp membership $m(x)$ for each input $x \in X$, where

$$m(x) = e^{-\frac{1}{2}(\frac{x-M}{\sigma_x})^2} \tag{1.3}$$

This is depicted in Fig. 1.3. Now, imagine that this membership of $x$ is a fuzzy set. Let us call the domain-elements of this set *primary memberships* of $x$ (denoted by $\mu_1$) and membership grades of these primary memberships *secondary memberships* of $x$ [denoted by $\mu_2(x, \mu_1)$]. So, for a fixed $x$, we get a type-1 fuzzy set whose domain-elements are primary memberships of $x$ and whose corresponding membership grades are secondary memberships of $x$. If we assume that the secondary memberships follow a Gaussian with mean $m(x)$ and standard deviation $\sigma_m$, as in Fig. 1.3, we can describe the secondary membership function for each $x$ as

$$\mu_2(x, \mu_1) = e^{-\frac{1}{2}(\frac{\mu_1 - m(x)}{\sigma_m})^2} \tag{1.4}$$

where $\mu_1 \in [0,1]$ and $m$ is as in (1.3). Equations (1.3) and (1.4) can be combined as

$$\mu_2(x,\mu_1) = e^{-\frac{1}{2}\left(\frac{\mu_1 - e^{-\frac{1}{2}\left(\frac{x-M}{\sigma_x}\right)^2}}{\sigma_m}\right)^2} ; \tag{1.5}$$

where $\mu_1 \in [0,1]$. Equation (1.5) stresses the fact that the secondary membership function can be viewed as a real function of two variables, $x$ and $\mu_1$. The membership grade for each $x$, $\mu(x)$, which represents all the primary memberships and their corresponding secondary memberships taken together, can be written as

$$\mu(x) = \int_{\mu_1 \in [0,1]} \mu_2(x,\mu_1)/\mu_1 ; \quad x \in X \tag{1.6}$$

where $\mu_2(x,\mu_1)$ is as in (1.5).

Observe that Example 1.3 is different than Examples 1.1 and 1.2 in that in Example 1.3 we are explicitly stating the secondary membership function. Actual values of the secondary membership grades were not defined in Examples 1.1 and 1.2. We return to Example 1.3 in Section 1.3. □

Now, let's see a situation in real life which needs to be described using type-2 fuzzy sets.

**Example 1.4** Consider classes of people with *below average*, *average* and *above average* earnings. These sets, of course, are fuzzy. Now, if we ask someone what memberships s/he would have in these three fuzzy sets, most likely we are going to get an answer of the form "a *high* membership in *above average earnings* and low in others", rather than crisp numbers as memberships. This means that the membership grade is a fuzzy set or in other words, the aforementioned three fuzzy sets are of type-2 ! Observe that in this example, the person who is asked the question, knows her/his own income exactly, but the uncertainty in the membership grade arises due to the fact that s/he doesn't know the exact parameters of the membership functions for these 3 sets. (If Gaussian membership functions are used, this is analogous to ambiguity in mean and/or variance.)

In this example, "a *high* membership in *above average*" cannot be rephrased as "highly above average". In "highly above average", "highly" is a *hedge* on "above average", but after the application of this hedge, all we get is another type-1 fuzzy set, "highly above average". Now, if the same person were asked what would be her/his membership in the set "highly above average", s/he would probably say *medium*, which means the membership is again fuzzy and the set is still best described as a type-2 fuzzy set. If the person could not give a crisp membership for the "above average" set, the person would definitely not be able to give a crisp membership for the "highly above average" set [this new set is just a (possibly) non-linear transformation of the original one]. □

From now on, we will use the membership terminology introduced in Example 1.3. *Membership grade* is a synonym for "degree of membership", which is a crisp number for type-1 sets, a type-1 set for type-2 sets, and in general, a type-$k$ set for type-$(k+1)$ sets. In the case of type-2 sets, *primary memberships* are the domain-elements of a membership grade and *secondary memberships* are membership grades of primary memberships. For example, in (1.5) and (1.6), $\mu_1$ indicates the primary memberships; $\mu_2(x, \mu_1)$ indicates the secondary memberships; and $\mu(x)$, which represents all the primary and secondary memberships taken together, indicates the membership grade of $x \in X$. Thus, for a type-1 set, $\mu(x)$ is short for $\mu_1(x)$ and $\mu_2(x, \mu_1) = 1$; and, for a type-2 set, $\mu(x)$ indicates the type-1 set $\int_{\mu_1} \mu_2(x, \mu_1)/\mu_1$.

A type-2 fuzzy set can also be thought of as a fuzzy valued function, which assigns to every $x \in X$, a type-1 fuzzy membership grade. In this sense, we will call $X$ the *domain* of the type-2 fuzzy set.

## 1.2 Some Useful Type-2 Sets

Here we formally define three kinds of type-2 sets that we will talk about often in this dissertation :

**1.** A *Gaussian type-2 set* is one in which the membership grade of every domain point is a Gaussian type-1 set contained in $[0, 1]$.

Example 1.3 shows an example of a Gaussian type-2 set. Note that it is not necessary for the principal membership function of a Gaussian type-2 set to also be a Gaussian, as is the case in Example 1.3. Figure 1.9 shows an example of a Gaussian type-2 set having a triangular principal membership function, using the 2-D pictorial representation described in Section 1.3.

**2.** An *interval type-2 set* is one in which the membership grade of every domain point is a crisp set whose domain is some interval conatined in $[0, 1]$.

In Example 1.1, if we attach equal degree of uncertainty to every value of $\sigma$ in $[\sigma_1, \sigma_2]$, i.e., if we let the standard deviation $\sigma$ be a crisp set with domain $[\sigma_1, \sigma_2]$, we can set all the secondary memberships of the resulting type-2 set equal to 1. The membership grade corresponding to every $x$ in this type-2 set, now, becomes a crisp set, and the type-2 set becomes an interval type-2 set (see Section 1.3).

Note that, although every membership grade of an interval type-2 set is a crisp set, the set itself is type-2, because the memberships are *sets* rather than crisp numbers. Interval type-2 sets are the simplest kind of type-2 sets to deal with, since all the secondary memberships are unity; and, we will often discuss them, though the main focus of our work is Gaussian type-2 sets. We will refer to the membership grades of an interval type-2 set as "interval type-1 sets".

**3.** A *triangular type-2 set* is one in which the membership grade of every domain point is a triangular type-1 set contained in $[0, 1]$.

Unless otherwise specified, a "triangle" will always mean a "symmetrical triangle" in our work. The results for triangular type-2 sets are collected in Appendix E.

## 1.3 Pictorial Representation

Now, let's try to represent a type-2 membership function pictorially. Observe that our earlier pictorial representations using 2-D plots (Figs. 1.1 and 1.2) did not indicate the numerical values of secondary memberships. All that one can see from those diagrams is just the set of primary memberships corresponding to each $x$. So, these representations do not contain all the information that we have. Recall the example of the Gaussian fuzzy set with uncertain mean. The 2-D diagram in Fig. 1.2 does not depend on the actual shape of the membership function for the fuzzy mean. It will remain the same as long as the support of the fuzzy set for the mean is $[m_1, m_2]$, which shows that these diagrams are not unique, i.e., we can get the same diagrammatic representation for two or more distinct situations. For example, Fig. 1.2 would remain unchanged if the fuzzy set for the mean followed a

Gaussian membership curve or a triangular membership curve as long as the support is $[m_1, m_2]$. This indicates that the earlier pictorial representations are not "complete".

A type-2 membership function can be viewed as a function of two variables. For each input $x$ and a primary membership $\mu_1$, we get a secondary membership, which is a crisp number. Let's call this secondary membership $\mu_2$. So, the membership function of a type-2 set can be represented as

$$\mu_2(x, \mu_1) \;: X \times [0,1] \to [0,1] \tag{1.7}$$

where $X$ is the space of all inputs $x$. Pictorially, we can display this function as a 3-D diagram with $x$ and $\mu_1$ as independent variables and $\mu_2$ as the dependent variable.

Recall Example 1.1 . Suppose that the degree of uncertainty that we attach to each value of $\sigma$ in the range $[\sigma_1, \sigma_2]$ is the same; in other words, let the standard deviation $\sigma$ be a *crisp* set with domain $[\sigma_1, \sigma_2]$. Since, each value of standard deviation is equally uncertain, we set all the secondary memberships of the resulting type-2 set equal to 1, i.e., the membership grade corresponding to each $x$ is an interval in $[0, 1]$ (the resulting type-2 set is an interval type-2 set). Figure 1.4 (a) shows a 3-D representation of this type-2 set, assuming $\sigma_1 = 0.1$ and $\sigma_2 = 0.2$, and Fig. 1.4 (b) shows the membership grade for $x = 0.65$; the domain of this membership grade is indicated in Fig. 1.1. Figure 1.5 (a) shows the 3-D diagram for Example 1.2, drawn by assuming that the mean $m$ is a crisp set with domain $[m_1, m_2] = [0.4, 0.6]$. Figure 1.5 (b) shows the membership grade corresponding to $x = 0.65$ in this interval type-2 set. See Appendix A for examples which let the standard deviation of the Gaussian in Example 1.1 and the mean of the Gaussian in Example 1.2 be Gaussian type-1 sets.

Figure 1.6 (a) depicts a 3-D representation of (1.5) and Fig. 1.6 (b) depicts the fuzzy type-1 set $\mu(x)$ for an arbitrary value of $x$ (obtained by taking a cross-section of Fig. 1.6 parallel to the $\mu_1 - \mu_2$ axes). $\mu(x)$ is a Gaussian, because we constructed it that way. Observe the similarity with a type-1 pictorial representation, where we display the membership function of a type-1 set as a 2-D picture (function of one variable, $x$).

Although the 3-D representation of a type-2 set conveys all the information that we have about the set, it is not very helpful to use these 3-D diagrams when we have to show more than one set on the same axes. Additionally, they can be quite complicated to construct. So, in spite of the aforementioned "incompleteness" of 2-D representations, we continue to use them in our analyses of type-2 sets. If there is a need to show the secondary membership functions explicitly, we will use a 3-D representation.

Figure 1.7 shows a 2-D representation of the Gaussian type-2 set depicted in Fig. 1.6 (a). We call the set of primary memberships that have secondary membership grades equal to 1, the *principal* membership function of the type-2 set (shown with a bold line in Fig. 1.7). From (1.4), we can see that $m(x)$ in (1.3) is the principal membership function. Since we are using Gaussian secondary membership functions for each input, only one primary membership has a secondary membership equal to 1. This seems reasonable, because the secondary membership function indicates the uncertainty in determining the membership grade for a particular input. We will generally use secondary membership functions like Gaussians or triangles, which assign unity membership to only one point in their domain. Observe that this is not true for the crisp secondary membership functions shown in Figs. 1.4 and 1.5.

The concept of a principal membership function also illustrates the fact that a type-1 fuzzy set can be thought of as a special case of a type-2 fuzzy set. We can think of a type-1 fuzzy set as a type-2 fuzzy set whose membership grades are type-1 fuzzy singletons, having secondary membership equal to unity for only one primary membership and zero for all others, i.e., we can think of the principal membership function of a type-2 set as an embedded type-1 set. *Our fundamental design requirement in this work is that our type-2 system results reduce to type-1 results when we replace all the type-2 sets by their principal membership functions.*

Figure 1.8 (a) shows a 3-D representation of a Gaussian type-2 set having a *triangular* principal membership function, and Fig. 1.8 (b) shows the membership grade for $x = 6.5$. The 2-D representation of this set is depicted in Fig. 1.9. The difference between the two Gaussian type-2 sets, in Figs. 1.6 (a) and 1.8 (a), is seen more clearly in the 2-D representation.

The secondary membership functions for the Gaussian type-2 sets depicted in Figs. 1.6 and 1.8 have constant standard deviations, implying that the uncertainty in the membership grades remains constant for all $x$. Intuitively, however, it seems more appropriate that membership values near zero should have less uncertainty associated with them than membership values near 1. In other words, it seems more appropriate that the uncertainty in a membership value be expressible as some percentage of it. Such a type-2 set (Gaussian type-2 with Gaussian principal membership function) is depicted in Fig. 1.10. The secondary membership functions of this set have decreasing standard deviations, implying that the uncertainty in the membership grades decreases as $x$ moves away from the mean of the principal membership function.

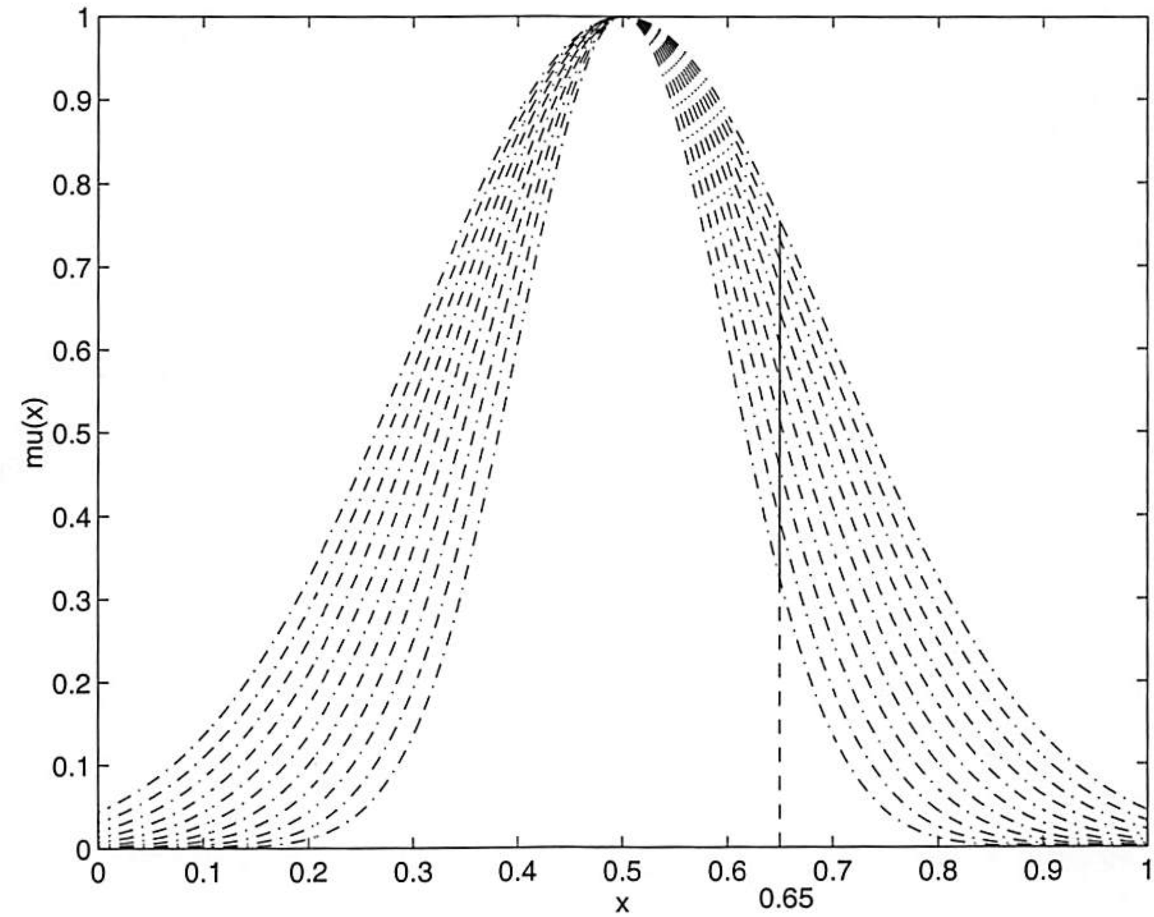



Figure 1.1: A type-2 fuzzy set representing a type-1 fuzzy set with uncertain standard deviation. The standard deviation is uncertain in the interval $[0.1, 0.2]$. The figure also shows the domain of the type-1 fuzzy set corresponding to $x = 0.65$; however, the membership grades in this type-1 fuzzy set are not shown.

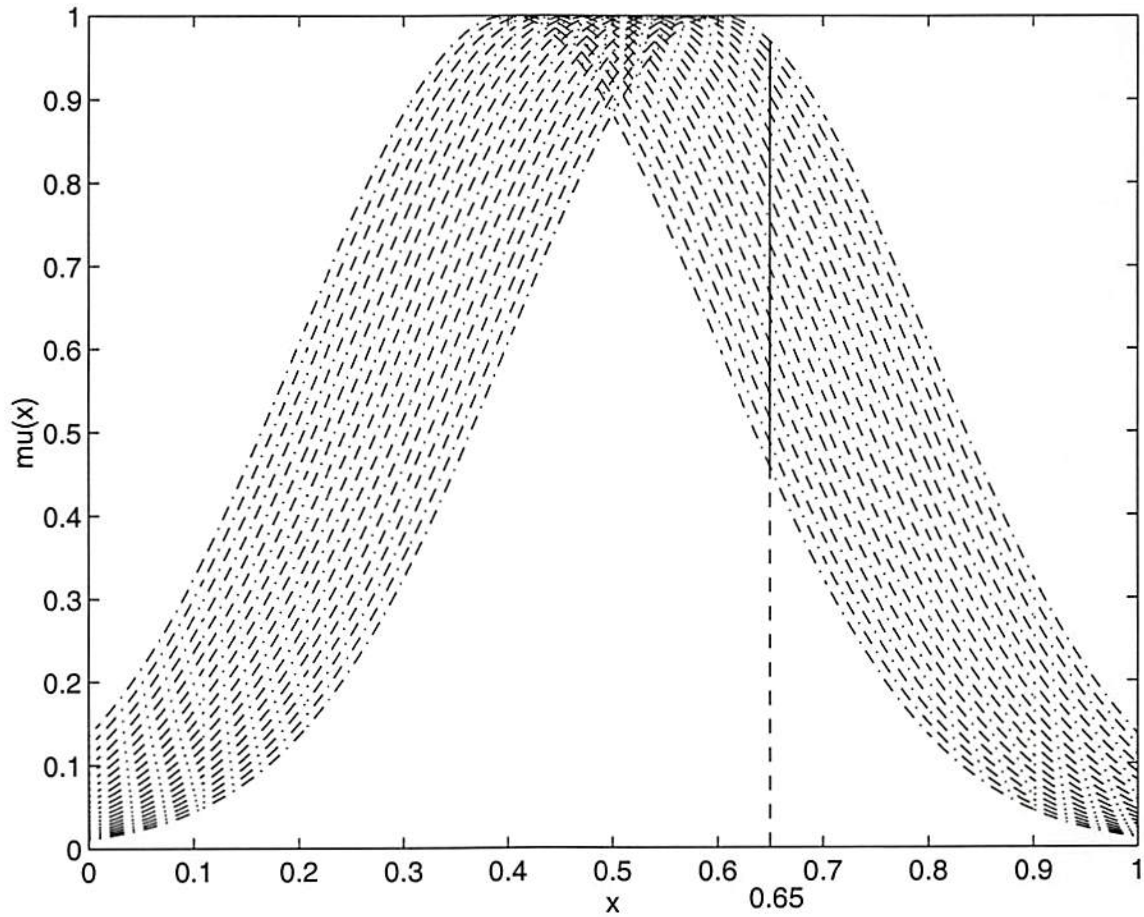



Figure 1.2: A type-2 fuzzy set representing a type-1 fuzzy set with uncertain mean. The mean is uncertain in the interval [0.4, 0.6]. The figure also shows the domain of the type-1 fuzzy set corresponding to x = 0.65; however, the membership grades in this type-1 fuzzy set are not shown.

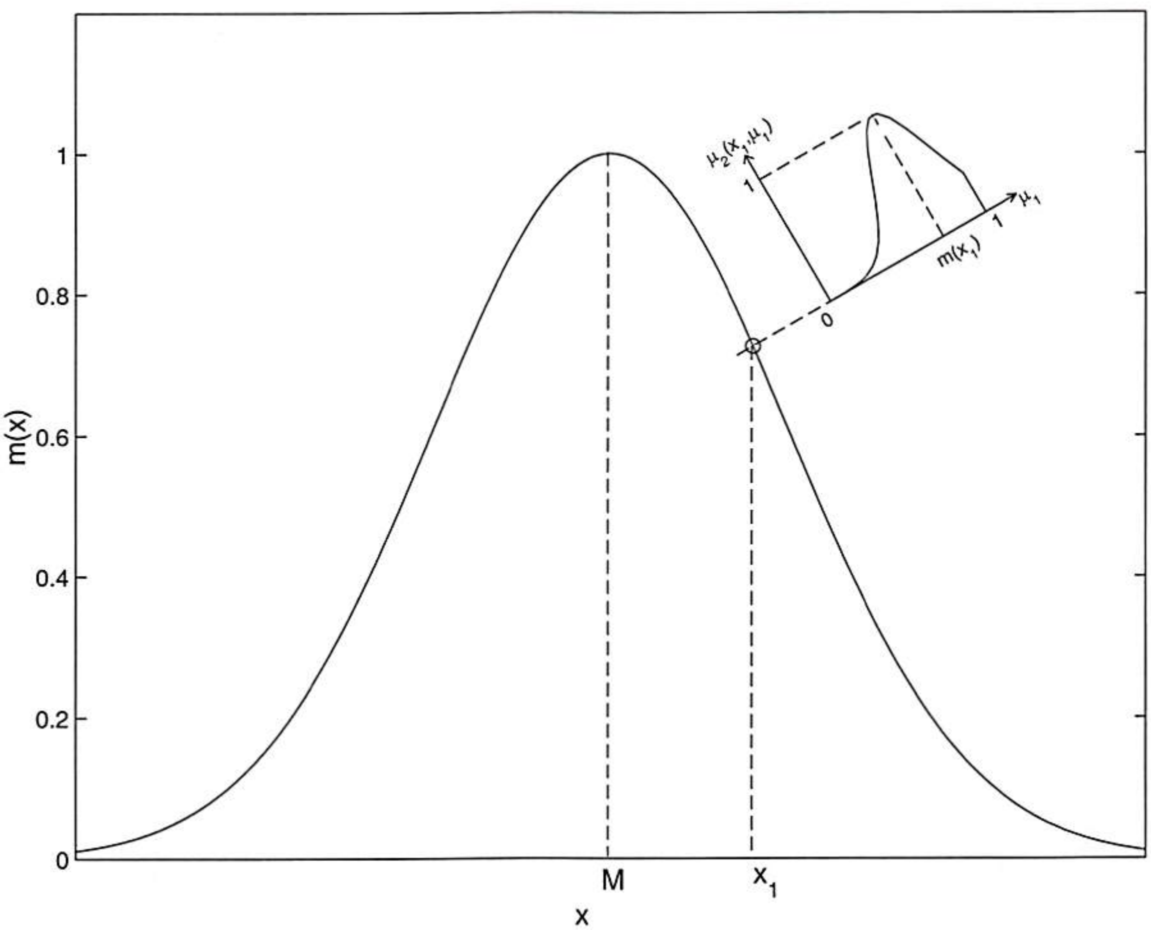



Figure 1.3: Figure for Example 1.3. The Gaussian $m(x)$ and membership grade corresponding to $x = x_1$ are shown. The membership grade is a Gaussian type-1 fuzzy set contained in $[0, 1]$ with mean $m(x_1)$.

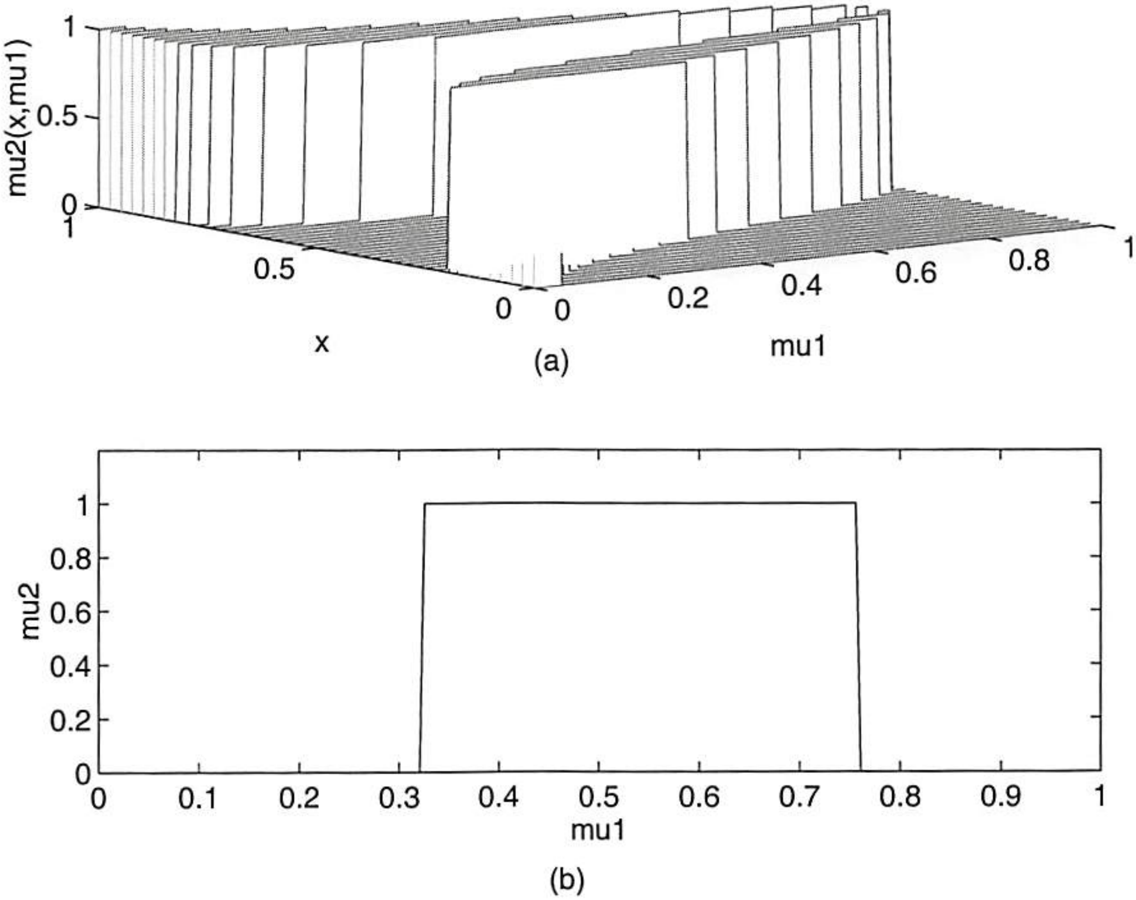



Figure 1.4: (a) Three dimensional representation of the type-2 set in Example 1.1, assuming that the standard deviation is a crisp set with domain $[\sigma_1, \sigma_2] = [0.1, 0.2]$. The membership grade for each $x$ is a crisp set. (b) The membership grade corresponding to $x = 0.65$.

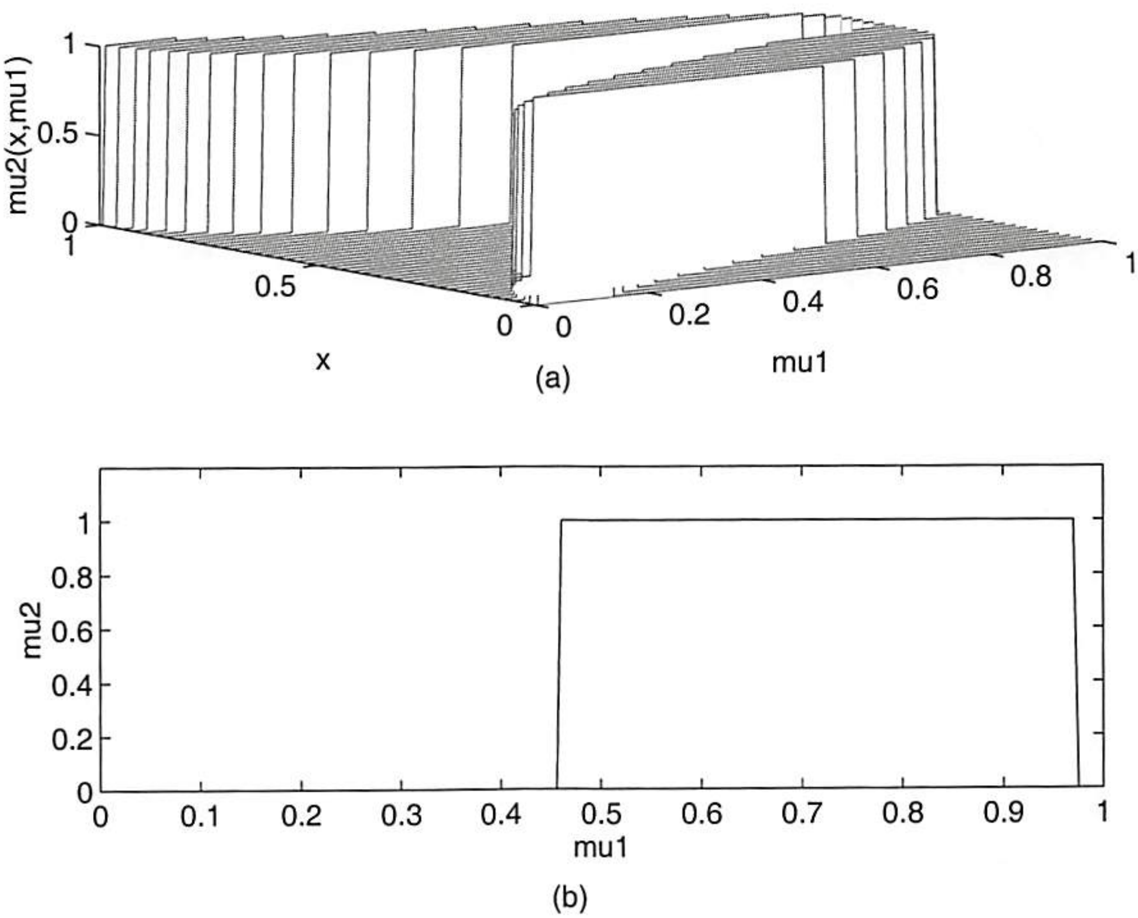



Figure 1.5: (a) Three dimensional representation of the type-2 set in Example 1.2, assuming that the mean is a crisp set with domain $[m_1, m_2] = [0.4, 0.6]$. The membership grade for each $x$ is a crisp set. (b) The membership grade corresponding to $x = 0.65$.

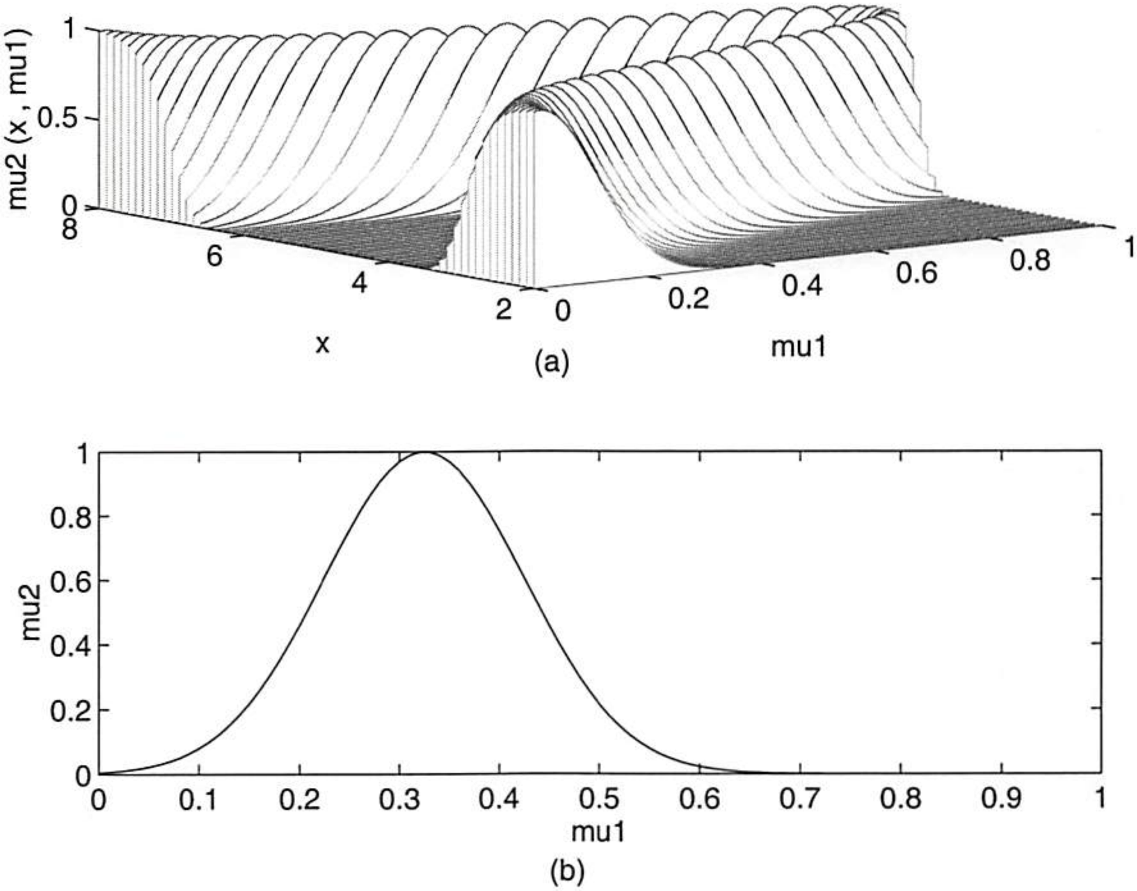



Figure 1.6: (a) Three dimensional representation of a Gaussian type-2 fuzzy set, having a Gaussian principal membership function. The membership grade for each $x$ is Gaussian by construction. All these Gaussians have the same standard deviation. (b) The membership grade corresponding to $x = 6.5$.

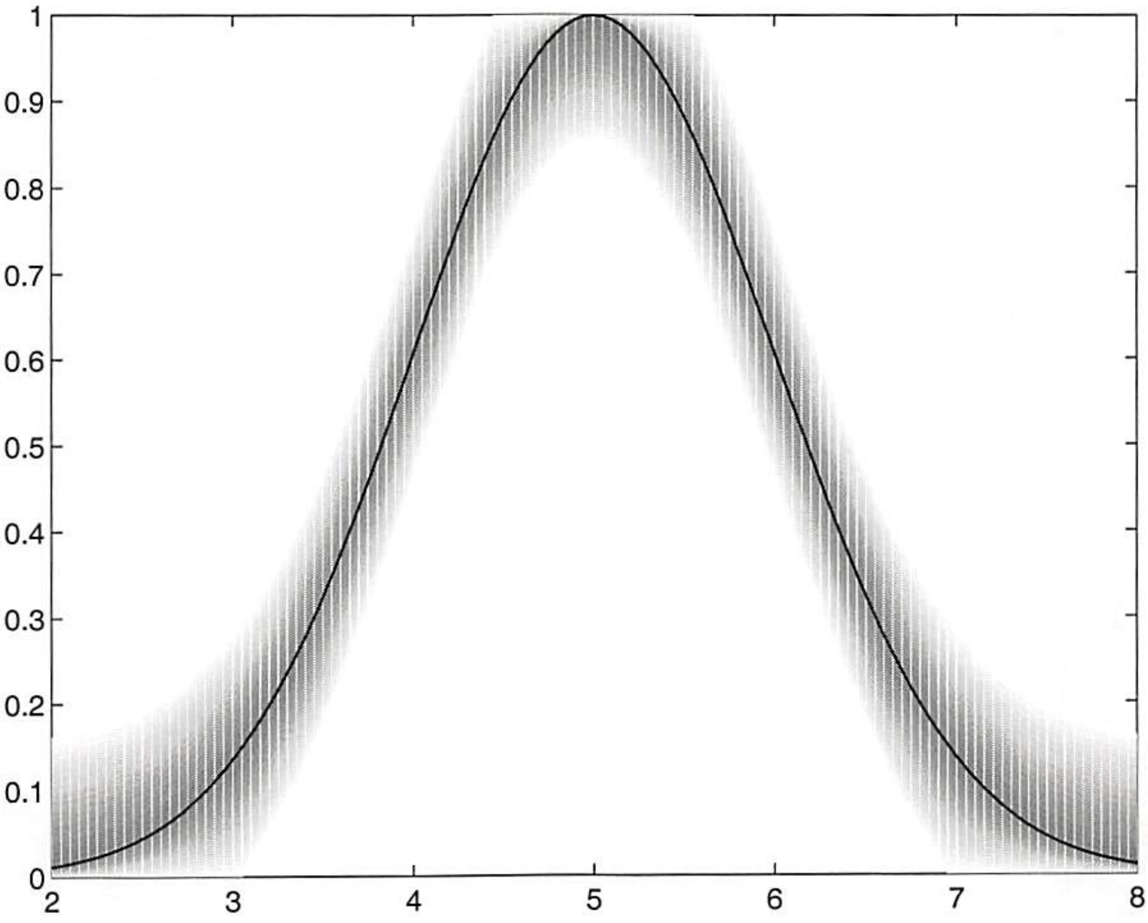



Figure 1.7: Two-dimensional representation of the Gaussian type-2 set depicted in Fig. 1.6 (a). The standard deviations of the secondary Gaussians are constant. The *principal* membership function, i.e., the set of primary memberships having secondary membership equal to 1, is indicated with a thick line. This principal membership function is a Gaussian because of the way the set is constructed. Intensity of the shading is approximately proportional to secondary membership grades. Darker areas indicate higher secondary memberships. The flat portion near the center and near the two ends, appears because primary memberships cannot be less than 0 or greater than 1 and so the Gaussians have to be "clipped".

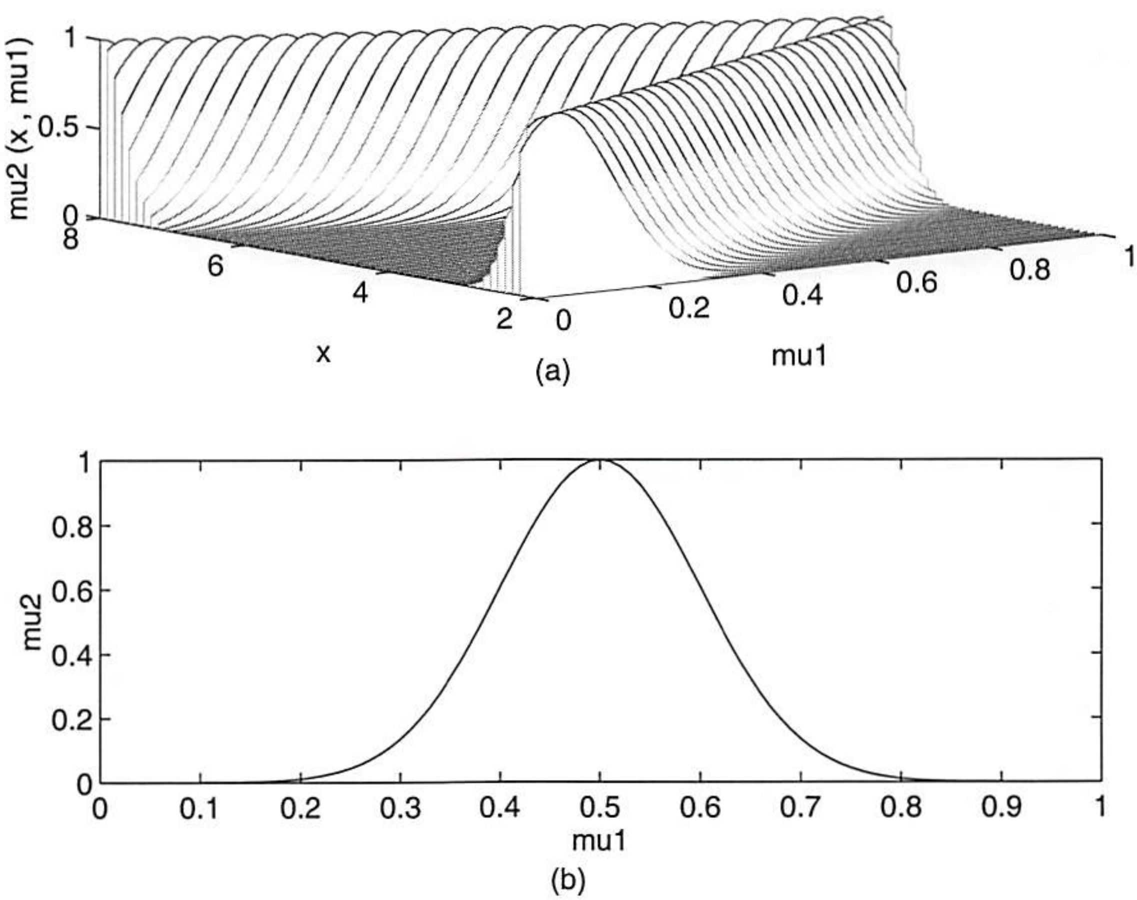



Figure 1.8: (a) Three dimensional representation of a Gaussian type-2 fuzzy set, having a triangular principal membership function. The membership grade for each $x$ is Gaussian by construction. All these Gaussians have the same standard deviation. (b) The membership grade corresponding to $x = 6.5$.

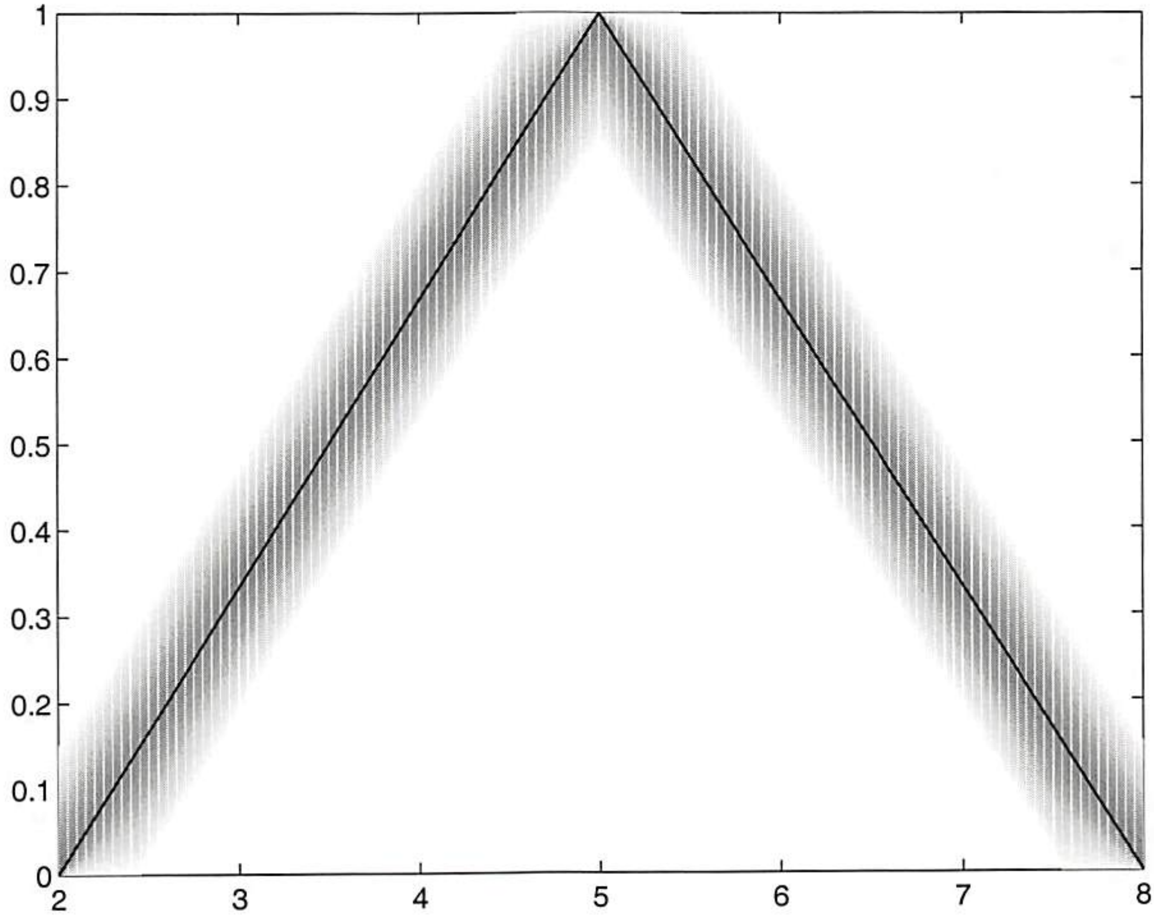



Figure 1.9: Two-dimensional representation of the Gaussian type-2 set depicted in Fig. 1.8 (a). The *principal* membership function is triangular. The standard deviations of the secondary Gaussians are constant.

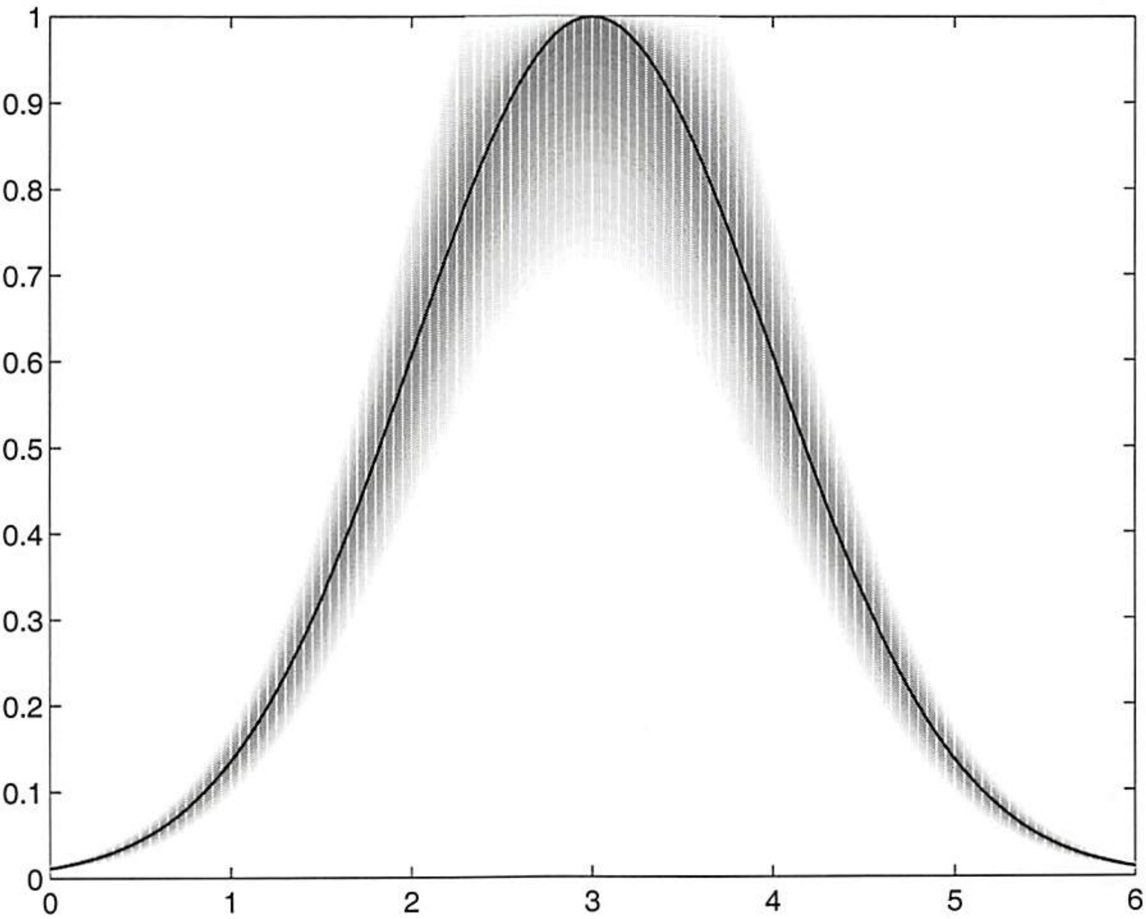



Figure 1.10: Two-dimensional representation of a Gaussian type-2 set, where standard deviations of the secondary Gaussians decrease by design, as $x$ moves away from the mean of the principal membership function.

# Chapter 2

# Operations on Type-2 Sets

In this chapter, we examine set theoretic operations on type-2 sets. We use the following notation. A type-1 fuzzy set $P$ is denoted as $\tilde{\mathrm{P}}$. A type-2 fuzzy set $A$ is denoted as $\tilde{\tilde{\mathrm{A}}}$. Consequently, if $x_0$ is an element of $\tilde{\tilde{\mathrm{A}}}$, the membership grade of $x_0$ in $\tilde{\tilde{\mathrm{A}}}$ is denoted as $\tilde{\mu}_{\tilde{\tilde{\mathrm{A}}}}(x_0)$. Recall that $\tilde{\mu}_{\tilde{\tilde{\mathrm{A}}}}(x_0)$ is itself a type-1 fuzzy set whose elements and their memberships are, respectively, the primary and secondary memberships of $x_0$.

## 2.1 Set Theoretic Operations

To begin, we recall some facts about type-1 sets. A fuzzy subset $\tilde{\mathrm{A}}$ of a set $X$ is represented as follows :

$$\begin{aligned}\tilde{\mathrm{A}} &= \mu_{\tilde{\mathrm{A}}}(x_1)/x_1 + \mu_{\tilde{\mathrm{A}}}(x_2)/x_2 + \ldots + \mu_{\tilde{\mathrm{A}}}(x_n)/x_n \\ &= \sum_i \mu_{\tilde{\mathrm{A}}}(x_i)/x_i, \quad x_i \in X \qquad (2.1)\end{aligned}$$

where the sum represents *union.* If the support of $\tilde{\mathrm{A}}$ is a continuum, we write

$$\tilde{\mathrm{A}} = \int_X \mu_{\tilde{\mathrm{A}}}(x)/x \qquad (2.2)$$

Suppose, we have 2 type-1 fuzzy sets $\tilde{\mathrm{F}}_1$ and $\tilde{\mathrm{F}}_2$ characterized by membership functions $\theta_1$ and $\theta_2$, as follows :

$$\tilde{\mathrm{F}}_1 = \sum_i \theta_1(y_i)/y_i \qquad (2.3)$$

$$\tilde{F}_2 = \sum_i \theta_2(y_i)/y_i \tag{2.4}$$

Using *max* $t$-conorm and *min* $t$-norm, the membership functions of the union, intersection and complement of these sets are given as [6] :

$$\mu_{\tilde{F}_1 \cup \tilde{F}_2}(y_i) = \max\{\theta_1(y_i), \theta_2(y_i)\} \ \forall i \tag{2.5}$$

$$\mu_{\tilde{F}_1 \cap \tilde{F}_2}(y_i) = \min\{\theta_1(y_i), \theta_2(y_i)\} \ \forall i \tag{2.6}$$

$$\mu_{\bar{\tilde{F}}_1}(y_i) = 1 - \theta_1(y_i) \ \forall i \tag{2.7}$$

$$\mu_{\bar{\tilde{F}}_2}(y_i) = 1 - \theta_2(y_i) \ \forall i \tag{2.8}$$

Since $\tilde{F}_1$ and $\tilde{F}_2$ are fuzzy sets of type-1, their membership grades $\theta_1(y_i)$ and $\theta_2(y_i)$ are crisp numbers and therefore, for each $y_i$, we can perform all the operations on the RHSs of Eqs. (2.5) - (2.8) in one step.

Now, suppose that $\tilde{\tilde{F}}_1$ and $\tilde{\tilde{F}}_2$ are type-2 fuzzy sets, so that the membership grades $\tilde{\theta}_1(y_i)$ and $\tilde{\theta}_2(y_i)$ are type-1 fuzzy sets. In order to compute the union, intersection and complement of $\tilde{\tilde{F}}_1$ and $\tilde{\tilde{F}}_2$, we need to extend the binary operations of *min* and *max*, and the unary operation of *negation* to fuzzy sets. We use Zadeh's *Extension Principle* for this purpose, which we state here for reference purposes.

**The Extension Principle** [2, 9] : Let $X$ be a Cartesian product of universes, $X = X_1 \times X_2 \times \cdots \times X_r$, and $\tilde{A}_1$, $\tilde{A}_2$,...,$\tilde{A}_r$ be fuzzy sets in $X_1$, $X_2$, ...,$X_r$, respectively. Let $f$ be a mapping from $X$ to a universe $Y$ such that $y = f(x_1, ..., x_r) \in Y$. Zadeh's Extension Principle allows us to induce from the $r$ fuzzy sets $\tilde{A}_i$, a fuzzy set $\tilde{B}$ on $Y$, through $f$, such that

$$\mu_{\tilde{B}}(y) = \sup_{x_1,\ldots,x_r \,:y=f(x_1,\ldots,x_r)} \min\{\mu_{\tilde{A}_1}(x_1), \ldots, \mu_{\tilde{A}_r}(x_r)\} \tag{2.9}$$

$$\mu_{\tilde{B}}(y) = 0 \text{ if } f^{-1}(y) = \emptyset$$

where $f^{-1}(y)$ is the inverse image of $y$ under $f$. □

Equation (2.9) makes use of the *max* $t$-conorm. If for any primary membership in the union, we get more than one choice of secondary memberships, the effective secondary membership is taken to be the maximum of all these choices.

Equation (2.9) assumes that $x_1, \ldots, x_n$ are non-interactive or that there is no joint constraint on $x_1, \ldots, x_n$. For more discussion about this, see Appendix B.

Zadeh defined the Extension Principle using *min* $t$-norm and *max* $t$-conorm. The use of these operations is implicit in (2.10) and (2.9). There have been attempts to use other $t$-norms and $t$-conorms in place of *min* and *max*, respectively, e.g. [8], [2]. We will work mostly with *min* or *product* $t$-norm and *max* $t$-conorm.

The Extension Principle can be viewed as a composition of fuzzy relations [2]. Let $\tilde{R}$ be the Cartesian product $\tilde{\mathrm{A}}_1 \times \cdots \times \tilde{\mathrm{A}}_r$ defined as [2]

$$\tilde{\mathrm{A}}_1 \times \cdots \times \tilde{\mathrm{A}}_r = \int_{X_1 \times \cdots \times X_r} \min\{\mu_{\tilde{\mathrm{A}}_1}(x_1), ..., \mu_{\tilde{\mathrm{A}}_r}(x_r)\}/(x_1, ..., x_r) \tag{2.10}$$

and let $S$ be the ordinary relation defined by $\mu_S(x_1, ..., x_r, y) = 1$ iff $y = f(x_1, ..., x_r)$. Then, we have $B = f(\tilde{\mathrm{A}}_1, ..., \tilde{\mathrm{A}}_r) = \tilde{R} \circ S$, i.e., the Extension Principle appears as a particular case of the composition of fuzzy relations.

Finally, when we replace *min* in (2.9) by another $t$-norm, we are replacing the sup-min composition by the more general sup-$\star$ composition. □

Consider two fuzzy sets of type-2, $\tilde{\tilde{\mathrm{A}}} \in X$ and $\tilde{\tilde{\mathrm{B}}} \in X$. Let $\tilde{\mu}_{\tilde{\tilde{\mathrm{A}}}}(x)$ and $\tilde{\mu}_{\tilde{\tilde{\mathrm{B}}}}(x)$ be two fuzzy grades (fuzzy sets in $J \subseteq [0,1]$) of these two sets, represented, for each $x \in X$, as

$$\begin{aligned} \tilde{\mu}_{\tilde{\tilde{\mathrm{A}}}}(x) &= f_x(u_1)/u_1 + f_x(u_2)/u_2 + ... + f_x(u_m)/u_m \\ &= \sum_i f_x(u_i)/u_i, \quad ; \quad u_i \in J \end{aligned} \tag{2.11}$$

$$\begin{aligned} \tilde{\mu}_{\tilde{\tilde{\mathrm{B}}}}(x) &= g_x(w_1)/w_1 + g_x(w_2)/w_2 + ... + g_x(w_n)/w_n \\ &= \sum_j g_x(w_j)/w_j, \quad ; \quad w_j \in J \end{aligned} \tag{2.12}$$

Observe that in (2.11) and (2.12), $u_i$ and $w_j$ are just dummy variables used to differentiate between the different primary memberships of $x$ in $\tilde{\tilde{\mathrm{A}}}$ and $\tilde{\tilde{\mathrm{B}}}$, respectively.

Using the Extension Principle, the membership grades for union, intersection and negation of type-2 fuzzy sets $\tilde{\tilde{\mathrm{A}}}$ and $\tilde{\tilde{\mathrm{B}}}$ can be defined as follows [8] :

*Union*

$$\tilde{\tilde{\mathrm{A}}} \cup \tilde{\tilde{\mathrm{B}}} \Leftrightarrow \tilde{\mu}_{\tilde{\tilde{\mathrm{A}}} \cup \tilde{\tilde{\mathrm{B}}}}(x) = \tilde{\mu}_{\tilde{\tilde{\mathrm{A}}}}(x) \sqcup \tilde{\mu}_{\tilde{\tilde{\mathrm{B}}}}(x) \quad ; \quad x \in X$$

$$\begin{aligned}
&= \Big(\sum_i f_x(u_i)/u_i\Big) \sqcup \Big(\sum_j g_x(w_j)/w_j\Big) \\
&= \sum_{i,j} \Big(f_x(u_i) \wedge g_x(w_j)\Big)/(u_i \vee w_j) \qquad (2.13)
\end{aligned}$$

*Intersection*

$$\begin{aligned}
\tilde{\tilde{\mathrm{A}}} \cap \tilde{\tilde{\mathrm{B}}} \Leftrightarrow \tilde{\mu}_{\tilde{\tilde{\mathrm{A}}} \cap \tilde{\tilde{\mathrm{B}}}}(x) &= \tilde{\mu}_{\tilde{\tilde{\mathrm{A}}}}(x) \sqcap \tilde{\mu}_{\tilde{\tilde{\mathrm{B}}}}(x) \quad ; \quad x \in X \\
&= \Big(\sum_i f_x(u_i)/u_i\Big) \sqcap \Big(\sum_j g_x(w_j)/w_j\Big) \\
&= \sum_{i,j} \Big(f_x(u_i) \wedge g_x(w_j)\Big)/(u_i \wedge w_j) \qquad (2.14)
\end{aligned}$$

*Complement*

$$\begin{aligned}
\bar{\tilde{\tilde{\mathrm{A}}}} \Leftrightarrow \tilde{\mu}_{\bar{\tilde{\tilde{\mathrm{A}}}}}(x) &= \neg\tilde{\mu}_{\tilde{\tilde{\mathrm{A}}}}(x) \quad ; \quad x \in X \\
&= \sum_i f_x(u_i)/(1 - u_i) \qquad (2.15)
\end{aligned}$$

where $\vee$ and $\wedge$ represent *max* and *min* respectively. As in [8], in the sequel, we refer to the operations $\sqcup$, $\sqcap$ and $\neg$ as *join*, *meet* and *negation*, respectively. In the continuous case, we use a notation similar to that in (2.2) and get expressions similar to those in (2.13), (2.14) and (2.15).

*In order to compute the union (or intersection) of $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$, we perform the join (or meet) operation between the membership grades of $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$ at every domain point $x \in X$; and, in order to compute the complement of $\tilde{\tilde{A}}$ (or $\tilde{\tilde{B}}$), we perform the negation operation on the membership grade of $\tilde{\tilde{A}}$ (or $\tilde{\tilde{B}}$) at every $x \in X$.*

It can be easily shown that these extended operations reduce to the original ones when we deal with type-1 sets. In case of type-1 sets, $f_x(u_i)$ $[g_x(w_j)]$ will have a value equal to 1 for only one of the indices, say $i_1$ $(j_1)$; the rest of $f_x(u_i)$s and $g_x(w_j)$s will all be zero (since the membership grades are not fuzzy). Consider the *join* operation. When we find the minima between all the $f_x(u_i)$s and $g_x(w_j)$s, the only pair that will give a non-zero answer is $\{f_x(u_{i_1}),\ g_x(w_{j_1})\}$, and their minimum value will be equal to 1. All other minima will be 0. So, the union of the two sets will consist of only one element $u_{i_1} \vee w_{j_1}$ or $\max\{u_{i_1}, w_{j_1}\}$, which is what we would expect. The same applies to the *meet* operation. The *negation* is even easier to see. If $u_{i_1}$ has a membership of 1 in $\tilde{\mu}_{\tilde{\tilde{A}}}(x)$ (the rest of the memberships being zero), $1 - u_{i_1}$ will have a membership of 1 in $\tilde{\mu}_{\bar{\tilde{\tilde{A}}}}(x)$.

**Example 2.1** Consider two type-2 sets $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$, and, for a particular element $x$, let the membership grades in these two sets be given as

$$\begin{aligned} \tilde{\mu}_{\tilde{\tilde{A}}}(x) &= 0.5/0 + 0.7/0.1 \\ \tilde{\mu}_{\tilde{\tilde{B}}}(x) &= 0.3/0.4 + 0.9/0.8 \end{aligned}$$

Then, from (2.13), we have

$$\begin{aligned} \tilde{\mu}_{\tilde{\tilde{A}}\cup\tilde{\tilde{B}}}(x) &= \tilde{\mu}_{\tilde{\tilde{A}}}(x) \sqcup \tilde{\mu}_{\tilde{\tilde{B}}}(x) \\ &= (0.5/0 + 0.7/0.1) \sqcup (0.3/0.4 + 0.9/0.8) \\ &= \frac{0.5 \wedge 0.3}{0 \vee 0.4} + \frac{0.5 \wedge 0.9}{0 \vee 0.8} + \frac{0.7 \wedge 0.3}{0.1 \vee 0.4} + \frac{0.7 \wedge 0.9}{0.1 \vee 0.8} \\ &= 0.3/0.4 + 0.5/0.8 + 0.3/0.4 + 0.7/0.8 \\ &= \max\{0.3, 0.3\}/0.4 + \max\{0.5, 0.7\}/0.8 \\ &= 0.3/0.4 + 0.7/0.8 \end{aligned}$$

Additionally, from (2.14), we have

$$\begin{aligned} \tilde{\mu}_{\tilde{\tilde{A}}\cap\tilde{\tilde{B}}}(x) &= \tilde{\mu}_{\tilde{\tilde{A}}}(x) \sqcap \tilde{\mu}_{\tilde{\tilde{B}}}(x) \\ &= (0.5/0 + 0.7/0.1) \sqcap (0.3/0.4 + 0.9/0.8) \\ &= \frac{0.5 \wedge 0.3}{0 \wedge 0.4} + \frac{0.5 \wedge 0.9}{0 \wedge 0.8} + \frac{0.7 \wedge 0.3}{0.1 \wedge 0.4} + \frac{0.7 \wedge 0.9}{0.1 \wedge 0.8} \\ &= 0.3/0 + 0.5/0 + 0.3/0.1 + 0.7/0.1 \\ &= \max\{0.3, 0.5\}/0 + \max\{0.3, 0.7\}/0.1 \\ &= 0.5/0 + 0.7/0.1 \end{aligned}$$

Finally, from (2.15), we have

$$\begin{aligned} \tilde{\mu}_{\bar{\tilde{\tilde{A}}}}(x) &= \neg\tilde{\mu}_{\tilde{\tilde{A}}}(x) \\ &= 0.5/(1-0) + 0.7/(1-0.1) \\ &= 0.5/1 + 0.7/0.9 \end{aligned}$$

□

*Algebraic product* is another popular $t$-norm operation, especially in engineering applications [7]. The union and intersection of type-2 fuzzy sets under *product* $t$-norm and *max* $t$-conorm can be defined as follows :

*Union*

$$\begin{aligned}\tilde{\tilde{A}} \cup \tilde{\tilde{B}} \Leftrightarrow \tilde{\mu}_{\tilde{\tilde{A}}\cup\tilde{\tilde{B}}}(x) &= \tilde{\mu}_{\tilde{\tilde{A}}}(x) \sqcup \tilde{\mu}_{\tilde{\tilde{B}}}(x) \\ &= \Big(\sum_i f_x(u_i)/u_i\Big) \sqcup \Big(\sum_j g_x(w_j)/w_j\Big) \\ &= \sum_{i,j} \big(f_x(u_i)g_x(w_j)\big)/(u_i \vee w_j) \qquad (2.16)\end{aligned}$$

*Intersection*

$$\begin{aligned}\tilde{\tilde{A}} \cap \tilde{\tilde{B}} \Leftrightarrow \tilde{\mu}_{\tilde{\tilde{A}}\cap\tilde{\tilde{B}}}(x) &= \tilde{\mu}_{\tilde{\tilde{A}}}(x) \sqcap \tilde{\mu}_{\tilde{\tilde{B}}}(x) \\ &= \Big(\sum_i f_x(u_i)/u_i\Big) \sqcap \Big(\sum_j g_x(w_j)/w_j\Big) \\ &= \sum_{i,j} \big(f_x(u_i)g_x(w_j)\big)/(u_i w_j) \qquad (2.17)\end{aligned}$$

Observe that, we use the same symbols for *join* and *meet* operations, as we used in case of the *min* $t$-norm. The definition of complement does not change.

Next, we take a closer look at the operations of *join*, *meet* and *negation*, under both *min* and *product* $t$-norms.

## 2.2 A Closer Look at Type-2 Set Theoretic Operations

From Section 2.1, we see that the membership grade of any point in the union or intersection of two type-2 fuzzy sets is obtained by the *join* or *meet* of the membership grades of that point, respectively. Now, we look more closely at these two operations. Most of the discussion below concerns the *join* or *meet* of two sets at a time; however, we also state generalized versions of our results when more than two sets are involved in the *join* or *meet* operations.

We will generally deal with *real fuzzy sets*, i.e., fuzzy subsets of the real line, which are *convex* and *normal.* Such sets are also known as *fuzzy numbers* [2, 5]; therefore, sometimes we will use the terms fuzzy sets and fuzzy numbers interchangeably.

### 2.2.1 Join and Meet under Minimum $t$-norm

As Theorem 2.1 below illustrates, *join* and *meet* operations with *min* $t$-norm give particularly simple results, if the participating type-1 sets are convex and normal.

In part (a) of Theorem 2.1, we talk about type-1 fuzzy sets $\tilde{\text{F}}$ and $\tilde{\text{G}}$ having membership functions $f$ and $g$. To connect this with our earlier discussion, we can think of two type-2 fuzzy sets $\tilde{\tilde{\text{A}}}$ and $\tilde{\tilde{\text{B}}}$ as in (2.11) and (2.12). Then, for an arbitrary input $x_0$, if we rename $\tilde{\mu}_{\tilde{\tilde{\text{A}}}}(x_0)$ as $\tilde{\text{F}}$ and $\tilde{\mu}_{\tilde{\tilde{\text{B}}}}(x_0)$ as $\tilde{\text{G}}$, and also drop the subscript $x_0$ on the membership functions $f_{x_0}$ and $g_{x_0}$, we can apply Theorem 2.1 to compute $\tilde{\mu}_{\tilde{\tilde{\text{A}}}\sqcup\tilde{\tilde{\text{B}}}}(x_0)$ and $\tilde{\mu}_{\tilde{\tilde{\text{A}}}\sqcap\tilde{\tilde{\text{B}}}}(x_0)$. Part (b) of Theorem 2.1 generalizes the results in part (a) to the *join*/*meet* of more than two sets.

**Theorem 2.1** *(a) Suppose that we have two convex, normal, type-1 real fuzzy sets $\tilde{F}$ and $\tilde{G}$ characterized by membership functions $f$ and $g$, respectively. Let $v_0 \in \Re$ and $v_1 \in \Re$ be such that $v_0 \leq v_1$ and $f(v_0) = g(v_1) = 1$. Then the membership functions of the join and meet of $\tilde{F}$ and $\tilde{G}$, using max t-conorm and min t-norm, can be expressed as*

$$\mu_{\tilde{\text{F}}\sqcup\tilde{\text{G}}}(\theta) = \begin{cases} f(\theta) \wedge g(\theta) & ; \quad \theta < v_0 \\ g(\theta) & ; \quad v_0 \leq \theta \leq v_1 \\ f(\theta) \vee g(\theta) & ; \quad \theta > v_1 \end{cases} \tag{2.18}$$

*and*

$$\mu_{\tilde{\text{F}}\sqcap\tilde{\text{G}}}(\theta) = \begin{cases} f(\theta) \vee g(\theta) & ; \quad \theta < v_0 \\ f(\theta) & ; \quad v_0 \leq \theta \leq v_1 \\ f(\theta) \wedge g(\theta) & ; \quad \theta > v_1 \end{cases} \tag{2.19}$$

*(b) Suppose that we have $n$ convex, normal, type-1 real fuzzy sets $\tilde{F}_1, \ldots, \tilde{F}_n$ characterized by membership functions $f_1, \ldots, f_n$, respectively. Let $v_1, v_2, \ldots, v_n$ be real*

*numbers such that* $v_1 \leq v_2 \leq \cdots \leq v_n$ *and* $f_1(v_1) = f_2(v_2) = \cdots = f_n(v_n) = 1$. *Then, using max t-conorm and min t-norm,*

$$\mu_{\sqcup_{i=1}^{n}\tilde{F}_i}(\theta) = \begin{cases} \bigwedge_{i=1}^{n} f_i(\theta) & ; \quad \theta < v_1 \\ \bigwedge_{i=k+1}^{n} f_i(\theta) & ; \quad v_k \leq \theta \leq v_{k+1} \quad ; \quad 1 \leq k \leq n-1 \\ \bigvee_{i=1}^{n} f_i(\theta) & ; \quad \theta > v_n \end{cases} \tag{2.20}$$

*and*

$$\mu_{\sqcap_{i=1}^{n}\tilde{F}_i}(\theta) = \begin{cases} \bigvee_{i=1}^{n} f_i(\theta) & ; \quad \theta < v_1 \\ \bigwedge_{i=1}^{k} f_i(\theta) & ; \quad v_k \leq \theta \leq v_{k+1} \quad ; \quad 1 \leq k \leq n-1 \\ \bigwedge_{i=1}^{n} f_i(\theta) & ; \quad \theta > v_n \end{cases} \tag{2.21}$$

□

See Appendix C.1 for the proof of Theorem 2.1.

**NOTE** : Dubois and Prade present the same result given in part (a) of Theorem 2.1 in a different context and a different manner in [1]. They present it in the context of fuzzification of *max* and *min* operations. Though their method of proof is very similar to ours, they prove the result for a special case, where $f$ and $g$ have at most three points of intersection and one needs to keep track of the points of intersection of $f$ and $g$ to use their theorem. We reprove this theorem in a general setting in Appendix C.1. We believe that our statements of $\mu_{\tilde{F}\sqcup\tilde{G}}(\theta)$ and $\mu_{\tilde{F}\sqcap\tilde{G}}(\theta)$ are more amenable to computer implementations than those of Dubois and Prade. Generalization to more than two sets [part (b) of Theorem 2.1] is also difficult in case of Dubois and Prade's result.

Figures 2.1 and 2.2 show examples of application of Theorem 2.1. As a consequence of Theorem 2.1, we have the following important result :

**Corollary 2.1** *(a) If* $f(\theta)$ *is the membership function of a convex, normal type-1 real fuzzy set* $\tilde{F}$, *and if* $\tilde{G}$ *is another type-1 set with membership function* $f(\theta - k)$, *where* $k$ *is a positive constant, then,* $\tilde{F} \sqcup \tilde{G} = \tilde{G}$ *and* $\tilde{F} \sqcap \tilde{G} = \tilde{F}$.
*(b) If we have* $n$ *convex, normal, type-1 fuzzy sets* $\tilde{F}_1, \ldots, \tilde{F}_n$ *characterized by membership functions* $f_1, \ldots, f_n$, *respectively, such that* $f_i(\theta) = f_1(\theta - k_i)$, *and* $0 = k_1 \leq k_2 \cdots \leq k_n$; *then* $\sqcup_{i=1}^{n} \tilde{F}_i = \tilde{F}_n$ *and* $\sqcap_{i=1}^{n} \tilde{F}_i = \tilde{F}_1$. □

See Appendix C.3 for the proof of this corollary. Figures 2.3 and 2.4 illustrate Corollary 2.1.

It can be observed, by applying Theorem 2.1, that, if two type-1 fuzzy sets are such that their membership functions do not touch, then Corollary 2.1 also holds true, i.e., $\tilde{F} \sqcup \tilde{G} = \tilde{G}$ and $\tilde{F} \sqcap \tilde{G} = \tilde{F}$. In the case of membership functions which extend infinitely (e.g., Gaussians), if the two membership functions (their centers) are far away from each other, then Corollary 2.1 approximately holds true, i.e., $\tilde{F} \sqcup \tilde{G} \approx \tilde{G}$ and $\tilde{F} \sqcap \tilde{G} \approx \tilde{F}$. Examples of this can be observed while working out the union and intersection of the Gaussian type-2 sets depicted in Figs. 2.7 (a), as explained later in this section.

Kaufmann and Gupta [5] give a result that is a bit more general than Corollary 2.1. They give this result in terms of $\alpha$-cuts as follows : Consider two fuzzy numbers $\tilde{A}$ and $\tilde{B}$, such that

$$\begin{aligned} A_\alpha &= [a_1^{(\alpha)}, a_2^{(\alpha)}] \text{ and} \\ B_\alpha &= [b_1^{(\alpha)}, b_2^{(\alpha)}] \end{aligned}$$

where $\alpha$-cuts, $A_\alpha$ and $B_\alpha$ are crisp sets

$$\begin{aligned} A_\alpha = \{x | \mu_{\tilde{A}}(x) \geq \alpha\}, \;\; \alpha \in [0,1] \\ B_\alpha = \{x | \mu_{\tilde{B}}(x) \geq \alpha\}, \;\; \alpha \in [0,1] \end{aligned}$$

If $\forall \alpha \in [0,1]$, $a_1^{(\alpha)} \leq b_1^{(\alpha)}$ and $a_2^{(\alpha)} \leq b_2^{(\alpha)}$, then $\tilde{A} \leq \tilde{B}$.

In our work, we frequently deal with normalized Gaussian membership functions. Corollary 2.1 takes a particularly simple form when the sets involved are Gaussians. For two Gaussians having the *same standard deviation*, the result of the join operation between them is the Gaussian with the larger mean, and the result of the meet operation is the Gaussian with the smaller mean. Gaussians having *different standard deviations* cannot be expressed as shifted versions of each other and hence Corollary 2.1 does not apply to them. Theorem 2.1 can, of course, be used in this case. Figures 2.2 and 2.6 show examples of the *join* and *meet* operations between Gaussians having different standard deviations, under the *min* $t$-norm.

Similar results hold for triangular, trapezoidal, or, for that matter, any other convex membership function.

From the definition of the *negation* operation, it follows that :

**Theorem 2.2** *If a type-1 fuzzy set $\tilde{F}$ has a membership function $f(\theta)$ ($\theta \in \Re$), $\neg\tilde{F}$ has a membership function $f(1-\theta)$ ($\theta \in \Re$).* □

See Appendix C.4 for the proof. Figure 2.5 shows an example of the *negation* operation.

So, we perform *join*, *meet* and *negation* operations on membership grades of type-2 sets while finding unions, intersections and complements of type-2 sets. Having seen how the results of these individual operations look, let's see how the overall type-2 set looks as a result of these operations. Figures 2.7 and 2.8 show examples of union, intersection and complement of Gaussian fuzzy sets using the 2-D representation introduced in Section 1.3 (see Fig. 1.10). In Fig. 2.7 (a), if we draw a vertical line at any $x$, we get the membership grades of $x$ in the two participating Gaussian type-2 sets. These membership grades are, of course, themselves Gaussian fuzzy sets confined to the interval $[0, 1]$. To these two type-1 sets, we apply Theorem 2.1 and get the results for union and intersection depicted in Figs. (b) and (c) respectively. Of course, while applying the theorem, we should be careful to see that $\tilde{\text{F}}$ and $\tilde{\text{G}}$ do satisfy all its requirements. Similarly, in Fig. 2.8 (a), we project upwards from $x$ to obtain it's membership grade and then apply Theorem 2.2 to it.

Observe that, in Figs. 2.7 and 2.8, if we look just at the *principal* membership functions, we can see that *the principal membership function of the result of an operation (union, intersection or complement) can be obtained by performing that operation on the principal membership functions of the participating type-2 sets.* So, if we replace all the type-2 sets by type-1 sets, which have the principal membership functions of the type-2 sets as their type-1 membership functions, all our results remain valid. This demonstrates the fact that all our type-2 operations collapse to the correct type-1 operations.

Theorem 2.1 considers the *join* and *meet* operations under *min* $t$-norm. Now, we examine these operations under the *product* $t$-norm, the $t$-conorm being *max*. This case was not considered by Dubois and Prade in [1].

### 2.2.2 Join under Product $t$-norm

The *join* operation with *product* $t$-norm gives a result very similar to that in Theorem 2.1. Consider the two convex normal type-1 fuzzy sets $\tilde{\text{F}}$ and $\tilde{\text{G}}$ used in Theorem 2.1. The membership function of the *join* of $\tilde{\text{F}}$ and $\tilde{\text{G}}$ using the *max* $t$-conorm and *product* $t$-norm, can be expressed as

$$\mu_{\tilde{\text{F}} \sqcup \tilde{\text{G}}}(\theta) = \begin{cases} f(\theta)g(\theta) & ; \quad \theta < v_0 \\ g(\theta) & ; \quad v_0 \leq \theta \leq v_1 \\ f(\theta) \vee g(\theta) & ; \quad \theta > v_1 \end{cases} \tag{2.22}$$

Figure 2.9 (b) shows an example of this operation. Comparing Figs. 2.9 (b) and 2.1 (b), we see that results of the two *join* operations (with *min* as well as *product* $t$-norm are exactly the same. This can be explained as follows. From (2.18) and (2.22), we see that the two results can differ only for $\theta < v_0$. In this range, the *min* $t$-norm gives $f(\theta) \wedge g(\theta)$ and *product* $t$-norm gives $f(\theta)g(\theta)$. In the example we have chosen, $f(\theta) = 1$ for $\theta < v_0$; therefore, for our example, these two results turn out the same. Generalization to more than two sets is also very similar to that in (2.20). It can be obtained by replacing the minima in (2.20) with products as follows :

$$\mu_{\sqcup_{i=1}^{n} \tilde{\text{F}}_i}(\theta) = \begin{cases} \prod_{i=1}^{n} f_i(\theta) & ; \quad \theta < v_1 \\ \prod_{i=k+1}^{n} f_i(\theta) & ; \quad v_k \leq \theta \leq v_{k+1} \quad ; \quad 1 \leq k \leq n-1 \\ \bigvee_{i=1}^{n} f_i(\theta) & ; \quad \theta > v_n \end{cases} \tag{2.23}$$

where we take $\prod_{i=n}^{n} f_i(\theta)$ to mean $f_n$. See Appendix C.5 for the proofs of (2.22) and (2.23).

Observe that (2.22) is very similar to (2.18). In fact, the information in both these pairs of equations can be conveyed as follows : For two type-1 sets $\tilde{\text{F}}$ and $\tilde{\text{G}}$ described in part (a) of Theorem 2.1,

$$\mu_{\tilde{\text{F}} \sqcup \tilde{\text{G}}}(\theta) = \begin{cases} f(\theta) \star g(\theta) & ; \quad \theta < v_0 \\ g(\theta) & ; \quad v_0 \leq \theta \leq v_1 \\ f(\theta) \vee g(\theta) & ; \quad \theta > v_1 \end{cases} \tag{2.24}$$

where $\star$ denotes the $t$-norm operation, which corresponds to *min* in (2.18) and *product* in (2.22).

Similarly comparing (2.20) and (2.23), for $n$ type-1 sets $\tilde{\mathrm{F}}_1, \ldots, \tilde{\mathrm{F}}_n$ described in part (b) of Theorem 2.1,

$$\mu_{\sqcup_{i=1}^n \tilde{\mathrm{F}}_i}(\theta) = \begin{cases} \mathcal{T}_{i=1}^n f_i(\theta) & ; \quad \theta < v_1 \\ \mathcal{T}_{i=k+1}^n f_i(\theta) & ; \quad v_k \leq \theta \leq v_{k+1} \quad ; \quad 1 \leq k \leq n-1 \\ \bigvee_{i=1}^n f_i(\theta) & ; \quad \theta > v_n \end{cases} \tag{2.25}$$

where $\mathcal{T}$ indicates the $t$-norm used, min or product.

We state this result formally as :

**Theorem 2.3** *(a) Suppose that we have two convex, normal, type-1 real fuzzy sets $\tilde{F}$ and $\tilde{G}$ characterized by membership functions $f$ and $g$, respectively. Let $v_0 \in \Re$ and $v_1 \in \Re$ be such that $v_0 \leq v_1$ and $f(v_0) = g(v_1) = 1$. Then the membership functions of the join of $\tilde{F}$ and $\tilde{G}$, using max t-conorm, can be expressed as*

$$\mu_{\tilde{\mathrm{F}} \sqcup \tilde{\mathrm{G}}}(\theta) = \begin{cases} f(\theta) \star g(\theta) & ; \quad \theta < v_0 \\ g(\theta) & ; \quad v_0 \leq \theta \leq v_1 \\ f(\theta) \vee g(\theta) & ; \quad \theta > v_1 \end{cases} \tag{2.26}$$

*where $\star$ denotes the t-norm operation used, min or product.*
*(b) Suppose that we have $n$ convex, normal, type-1 real fuzzy sets $\tilde{F}_1, \ldots, \tilde{F}_n$ characterized by membership functions $f_1, \ldots, f_n$, respectively. Let $v_1, v_2, \ldots, v_n$ be real numbers such that $v_1 \leq v_2 \leq \cdots \leq v_n$ and $f_1(v_1) = f_2(v_2) = \cdots = f_n(v_n) = 1$. Then, the membership function of $\sqcup_{i=1}^n \tilde{F}_i$ using max t-conorm, can be expressed as*

$$\mu_{\sqcup_{i=1}^n \tilde{\mathrm{F}}_i}(\theta) = \begin{cases} \mathcal{T}_{i=1}^n f_i(\theta) & ; \quad \theta < v_1 \\ \mathcal{T}_{i=k+1}^n f_i(\theta) & ; \quad v_k \leq \theta \leq v_{k+1} \quad ; \quad 1 \leq k \leq n-1 \\ \bigvee_{i=1}^n f_i(\theta) & ; \quad \theta > v_n \end{cases} \tag{2.27}$$

*where $\mathcal{T}$ indicates the t-norm used, min or product.* □

Figures 2.10, 2.12 (b) and 2.13 (b) show results of *join* operations on Gaussians under *product* $t$-norm. Note that Corollary 2.1 is not valid under *product* $t$-norm.

### 2.2.3 Meet under Product $t$-norm

The *meet* operation between $\tilde{\mathrm{F}}$ and $\tilde{\mathrm{G}}$ (convex, normal, type-1 fuzzy sets used in Theorem 2.1), under the *product* $t$-norm can be represented as

$$\tilde{\mathrm{F}} \sqcap \tilde{\mathrm{G}} = \int_{v\in\Re}\int_{w\in\Re}[f(v)g(w)]/(vw) \tag{2.28}$$

Observe that this equation involves the product of primary memberships $v$ and $w$ rather than a *min* or *max* operation between them; hence, the analysis of the *meet* operation under *product* $t$-norm is quite different than that of *join* or *meet* operations previously discussed.

Equation (2.28) simplifies considerably when $\tilde{\mathrm{F}}$ and $\tilde{\mathrm{G}}$ are interval type-1 sets, as we show with an example next. (Recall, from Chapter 1, that interval type-1 sets are crisp sets whose domains are intervals on the real line.)

**Example 2.2** Let $F$ and $G$ be two interval type-1 sets with domains $[l_f, r_f]$ and $[l_g, r_g]$, respectively. (We drop the tilde, since the sets are crisp.) Using (2.28), the *meet* between $F$ and $G$, under product $t$-norm, can be obtained as

$$F \sqcap G = \int_{v\in F}\int_{w\in G}(1\times 1)/(vw) \tag{2.29}$$

Observe, from (2.29), that

- each term in $F \sqcap G$ is equal to the product $vw$ for some $v \in F$ and $w \in G$, the smallest term being $l_f l_g$ and the largest $r_f r_g$; and,
- since both $F$ and $G$ have continuous domains, $F \sqcap G$ also has a continuous domain;

consequently, $F \sqcap G$ is an interval type-1 set with domain $[l_f l_g, r_f r_g]$, i.e.,

$$F \sqcap G = \int_{v\in[l_f l_g, r_f r_g]} 1/v \tag{2.30}$$

In a similar manner, the *meet*, $\sqcap_{i=1}^{n} F_i$, of $n$ interval type-1 sets $F_1, \ldots, F_n$, having domains $[l_1, r_1], \ldots, [l_n, r_n]$, respectively, is an interval set with domain $[\prod_{i=1}^{n} l_i, \prod_{i=1}^{n} r_i]$.

[5] gives a similar result while discussing multiplication of fuzzy numbers (see Section 2.4 for algebraic operations on fuzzy sets). □

If the sets involved in the *meet* operation are not interval type-1 sets, generally a direct application of (2.29) does not give such a nice result. We, then, analyze this operation as follows. If $\theta$ is an element of $\tilde{\mathrm{F}} \sqcap \tilde{\mathrm{G}}$, then the membership grade of $\theta$ can be found by finding all the pairs $\{v, w\}$ such that $v \in \tilde{\mathrm{F}}$, $w \in \tilde{\mathrm{G}}$ and $vw = \theta$; multiplying the membership grades of $v$ and $w$ in each pair; and then finding the maximum of these products of membership grades. The possible admissible $\{v, w\}$ pairs whose product is $\theta$ are $\{v, \theta/v\}$ ($v \in \Re$, $v \neq 0$) for $\theta \neq 0$ and $\{v, 0\}$ or $\{0, w\}$, where $v, w \in \Re$ for $\theta = 0$. We find the products of membership grades of $v$ and $w$ from each such pair and take the maximum of all these products as the membership grade of $\theta$, i.e.,

$$
\begin{aligned}
\mu_{\tilde{\mathrm{F}} \sqcap \tilde{\mathrm{G}}}(\theta) &= \sup_{v \in \Re, v \neq 0} f(v) g\left(\frac{\theta}{v}\right) ; \; \theta \in \Re, \; \theta \neq 0 \\
\mu_{\tilde{\mathrm{F}} \sqcap \tilde{\mathrm{G}}}(0) &= [\sup_{v \in \Re} f(v) g(0)] \vee [\sup_{w \in \Re} f(0) g(w)]
\end{aligned}
\tag{2.31}
$$

Observe that

$$
\begin{aligned}
\sup_{v \in \Re} f(v) g(0) &= g(0) \sup_{v \in \Re} f(v) \\
&= g(0) \times 1 \\
&= g(0)
\end{aligned}
\tag{2.32}
$$

and similarly,

$$
\sup_{w \in \Re} f(0) g(w) = f(0) \; ; \tag{2.33}
$$

therefore, summarizing the above discussion, we have that for two convex, normal, type-1 fuzzy sets $\tilde{\mathrm{F}}$ and $\tilde{\mathrm{G}}$ (satisfying conditions of Theorem 2.1), the membership function of the *meet* under *product* $t$-norm can be expressed as

$$
\begin{aligned}
\mu_{\tilde{\mathrm{F}} \sqcap \tilde{\mathrm{G}}}(\theta) &= \sup_{v \in \Re, v \neq 0} f(v) g\left(\frac{\theta}{v}\right) \; ; \; \theta \neq 0 \\
\mu_{\tilde{\mathrm{F}} \sqcap \tilde{\mathrm{G}}}(0) &= f(0) \vee g(0)
\end{aligned}
\tag{2.34}
$$

If we substitute $\theta/v = w$ in (2.34), we get a similar expression in terms of $f(\theta/w)g(w)$. Since the *meet* operation is commutative [4], we get the same result whether we substitute $\theta/w = v$ or $\theta/v = w$.

As is apparent from (2.34), the result is very much dependent on functions $f$ and $g$ and does not easily generalize like the *join* and *meet* operations considered earlier, and generally, it is very difficult to obtain a closed form expression for the result of the *meet* operation [which is why we have not stated (2.34) as a theorem]. Even if both the fuzzy sets involved have Gaussian membership functions, it is difficult to obtain a nice closed form expression for the result of the *meet* operation. See Appendix C.6 for more discussion about *meet* between Gaussians under product $t$-norm.

Figure 2.9 (c) shows an example of this operation. To determine the membership of a particular point $\theta$ in $\tilde{F} \sqcap \tilde{G}$, we find all the pairs $\{v, w\}$ such that $v \in \Re$, $w \in \Re$ and $vw = \theta$; and multiply the memberships of each pair. The membership grade of $\theta$ is given by the supremum of the set of all these products. For example, if $\theta = 20$, all the pairs $\{v, w\}$ that give 20 as their product are $v$ and $\frac{20}{v}$ $(v \in \Re, v \neq 0)$. So, the membership grade of 20 is given by the supremum of the set of all the products $f(v)g(\frac{20}{v})$ $(v \in \Re, v \neq 0)$. Figure 2.11 shows how $f(v)g(\frac{20}{v})$ looks for $\tilde{F}$ and $\tilde{G}$ depicted in Fig. 2.9 (a). Clearly, it is no easy matter to represent (2.34) visually.

One situation when the result of the *meet* operation in (2.34) simplifies considerably is when either one of $\tilde{F}$ or $\tilde{G}$ is a fuzzy singleton. For example, assume that $\tilde{F}$ is a fuzzy singleton, such that $f(v_0) = 1$ and $f(v) = 0$ for $v \neq v_0$. Now, $f(v)g(\frac{\theta}{v})$ is non-zero only at $v = v_0$, implying that $\mu_{\tilde{F}\sqcap\tilde{G}}(\theta) = g(\frac{\theta}{v_0})$. Similarly, if $\tilde{G}$ is a fuzzy singleton, such that $g(v_1) = 1$ and $g(w) = 0$ for $w \neq v_1$, $f(v)g(\frac{\theta}{v})$ is non-zero only at $\frac{\theta}{v} = v_1$, i.e., only when $v = \frac{\theta}{v_1}$, implying that $\mu_{\tilde{F}\sqcap\tilde{G}}(\theta) = f(\frac{\theta}{v_1})$.

Because the *meet* under *product* $t$-norm will be heavily used by us in the sequel (see [4]), we seek approximations to it that will make it practical.

## 2.3 Approximations for Meet under Product $t$-norm

In this section, we discuss some approximations to the *meet* under product $t$-norm that will help us make the *meet* calculations more efficient. Subsection 2.3.1 discusses an ad hoc approximation that can be used with any normal membership functions. Subsection 2.3.2 discusses a Gaussian approximation for Gaussian membership functions, and Appendix E.1 discusses a triangular approximation for symmetrical triangular membership functions. Both, the Gaussian and the triangular approximations have the following two very desirable properties : 1) they can be computed very easily from the membership functions of the type-1 sets involved in the *meet* operation; and 2) they are easily generalizable to the *meet* of more than two fuzzy sets.

### 2.3.1 First Approximation

As explained earlier, if one of the two fuzzy sets is a fuzzy singleton, the *meet* operation simplifies a lot, e.g., if $\tilde{\mathrm{F}}$ is a singleton, having membership equal to 1 at $v_0$ and zero at all other points, as explained at the end of Section 2.2.3, the result of the *meet* operation is (assuming $v_0 \neq 0$)

$$\mu_{\tilde{\mathrm{F}} \sqcap \tilde{\mathrm{G}}}(\theta) = g\Big(\frac{\theta}{v_0}\Big) \;\; ; \; \theta \in \Re \tag{2.35}$$

If $v_0 = 0$, $\tilde{\mathrm{F}} = 1/0$. Then, from (2.28), we have

$$\begin{aligned} \tilde{\mathrm{F}} \sqcap \tilde{\mathrm{G}} &= \int_{w \in \Re} [f(0) g(w)]/0 \\ &= \int_{w \in \Re} g(w)/0 \\ &= [\sup_w g(w)]/0 \\ &= 1/0 \end{aligned} \tag{2.36}$$

where we have made use of the facts that the integrals in (2.28) denote union, the $t$-conorm used is maximum, and $\tilde{\mathrm{G}}$ is normal.

Similarly, if $\tilde{G}$ is a singleton, having membership equal to 1 at $v_1$ and zero at all other points, the result of the *meet* operation is (assuming $v_1 \neq 0$)

$$\mu_{\tilde{F}\sqcap\tilde{G}}(\theta) = f\left(\frac{\theta}{v_1}\right) \; ; \; \theta \in \Re \tag{2.37}$$

This motivates the following ad hoc approximation for *meet* under *product t-norm*,

$$\mu_{\tilde{F}\sqcap\tilde{G}}(\theta) \approx f\left(\frac{\theta}{v_1}\right) \vee g\left(\frac{\theta}{v_0}\right) \; ; \; \theta \in \Re \tag{2.38}$$

This expression does not take into account the possibility that $\tilde{F}$ or $\tilde{G}$ may have more than one point with membership grade equal to 1 in their support (an example of such a fuzzy set is $F$ in Fig. 2.1). To account for this case, we generalize (2.38) as follows :

$$\mu_{\tilde{F}\sqcap\tilde{G}}(\theta) \approx \int_{v\in V} f\left(\frac{\theta}{v}\right) \vee \int_{w\in W} g\left(\frac{\theta}{w}\right) \tag{2.39}$$

where $V$ is the crisp (non-fuzzy) set of all points having a membership grade equal to 1 in $\tilde{F}$ and $W$ is the crisp set of all points having a membership grade equal to 1 in $\tilde{G}$. Figure 2.9 (c) shows the above approximation along with the actual result. We do not claim that this approximation is optimal in any sense; however, it looks intuitively reasonable and is much easier to compute than the actual result [particularly because, as mentioned earlier, we will mostly deal with type-1 fuzzy sets (membership grades of type-2 sets) that have only one point at which the secondary membership reaches 1, so that we can use Eq. (2.38)]. Additionally, as we show next, it collapses to the correct type-1 result if fuzzy memberships are replaced by appropriate crisp memberships, i.e., if we replace the type-2 sets by appropriate type-1 sets, the results remain valid. (Given a type-2 set, an "appropriate" type-1 set is one which has a membership function equal to the principal membership function of the type-2 set.)

Recall, that

$$\tilde{\mu}_{\tilde{\tilde{A}}\cap\tilde{\tilde{B}}}(x_0) = \tilde{\mu}_{\tilde{\tilde{A}}}(x_0) \sqcap \tilde{\mu}_{\tilde{\tilde{B}}}(x_0) \tag{2.40}$$

In our analysis, we denote $\tilde{\mu}_{\tilde{\tilde{A}}}(x_0)$ and $\tilde{\mu}_{\tilde{\tilde{B}}}(x_0)$ by $\tilde{F}$ and $\tilde{G}$, respectively. We have assumed that the membership functions of $\tilde{F}$ and $\tilde{G}$, namely $f$ and $g$, are such that $f(v_0) = g(v_1) = 1$. If all the type-2 sets are replaced by the appropriate type-1 sets,

$\tilde{F}$ and $\tilde{G}$ reduce to singletons $1/v_0$ and $1/v_1$, respectively; so that $\tilde{F} \sqcap \tilde{G} = 1/(v_0 v_1) = v_0 v_1$.

Now let's see how the approximation in (2.38) reduces to this result.

$$\begin{aligned} f(\theta) &= \begin{cases} 1 \;; & \theta = v_0 \\ 0 \;; & \text{otherwise} \end{cases} \\ \Rightarrow f(\frac{\theta}{v_1}) &= \begin{cases} 1 \;; & \theta = v_0 v_1 \\ 0 \;; & \text{otherwise} \end{cases} \end{aligned} \tag{2.41}$$

and

$$\begin{aligned} g(\theta) &= \begin{cases} 1 \;; & \theta = v_1 \\ 0 \;; & \text{otherwise} \end{cases} \\ \Rightarrow g(\frac{\theta}{v_0}) &= \begin{cases} 1 \;; & \theta = v_0 v_1 \\ 0 \;; & \text{otherwise} \end{cases} \end{aligned} \tag{2.42}$$

From (2.38), (2.41) and (2.42), we can see that when both $\tilde{F}$ and $\tilde{G}$ are singletons, the result of the first approximation is equal to $v_0 v_1$, which is the true result of the *meet*.

In Theorem 2.1 and our discussion about the *product* $t$-norm, we have considered *join* and *meet* operations between general fuzzy sets, which have the real line as their support; however, when dealing with type-2 sets, we use these operations between fuzzy membership grades, which are type-1 fuzzy sets supported in $[0, 1]$; hence, results of all the *join* and *meet* operations, for both *min* as well as *product* $t$-norms are again type-1 fuzzy sets supported in $[0, 1]$. Additionally we will be interested primarily in Gaussian membership functions.

**Example 2.3** Let's see how the above approximation for *meet* looks in the case of Gaussian fuzzy sets. Since Gaussians reach unity height at only a single point, we can use (2.38). If $f$ and $g$ are Gaussians with means $m_f$ and $m_g$, and standard deviations $\sigma_f$ and $\sigma_g$ respectively, then from (2.38), we have

$$\mu_{\tilde{F} \sqcap \tilde{G}}(\theta) \approx e^{-\frac{1}{2}\left(\frac{\frac{\theta}{m_g} - m_f}{\sigma_f}\right)^2} \vee e^{-\frac{1}{2}\left(\frac{\frac{\theta}{m_f} - m_g}{\sigma_g}\right)^2} \;; \theta \in [0, 1]$$

$$= e^{-\frac{1}{2}\left(\frac{\theta - m_f m_g}{m_g \sigma_f}\right)^2} \vee e^{-\frac{1}{2}\left(\frac{\theta - m_f m_g}{m_f \sigma_g}\right)^2} \quad ; \theta \in [0,1] \qquad (2.43)$$

On the RHS of (2.43), we are comparing two Gaussians with equal means. Obviously, their maximum will equal the Gaussian with the larger value of $m_g\sigma_f$ or $m_f\sigma_g$; therefore,

$$\mu_{\tilde{F}\sqcap\tilde{G}}(\theta) \approx e^{-\frac{1}{2}\left(\frac{\theta - m_f m_g}{\max\{m_f\sigma_g, m_g\sigma_f\}}\right)^2} \quad ; \theta \in [0,1] \qquad (2.44)$$

So, *the approximation of* meet *between two Gaussians is a Gaussian, whose mean is equal to the product of the means of the two participating Gaussians.* Figure 2.12 (c) depicts an example of *meet* (approximation with the actual result) of Gaussians under *product t*-norm and *max t*-conorm. Figure 2.13 (c) depicts a similar result for Gaussians with equal standard deviations.

Let's see how this generalizes to more than two Gaussians at a time. Consider the meet between three Gaussians $\tilde{F}_1$, $\tilde{F}_2$, $\tilde{F}_3$ with means $m_1$, $m_2$, $m_3$ and standard deviations $\sigma_1$, $\sigma_2$, $\sigma_3$. Since the *meet* operation is associative under *product t*-norm and *max t*-conorm [4], we can first find the *meet* between $\tilde{F}_1$ and $\tilde{F}_2$ and then find the *meet* of the resulting function with $\tilde{F}_3$. Using (2.44), we see that the approximation of $\tilde{F}_1 \sqcap \tilde{F}_2$ is a Gaussian, say $\tilde{F}_{12}$ with mean $m_{12} = m_1 m_2$ and standard deviation $\sigma_{12} = \max\{m_1\sigma_2, m_2\sigma_1\}$. Using (2.44) again, we see that the approximation of $\tilde{F}_{12} \sqcap \tilde{F}_3$ is again a Gaussian, say $\tilde{F}_{123}$ with mean $m_{123} = m_{12}m_3 = m_1m_2m_3$ and standard deviation

$$\sigma_{123} = \max\{m_{12}\sigma_3, m_3\sigma_{12}\} = \max\{m_1m_2\sigma_3, m_1m_3\sigma_2, m_2m_3\sigma_1\} \qquad (2.45)$$

The generalization of this result is straightforward. If there are $n$ Gaussian fuzzy sets $\tilde{F}_1, \tilde{F}_2, \cdots, \tilde{F}_n$ with means $m_1, m_2, \cdots, m_n$ and standard deviations $\sigma_1, \sigma_2, \cdots, \sigma_n$, respectively, then repeated application of (2.38) yields

$$\mu_{\tilde{F}_1\sqcap\tilde{F}_2\sqcap\cdots\sqcap\tilde{F}_n}(\theta) \approx e^{-\frac{1}{2}\left(\frac{\theta - m_1 m_2 \cdots m_n}{\sigma}\right)^2} \qquad (2.46)$$

where

$$\bar{\sigma} = \max\Big\{\sigma_1 \prod_{i\,;\,i\neq 1} m_i, \sigma_2 \prod_{i\,;\,i\neq 2} m_i, \cdots, \sigma_j \prod_{i;i\neq j} m_i, \cdots, \sigma_n \prod_{i;i\neq n} m_i\Big\}\ ;\ i = 1, 2, \cdots, n \tag{2.47}$$

Figure 2.14 shows an example of *meet* of more than two Gaussians and compares the actual result with the approximation.

Note that the Gaussians are contained in $[0, 1]$ and may, therefore, be clipped (see Appendix C.8.2). In this example, we did not consider effects of clipping; however, we consider them in Appendix C.8.3 while calculating a lower bound for the Gaussian approximation derived in Section 2.3.2. The process of finding a lower bound on the Gaussian approximation is very similar to computing our first approximation to the *meet* between Gaussians. □

### 2.3.2 A Gaussian Approximation

The approximation in (2.44) was motivated by a general membership function, not necessarily Gaussian [see (2.38) and (2.39)]. If we focus on Gaussian fuzzy sets, we can come up with a better approximation for *meet* under the product $t$-norm. Observe that the *meet* operation is performed between membership grades of type-2 sets; therefore in the following, we require that the *secondary* membership functions of the type-2 sets involved be Gaussians. *Their principal membership functions, however, can have any shape (e.g., triangular, Gaussian, trapezoidal).*

Consider the case when $f(v)$ and $g(w)$ are Gaussians with support $[0, 1]$ with means $m_f$, $m_g$ and standard deviations $\sigma_f$, $\sigma_g$, respectively. Then,

$$\tilde{\mathrm{F}} \sqcap \tilde{\mathrm{G}} = \int_v \int_w e^{-\frac{1}{2}\left(\frac{v-m_f}{\sigma_f}\right)^2} e^{-\frac{1}{2}\left(\frac{w-m_g}{\sigma_g}\right)^2} / (vw) \tag{2.48}$$

Recall that the integral in the above equation denotes union in the continuum. If $\theta$ is an element of $\tilde{\mathrm{F}} \sqcap \tilde{\mathrm{G}}$, then the membership grade of $\theta$ can be found by : finding all the pairs $\{v, w\}$ such that $v \in \tilde{\mathrm{F}}$, $w \in \tilde{\mathrm{G}}$ and $vw = \theta$; multiplying the membership grades of $v$ and $w$ in each pair; and then finding the maximum of these products of membership grades. That is,

$$\mu_{\tilde{\mathrm{F}}\sqcap\tilde{\mathrm{G}}}(\theta) = \sup\left\{e^{-\frac{1}{2}\left(\frac{v-m_f}{\sigma_f}\right)^2} e^{-\frac{1}{2}\left(\frac{w-m_g}{\sigma_g}\right)^2} ; vw = \theta; v \in \tilde{\mathrm{F}}; w \in \tilde{\mathrm{G}}\right\} \tag{2.49}$$

Given any $v$ (assuming $v \neq 0$), the constraint $vw = \theta$ gives us $w = \theta/v$. Further, since $w \in [0,1]$, it follows that $\theta/v \leq 1$ or $v \geq \theta$. So, given any $\theta \in (0,1]$, the acceptable $\{v,w\}$ pairs that can give $\theta$ as the result of the product operation are $\{(v, \frac{\theta}{v}); 0 < \theta \leq v \leq 1\}$; therefore, from (2.49), we have

$$\begin{aligned}
\mu_{\tilde{\mathrm{F}}\sqcap\tilde{\mathrm{G}}}(\theta) &= \sup_{v\in[\theta,1]} e^{-\frac{1}{2}\left[\left(\frac{v-m_f}{\sigma_f}\right)^2 + \left(\frac{\frac{\theta}{v}-m_g}{\sigma_g}\right)^2\right]}, \ \theta \neq 0 \\
&= \sup_{v\in[\theta,1]} e^{-\frac{1}{2}\left[\left(\frac{v-m_f}{\sigma_f}\right)^2 + \left(\frac{\theta-vm_g}{v\sigma_g}\right)^2\right]}, \ \theta \neq 0
\end{aligned} \tag{2.50}$$

When $\theta = 0$, either $v = 0$ and $w$ is any number in $[0,1]$, or $w = 0$ and $v$ is any number in $[0,1]$; therefore, from (2.49), we have

$$\begin{aligned}
\mu_{\tilde{\mathrm{F}}\sqcap\tilde{\mathrm{G}}}(0) &= \sup_{w\in[0,1]} e^{-\frac{1}{2}\left(\frac{m_f}{\sigma_f}\right)^2} e^{-\frac{1}{2}\left(\frac{w-m_g}{\sigma_g}\right)^2} \vee \sup_{v\in[0,1]} e^{-\frac{1}{2}\left(\frac{v-m_f}{\sigma_f}\right)^2} e^{-\frac{1}{2}\left(\frac{m_g}{\sigma_g}\right)^2} \\
&= e^{-\frac{1}{2}\left(\frac{m_f}{\sigma_f}\right)^2} \vee e^{-\frac{1}{2}\left(\frac{m_g}{\sigma_g}\right)^2}
\end{aligned} \tag{2.51}$$

Solving the optimization problem in (2.50), in general, is quite complicated and does not lead to a closed-form expression (see Appendix C.6). Also, since the final result is non-Gaussian, it can not be easily generalized to the case of the *meet* of more than two Gaussians at a time; therefore, we now try to find a Gaussian approximation to this result.

The supremum in (2.50) can be obtained by minimizing the exponent on the RHS of (2.50). Let us call the exponent $J(v)$. So, we want to minimize

$$J(v) = \left(\frac{v - m_f}{\sigma_f}\right)^2 + \left(\frac{\theta - m_g v}{\sigma_g v}\right)^2, \ \theta \neq 0 \tag{2.52}$$

with the constraint $v \in [\theta, 1]$. Since the second term on the RHS of (2.52) has $v$ in its denominator, $J$ is non-convex and is difficult to minimize. The actual function resulting from the minimization of $J$ is non-Gaussian. In order to find a Gaussian approximation for $\tilde{\mathrm{F}} \sqcap \tilde{\mathrm{G}}$, we simplify the problem a bit.

Equation (2.48) can be interpreted as follows. Each element $v$ of set $\tilde{\mathrm{F}}$ multiplies every element $w$ of set $\tilde{\mathrm{G}}$, and, at the same time, the membership grade of $v$ in $\tilde{\mathrm{F}}$ multiplies the membership grade of $w$ in $\tilde{\mathrm{G}}$. So, given a particular element $v_1$ of $\tilde{\mathrm{F}}$, what we get as a result of these multiplications is a scaled version of the membership function of $\tilde{\mathrm{G}}$ (scaled along both the axes : along the independent axis by $v_1$ and along the dependent axis by $e^{-\frac{1}{2}(\frac{v_1 - m_f}{\sigma_f})^2}$). This process is repeated for every element of $\tilde{\mathrm{F}}$ and finally, the meet of $\tilde{\mathrm{F}}$ and $\tilde{\mathrm{G}}$ is given by the envelope of all the above scaled Gaussians. The expression for the membership of an element $\theta$ in $\tilde{\mathrm{F}} \sqcap \tilde{\mathrm{G}}$ is given by (2.50) and (2.51).

In order to simplify the problem, we replace the $v$ in the denominator of the second term on the RHS of (2.52) by a constant $k$. By solving this simplified optimization problem, we get an approximation to $\tilde{\mathrm{F}} \sqcap \tilde{\mathrm{G}}$. Let's call it $\tilde{\mathrm{E}}$, so that

$$\begin{aligned} \mu_{\tilde{\mathrm{E}}}(\theta) &= \sup_{v \in [\theta, 1]} e^{-\frac{1}{2}\left(\frac{v - m_f}{\sigma_f}\right)^2} e^{-\frac{1}{2}\left(\frac{\theta - m_g v}{k \sigma_g}\right)^2} \\ &= \sup_{v \in [\theta, 1]} e^{-\frac{1}{2}\left[\left(\frac{v - m_f}{\sigma_f}\right)^2 + \left(\frac{\theta - m_g v}{k \sigma_g}\right)^2\right]} \end{aligned} \qquad (2.53)$$

Observe that the only difference between (2.53) and (2.50) is that the standard deviation of the second Gaussian in (2.53) is a constant ($k\sigma_g$), whereas that in (2.50) is proportional to $v$ ($v\sigma_g$).

To see the dependence of $\mu_{\tilde{\mathrm{E}}}(\theta)$ on $k$, let

$$H(v, k) = \Big(\frac{v - m_f}{\sigma_f}\Big)^2 + \Big(\frac{\theta - m_g v}{k \sigma_g}\Big)^2 \quad ; \quad k \in (0, 1] \qquad (2.54)$$

Obviously,

$$\begin{array}{rcccccl} & H(v,\epsilon) & \geq & H(v,k) & \geq & H(v,1) & ;\ 0 < \epsilon \leq k \leq 1 \\ \Rightarrow & \inf_{v \in [\theta,1]} H(v,\epsilon) & \geq & \inf_{v \in [\theta,1]} H(v,k) & \geq & \inf_{v \in [\theta,1]} H(v,1) & ;\ 0 < \epsilon \leq k \leq 1 \\ \Rightarrow & \mu_{\tilde{\mathrm{E}}}(\theta)\Big|_{k=\epsilon} & \leq & \mu_{\tilde{\mathrm{E}}}(\theta) & \leq & \mu_{\tilde{\mathrm{E}}}(\theta)\Big|_{k=1} & ;\ 0 < \epsilon \leq k \leq 1 \end{array} \qquad (2.55)$$

Observe that as $\epsilon \to 0$, $H(v, \epsilon) \to \infty$ and $\mu_{\tilde{E}}(\theta) \to 0$. Since $k$ in (2.53) replaces $v$ in the actual problem (which varies between 0 and 1), it is apparent from (2.55), that $\lim_{k\to 0} \mu_{\tilde{E}}(\theta)$ gives a lower bound on $\mu_{\tilde{F}\sqcap\tilde{G}}(\theta)$ and $\mu_{\tilde{E}}(\theta)\Big|_{k=1}$ gives an upper bound on $\mu_{\tilde{F}\sqcap\tilde{G}}(\theta)$.

Recall that our earlier approximation in (2.44), which was a Gaussian with standard deviation equal to $\max\{m_f\sigma_g, m_g\sigma_f\}$ , was motivated by considering the case where one of the Gaussians was a singleton, i.e., one of the Gaussians has a zero standard deviation, which is analogous to assuming that $k = 0$ in (2.53). Therefore, from the discussion above, it is clear that this earlier approximation acts as a lower bound on $\mu_{\tilde{F}\sqcap\tilde{G}}(\theta)$. [Figure 2.15 (f) depicts a result that seems to contradict this statement; however, it is obtained because only half of one of the participating Gaussians is contained in $[0, 1]$. See the discussion at the end of this Section.]

From this discussion, it seems conceivable that choosing some $\mu_{\tilde{E}}(\theta)$ between the upper and the lower bounds, i.e., substituting some value of $k \in (0, 1)$ into (2.53), should enable us to obtain a good approximation to $\mu_{\tilde{F}\sqcap\tilde{G}}(\theta)$. Our criterion for choosing $k$ is that the approximation should be commutative (i.e., if we switch $\tilde{F}$ and $\tilde{G}$, we should still get the same result), because the true result is commutative. Considering both these factors, we choose $k = m_f$. Refer to Appendix C.6 for details of the solution of (2.53) and the choice of $k$. We state the result here.

If $\tilde{F}$ and $\tilde{G}$ are two Gaussian type-1 fuzzy sets in $[0, 1]$ having means $m_f$ and $m_g$ and standard deviations $\sigma_f$ and $\sigma_g$, then the membership function for the *meet* of $\tilde{F}$ and $\tilde{G}$ can be approximated as

$$\mu_{\tilde{F}\sqcap\tilde{G}}(\theta) \approx e^{-\frac{1}{2}\left(\frac{\theta - m_f m_g}{\sqrt{m_g^2\sigma_f^2 + m_f^2\sigma_g^2}}\right)^2} \tag{2.56}$$

Generalization to the case of more than two Gaussians is straightforward. Assume that $\tilde{L}$ is a Gaussian fuzzy set with mean $m_l$ and standard deviation $\sigma_l$. Using the associative property, we have $\tilde{F} \sqcap \tilde{G} \sqcap \tilde{L} = (\tilde{F} \sqcap \tilde{G}) \sqcap \tilde{L}$; hence, using (2.56), we have

$$\mu_{\tilde{F}\sqcap\tilde{G}}(\theta) \approx e^{-\frac{1}{2}\left(\frac{\theta - m_{fg}}{\sigma_{fg}}\right)^2} \tag{2.57}$$

where $m_{fg} = m_f m_g$ and $\sigma_{fg} = \sqrt{m_g^2\sigma_f^2 + m_f^2\sigma_g^2}$. Using (2.56) again, we have

$$\begin{aligned}\mu_{\tilde{F}\sqcap\tilde{G}\sqcap\tilde{L}}(\theta) &\approx e^{-\frac{1}{2}\left(\frac{\theta - m_{fg}m_l}{\sqrt{m_l^2\sigma_{fg}^2 + m_{fg}^2\sigma_l^2}}\right)^2} \\ &= e^{-\frac{1}{2}\left(\frac{\theta - m_f m_g m_l}{\sqrt{m_l^2 m_g^2\sigma_f^2 + m_l^2 m_f^2\sigma_g^2 + m_f^2 m_g^2\sigma_l^2}}\right)^2} \qquad (2.58)\end{aligned}$$

If there are $n$ Gaussian fuzzy sets $\tilde{F}_1, \tilde{F}_2, \cdots, \tilde{F}_n$ with means $m_1, m_2, \cdots, m_n$ and standard deviations $\sigma_1, \sigma_2, \cdots, \sigma_n$, respectively, then repeated application of (2.56) yields

$$\mu_{\tilde{F}_1\sqcap\tilde{F}_2\sqcap\cdots\sqcap\tilde{F}_n}(\theta) \approx e^{-\frac{1}{2}\left(\frac{\theta - m_1 m_2\cdots m_n}{\bar{\sigma}}\right)^2} \qquad (2.59)$$

where

$$\bar{\sigma} = \sqrt{\sigma_1^2 \prod_{i\,;\,i\neq 1} m_i^2 + \sigma_2^2 \prod_{i\,;\,i\neq 2} m_i^2 + \cdots + \sigma_j^2 \prod_{i;i\neq j} m_i^2 + \cdots + \sigma_n^2 \prod_{i;i\neq n} m_i^2}\ ;\ i = 1, 2, \cdots, n \qquad (2.60)$$

Comparing (2.59) with (2.46), we see that the two approximations have the same mean. Only the standard deviations are different. All the results and approximations that we have developed will finally be used for operations between membership grades of type-2 sets.

Recall that we require that all our results remain valid if we replace all the type-2 sets by corresponding type-1 sets (i.e., type-1 sets having the principal membership functions of the type-2 sets as their membership functions). In case of Gaussian type-2 sets, replacing type-2 sets by corresponding type-1 sets is analogous to reducing the standard deviations of all the secondary membership functions to zero. [Observe that a Gaussian with a zero standard deviation is like an impulse function. As we reduce the standard deviation of a Gaussian, keeping its mean constant, it grows narrower and narrower. The height of the Gaussian at the mean remains unchanged though. In the limit, as the standard deviation reduces to zero, only the mean of the Gaussian has a non-zero membership, which is equal to 1. Mathematically, if we have a Gaussian with mean $m$ and standard deviation $\sigma$, $\lim_{\sigma\to 0}(\frac{\theta - m}{\sigma}) \to \infty$ if $\theta \neq m$ and $\lim_{\sigma\to 0}(\frac{\theta - m}{\sigma}) = 0$ if $\theta = m$. So, in the limit $\exp\{-\frac{1}{2}(\frac{\theta - m}{\sigma})^2\}$ is equal to 1 if $\theta = m$ and is equal to 0 otherwise.] All that remains of a membership grade after doing this is a crisp number in $[0, 1]$ equal to the center of the Gaussian in the type-2 case; and the *meet* operation reduces to the product of all these crisp numbers. The actual type-2 result for the *meet* between Gaussians (Appendix C.6) as well as both our approximations [(2.59) and (2.46)] obey the same result. If we reduce the standard deviations of all the Gaussians involved to zero, the result of the *meet* operation is equal to that in the type-1 case.

Figures 2.15 (a) - (f) show some examples of this approximation. In general, if the Gaussians have $\pm 2$ (or more) standard deviations contained within $[0, 1]$, the results look quite good. In Fig. 2.15 (f), one of the Gaussians is centered at 1, so only half of this Gaussian (the part lying to the left of the mean) is contained in $[0, 1]$. Consequently, the result of the *meet* is much more "non-Gaussian" than earlier cases, i.e., the difference between our Gaussian approximation and the actual curve is larger than that in the other examples. Also, observe that the first approximation in (2.46) does not act as a lower bound on the result of the *meet* in this case.

Appendix C.6 derives bounds on the error between the Gaussian approximation and the result of the actual *meet* operation, by finding upper and lower bounds which contain both the approximation as well as the actual function.

### 2.3.3 A Triangular Approximation

Triangular membership funtions are widely used; therefore, we develop a triangular *meet* approximation similar to the Gaussian approximation derived in Section 2.3.2. See Appendix E.1 for details.

## 2.4 Algebraic Operations on Fuzzy Numbers

As already mentioned, convex and normal type-1 fuzzy subsets of the real line are also known as *fuzzy numbers* [2, 5]. Algebraic operations like addition and multiplication between fuzzy numbers can be defined using the Extension Principle, just as we defined the $t$-norm and $t$-conorm (i.e., *meet* and *join*) operations (see, for example, [2, 5]). The two operations of most interest to us are multiplication and addition.

### 2.4.1 Multiplication of Type-1 Fuzzy Numbers

The product of two fuzzy numbers $\tilde{F} = \int f(v)/v$ and $\tilde{G} = \int g(w)/w$ is defined as

$$\tilde{F} \times \tilde{G} = \int_v \int_w [f(v) \star g(w)]/[v \times w] \tag{2.61}$$

where $\star$ indicates the $t$-norm used.

Observe, from (2.28) and (2.61), that under product $t$-norm, the product of $\tilde{F}$ and $\tilde{G}$ is the same as the *meet* of $\tilde{F}$ and $\tilde{G}$, i.e., $\tilde{F} \times \tilde{G} = \tilde{F} \sqcap \tilde{G}$; so, all our earlier discussion about *meet* under product $t$-norm applies to multiplication of fuzzy numbers under product $t$-norm. We do not discuss multiplication under minimum $t$-norm, rather we focus on the addition of fuzzy numbers.

### 2.4.2 Addition of Type-1 Fuzzy Numbers

The addition of two fuzzy numbers $\tilde{F} = \int f(v)/v$ and $\tilde{G} = \int g(w)/w$ is defined as

$$\tilde{F} + \tilde{G} = \int_v \int_w [f(v) \star g(w)]/[v + w] \tag{2.62}$$

When $\tilde{F}$ and$\tilde{G}$ are interval type-1 sets, (2.62) simplifies considerably, as we show next.

#### 2.4.2.1 Addition of Interval Type-1 Numbers

Let $F$ and $G$ be two interval type-1 sets with domains $[l_f, r_f]$ and $[l_g, r_g]$, respectively. Using (2.62), the algebraic sum of $F$ and $G$, can be obtained as

$$F + G = \int_{v \in F} \int_{w \in G} (1 \star 1)/(v + w) \tag{2.63}$$

Observe, from (2.63), that

- each term in $F + G$ is equal to the sum $(v + w)$ for some $v \in F$ and $w \in G$, the smallest term being $(l_f + l_g)$ and the largest $(r_f + r_g)$; and,
- since both $F$ and $G$ have continuous domains, $F + G$ also has a continuous domain;

consequently, $F+G$ is an interval type-1 set with domain $[l_f+l_g, r_f+r_g]$, i.e.,

$$F+G = \int_{v\in[l_f+l_g, r_f+r_g]} 1/v \tag{2.64}$$

Similarly, the algebraic sum of $n$ interval type-1 numbers $F_1, \ldots, F_n$, having domains $[l_1, r_1], \ldots, [l_n, r_n]$, respectively, is an interval type-1 set with domain $[\sum_{i=1}^n l_i, \sum_{i=1}^n r_i]$. See [5] for a similar result.

See Theorem D.1 for an expression for an affine combination of interval type-1 sets.

**NOTE** : Observe, from (2.63) that while performing algebraic operations on interval type-1 sets, the choice of $t$-norm does not matter, since all the memberships involved are unity.

#### 2.4.2.2 Addition of Gaussian Type-1 Numbers

**Theorem 2.4** *Given $n$ type-1 Gaussian fuzzy numbers $\tilde{F}_1, \ldots, \tilde{F}_n$, with means $m_1, m_2, \ldots, m_n$ and standard deviations $\sigma_1, \sigma_2, \ldots, \sigma_n$, their affine combination $\sum_{i=1}^n \alpha_i \tilde{F}_i + \beta$, where $\alpha_i$ $(i = 1, \ldots, n)$ and $\beta$ are crisp constants, is also a Gaussian fuzzy number with mean $\sum_{i=1}^n \alpha_i m_i + \beta$, and standard deviation $\Sigma'$, where*

$$\Sigma' = \begin{cases} \sqrt{\sum_{i=1}^n \alpha_i^2 \sigma_i^2} & , \quad \textit{if product } t\textit{-norm is used} \\ \sum_{i=1}^n |\alpha_i \sigma_i| & , \quad \textit{if minimum } t\textit{-norm is used} \end{cases} \tag{2.65}$$

□

See Appendix C.9 for the proof of Theorem 2.4.

#### 2.4.2.3 Addition of Triangular Type-1 Numbers

For comparable results about triangular fuzzy numbers, see Appendix E.2.

## 2.5 Centroid of a Type-2 Set

In this section, we extend the concept of the centroid of a fuzzy set from type-1 to type-2. The centroid of a type-1 set $\tilde{\mathrm{A}}$ whose domain is discrete with $N$ points or is discretized into $N$ points, is given as

$$C_{\tilde{\mathrm{A}}} = \frac{\sum_{i=1}^{N} x_i \mu_{\tilde{\mathrm{A}}}(x_i)}{\sum_{i=1}^{N} \mu_{\tilde{\mathrm{A}}}(x_i)} \tag{2.66}$$

Similarly, the centroid of a type-2 set $\tilde{\tilde{\mathrm{A}}}$, whose domain is discretized into $N$ points (see Chapter 1 for the definition of the domain of a type-2 set), can be defined using the Extension Principle [see Appendix B, especially (B.13)] as follows. If we let $\tilde{\mathrm{D}}_i = \tilde{\mu}_{\tilde{\tilde{\mathrm{A}}}}(x_i)$, then

$$\tilde{C}_{\tilde{\tilde{\mathrm{A}}}} = \int_{\theta_1} \cdots \int_{\theta_N} [\mu_{\tilde{\mathrm{D}}_1}(\theta_1) \star \cdots \star \mu_{\tilde{\mathrm{D}}_N}(\theta_N)] \Big/ \frac{\sum_{i=1}^{N} x_i \theta_i}{\sum_{i=1}^{N} \theta_i} \tag{2.67}$$

where $\theta_i \in \tilde{\mathrm{D}}_i$.

Equation (2.67) can be described in words as follows. Each point $x_i$ of $\tilde{\tilde{\mathrm{A}}}$ has a type-1 fuzzy membership grade, $\tilde{\mathrm{D}}_i = \tilde{\mu}_{\tilde{\tilde{\mathrm{A}}}}(x_i)$, associated with it. To find the centroid, we consider every possible combination $\{\theta_1, \ldots, \theta_N\}$ such that $\theta_i \in \tilde{\mathrm{D}}_i$. For every such combination, we perform the type-1 centroid calculation in (2.66) by using $\theta_i$'s in place of $\tilde{\mu}_{\tilde{\tilde{\mathrm{A}}}}(x_i)$'s; and, to each point in the centroid, we assign a membership grade equal to the $t$-norm of the membership grades of the $\theta_i$s in the $\tilde{\mathrm{D}}_i$s. If more than one combination of $\theta_i$'s gives us the same point in the centroid, we keep the one with the largest membership grade. If we let $\sum_{i=1}^{N} x_i \theta_i \big/ \sum_{i=1}^{N} \theta_i = x$, then (2.67) can also be written as

$$\tilde{C}_{\tilde{\tilde{\mathrm{A}}}} = \int_x \sup_{\{\theta_1, \ldots, \theta_N\}} [\mu_{\tilde{\mathrm{D}}_1}(\theta_1) \star \cdots \star \mu_{\tilde{\mathrm{D}}_N}(\theta_N)] \big/ x \tag{2.68}$$

where $\{\theta_1, \ldots, \theta_N\}$ are such that $\sum_{i=1}^{N} x_i \theta_i \big/ \sum_{i=1}^{N} \theta_i = x$. We will illustrate the calculation of $\tilde{C}_{\tilde{\tilde{\mathrm{A}}}}$ below, in Example 2.4. First, however, we provide some general insights into (2.67).

A type-2 set $\tilde{\tilde{A}}$ can be thought of as a collection of type-1 sets, which we call type-1 sets *embedded* in $\tilde{\tilde{A}}$.

**Definition** : A type-1 set $\tilde{A}$ embedded in a type-2 set $\tilde{\tilde{A}}$ is a type-1 set for which : (1) $x \in \tilde{\tilde{A}} \Leftrightarrow x \in \tilde{A}$, and (2) $\mu_{\tilde{A}}(x) \in \tilde{\mu}_{\tilde{\tilde{A}}}(x)\, \forall x \in \tilde{\tilde{A}}$.

In the 2-D representation of a type-2 set, an embedded type-1 set is one whose membership function lies inside the shaded region. Figure 2.16 shows an example of a type-1 set embedded in a type-2 set. For the type-2 set $\tilde{\tilde{A}}$ in (2.67), every combination $\{\theta_1, \ldots, \theta_N\}$ such that $\theta_i \in \tilde{D}_i$, corresponds to the membership function of an embedded type-1 set. The centroid of $\tilde{\tilde{A}}$, $\tilde{C}_{\tilde{\tilde{A}}}$ can be thought of as a type-1 set whose elements are the centroids of all the embedded type-1 sets in $\tilde{\tilde{A}}$. The membership grade of an embedded set centroid in $\tilde{C}_{\tilde{\tilde{A}}}$ is calculated as the $t$-norm of all the secondary memberships corresponding to $\{\theta_1, \ldots, \theta_N\}$ that make up that embedded set. When $\tilde{\tilde{A}}$ collapses to an embedded type-1 set $\tilde{A}$, which corresponds to the combination $\{\theta'_1, \ldots, \theta'_N\}$, each $\tilde{D}_i$ reduces to a fuzzy singleton, such that $\mu_{\tilde{D}_i}(\theta'_i) = 1$ and $\mu_{\tilde{D}_i}(\theta_i) = 0$ if $\theta_i \neq \theta'_i$; therefore, we get $\mathcal{T}_{i=1}^{N} \mu_{\tilde{D}_i}(\theta'_i) = 1$, and for all other $\{\theta_1, \ldots, \theta_N\}$ combinations $\mathcal{T}_{i=1}^{N} \mu_{\tilde{D}_i}(\theta_i) = 0$. Consequently, $\tilde{C}_{\tilde{\tilde{A}}}$ reduces to the crisp number $C_{\tilde{A}}$, the centroid of $\tilde{A}$.

Observe that if the domain of $\tilde{\tilde{A}}$ and/or $\tilde{\mu}_{\tilde{\tilde{A}}}(x)$ $(x \in \tilde{\tilde{A}})$ is continuous, the domain of $\tilde{C}_{\tilde{\tilde{A}}}$ is also continuous. The number of all the embedded type-1 sets in $\tilde{\tilde{A}}$, in this case, is uncountable; therefore, the domains of $\tilde{\tilde{A}}$ and each $\tilde{\mu}_{\tilde{\tilde{A}}}(x)$ $(x \in \tilde{\tilde{A}})$ have to be discretized for the calculation of $\tilde{C}_{\tilde{\tilde{A}}}$. Observe, from (2.67), that if the domain of each $\tilde{D}_l$ is discretized into $M$ points, the number of possible $\{\theta_1, \ldots, \theta_N\}$ combinations is $M^N$, which can be very large even for small $M$ and $N$. If, however, the membership functions of $\tilde{D}_l$'s have a regular structure (e.g., Gaussian, triangular, interval), we can approximate the centroid without having to do all the calculations. See Example 2.4 and Sections 2.5.2.1, 2.5.2.3, 2.5.2.2 for more details. In the case of an interval type-2 set, even the actual centroid can be obtained relatively easily by using the computational procedure described in Appendix D.

**Example 2.4** In this example, we show the centroid calculation for a type-2 set that results from a type-1 set with only location uncertainty, e.g., see Example 1.2. We focus on the special case of Gaussian membership functions with uncertain means, such that every value of the mean is equally uncertain. In this case, we set all the secondary memberships equal to 1, to indicate that the level of uncertainty associated with every primary membership is the same, so that the resulting set is an interval type-2 set.

Figure 2.17 (a) shows a type-2 set $\tilde{\tilde{A}}$ resulting from a Gaussian type-1 set with mean uniformly uncertain in the interval $[m_1, m_2]$. In the figure, $m_1 = 0.45$, $m_2 = 0.55$ and the standard deviation, $\sigma = 0.2$. All the secondary memberships are equal to 1.

Observe, from (2.67), that :

1. all the secondary memberships are equal to 1, so the membership of each point in the centroid is also equal to 1, i.e., $\mu_{\tilde{D}_1}(\theta_1) \star \cdots \star \mu_{\tilde{D}_N}(\theta_N) = 1$; hence, the centroid is a crisp set;

2. the mean varies on a continuous domain $[m_1, m_2]$, so the crisp set corresponding to the centroid will also have a continuous domain; and,

3. each Gaussian centered at $m \in [m_1, m_2]$ is an embedded set in $\tilde{\tilde{A}}$, so the centroid of each such Gaussian $\big($ i.e., each $m \in [m_1, m_2]\big)$ will be an element of the centroid.

From these three observations, we see that the centroid of $\tilde{\tilde{A}}$ is some interval, $[c_l, c_r]$, which contains $[m_1, m_2]$. Now, we have to find the end-points of this interval. To do this, we show how to compute the left end-point, $c_l$. Since the set is symmetrical, the calculation of $c_r$ will be similar.

It is easy to verify that the left end-point $c_l$ is the centroid of the embedded type-1 set which assigns the highest possible memberships to all the points to the left of its centroid and lowest possible memberships to all the points to the right of its centroid (see the computational procedure in Appendix D.1 for more discussion). Any change in this membership function will always cause its centroid to move towards the right, implying that the centroid of this embedded type-1 set is equal to $c_l$. An example of such an embedded type-1 set is shown by the thick dashed line in Fig. 2.17 (b).

Though we do not know the exact value of $c_l$, we can make an estimate by considering the embedded type-1 set shown in Fig. 2.17 (c). This type-1 set is formed by assigning the highest possible memberships to the points to the left of $m_1$ and the lowest possible memberships to the points to the right of $m_1$. The membership function of this set looks like a Gaussian with a small portion missing, and its centroid (which was calculated numerically) is equal to $c_1 = 0.44992$, which is a little bit to the left of $m_1$. The fact that $c_1$ is just slightly less than $m_1$ shows that the area to the left of $m_1$ in the type-1 set in Fig. 2.17 (c) is just slightly more than the area to its right; therefore, $c_l$ will also be just slightly less than $m_1$.

Similarly, the embedded type-1 set constructed by assigning highest possible memberships to all the points to the right of $m_2$ and lowest possible memberships to the points to the left of $m_2$, has a centroid $c_2 = 0.55008$, which is slightly larger than $m_2$; hence, we conclude that $c_r$ will be just a little bit larger than $m_2$. We can, therefore, say that $c_l \approx c_1$ and $c_r \approx c_2$. Figure 2.17 (d) shows $\tilde{\tilde{A}}$ with its centroid , which is a crisp set with domain $[c_l, c_r] \approx [c_1, c_2] \approx [m_1, m_2]$.

It can be shown that, if $(m_2 - m_1)$ is small compared to the standard deviation $(\sigma)$ of $\tilde{A}$, then $[c_l, c_r] \approx [m_1, m_2]$ (see Appendix C.10 for the proof).

If we increase $(m_2 - m_1)$, keeping $\sigma$ the same, the difference between the approximation and the true centroid (computed using the computational procedure in Appendix D.1) increases, e.g., for $\sigma = 0.2$, if $\{m_1, m_2\} = \{0.4, 0.5\}$, $\{c_1, c_2\} = \{0.39855, 0.60145\}$, and, if $\{m_1, m_2\} = \{0.3.0.7\}$, $\{c_1, c_2\} = \{0.28146, 0.71854\}$. We, therefore, recommend using the computational procedure described in Appendix D.1 to obtain the centroid, if $(m_2 - m_1)$ is not small compared to $\sigma$. $\square$

See Appendix D.1 for a computational procedure to compute the centroid of a general interval type-2 set. We next describe a problem that arises when one attempts to compute the centroid of a type-2 set having a continuous domain using product $t$-norm.

### 2.5.1 Centroid Calculation Using the Product $t$-norm

Calculation of the centroid, using product $t$-norm, of a type-2 set which has a continuous domain and not all of whose secondary memberships are unity, gives us an unexpected result. In this section, we concentrate on type-2 sets having a continuous domain whose secondary membership functions are such that, for any domain point, only one primary membership has a secondary membership equal to one, e.g., Gaussian or triangular type-2 sets. We first describe the problem and then discuss its cause and remedy.

### Problem

In the discussion associated with (2.67), we assumed that the domain of $\tilde{\tilde{A}}$ is discretized into $N$ points. The true centroid of $\tilde{\tilde{A}}$ (assuming $\tilde{\tilde{A}}$ has a continuous domain) is the limit of $\tilde{C}_{\tilde{\tilde{A}}}$ in (2.67) as $N \to \infty$. When we use the product $t$-norm $\lim_{N\to\infty} \mathcal{T}_{i=1}^{N} \mu_{\tilde{D}_i}(\theta_i) = \lim_{N\to\infty} \prod_{i=1}^{N} \mu_{\tilde{D}_i}(\theta_i)$.

Let $\tilde{B}$ be an embedded type-1 set in $\tilde{\tilde{A}}$. The centroid of $\tilde{B}$ is computed as

$$C_{\tilde{B}} = \frac{\sum_{i=1}^{N} x_i \mu_{\tilde{B}}(x_i)}{\sum_{i=1}^{N} \mu_{\tilde{B}}(x_i)} \tag{2.69}$$

and the membership of $C_{\tilde{B}}$ in $\tilde{C}_{\tilde{\tilde{A}}}$ [denoted as $\mu_{\tilde{C}}(C_{\tilde{B}})$] is

$$\mu_{\tilde{C}}(C_{\tilde{B}}) = \prod_{i=1}^{N} \mu_{\tilde{D}_i}(\theta_i) \tag{2.70}$$

where $\{\theta_1, \ldots, \theta_N\}$ are the primary memberships that make up the type-1 set $\tilde{B}$.

Let $\tilde{A}$ denote the principal membership function of $\tilde{\tilde{A}}$. Obviously, $\mu_{\tilde{C}}(C_{\tilde{A}}) = 1$.

Consider the case where the secondary membership functions are like Gaussians or triangles (having only one point with unity membership). We make two observations :

1. $\lim_{N\to\infty} \mu_{\tilde{C}}(C_{\tilde{B}})$ is non-zero only if $\tilde{B}$ differs from $\tilde{A}$ in *at most a finite number of points.* For all other embedded sets $\tilde{B}$, the product of an infinite number of quantities less than one will cause $\mu_{\tilde{C}}(C_{\tilde{B}})$ to go to zero as $N \to \infty$.

2. For any embedded set $\tilde{B}$, whose membership function differs from that of $\tilde{A}$ in only a finite number of points (i.e., when $\mu_{\tilde{B}}(x) \neq \mu_{\tilde{A}}(x)$, for only a finite number of points $x$), $C_{\tilde{B}} = C_{\tilde{A}}$. This can be explained as follows :

   The (true) centroid of $\tilde{B}$ is defined as

   $$C_{\tilde{B}} = \frac{\int_x x \mu_{\tilde{B}}(x) dx}{\int_x \mu_{\tilde{B}}(x) dx} \tag{2.71}$$

   where $x \in \tilde{B}$. Since $\tilde{A}$ and $\tilde{B}$ share the same domain (both are embedded sets in $\tilde{\tilde{A}}$), $x \in \tilde{A} \Leftrightarrow x \in \tilde{B}$; and since $\mu_{\tilde{A}}(x)$ and $\mu_{\tilde{B}}(x)$ differ only in a finite number of points, $\int_x x\mu_{\tilde{B}}(x)dx = \int_x x\mu_{\tilde{A}}(x)dx$ and $\int_x \mu_{\tilde{B}}(x)dx = \int_x \mu_{\tilde{A}}(x)dx$; therefore, $C_{\tilde{B}} = C_{\tilde{A}}$.

From these two observations, we can see that the only point having non-zero membership in $\tilde{C}_{\tilde{\tilde{A}}}$ is equal to $C_{\tilde{A}}$; and its membership grade is equal to the supremum of the membership grades of all the embedded type-1 sets which have the same centroid, which is equal to 1 [since $\mu_{\tilde{C}}(C_{\tilde{A}}) = 1$]. In other words, $\tilde{C}_{\tilde{\tilde{A}}} = 1/C_{\tilde{A}} = C_{\tilde{A}}$, i.e., the centroid of $\tilde{\tilde{A}}$ will be equal to a crisp number ..... the centroid of its principal membership function !

### Cause

The above problem occurs because, under the product $t$-norm, $\lim_{N\to\infty} \mathcal{T}_{i=1}^{N} \mu_{\tilde{D}_i}(\theta_i) = \lim_{N\to\infty} \prod_{i=1}^{N} \mu_{\tilde{D}_i}(\theta_i) = 0$, unless only a finite number of $\mu_{\tilde{D}_i}(\theta_i)$'s are less than 1. The minimum $t$-norm does not cause such a problem.

### Remedy

One obvious way to deal with the problem explained above is to *not* use product $t$-norm for centroid calculation. *From now on, we will always use the minimum $t$-norm to calculate the centroid of a type-2 set having a continuous domain.*

## 2.5.2 Approximations to Centroids of Certain Type-2 Sets

In this section we develop approximations to the Centroids of Gaussian and triangular type-2 sets.

### 2.5.2.1 Centroid of a Gaussian Type-2 Set

We first prove a general result and then use it to find the centroid of a Gaussian type-2 set.

**Weighted Average of Gaussian Type-1 Sets :** Consider the weighted average

$$y(z_1, \ldots, z_M, w_1, \ldots, w_M) = \frac{\sum_{l=1}^{M} w_l z_l}{\sum_{l=1}^{M} w_l} \tag{2.72}$$

where $z_l \in \Re$ and $w_l \in [0,1]$ for $l = 1, \ldots, M$. If each $z_l$ is replaced by a type-1 fuzzy set $\tilde{Z}_l \subset \Re$ and each $w_l$ is replaced by a type-1 fuzzy set $\tilde{W}_l \subset [0,1]$, then the extension of (2.72) gives

$$\tilde{Y}(\tilde{Z}_1, \ldots, \tilde{Z}_M, \tilde{W}_1, \ldots, \tilde{W}_M) = \int_{z_1} \cdots \int_{z_M} \int_{w_1} \cdots \int_{w_M} \mathcal{T}_{l=1}^M \mu_{\tilde{Z}_l}(z_l) \star \mathcal{T}_{l=1}^M \mu_{\tilde{W}_l}(w_l) \Bigg/ \frac{\sum_{l=1}^M w_l z_l}{\sum_{l=1}^M w_l} \tag{2.73}$$

where $\mathcal{T}$ and $\star$ both indicate the $t$-norm used ... product or minimum, $w_l \in \tilde{W}_l$ and $z_l \in \tilde{Z}_l$ for $l = 1, \ldots, M$.

**Theorem 2.5** *If each $\tilde{Z}_l$ is a Gaussian type-1 set, with mean $m_l$ and standard deviation $\sigma_l$, and if each $\tilde{W}_l$ is also a Gaussian type-1 set with mean $h_l$ and standard deviation $\Delta_l$, then $\tilde{Y}$ is approximately a Gaussian type-1 set, with mean $\mathcal{M}$ and standard deviation $\Sigma$, where*

$$\mathcal{M} = \frac{\sum_{l=1}^M h_l m_l}{\sum_{l=1}^M h_l} \tag{2.74}$$

*and*

$$\Sigma = \begin{cases} \dfrac{\sqrt{\sum_{l=1}^M \left[(h_l\sigma_l)^2 + [m_l - \mathcal{M}]^2 \Delta_l^2\right]}}{\sum_{l=1}^M h_l}, & \text{if product } t\text{-norm is used} \\[2ex] \dfrac{\sum_{l=1}^M [(h_l\sigma_l) + |m_l - \mathcal{M}|\Delta_l]}{\sum_{l=1}^M h_l}, & \text{if minimum } t\text{-norm is used} \end{cases} \tag{2.75}$$

*provided that*

$$\frac{k \sum_{l=1}^M \Delta_l}{\sum_{l=1}^M h_l} \ll 1, \tag{2.76}$$

*where $k$ is the number of standard deviations of a Gaussian considered significant (generally, $k = 2$ or $3$). The Gaussian approximation improves as $k\left(\sum_{l=1}^M \Delta_l \big/ \sum_{l=1}^M h_l\right)$ grows smaller, and the result is exact when $\sum_{l=1}^M \Delta_l = 0$, i.e., when $\Delta_l = 0$ for $l = 1, \ldots, M$.* □

See Appendix C.11 for the proof. A sufficient condition that satisfies (2.76) is that the Gaussian $\tilde{W}_l$'s are narrow, i.e., $k\Delta_l/h_l \ll 1$ for $l = 1, \ldots, M$. Observe, however, that there is no condition on the standard deviations of the $\tilde{Z}_l$'s; consequently, when all the $\tilde{W}_l$'s are crisp numbers, the theorem gives an exact result. See the comments at the end of Appendix C.11 for bounds on the domain of $\tilde{Y}$.

Recall that we will use only minimum $t$-norm for the centroid calculation of a type-2 set with a continuous domain. (If the domain is discrete, however, product $t$-norm may be used.) From Theorem 2.5, we get the following result for the centroid of a Gaussian type-2 set.

**Corollary 2.2** *The centroid of a Gaussian type-2 set* $\tilde{\tilde{A}}$ *is approximately a Gaussian type-1 set with mean* $\mathcal{M}(\tilde{C}_{\tilde{\tilde{A}}})$ *[Eq. (2.77)] and standard deviation* $\Sigma(\tilde{C}_{\tilde{\tilde{A}}})$ *[Eq. (2.78)], if the standard deviations of the secondary memberships are small compared to their means, i.e., if (2.79) is satisfied.*

**Proof :** Observe that the $x_i$'s in (2.67), which are crisp numbers, correspond to the $z_l$'s in (2.73), the $\tilde{D}_i$'s in (2.67) $[\tilde{D}_i = \tilde{\mu}_{\tilde{\tilde{A}}}(x_i)]$ correspond to the $\tilde{W}_l$'s in (2.73), and the sum in (2.67) goes from 1 to $N$ instead of from 1 to $M$. If we denote the mean and the standard deviation of $\tilde{\mu}_{\tilde{\tilde{A}}}(x_i)$ as $m(x_i)$ and $\sigma(x_i)$, respectively, then using Theorem 2.5, $\tilde{C}_{\tilde{\tilde{A}}}$ is approximately a Gaussian type-1 set with mean $\mathcal{M}(\tilde{C}_{\tilde{\tilde{A}}})$ and standard deviation $\Sigma(\tilde{C}_{\tilde{\tilde{A}}})$, where

$$\mathcal{M}(\tilde{C}_{\tilde{\tilde{A}}}) = \frac{\sum_{i=1}^{N} x_i m(x_i)}{\sum_{i=1}^{N} m(x_i)} \tag{2.77}$$

and

$$\Sigma(\tilde{C}_{\tilde{\tilde{A}}}) = \frac{\sum_{i=1}^{N} |x_i - \mathcal{M}(\tilde{C}_{\tilde{\tilde{A}}})| \sigma(x_i)}{\sum_{i=1}^{N} m(x_i)} \tag{2.78}$$

provided that

$$k \frac{\sum_{i=1}^{N} \sigma(x_i)}{\sum_{i=1}^{N} m(x_i)} \ll 1 \tag{2.79}$$

where $k$ has the same meaning as in Theorem 2.5. Equation (2.79) is satisfied if standard deviations of the secondary membership functions are small compared to their means. □

**Comment 1 :** See Fig. 1.10 for an example of a type-2 set, which can be made to satisfy condition (2.79) easily. In this set the standard deviation of every membership grade is proportional to its mean. If we set the constant of proportionality to a small value, (2.79) can be satisfied. See Example 2.7 for an expression for the centroid of such a type-2 set.

**Comment 2 :** Because the membership grade of each $x \in \tilde{\tilde{A}}$ is a Gaussian type-1 set, the primary membership which has a secondary membership equal to unity is $m(x)$; and, since the principal membership function is the set of those primary memberships for which the secondary memberships are equal to 1, $m(x)$ *for* $x \in \tilde{\tilde{A}}$ *is the same as the principal membership function of* $\tilde{\tilde{A}}$. Observe, therefore, from (2.77), that the mean of the approximate centroid, $\mathcal{M}(\tilde{C}_{\tilde{\tilde{A}}})$, corresponds to the centroid of the principal membership function of $\tilde{\tilde{A}}$.

**Example 2.5** Consider the centroid calculation of a type-2 set [see (2.67)]. If the type-2 set is discretized into $N$ points $(x_1, \ldots, x_N)$ and if the membership grade of every $x_i$ is discretized into $M$ points, the total number of possible $\{\theta_1, \ldots, \theta_N\}$ combinations is $M^N$. This number can be very large even for modest values of $M$ and $N$, e.g., if $N = 10$ and $M = 5$, the number of possible combinations is $9,765,625$, i.e., about 10 million ! And for each of these combinations, we have to compute the weighted average $\sum_{i=1}^{N} x_i\theta_i \big/ \sum_{i=1} \theta_i$. On the other hand, if (2.79) is satisfied, all we have to do to compute the centroid is compute two weighted averages, one for the mean of the centroid [(2.77)] and one for its standard deviation [(2.78)]. This example demonstrates the significance of our Gaussian approximation results. □

**Example 2.6** Now, we demonstrate the use of Corollary 2.2 with an example. Consider a Gaussian type-2 set with a discrete domain consisting of only 3 points, $x_1 = 1$, $x_2 = 3$ and $x_3 = 5$ [see Fig. 2.18 (a)]. Suppose that $m(x_1) = 0.1$, $m(x_2) = 0.8$ and $m(x_3) = 0.6$. We consider three cases :

1. If $\sigma(x_i) = 0.05m(x_i)$ for $i = 1, 2, 3$, the membership grades of $x_1$, $x_2$ and $x_3$ are shown in Fig. 2.18 (b), (c) and (d), respectively; and, the true centroid and the approximation in Corollary 2.2 are as shown in Fig. 2.19 (a). In this case, when $k = 2$, $k\big[\sum_i \sigma(x_i)\big]\big/\big[\sum_i m(x_i)\big] = 0.1$.

2. If $\sigma(x_1) = 0.3m(x_1)$, $\sigma(x_2) = 0.1m(x_2)$ and $\sigma(x_3) = 0.2m(x_3)$, the membership grades of $x_1$, $x_2$ and $x_3$ are shown in Fig. 2.18 (e), (f) and (g), respectively; and, the true centroid and the approximation in Corollary 2.2 are as shown in Fig. 2.19 (b). In this case, when $k = 2$, $k\big[\sum_i \sigma(x_i)\big]\big/\big[\sum_i m(x_i)\big] = 0.3066$.

3. If $\sigma(x_i) = 0.5m(x_i)$ for $i = 1, 2, 3$, the membership grades of $x_1$, $x_2$ and $x_3$ are shown in Fig. 2.18 (h), (i) and (j), respectively; and, the true centroid and

the approximation in Corollary 2.2 are as shown in Fig. 2.19 (c). In this case, when $k = 2$, $k\left[\sum_i \sigma(x_i)\right] / \left[\sum_i m(x_i)\right] = 1$.

When computing the true centroids, only primary membership values between $m(x_i) \pm 2\sigma(x_i)$ were considered. Observe that though the domain of the type-2 set is discrete, that of its centroid is continuous, because the membership grades of $x_1$, $x_2$ and $x_3$ have continuous domains.

Observe that the approximation in the first two cases is much closer to the true centroid than that in the third case; however, though a smaller value for $\left[\sum_i \sigma(x_i)\right] / \left[\sum_i m(x_i)\right]$ will generally give a better approximation, it is not at all easy to predict how close the actual centroid of a given Gaussian type-2 set will be to its approximation. The same can be said about the approximation in Theorem 2.5, which allows the domain points $x_i$'s to be replaced by fuzzy sets. □

**Example 2.7** Consider a Gaussian type-2 set $\tilde{\tilde{A}} \subset X$. Let the principal membership function of $\tilde{\tilde{A}}$ be a Gaussian type-1 set with mean $M$ and standard deviation $\Sigma$; and, let the standard deviation of each secondary membership function of $\tilde{\tilde{A}}$ be proportional to the mean of that secondary membership function. Figure 1.10 shows an example of such a Gaussian type-2 set. In this example, we obtain an expression for the centroid of $\tilde{\tilde{A}}$, using Corollary 2.2.

Recall, from comment 2 at the end of Corollary 2.2, that $m(x)$ *for* $x \in X$ *is the same as the principal membership function of* $\tilde{\tilde{A}}$. The membership grade of every $x \in X$ in $\tilde{\tilde{A}}$ can, therefore, be described as

$$\tilde{\mu}_{\tilde{\tilde{A}}}(x) = G\left(m(x), \sigma(x)\right) \quad ; \quad x \in X \tag{2.80}$$

where

$$m(x) = e^{-\frac{1}{2}\left(\frac{x-M}{\Sigma}\right)^2} \tag{2.81}$$

and

$$\sigma(x) = cm(x) \tag{2.82}$$

where $G(m, \sigma)$ indicates a Gaussian with mean $m$ and standard deviation $\sigma$; and, $c$ is a constant, which is generally in $(0, 1)$.

Let us now find an expression for $\tilde{C}_{\tilde{\tilde{A}}}$, the centroid of $\tilde{\tilde{A}}$, in terms of $M$, $\Sigma$ and $c$. From Corollary 2.2, we know that $\tilde{C}_{\tilde{\tilde{A}}}$ is approximately a Gaussian type-1 set with mean $\mathcal{M}(\tilde{C}_{\tilde{\tilde{A}}})$ and standard deviation $\Sigma(\tilde{C}_{\tilde{\tilde{A}}})$, where

$$\mathcal{M}(\tilde{C}_{\tilde{\tilde{A}}}) = \frac{\sum_{i=1}^{N} x_i m(x_i)}{\sum_{i=1}^{N} m(x_i)} \tag{2.83}$$

and

$$\Sigma(\tilde{C}_{\tilde{\tilde{A}}}) = \frac{\sum_{i=1}^{N} |x_i - \mathcal{M}(\tilde{C}_{\tilde{\tilde{A}}})|\sigma(x_i)}{\sum_{i=1}^{N} m(x_i)} \tag{2.84}$$

provided that

$$k\frac{\sum_{i=1}^{N} \sigma(x_i)}{\sum_{i=1}^{N} m(x_i)} \ll 1 \tag{2.85}$$

where $k$ is the number of standard deviations considered significant, 2 or 3, and the domain of $\tilde{\tilde{A}}$ is assumed to be discretized into $N$ points.

Since $m(x)$ for $x \in X$ is the principal membership function of $\tilde{\tilde{A}}$, we see, from (2.83), that $\mathcal{M}(\tilde{C}_{\tilde{\tilde{A}}})$ is the same as the centroid of the principal membership function, which is equal to $M$, i.e.,

$$\mathcal{M}(\tilde{C}_{\tilde{\tilde{A}}}) = M \tag{2.86}$$

To find $\Sigma(\tilde{C}_{\tilde{\tilde{A}}})$, let us assume that $X$ is not discretized (i.e., it is continous), so that (2.84) can be rewritten as (using the fact that $\mathcal{M}(\tilde{C}_{\tilde{\tilde{A}}}) = M$)

$$\Sigma(\tilde{C}_{\tilde{\tilde{A}}}) = \frac{\int_{x\in X} |x - M|\sigma(x)dx}{\int_{x\in X} m(x)dx} \tag{2.87}$$

The fuzzy sets we deal with are generally subsets of the real line, so that $X = \Re$. Observe, from (2.81) and (2.87), that the denominator of (2.87) is the area under a Gaussian with mean $M$ and standard deviation $\Sigma$. Recall from probability theory that the area under a probability density function is unity; therefore,

$$\int_{x\in\Re} m(x)dx = \sqrt{2\pi}\Sigma \tag{2.88}$$

Let the numerator of (2.87) be equal to $I$. It can be computed as follows.

$$I = I_1 + I_2 \tag{2.89}$$

where

$$I_1 = \int_{x \leq M} (M - x)\sigma(x)dx \tag{2.90}$$

and

$$I_2 = \int_{x > M} (x - M)\sigma(x)dx \tag{2.91}$$

Substituting (2.81), (2.82) and $\frac{1}{2}[(x - M)/\Sigma]^2 = t$ into (2.90) and (2.91), it is easy to see that

$$I_1 = I_2 = c\Sigma^2 \int_0^\infty e^{-t}dt = c\Sigma^2 \tag{2.92}$$

Using (2.88), (2.89) and (2.92) in (2.87), we find that

$$\Sigma(\tilde{C}_{\tilde{\tilde{A}}}) = \frac{2c\Sigma^2}{\sqrt{2\pi}\Sigma} = \sqrt{\frac{2}{\pi}}c\Sigma \tag{2.93}$$

From (2.86) and (2.93), we see that $\tilde{C}_{\tilde{\tilde{A}}}$ is approximately a Gaussian type-1 set with mean $M$ and standard deviation $\sqrt{2/\pi}c\Sigma$. The condition (2.85) requires that $c \ll 1/k$ [since $\sigma(x) = cm(x)$].

Observe that, if we set $c = 0$, $\sigma(x) = 0$ for all $x \in X$, implying that $\tilde{\tilde{A}}$ collapses to its principal membership. Now, from (2.93) , we get $\Sigma(\tilde{C}_{\tilde{\tilde{A}}}) = 0$, which means that the centroid of $\tilde{\tilde{A}}$ collapses to a single point, equal to $M$. This is consistent with the fact that $M$ is the centroid of the principal membership function of $\tilde{\tilde{A}}$. □

#### 2.5.2.2 Centroid of an Interval Type-2 Set

Appendix D.1 describes a computational procedure to compute the exact result of a weighted average of interval type-1 sets. This procedure can be used to compute the centroid of an interval type-2 set. Appendix D.2 also gives a result similar to Theorem 2.5.

#### 2.5.2.3 Centroid of a Triangular Type-2 Set

For a result similar to Theorem 2.5 for triangular type-2 sets, see Appendix E.2.2.3.

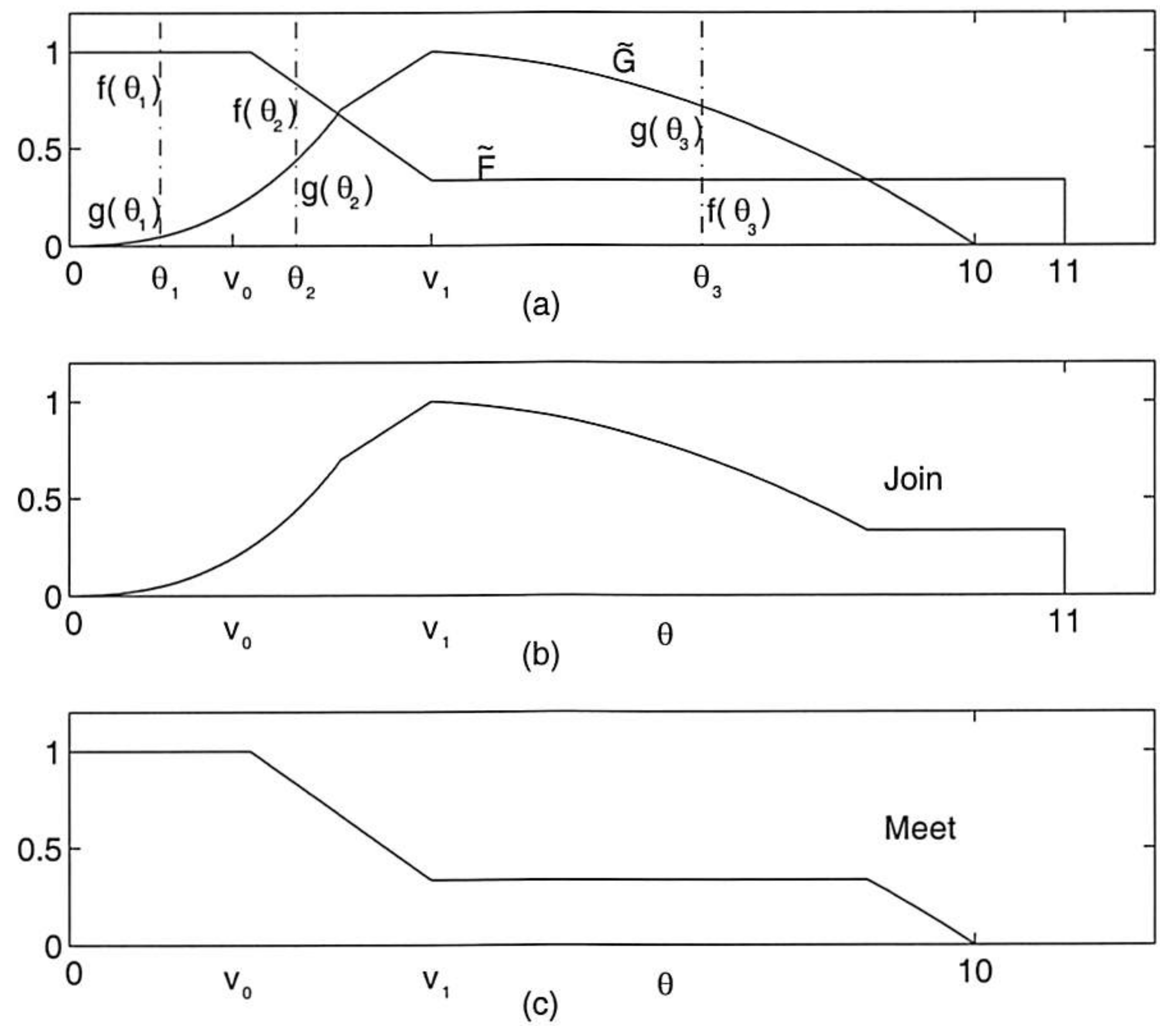



Figure 2.1: An example of two general membership functions, $f$ and $g$, that satisfy the requirements of Theorem 2.1, part (a). Observe that for set $\tilde{\mathrm{F}}$, any of the points at which $f$ attains its maximum value of unity may be chosen as $v_0$. We arbitrarily chose $v_0 = 1.8$. (a) The three possibilities : $\theta_1 < v_0$, $v_0 \leq \theta_2 < v_1$, $\theta_3 > v_1$. (b) Result of the *join* operation. (c) Result of the *meet* operation. The $t$-norm used is *min*.

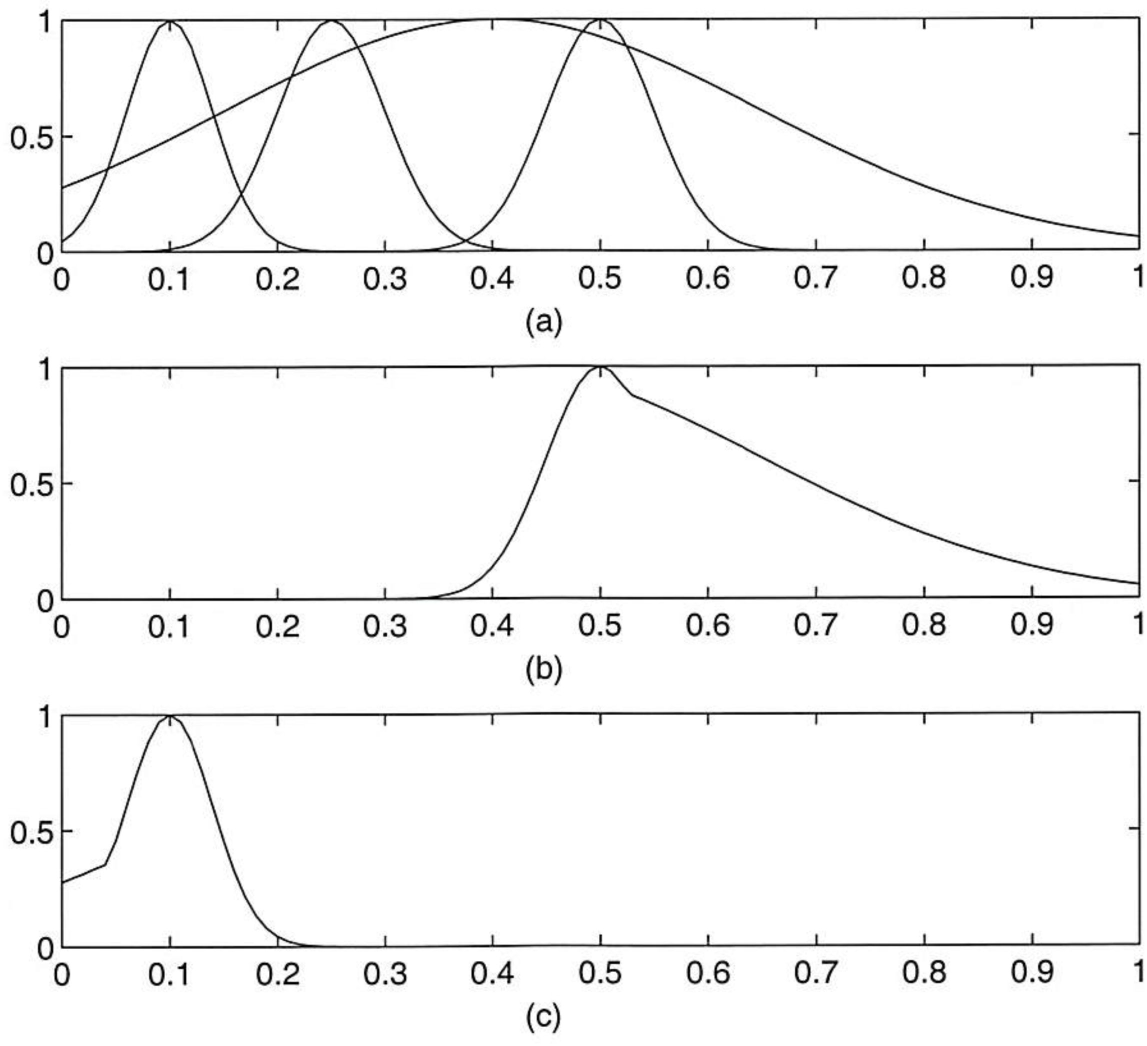



Figure 2.2: An illustration of Theorem 2.1, part (b), for Gaussians. (a) Participating Gaussians; (b) *join*; and (c) *meet*.

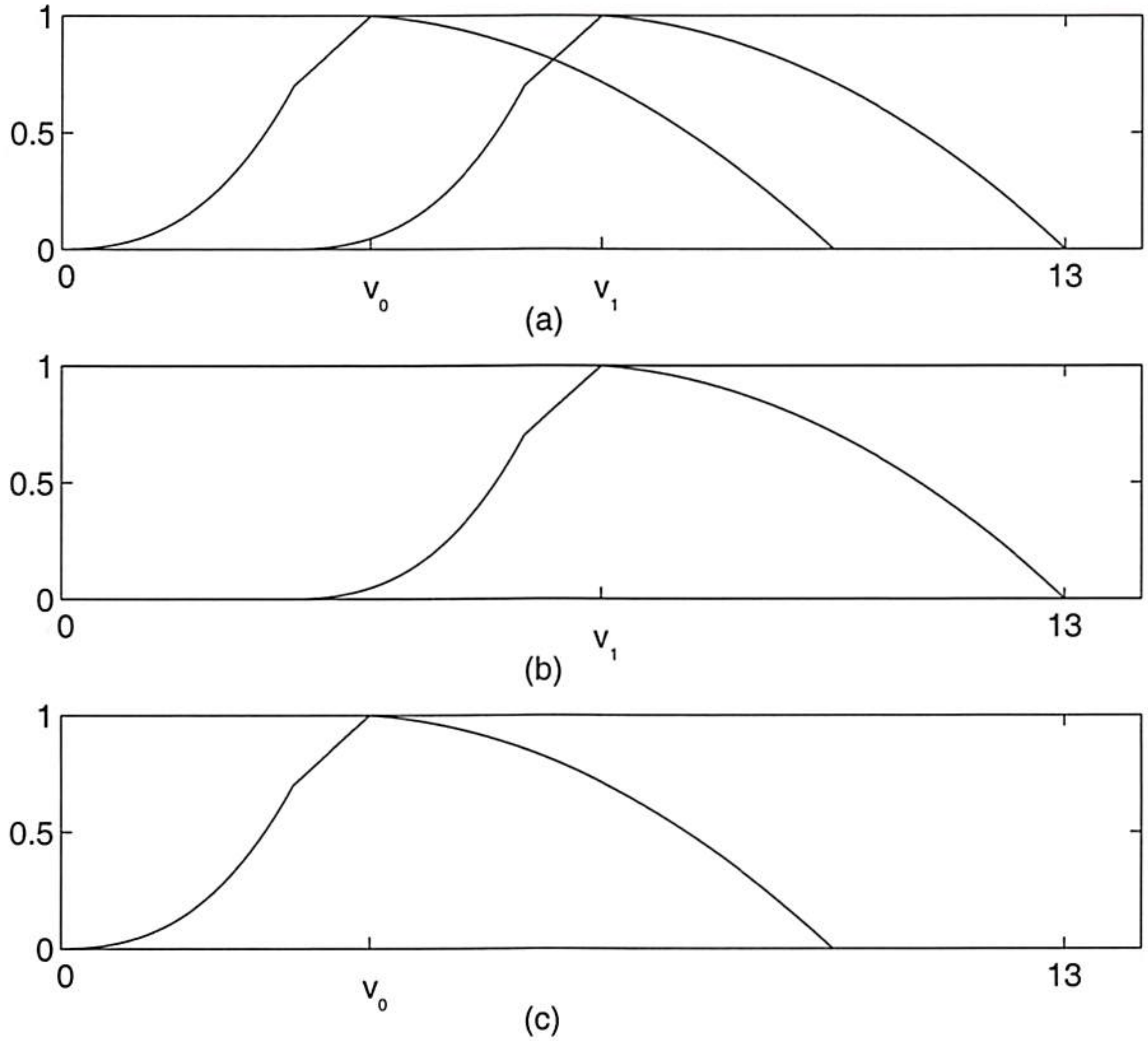



Figure 2.3: An illustration of Corollary 2.1. (a) Convex type-1 sets $\tilde{F}$ and $\tilde{G}$. The membership functions of $\tilde{F}$ and $\tilde{G}$ are shifted versions of each other. (b) $\tilde{F} \sqcup \tilde{G} = \tilde{G}$. (c) $\tilde{F} \sqcap \tilde{G} = \tilde{F}$.

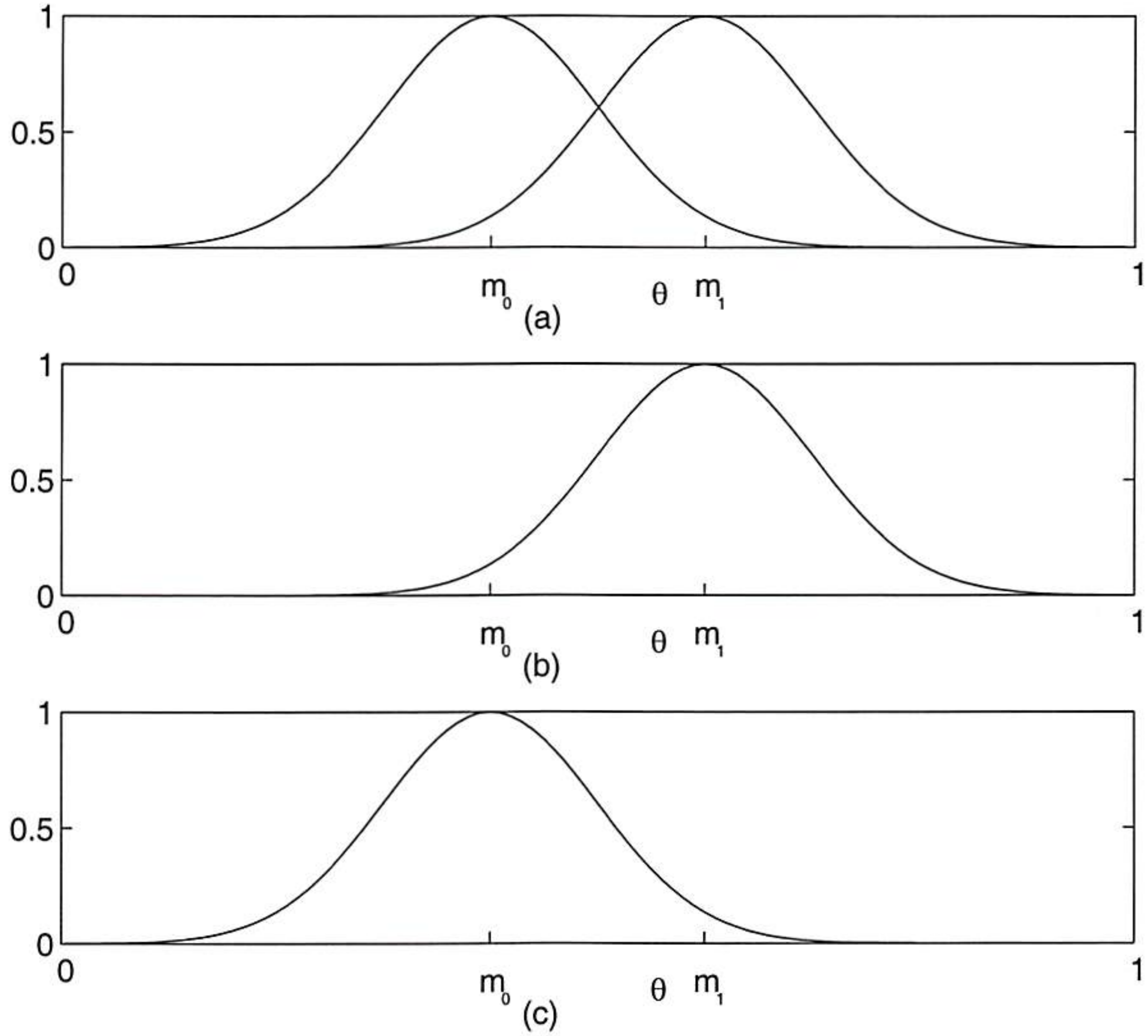



Figure 2.4: An illustration of Corollary 2.1 for the Gaussian case. (a) Participating Gaussians; (b) *join* is the Gaussian with larger mean; and (c) *meet* is the Gaussian with smaller mean.

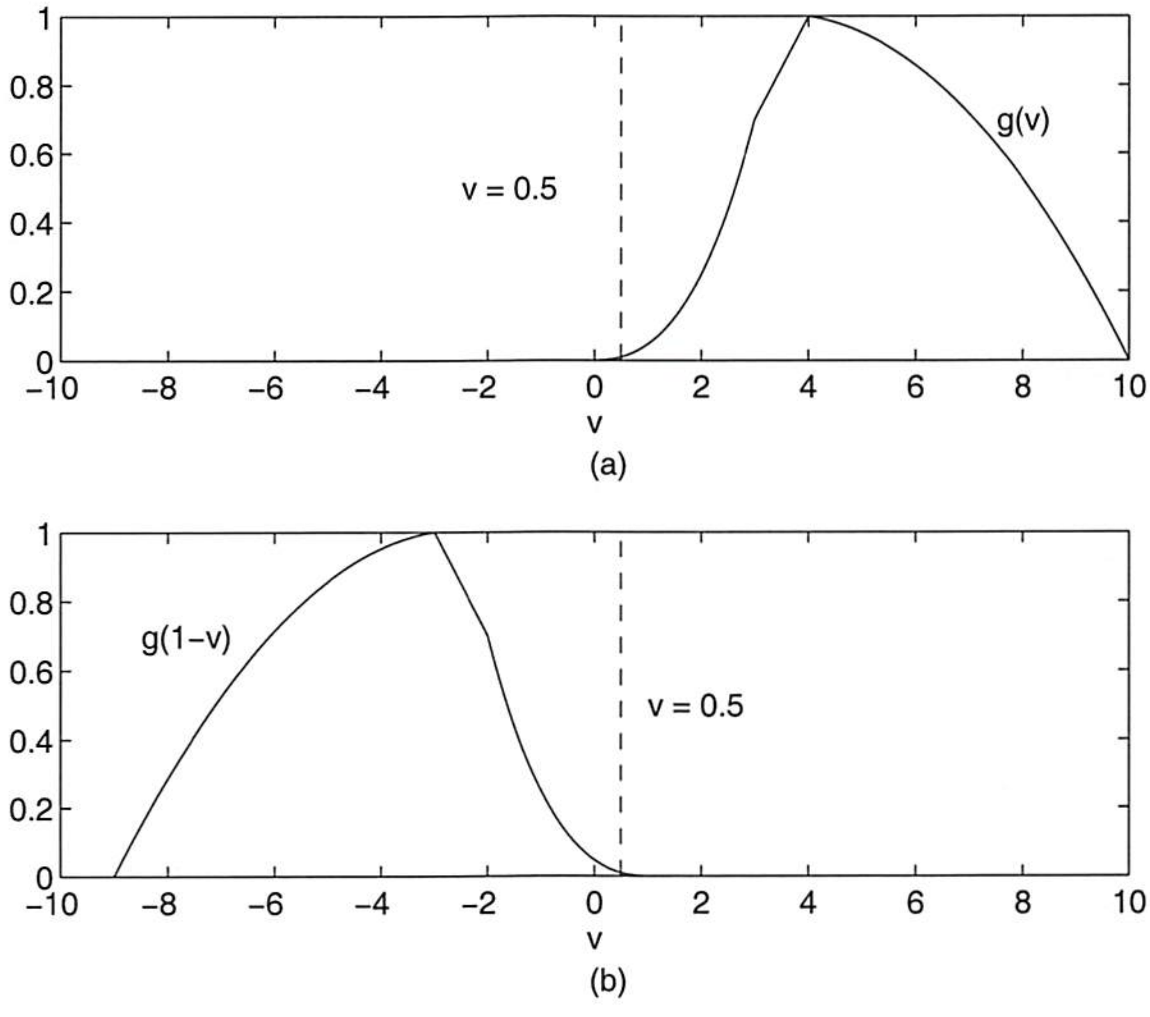



Figure 2.5: An illustration of the *negation* operation. (a) Type-1 fuzzy set $\tilde{G}$, (b) $\neg\tilde{G}$.

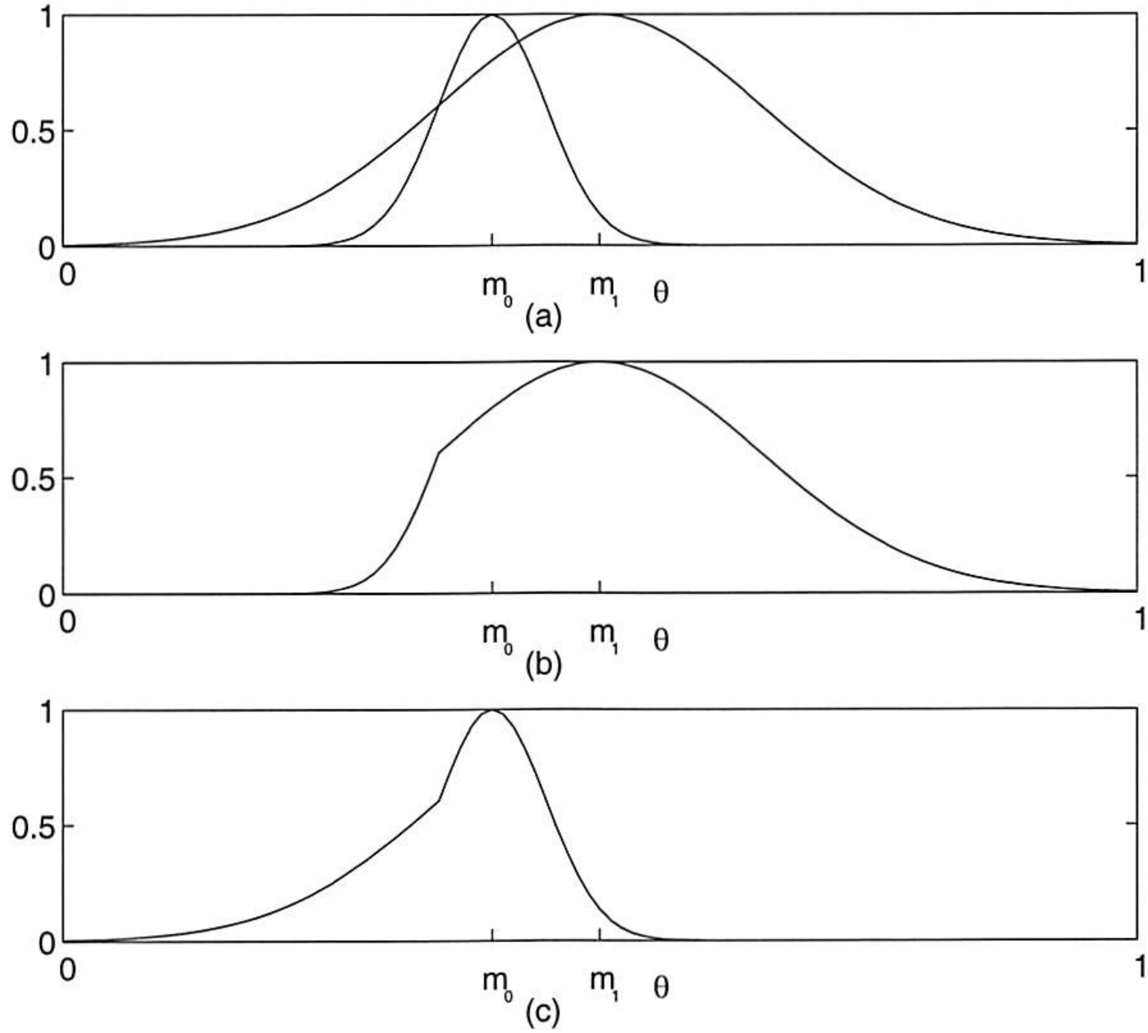



Figure 2.6: *Join* and *meet* operations between Gaussians under *min* $t$-norm. (a) Participating Gaussians; (b) *join*; and (c) *meet*.

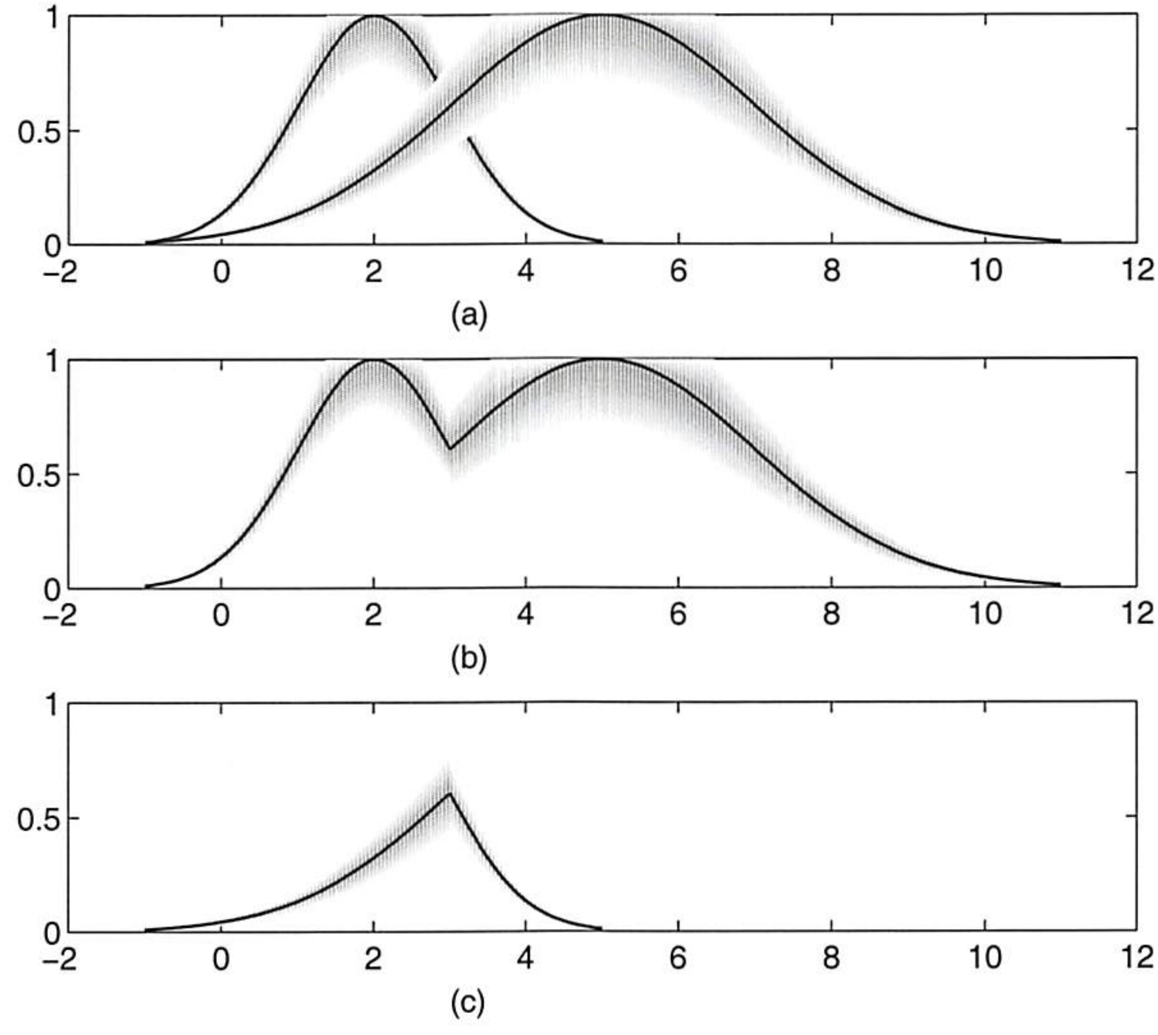



Figure 2.7: Union and Intersection of Gaussian type-2 fuzzy sets using the 2-D pictorial representation introduced in Fig. 1.10. (a) Participating sets; (b) union ; and (c) intersection.

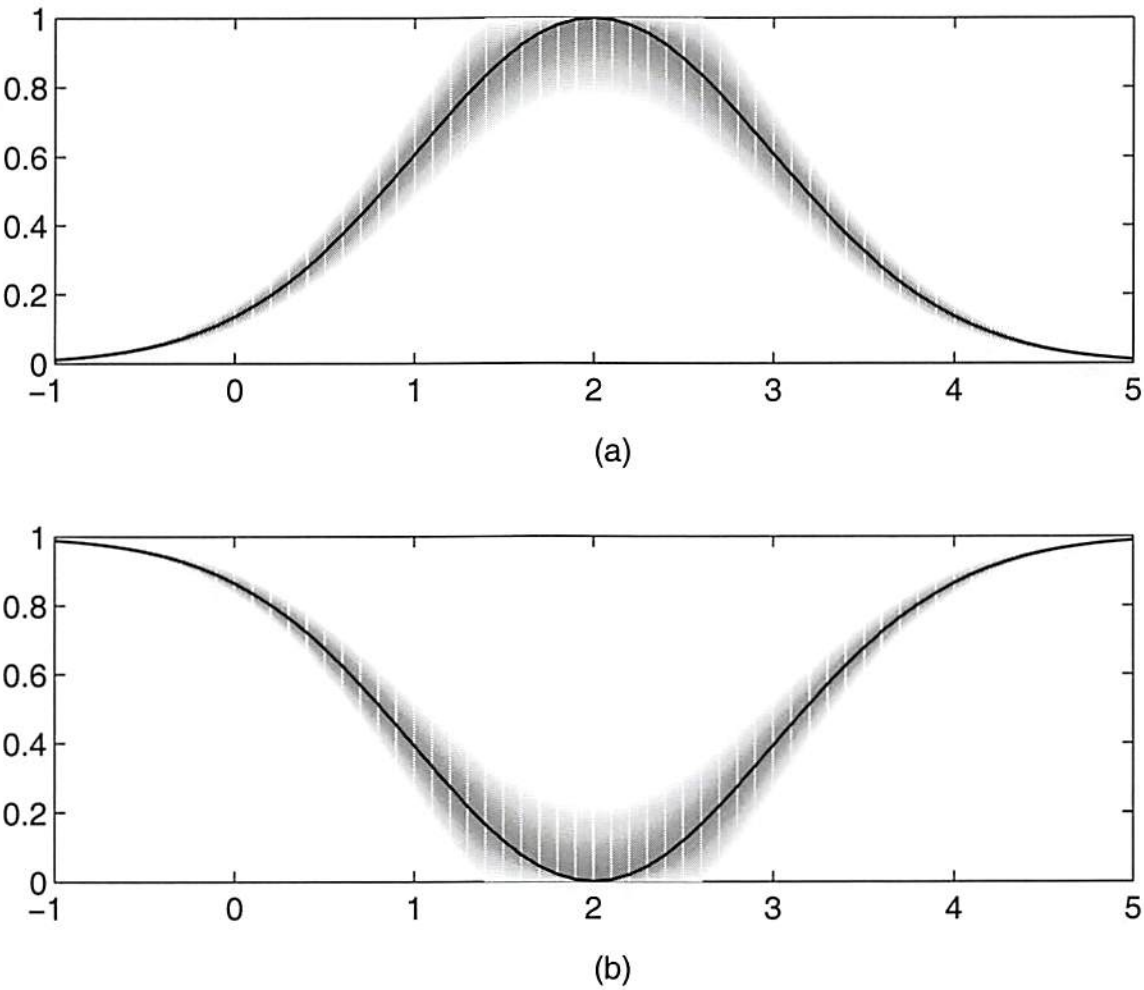



Figure 2.8: Complement of a Gaussian type-2 fuzzy set using the 2-D pictorial representation introduced in Fig. 1.10. (a) Gaussian type-2 set; (b)complement.

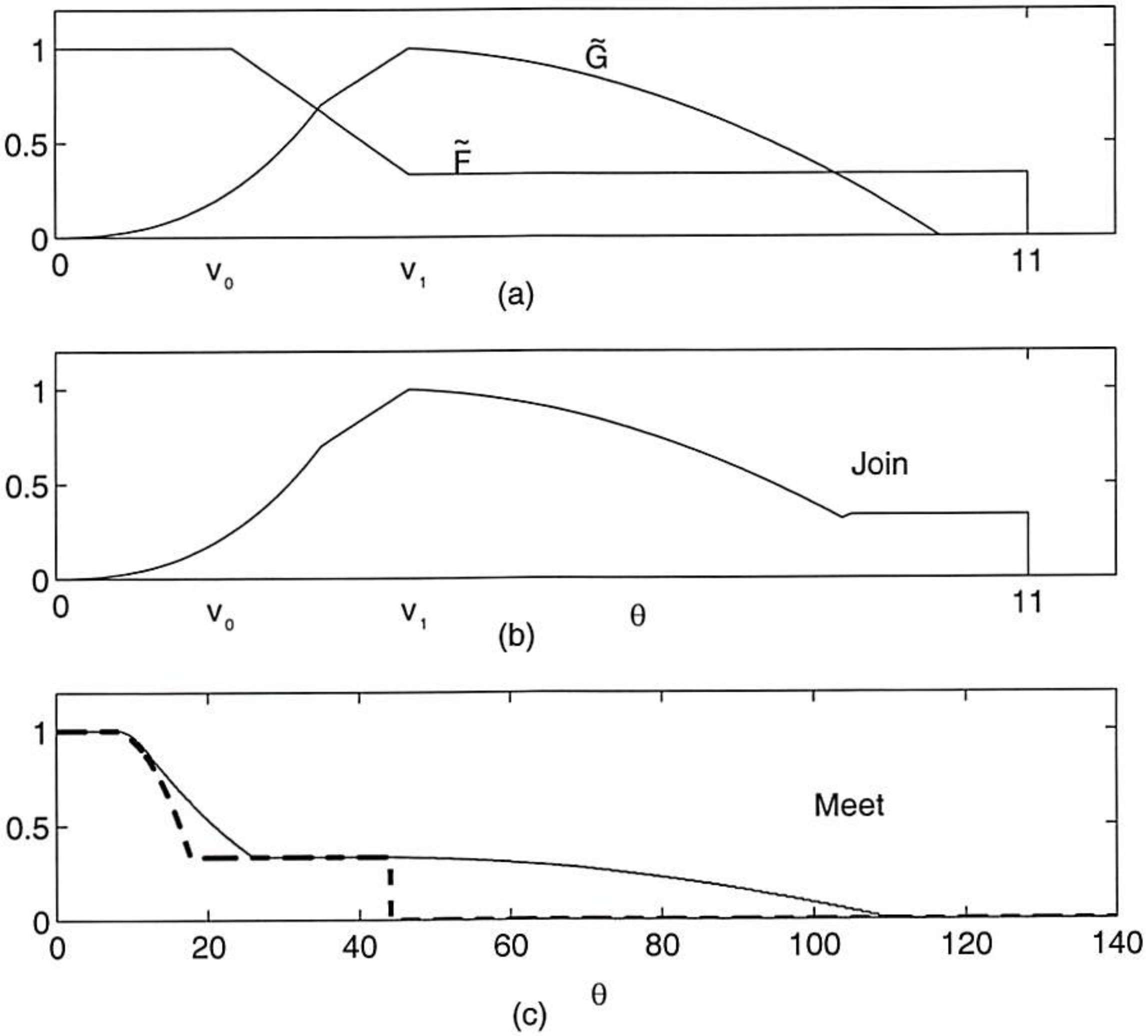



Figure 2.9: *Join* and *meet* operations under *product* $t$-norm. (a) Participating type-1 fuzzy sets; (b) *join* ; (c) *meet* : the actual result is shown with the thin solid line and the approximation in (2.39) with the thick dashed line.

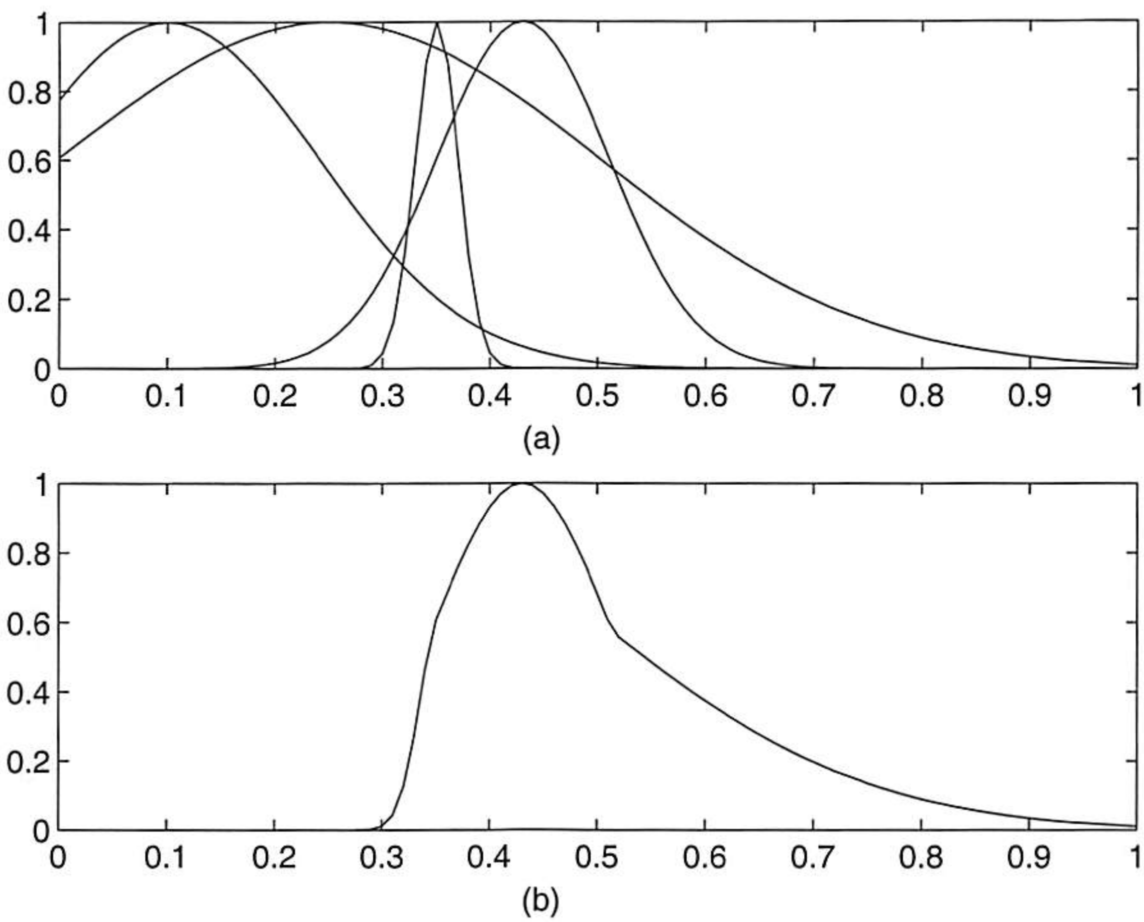



Figure 2.10: An illustration of (2.23) for the Gaussian case. (a) Participating Gaussians; (b) *join* under product $t$-norm.

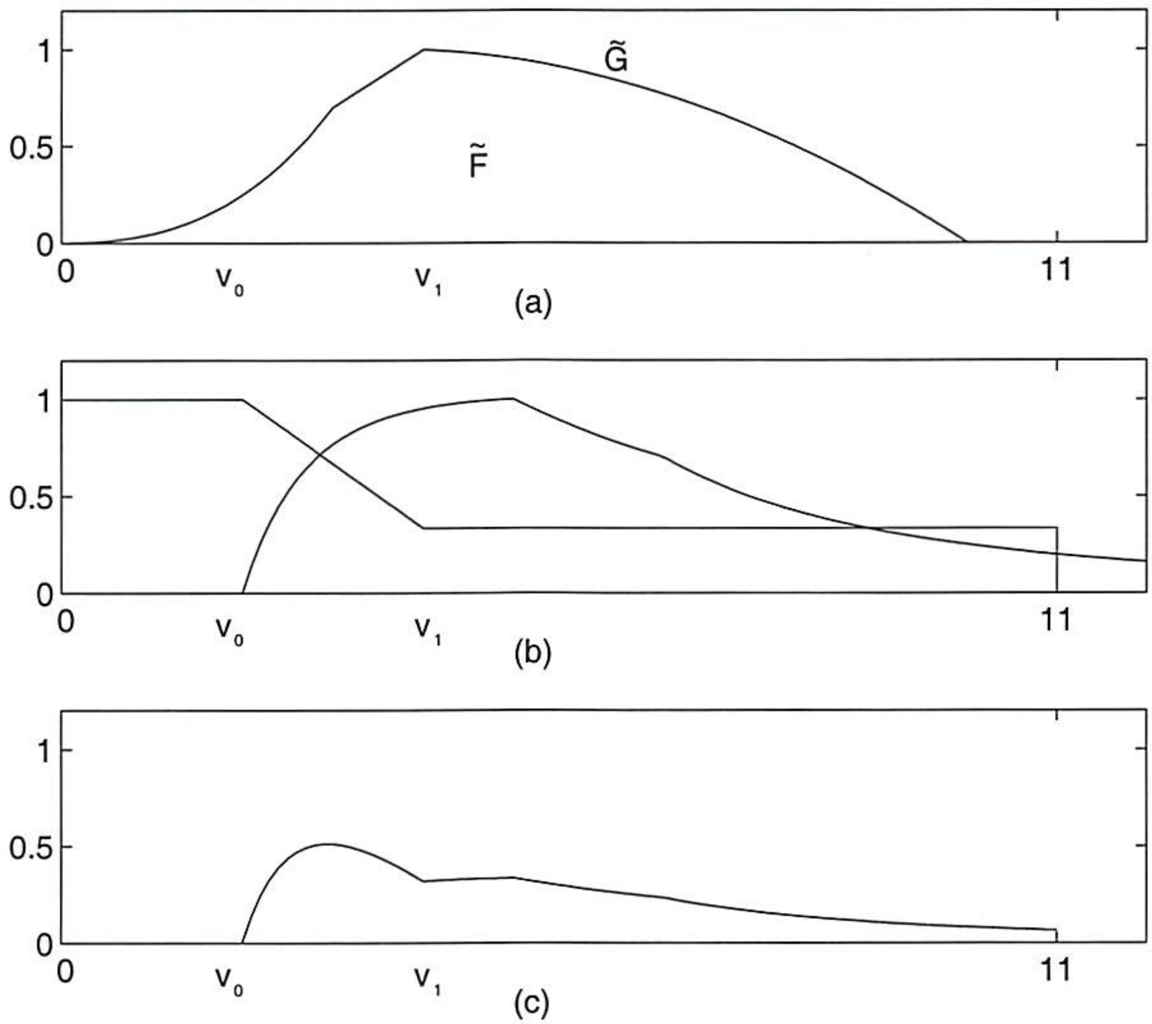



Figure 2.11: An example showing how $f(v)g(\frac{20}{v})$ looks for the curves we considered in the proof of Theorem 2.1. (a) The membership functions $f$ and $g$ of type-1 sets $\tilde{F}$ and $\tilde{G}$, respectively. (b) $f(v)$, which is the same as that in Fig. (a) and $g(\frac{20}{v})$. (c) The product $f(v)g(\frac{20}{v})$.

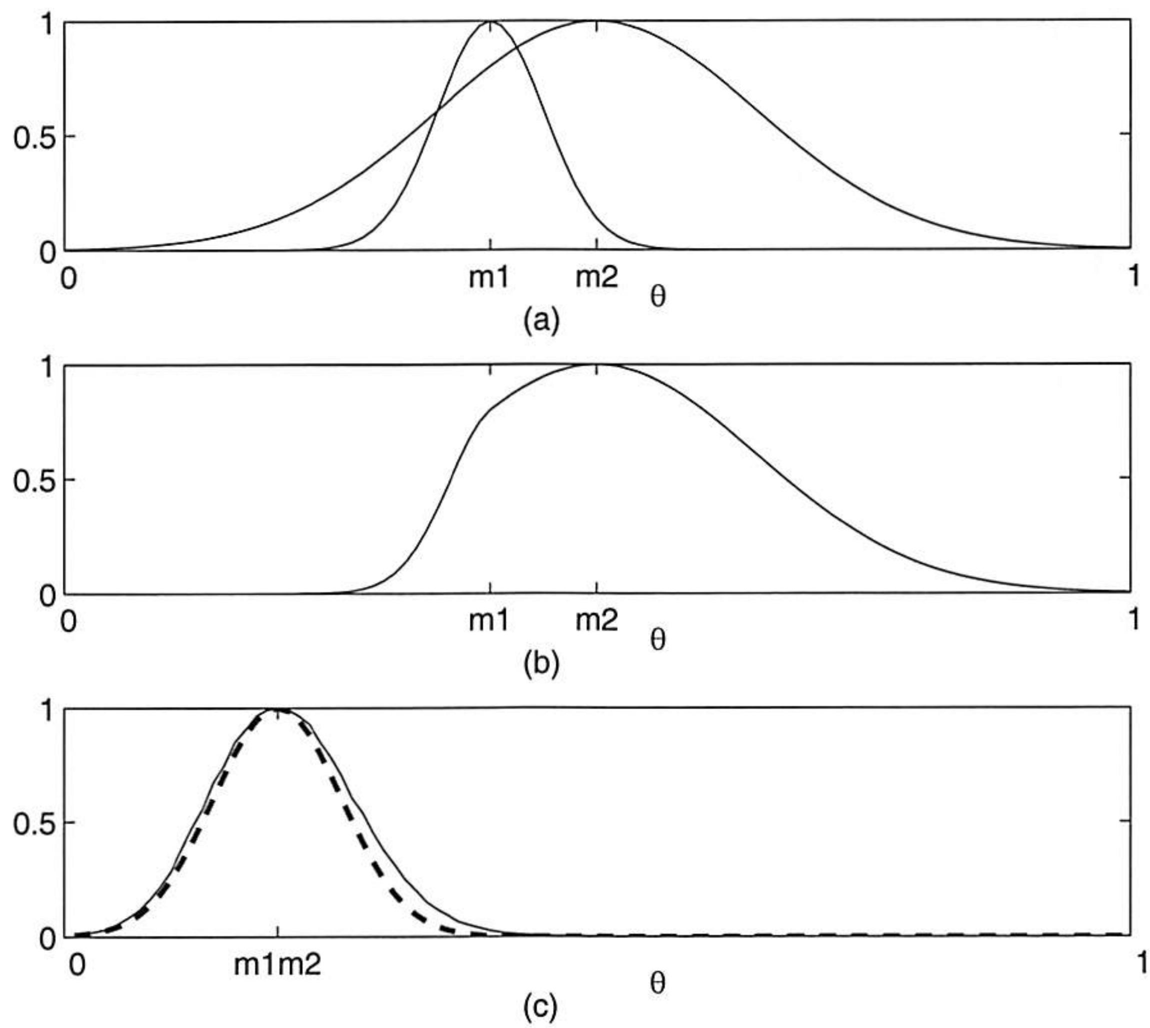



Figure 2.12: *Join* and *meet* operations between Gaussians under *product* $t$-norm. (a) Participating Gaussians; (b) *join*; and (c) *meet* : the thin solid line depicts the actual result and the thick dashed line shows the approximation in (2.44). Compare these results with those in Fig. 2.6 obtained using the *min* $t$-norm.

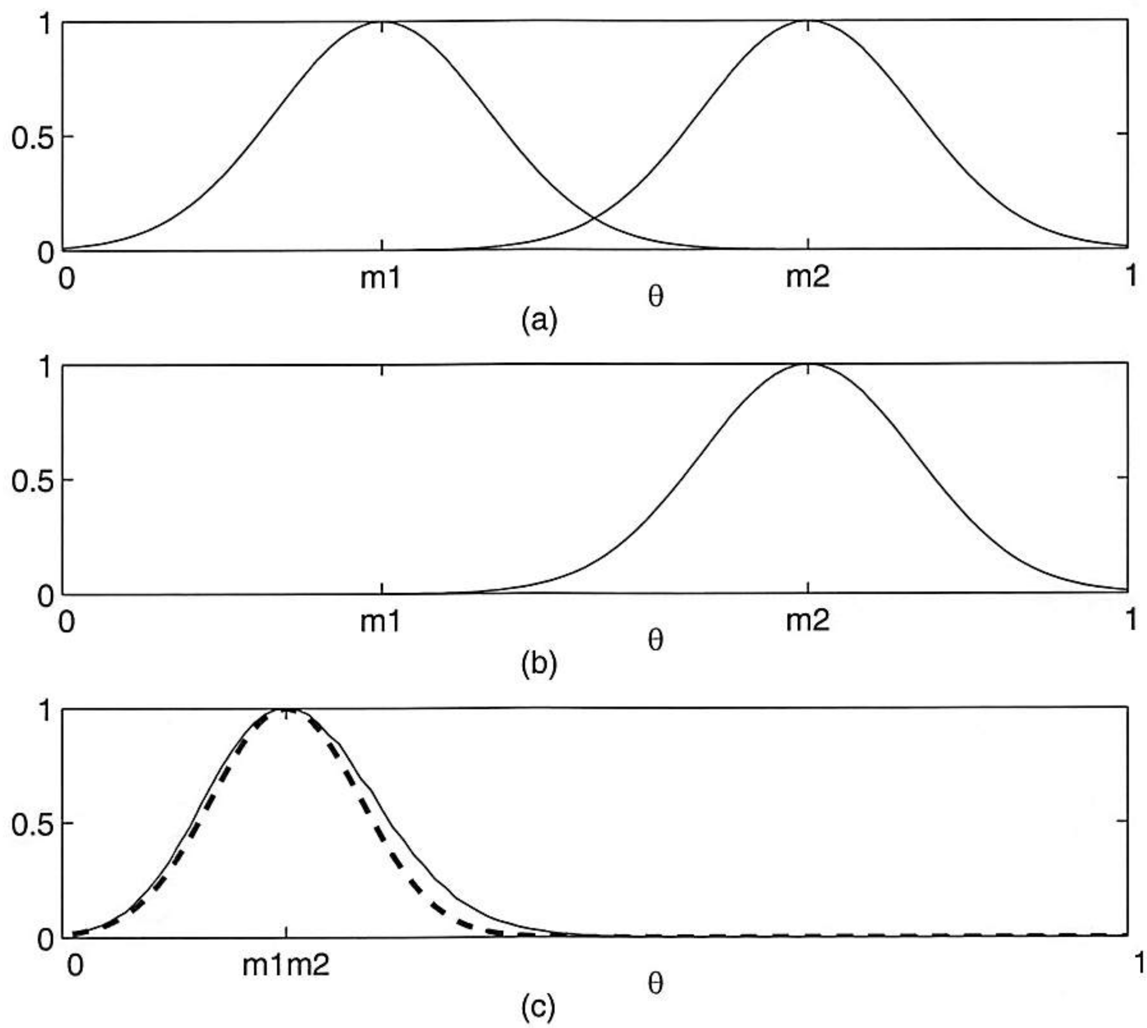



Figure 2.13: *Join* and *meet* operations between Gaussians, having *the same standard deviation*, under *product* $t$-norm. (a) Participating Gaussians; (b) *join*; and (c) *meet* : the thin solid line depicts the actual result and the thick dashed line shows the approximation in (2.44).

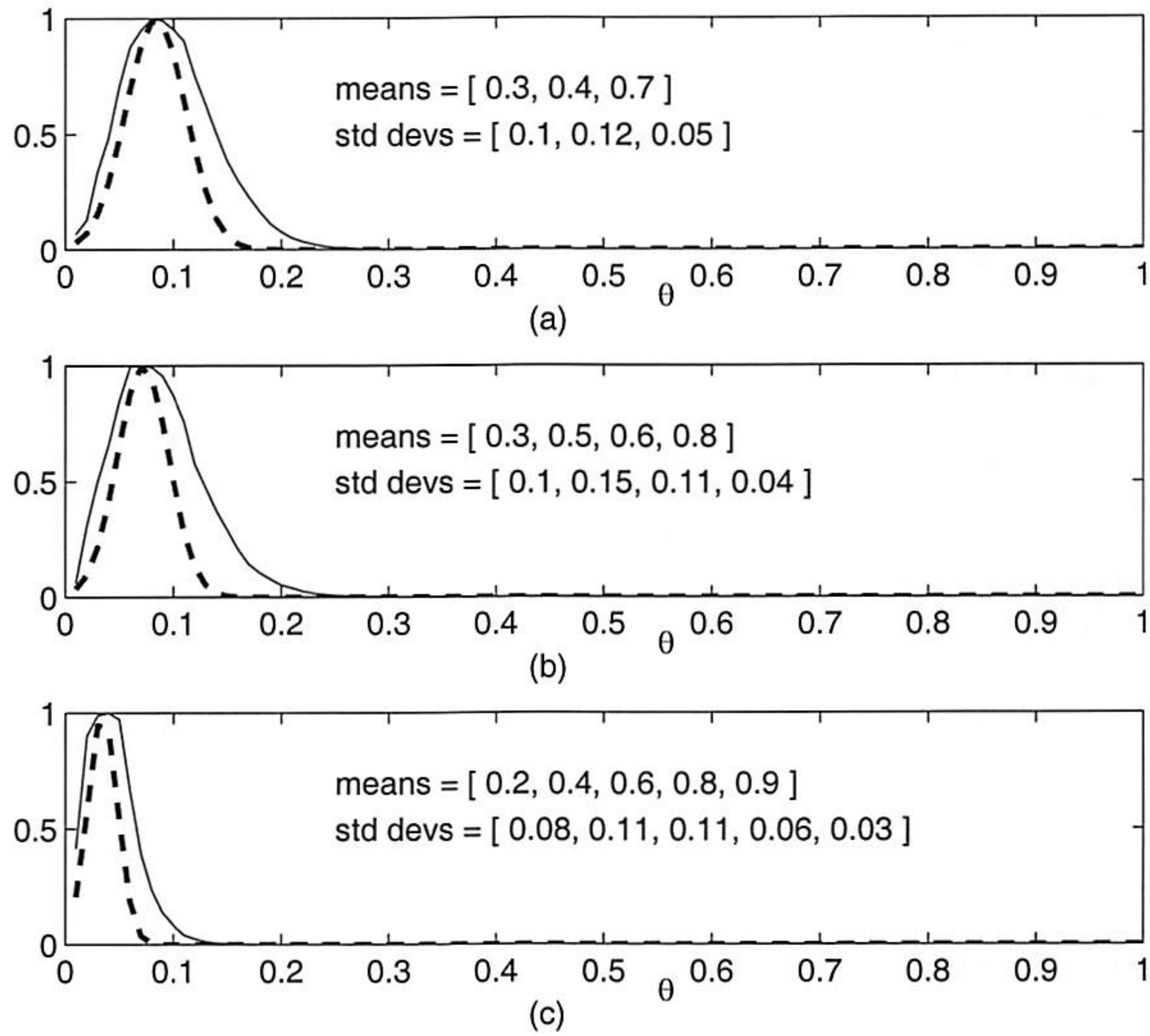



Figure 2.14: Examples of *meet* of more than two Gaussians at a time for *product* $t$-norm. The approximation in (2.46) is shown with the thick dashed line. The thin solid line shows the actual result, which was calculated numerically.

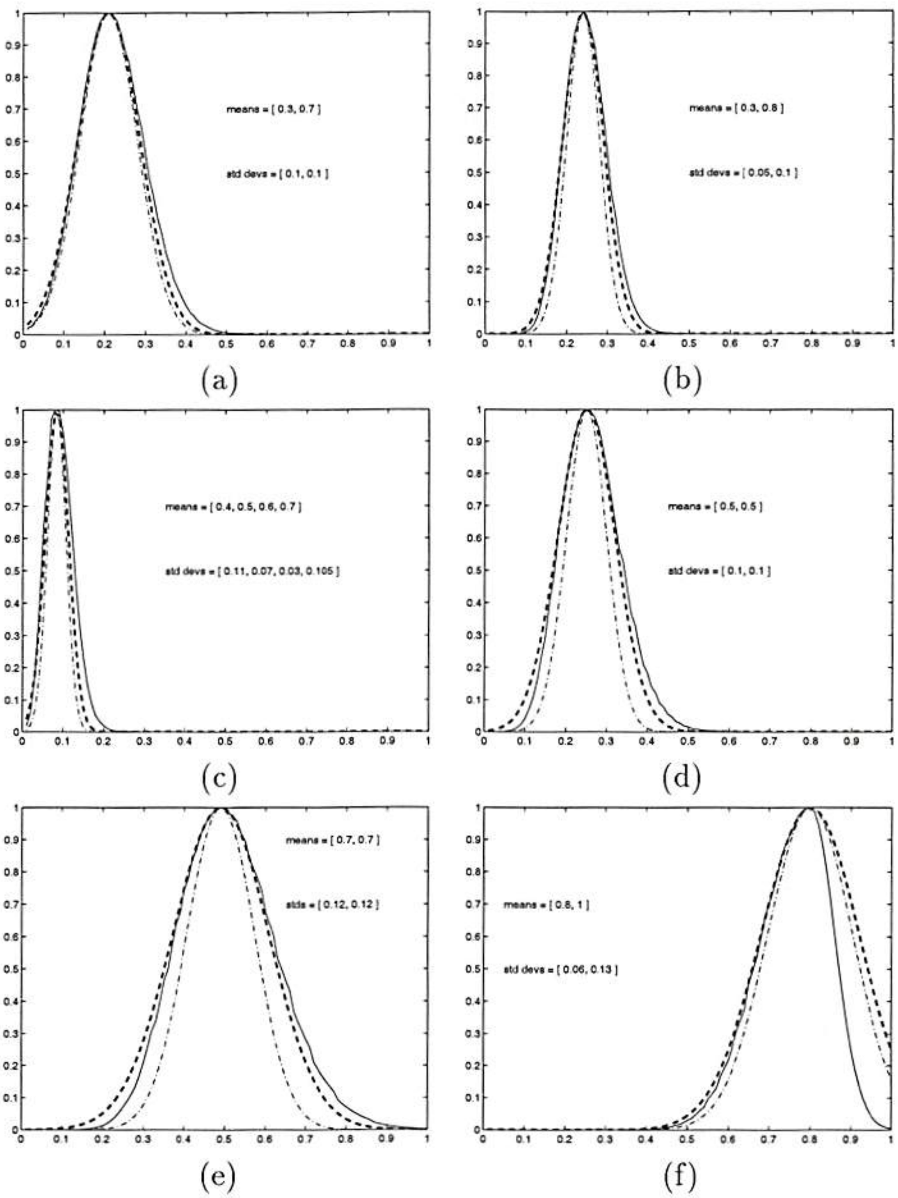



Figure 2.15: Actual and approximate results of the *meet* operation between Gaussians under product $t$-norm. The thin solid line shows the actual result computed numerically. The thin dash-dotted line shows the first approximation in (2.46). The approximation in (2.59) is shown by the thick dashed line. Means and standard deviations of the Gaussians are as indicated in the figure. In Figs. (d) and (e), the two Gaussians are coincident (the same curve). The first approximation does very poorly in this case. In Fig. (f), observe the difference between the approximation and the actual curve on the RHS of the mean. This is due to the fact that one of the Gaussians is centered at 1, i.e., only half of it lies in [0, 1]. This clipping effect is discussed in Appendix C.8.2.

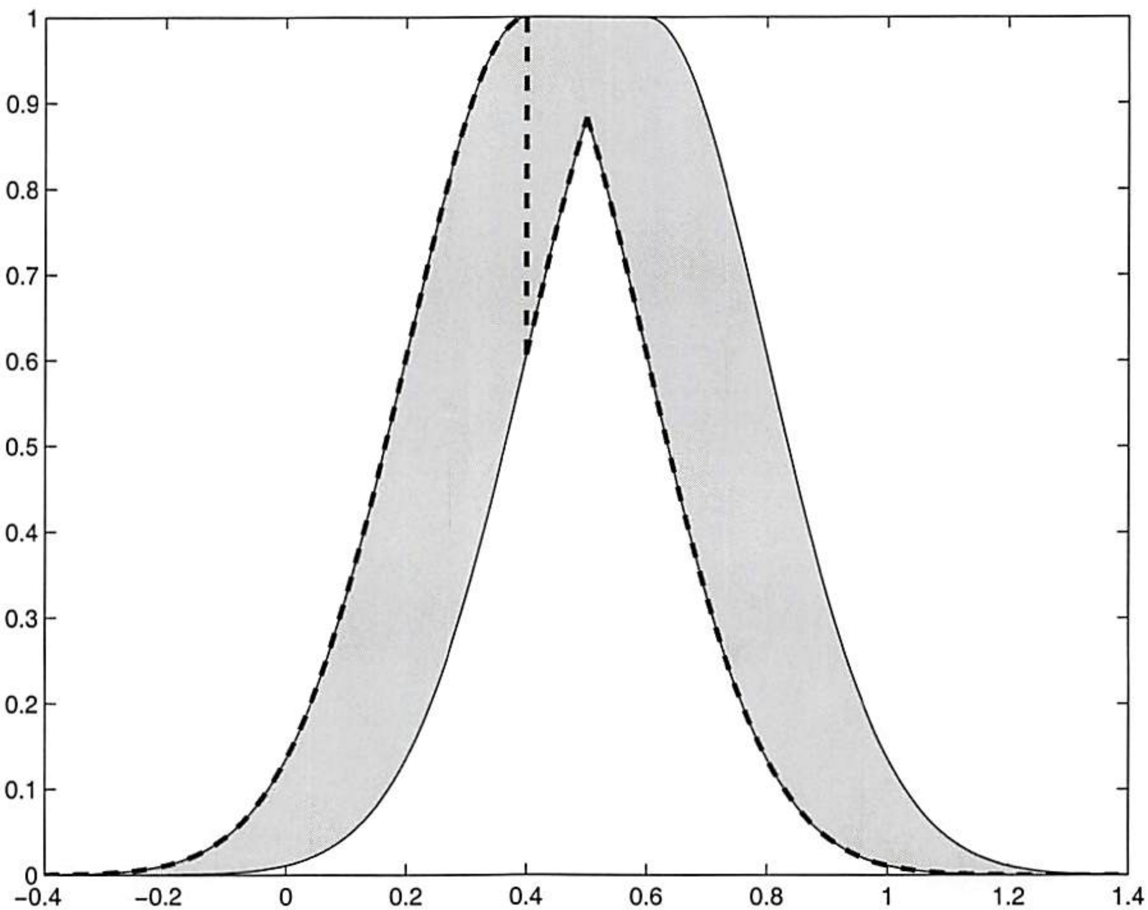



Figure 2.16: Example of a type-1 set, shown with the thick dashed line, embedded in a type-2 set.

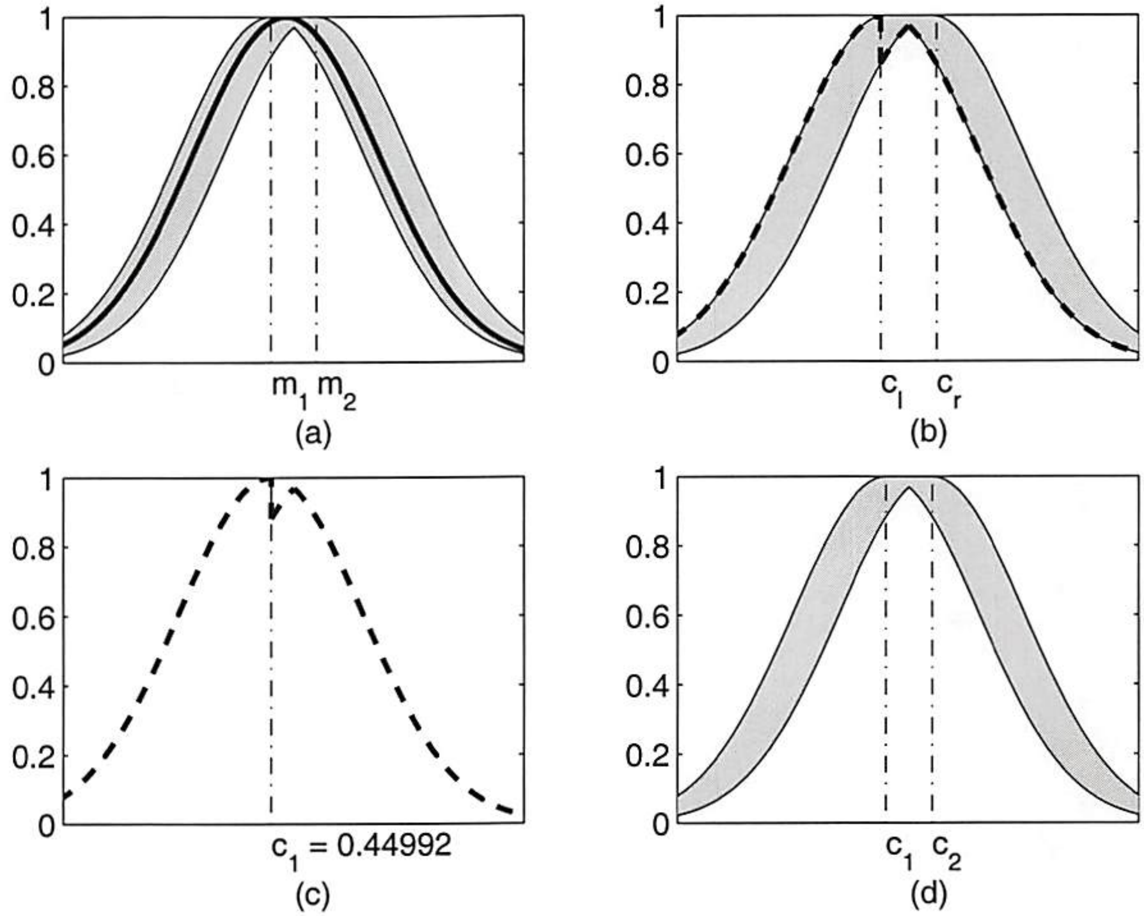



Figure 2.17: Figures for Example 2.4. (a) An interval type-2 set $\tilde{\tilde{A}}$ resulting from a Gaussian type-1 set with standard deviation equal to 0.2 and mean uniformly uncertain in the interval $[m_1, m_2] = [0.45, 0.55]$. The thick line shows an embedded Gaussian type-1 set. (b) The embedded type-1 set whose centroid equals $c_l$ is shown with a thick dashed line. (c) The type-1 set formed by assigning highest possible memberships to the points to the left of $m_1$ and lowest possible memberships to the points to the right of $m_2$. The centroid of this set is $c_1 = 0.44992 \approx c_l$. (d) The centroid of $\tilde{\tilde{A}}$ is the interval $[c_l, c_r] \approx [c_1, c_2] \approx [m_1, m_2]$.

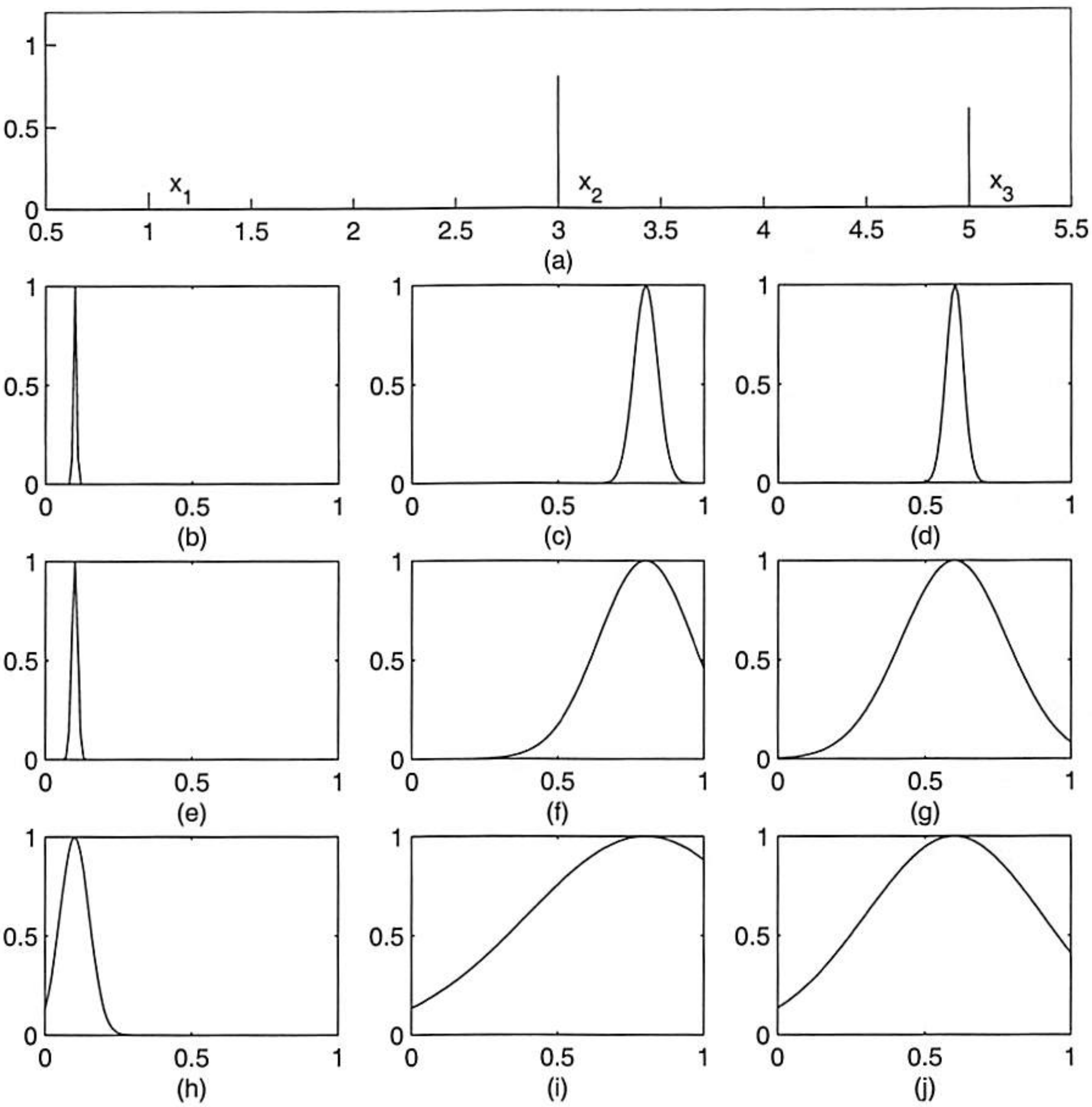



Figure 2.18: Figures for Example 2.6. The domain of the discrete Gaussian type-2 set having 3 points, $x_1 = 1$, $x_2 = 3$ and $x_3 = 5$, is depicted in (a). The membership grades of $x_1$, $x_2$ and $x_3$ for case 1 are depicted in (b), (c) and (d), respectively ; the membership grades for case 2 are depicted in (e), (f) and (g); and, those for case 3 are depicted in (h), (i) and (j). Each of the figures (b) to (j) show plots of primary versus secondary memberships. In each case, $m(x_1) = 0.1$, $m(x_2) = 0.8$ and $m(x_3) = 0.6$. For case 1, $\sigma(x_i) = 0.05m(x_i)$ for $i = 1, 2, 3$; for case 2, $\sigma(x_1) = 0.3m(x_1)$, $\sigma(x_2) = 0.1m(x_2)$ and $\sigma(x_3) = 0.2m(x_3)$; and, for case 3, $\sigma(x_i) = 0.5m(x_i)$ for $i = 1, 2, 3$.

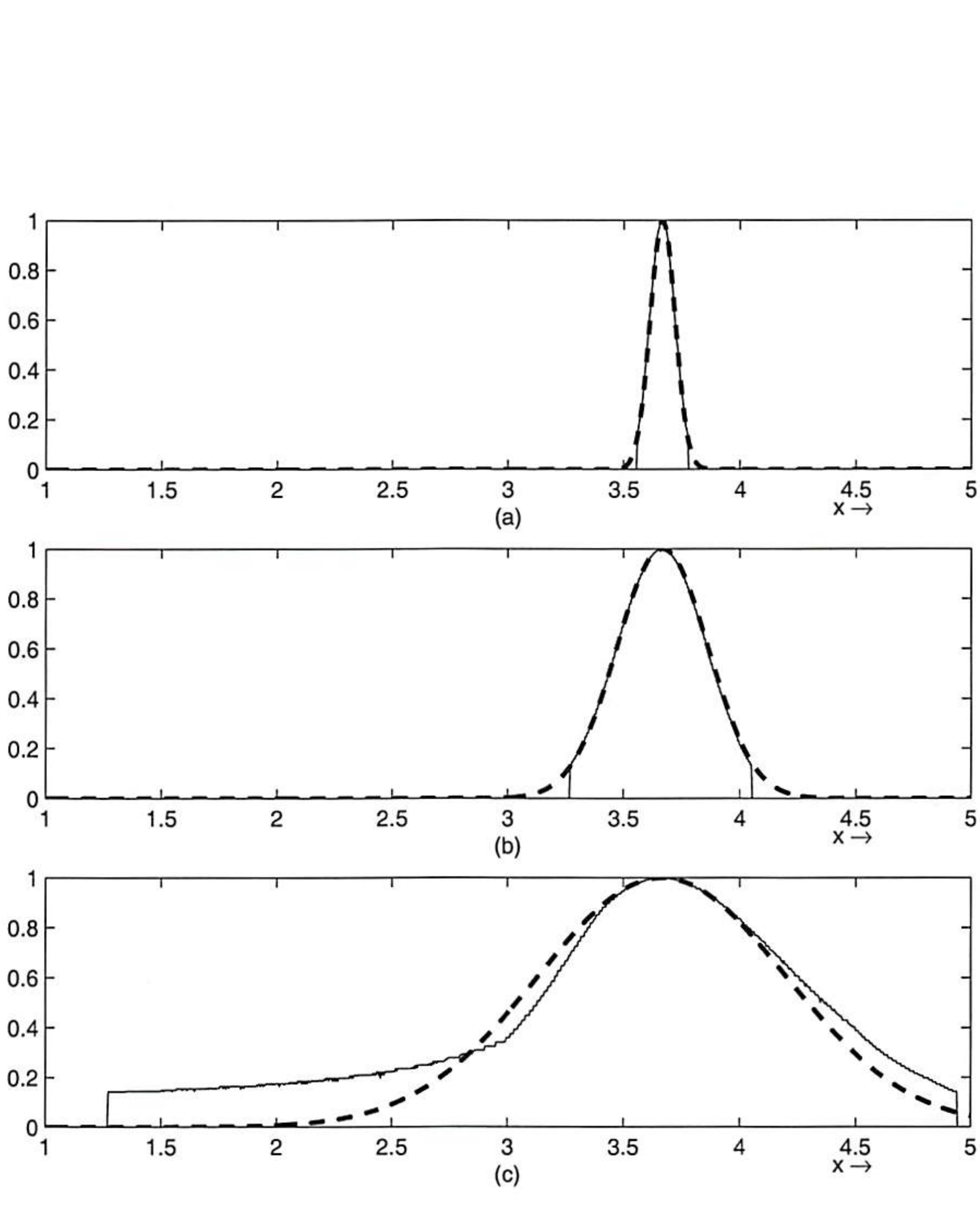



Figure 2.19: Figures for Example 2.6. Centroids of the Gaussian type-2 set depicted in Fig. 2.18 for the three choices of $\sigma(x_i)$ ($i = 1, 2, 3$). (a) $\sigma(x_i) = 0.05m(x_i)$ for $i = 1, 2, 3$. (b) $\sigma(x_1) = 0.3m(x_1)$, $\sigma(x_2) = 0.1m(x_2)$ and $\sigma(x_3) = 0.2m(x_3)$. (c) $\sigma(x_i) = 0.5m(x_i)$ for $i = 1, 2, 3$. When computing the true centroids only primary membership values between $m(x_i) \pm 2\sigma(x_i)$ are considered.

# Appendix A

# Examples for Chapter 1

In this Appendix, we provide 3-D representations for Examples 1.1 and 1.2, assuming that the standard deviation of the Gaussian in Example 1.1 and the mean of the Gaussian in Example 1.2 are Gaussian type-1 sets. We shall see that obtaining the 3-D representations in this case is fairly more complicated than obtaining the ones in Figs. 1.4 and 1.5, where the uncertain standard deviation and mean were assumed to be crisp sets.

**Example A.1** Consider Example 1.1. Suppose that the standard deviation of this Gaussian is a type-1 fuzzy set with domain $[\sigma_1, \sigma_2]$ that is characterized by a Gaussian membership function with mean $M_\sigma = \frac{\sigma_1+\sigma_2}{2}$ and standard deviation $\Sigma_\sigma = \frac{\sigma_2-\sigma_1}{4}$. These values for $M_\sigma$ and $\Sigma_\sigma$ were chosen for illustration purposes only. The membership grade for each $x$ still has the same domain as it had when all the values of the standard deviation were equally uncertain, but now, we assign secondary memberships as follows. For any $x$ (e.g., $x = 0.65$ in Fig. 1.1), if a primary membership $\mu_1 \in [0,1]$ is such that $\mu_1 = \exp\{-\frac{1}{2}\left(\frac{x-m}{\sigma'}\right)^2\}$ for some $\sigma' \in [\sigma_1, \sigma_2]$ ($\sigma_1 = 0.1$ and $\sigma_2 = 0.2$ in Fig. 1.1), then we set the secondary membership corresponding to this $x$ and $\mu_1$, $\mu_2(x, \mu_1)$, equal to the membership of $\sigma'$ in the fuzzy set $\sigma$, i.e., we set

$$\mu_2(x,\mu_1) = e^{-\frac{1}{2}\left(\frac{\sigma'-M_\sigma}{\Sigma_\sigma}\right)^2} \quad \text{where} \quad \mu_1 = e^{-\frac{1}{2}\left(\frac{x-m}{\sigma'}\right)^2} \tag{A.1}$$

In Fig. 1.1, for $x = 0.65$, this occurs for $\mu_1 \in [0.3247, 0.7548]$. If a primary membership $\mu_1 \in [0,1]$ is such that no $\sigma' \in [\sigma_1, \sigma_2]$ satisfies $\mu_1 = \exp\{-\frac{1}{2}\left(\frac{x-m}{\sigma'}\right)^2\}$, we set $\mu_2(x, \mu_1) = 0$. In Fig. 1.1, for $x = 0.65$, this occurs for $\mu_1 \notin [0.3247, 0.7548]$. Note that the above choice of $\mu_2(x,\mu_1)$ was quite arbitrary. One may choose $\mu_2(x,\mu_1)$ to be any suitable function of $\sigma'$.

Note that $\mu_1 = \exp\left\{-\frac{1}{2}\left(\frac{x-m}{\sigma'}\right)^2\right\} \Rightarrow \sigma' = |x-m|/\sqrt{-2\ln(\mu_1)}$, where we have made use of the fact that $\sigma'$, being the standard deviation of a Gaussian, is positive. Consequently, we can rewrite (A.1) explicitly in terms of $x$ and $\mu_1$ as follows : When $x \neq m$,

$$\mu_2(x,\mu_1) = \begin{cases} \exp\left\{-\frac{1}{2}\left(\frac{\frac{|x-m|}{\sqrt{-2\ln(\mu_1)}}-M_\sigma}{\Sigma_\sigma}\right)^2\right\} & ; \ \mu_1 \in \left[\exp\left\{-\frac{1}{2}\left(\frac{x-m}{\sigma_1}\right)^2\right\}, \exp\left\{-\frac{1}{2}\left(\frac{x-m}{\sigma_2}\right)^2\right\}\right] \\ 0 & ; \ \text{otherwise} \end{cases} \tag{A.2}$$

When $x = m$ ($m = 0.5$ in Fig. 1.1), every $\sigma' \in [\sigma_1, \sigma_2]$ gives $\mu_1 = \exp\{-\frac{1}{2}\left(\frac{x-m}{\sigma'}\right)^2\} = 1$. In this case, we set $\sigma'$ equal to that value of $\sigma$ which maximizes $\mu_2(x,\mu_1)|_{(m,1)}$, i.e., we set $\sigma' = M_\sigma$; consequently, $\mu_2(m,1) = 1$ and $\mu_2(m,\mu_1) = 0$ for $\mu_1 \neq 1$.

The membership grade in (A.2) is depicted in Fig. A.1 (b). Figure A.1 (a) shows a 3-D representation of this type-2 set. Observe, from (A.2), that the membership grade corresponding to any $x$ (i.e., $\mu_1 - \mu_2$ plot for a fixed $x$) is generally non-Gaussian. Each of the slices in the 3-D plot was constructed by evaluating (A.2) for different values of $x$. □

**Example A.2** Consider Example 1.2. Suppose that the mean of this Gaussian is a type-1 fuzzy set with domain $[m_1, m_2]$ that is characterized by a Gaussian membership function with mean $M_m = \frac{m_1+m_2}{2}$ and standard deviation $\Sigma_m = \frac{m_2-m_1}{4}$. Figure A.2 (a) shows 3-D diagrams for Example 1.2, when $m_1 = 0.4$ and $m_2 = 0.6$. The secondary memberships are computed as follows.

For any $x$ (e.g., $x = 0.65$ in Fig. 1.2), if a primary membership $\mu_1 \in [0,1]$ is such that $\mu_1 = \exp\{-\frac{1}{2}\left(\frac{x-m'}{\sigma}\right)^2\}$ for some $m' \in [m_1, m_2]$ ($m_1 = 0.4$ and $m_2 = 0.6$ in Fig. 1.2), then the corresponding secondary membership $\mu_2(x,\mu_1)$ is set equal to the membership of $m'$ in the type-1 fuzzy set $m$, i.e., we set

$$\mu_2(x,\mu_1) = e^{-\frac{1}{2}\left(\frac{m'-M_m}{\Sigma_m}\right)^2} \quad \text{where} \quad \mu_1 = e^{-\frac{1}{2}\left(\frac{x-m'}{\sigma}\right)^2} \tag{A.3}$$

In Fig. 1.2, for $x = 0.65$, this occurs for $\mu_1 \in [0.4578, 0.9692]$. If $\mu_1$ is such that no $m \in [m_1, m_2]$ satisfies $\mu_1 = \exp\{-\frac{1}{2}\left(\frac{x-m'}{\sigma}\right)^2\}$, we set $\mu_2(x, \mu_1) = 0$. In Fig. 1.2, for $x = 0.65$, this occurs for $\mu_1 \notin [0.4578, 0.9692]$. Observe also, from Fig. 1.2, that in the interval $[m_1, m_2]$, there may be more than one value of $m'$ which satisfies (A.3). In this case, we choose that value for $m'$ which maximizes $\mu_2(x, \mu_1)$. In Fig. 1.2, this occurs for $x \in [0.4, 0.6]$.

Note that $\mu_1 = \exp\left\{-\frac{1}{2}\left(\frac{x-m'}{\sigma}\right)^2\right\}$ implies that

$$m' = \begin{cases} x + \sigma\sqrt{-2\ln(\mu_1)} & ; \quad x < m_1 \\ x \pm \sigma\sqrt{-2\ln(\mu_1)} & ; \quad m_1 \leq x \leq m_2 \\ x - \sigma\sqrt{-2\ln(\mu_1)} & ; \quad x > m_2 \end{cases} \tag{A.4}$$

Consequently, using (A.4), (A.3) can be rewritten as follows (see Fig. 1.2) : For $x < m_1$,

$$\mu_2(x, \mu_1) = \begin{cases} e^{-\frac{1}{2}\left(\frac{x+\sigma\sqrt{-2\ln(\mu_1)}-M_m}{\Sigma_m}\right)^2} & ; \quad \mu_1 \in [\mu_1^2, \mu_1^1] \\ 0 & ; \quad \text{otherwise} \end{cases} \tag{A.5}$$

For $m_1 \leq x \leq (m_1 + m_2)/2$,

$$\mu_2(x, \mu_1) = \begin{cases} e^{-\frac{1}{2}\left(\frac{x+\sigma\sqrt{-2\ln(\mu_1)}-M_m}{\Sigma_m}\right)^2} & ; \quad \mu_1 \in [\mu_1^2, \mu_1^1] \\ \max\left\{e^{-\frac{1}{2}\left(\frac{x+\sigma\sqrt{-2\ln(\mu_1)}-M_m}{\Sigma_m}\right)^2}, e^{-\frac{1}{2}\left(\frac{x-\sigma\sqrt{-2\ln(\mu_1)}-M_m}{\Sigma_m}\right)^2}\right\} & ; \quad \mu_1 \in [\mu_1^1, 1] \\ 0 & ; \quad \text{otherwise} \end{cases} \tag{A.6}$$

For $(m_1 + m_2)/2 \leq x \leq m_2$,

$$\mu_2(x, \mu_1) = \begin{cases} e^{-\frac{1}{2}\left(\frac{x-\sigma\sqrt{-2\ln(\mu_1)}-M_m}{\Sigma_m}\right)^2} & ; \quad \mu_1 \in [\mu_1^1, \mu_1^2] \\ \max\left\{e^{-\frac{1}{2}\left(\frac{x+\sigma\sqrt{-2\ln(\mu_1)}-M_m}{\Sigma_m}\right)^2}, e^{-\frac{1}{2}\left(\frac{x-\sigma\sqrt{-2\ln(\mu_1)}-M_m}{\Sigma_m}\right)^2}\right\} & ; \quad \mu_1 \in [\mu_1^2, 1] \\ 0 & ; \quad \text{otherwise} \end{cases} \tag{A.7}$$

For $x > m_2$,

$$\mu_2(x, \mu_1) = \begin{cases} e^{-\frac{1}{2}\left(\frac{x-\sigma\sqrt{-2\ln(\mu_1)}-M_m}{\Sigma_m}\right)^2} & ; \quad \mu_1 \in [\mu_1^1, \mu_1^2] \\ 0 & ; \quad \text{otherwise} \end{cases} \tag{A.8}$$

where

$$\mu_1^1 = e^{-\frac{1}{2}\left(\frac{x-m_1}{\sigma}\right)^2} \tag{A.9}$$

and

$$\mu_1^2 = e^{-\frac{1}{2}\left(\frac{x-m_2}{\sigma}\right)^2} \tag{A.10}$$

Figure A.2 (b) shows the membership grade corresponding to $x = 0.65$. Observe, from (A.5) - (A.8), that the membership grade corresponding to any $x$ is generally non-Gaussian. Each of the slices in the 3-D plot was constructed by evaluating (A.5) - (A.8) for different values of $x$. □

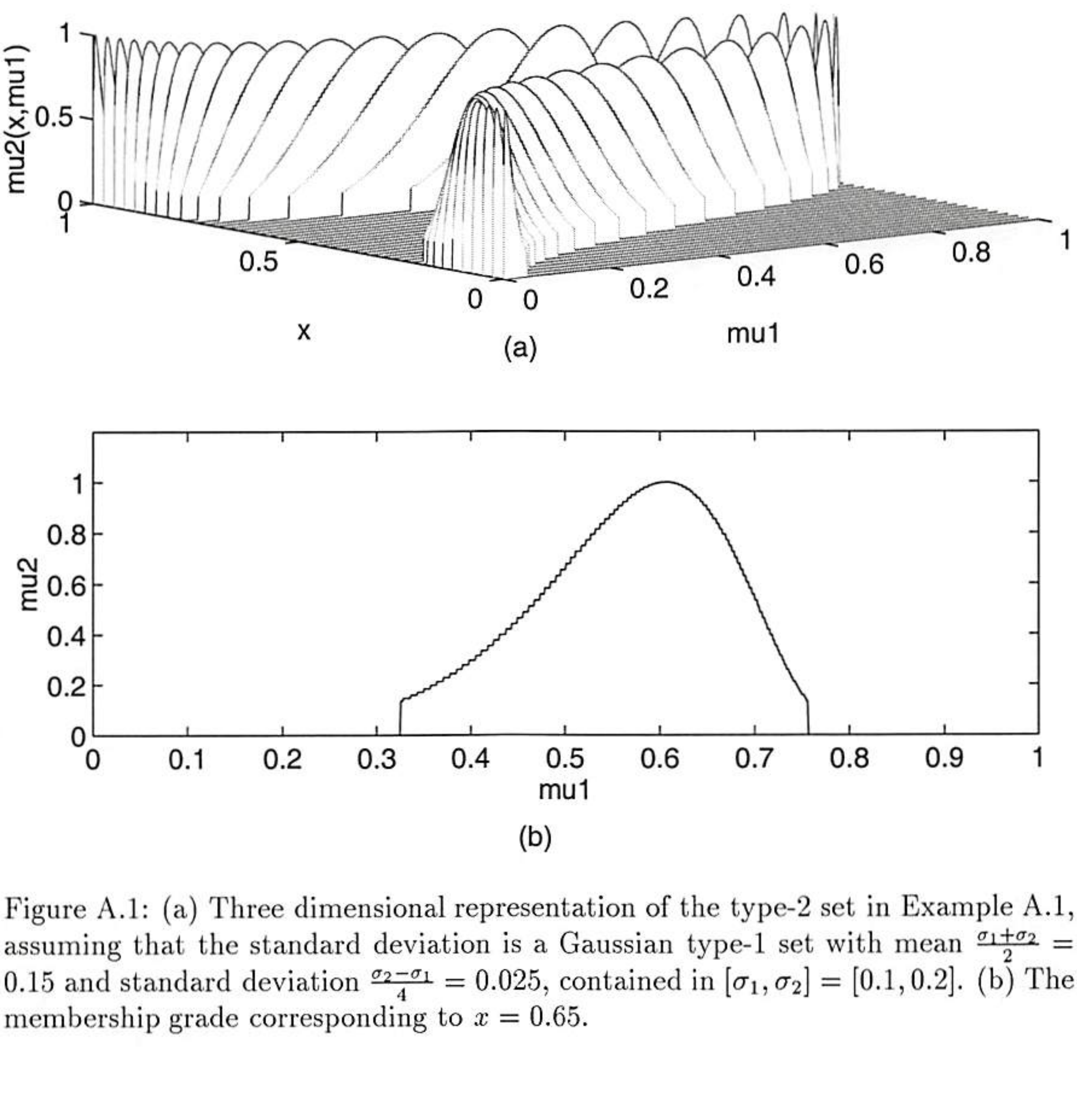
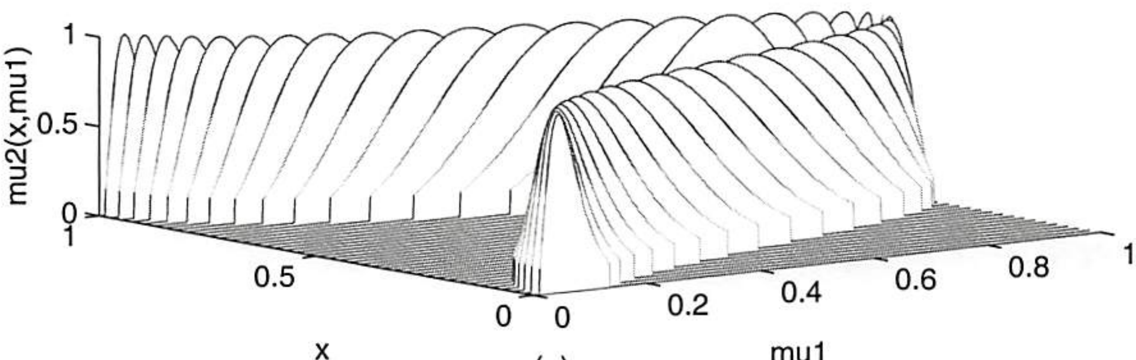
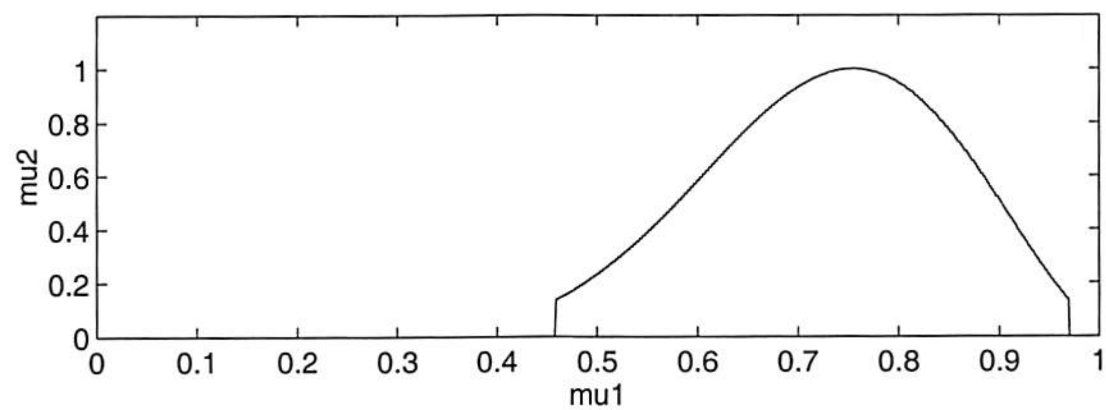



Figure A.1: (a) Three dimensional representation of the type-2 set in Example A.1, assuming that the standard deviation is a Gaussian type-1 set with mean $\frac{\sigma_1+\sigma_2}{2} = 0.15$ and standard deviation $\frac{\sigma_2-\sigma_1}{4} = 0.025$, contained in $[\sigma_1, \sigma_2] = [0.1, 0.2]$. (b) The membership grade corresponding to $x = 0.65$.




(a)




(b)

Figure A.2: (a) Three dimensional representation of the type-2 set in Example A.2, assuming that the mean is a Gaussian type-1 set with mean $\frac{m_1+m_2}{2} = 0.5$ and standard deviation $\frac{m_2-m_1}{4} = 0.05$, contained in $[m_1, m_2] = [0.4, 0.6]$. (b) The membership grade corresponding to $x = 0.65$.

# Appendix B

# A Note on the Extension Principle

The Extension Principle [9] allows the domain of definition of a mapping or a relation to be extended from points in $U$ to fuzzy subsets of $U$. If $f$ is a mapping from $U$ to $V$ and $\tilde{A}$ is a fuzzy subset of $U$, such that

$$\tilde{A} = \sum_{i=1}^{n} \mu_i / u_i \,, \tag{B.1}$$

then

$$f(\tilde{A}) = \sum_{i=1}^{n} \mu_i / f(u_i) \tag{B.2}$$

If $f$ is a mapping from a Cartesian product $U_1 \times U_2 \times \cdots \times U_n$ to $V$ and if $\tilde{A}$ is a fuzzy set (relation) in $U_1 \times U_2 \times \cdots \times U_n$ characterized by the membership function $\mu_{\tilde{A}}(u_1, \ldots, u_n)$, where $u_i \in U_i$, then

$$f(\tilde{A}) = \int_V \mu_{\tilde{A}}(u_1, \ldots, u_n) / f(u_1, \ldots, u_n) \tag{B.3}$$

Many times, we don't know $\tilde{A}$, but instead only know projections of $\tilde{A}$, $\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n$ on $U_1, U_2, \ldots, U_n$, respectively. If $\tilde{A} = \tilde{A}_1 \times \cdots \times \tilde{A}_n$, we can use the following expression for $\mu_{\tilde{A}}(u_1, \ldots, u_n)$ [Zadeh uses only the minimum $t$-norm]

$$\mu_{\tilde{A}}(u_1, \ldots, u_n) = \mu_{\tilde{A}_1}(u_1) \star \mu_{\tilde{A}_2}(u_2) \star \cdots \star \mu_{\tilde{A}_n}(u_n) \tag{B.4}$$

Let us consider the case $n = 2$, for which $*$ is a binary operation defined on $U \times V$ with values in $W$, i.e., if $u \in U$ and $v \in V$, then $w = u * v \in W$. Now, if $\tilde{A} = \sum_{i=1}^{n} \mu_i/u_i$ and $\tilde{B} = \sum_{j=1}^{m} \nu_j/v_j$ are fuzzy sets in $U$ and $V$, respectively, then

$$\begin{aligned} \tilde{A} * \tilde{B} &= \Big(\sum_{i=1}^{n} \mu_i/u_i\Big) * \Big(\sum_{j=1}^{m} \nu_j/v_j\Big) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{m} (\mu_i \star \nu_j)\Big/(u_i * v_j) \end{aligned} \tag{B.5}$$

The validity of (B.5) depends on the assumption that $u_i$ and $v_j$ are "non-interactive", or that there is no constraint on $(u_i, v_j)$ [we can think of $u_i$ and $v_j$ as being "independent" in some sense]. If there is a constraint on $(u, v)$, which is expressed as a relation $\tilde{R}$ with a membership function $\mu_{\tilde{R}}$, then the expression for $\tilde{A} * \tilde{B}$ should be written as

$$\begin{aligned} \tilde{A} * \tilde{B} &= \Big(\sum_{i=1}^{n} \mu_i/u_i\Big) * \Big(\sum_{j=1}^{m} \nu_j/v_j\Big) \cap \tilde{R} \\ &= \sum_{i=1}^{n}\sum_{j=1}^{m} [\mu_i \star \nu_j \star \mu_{\tilde{R}}(u_i, v_j)]\Big/(u_i * v_j) \end{aligned} \tag{B.6}$$

If $\tilde{R}$ is a crisp relation [i.e., if the constraint on $(u_i, v_j)$ is expressible as a crisp relation $R$], then the right-hand side of (B.6) will contain only those terms which satisfy the constraint.

**Example B.1** Let $U = 1 + \cdots + 10$ and let $\tilde{A}$ be a fuzzy subset of $U$ defined as

$$\tilde{A} = 1/1 + 0.6/4 + 0.4/5 \tag{B.7}$$

$\tilde{A}^2$ can be found in two ways. If we take $f$ as the operation of squaring, then using (B.2),

$$\tilde{A}^2 = 1/1 + 0.6/16 + 0.4/25 \tag{B.8}$$

If we write $\tilde{A}^2$ as $\tilde{A} \times \tilde{A}$, (B.5) gives us (assuming minimum $t$-norm)

$$\begin{aligned} \tilde{A} \times \tilde{A} &= (1/1 + 0.6/4 + 0.4/5) \times (1/1 + 0.6/4 + 0.4/5) \\ &= (1/1) \times (1/1 + 0.6/4 + 0.4/5) + (0.6/4) \times (1/1 + 0.6/4 + 0.4/5) \end{aligned}$$

$$
\begin{aligned}
&+(0.4/5) \times (1/1 + 0.6/4 + 0.4/5) \\
= \; & 1/1 + 0.6/4 + 0.4/5 + 0.6/4 + 0.6/16 \\
&+0.4/20 + 0.4/5 + 0.4/20 + 0.4/25 \\
= \; & 1/1 + 0.6/4 + 0.4/5 + 0.6/16 + 0.4/20 + 0.4/25
\end{aligned} \tag{B.9}
$$

From (B.8) and (B.9), we see that using (B.5), $\tilde{A}^2 \neq \tilde{A} \times \tilde{A}$ ! This happened, because we did not use the right form of the Extension Principle. In order to get $\tilde{A}^2 = \tilde{A} \times \tilde{A}$, we have to use the restricted form of the Extension Principle [i.e., (B.6)] to evaluate $\tilde{A} \times \tilde{A}$. The restriction is crisp in this case and can be expressed as

$$
\mu_R(u_i, v_j) = \begin{cases} 1 \; ; & u_i = v_j \\ 0 \; ; & \text{otherwise} \end{cases} \tag{B.10}
$$

Using (B.6) with (B.10), we get

$$
\begin{aligned}
\tilde{A} \times \tilde{A} = \; & (1/1 + 0.6/4 + 0.4/5) \times (1/1 + 0.6/4 + 0.4/5) \cap R \\
= \; & [(1/1) \times (1/1 + 0.6/4 + 0.4/5) + (0.6/4) \times (1/1 + 0.6/4 + 0.4/5) \\
&+(0.4/5) \times (1/1 + 0.6/4 + 0.4/5)] \cap R \\
= \; & [1 \wedge 1 \wedge \mu_R(1,1)]/(1 \times 1) + [1 \wedge 0.6 \wedge \mu_R(1,4)]/(1 \times 4) \\
&+[1 \wedge 0.4 \wedge \mu_R(1,5)]/(1 \times 5) + [0.6 \wedge 1 \wedge \mu_R(4,1)]/(4 \times 1) \\
&+[0.6 \wedge 0.6 \wedge \mu_R(4,4)]/(4 \times 4) + [0.6 \wedge 0.4 \wedge \mu_R(4,5)]/(4 \times 5) \\
&+[0.4 \wedge 1 \wedge \mu_R(5,1)]/(5 \times 1) + [0.4 \wedge 0.6 \wedge \mu_R(5,4)]/(5 \times 4) \\
&+[0.4 \wedge 0.4 \wedge \mu_R(5,5)]/(5 \times 5) \\
= \; & 1/1 + 0/4 + 0/5 + 0/4 + 0.6/16 + 0/20 + 0/5 + 0/20 + 0.4/25 \\
= \; & 1/1 + 0.6/16 + 0.4/25 \\
= \; & \tilde{A}^2
\end{aligned} \tag{B.11}
$$

□

Observe that using the restricted form of the Extension Principle can complicate computations quite a lot. There are also a few problems/difficulties associated with using the restricted form of the Extension Principle.

1. If we have to perfom an operation between two given fuzzy sets $\tilde{A}$ and $\tilde{B}$, it may not always be easy to define a restriction between the two. For example, if we have to find $\tilde{A} \sqcup \tilde{B}$, where $\tilde{A} = 1/1 + 0.6/4 + 0.4/5$ and $\tilde{B} = 1/1 + 0.7/3$, without any other information about $\tilde{A}$ and $\tilde{B}$, there is no fixed way of defining a relation between $\tilde{A}$ and $\tilde{B}$; in fact, it may not even be possible to tell if $\tilde{A}$ and $\tilde{B}$ are related at all.

2. When performing operations like $\tilde{A} \sqcap \tilde{B} \sqcup \tilde{A}$, if we are given a restriction on $\tilde{A}$ and $\tilde{B}$, it may be easy to use the restricted from of the Extension Principle; however, if we are given $\tilde{A} \sqcap \tilde{B}$ and $\tilde{A}$, it may not be easy (or it may not even be possible) to define a restriction on the elements of $\tilde{A} \sqcap \tilde{B}$ and $\tilde{A}$ so as to use the restricted form.

3. When we use the product $t$-norm, even using the restricted form of the Extension Principle may not give us equalities in cases like the one considered in Example B.1, as demonstrated in the following example :

   **Example B.2** Consider the same type-1 set $\tilde{A} \subset U$ considered in Example B.1. Computing $\tilde{A}^2$ by considering squaring as an operation on $\tilde{A}$ gives us the same $\tilde{A}^2$ as in (B.8); however, computing $\tilde{A} \times \tilde{A}$ using the product $t$-norm, gives us [where we use the same restriction $R$ as in (B.10)]

$$\begin{aligned}
\tilde{A} \times \tilde{A} &= (1/1 + 0.6/4 + 0.4/5) \times (1/1 + 0.6/4 + 0.4/5) \cap R \\
&= [(1/1) \times (1/1 + 0.6/4 + 0.4/5) + (0.6/4) \times (1/1 + 0.6/4 + 0.4/5) \\
&\quad +(0.4/5) \times (1/1 + 0.6/4 + 0.4/5)] \cap R \\
&= [1 \times 1 \times \mu_R(1,1)]/(1 \times 1) + [1 \times 0.6 \times \mu_R(1,4)]/(1 \times 4) \\
&\quad +[1 \times 0.4 \times \mu_R(1,5)]/(1 \times 5) + [0.6 \times 1 \times \mu_R(4,1)]/(4 \times 1) \\
&\quad +[0.6 \times 0.6 \times \mu_R(4,4)]/(4 \times 4) + [0.6 \times 0.4 \times \mu_R(4,5)]/(4 \times 5) \\
&\quad +[0.4 \times 1 \times \mu_R(5,1)]/(5 \times 1) + [0.4 \times 0.6 \times \mu_R(5,4)]/(5 \times 4) \\
&\quad +[0.4 \times 0.4 \times \mu_R(5,5)]/(5 \times 5) \\
&= 1/1 + 0/4 + 0/5 + 0/4 + 0.36/16 + 0/20 + 0/5 + 0/20 + 0.16/25 \\
&= 1/1 + 0.36/16 + 0.16/25 \\
&\neq \tilde{A}^2 \qquad \text{(B.12)}
\end{aligned}$$

□

Though, as shown in Example B.2, product $t$-norm does not give intuitive results with the restricted form of the Extension Principle, in view of the desirable properties of the product $t$-norm, we continue to use it in our work. In order to avoid the above mentioned difficulties with the restricted form of the Extension Principle, we adopt the following approach.

When we need to extend an operation of the form $f(\theta_1, \ldots, \theta_n)$ to an operation $f(\tilde{A}_1, \ldots, \tilde{A}_n)$, we will not extend the individual operations, like multiplication, addition, etc. involved in $f$; rather, we will use the following definition :

$$f(\tilde{A}_1, \ldots, \tilde{A}_n) = \int_{\theta_1} \cdots \int_{\theta_n} \mu_{\tilde{A}_1}(\theta_1) \star \cdots \star \mu_{\tilde{A}_n}(\theta_n) \Big/ f(\theta_1, \ldots, \theta_n) \tag{B.13}$$

where $\theta_i \in \tilde{A}_i$ for $i = 1, \ldots, n$. For example, if $f(\theta_1, \theta_2) = [\theta_1\theta_2] \big/ [\theta_1 + \theta_2]$, we write the extension of $f$ to type-1 sets $\tilde{A}_1$ and $\tilde{A}_2$ as

$$f(\tilde{A}_1, \tilde{A}_2) = \int_{\theta_1} \int_{\theta_2} \mu_{\tilde{A}_1}(\theta_1) \star \mu_{\tilde{A}_2}(\theta_2) \Big/ \frac{\theta_1\theta_2}{\theta_1 + \theta_2} \tag{B.14}$$

where $\theta_i \in \tilde{A}_i$ for $i = 1, 2$; and <u>**not**</u> as

$$f(\tilde{A}_1, \tilde{A}_2) = \frac{\tilde{A}_1 \times \tilde{A}_2}{\tilde{A}_1 + \tilde{A}_2} \tag{B.15}$$

When discussing properties of membership grades in [4], however, we use the *unrestricted* form of the Extension Principle, just like Mizumoto and Tanaka do in [8].

# Appendix C

# Proofs in Chapter 2

## C.1 Proof of Theorem 2.1

In the proof of Theorem 2.1, given next, we represent fuzzy sets $\tilde{F}$ and $\tilde{G}$ as follows :

$$\tilde{F} = \int_{v\in\Re} f(v)/v \tag{C.1}$$

$$\tilde{G} = \int_{w\in\Re} g(w)/w \tag{C.2}$$

As is apparent from (C.1) and (C.2), fuzzy sets $\tilde{F}$ and $\tilde{G}$ can, in general, have the real line as their domain. If a real number $w_0$ is not in $\tilde{F}$ (or $\tilde{G}$), $f(w_0)$ [or $g(w_0)$] will be zero. With this understanding, we will sometimes use the notation $v \in \Re$ and $v \in \tilde{F}$ or $w \in \Re$ and $w \in \tilde{G}$ interchangeably.

**Proof :**

**(a-I)** The *join* operation between $\tilde{F}$ and $\tilde{G}$ can be expressed, as

$$\tilde{F} \sqcup \tilde{G} = \int_{v\in\Re}\int_{w\in\Re}[f(v) \wedge g(w)]/(v \vee w) \tag{C.3}$$

Let's see what operations are involved here. For every pair of points $\{v, w\}$, such that $v \in \tilde{F}$ and $w \in \tilde{G}$, we find the maximum of $v$ and $w$ and the minimum of their memberships, so that $v \vee w$ is an element of $\tilde{F} \sqcup \tilde{G}$ and $f(v) \wedge g(w)$ is the corresponding membership grade. If more than one $\{v, w\}$ pair gives the same maximum (i.e., the same element in $\tilde{F} \sqcup \tilde{G}$), we use the maximum of all the corresponding membership grades as the membership of this element. So, every element of the resulting set is obtained as a result of the *max* operation on one or more $\{v, w\}$ pairs, and it's membership is the maximum of all the results of the *min* operation on memberships of $v$ and $w$.

We analyze the *join* operation by picking a point in $\tilde{\mathrm{F}} \sqcup \tilde{\mathrm{G}}$ and finding it's membership grade. Figures 2.1 (a) and (b) depict an example of the *join* operation. Let $\theta \in \tilde{\mathrm{F}} \sqcup \tilde{\mathrm{G}}$. As noted in the preceding paragraph, $\theta$ must be the result of the *max* operation on one or more $\{v, w\}$ pairs; hence, the possible admissible pairs can only be $\{v, \theta\}$ where $v \in (-\infty, \theta]$ and $\{\theta, w\}$ where $w \in (-\infty, \theta]$. To find the membership of $\theta$, we have to perform the *min* operation between the memberships of all these possible pairs $\{v, w\}$ and then take the maximum of them. For example, to find the membership grade of the point $\theta = 3$ in the union of $\tilde{\mathrm{F}}$ and $\tilde{\mathrm{G}}$, first we compare $g(3)$ with each $f(v)$ for $v \in (-\infty, 3]$ or $v \leq 3$, find the minimum in each of these comparisons and finally find the maximum of all these answers; then we compare $f(3)$ with all $g(w)$ for $w \in (-\infty, 3]$ or $w \leq 3$ and do a similar minimax operation, and finally find the maximum of the results of these two minimax operations.

We break this process into three steps : (1) find the minima between the memberships of all the pairs $\{v, \theta\}$ such that $v \in (-\infty, \theta]$ and then find their supremum; (2) do the same with all the pairs $\{\theta, w\}$ such that $w \in (-\infty, \theta]$; and, (3) find the maximum of the two suprema, i.e.,

$$\mu_{(\tilde{\mathrm{F}} \sqcup \tilde{\mathrm{G}})}(\theta) = \phi_1(\theta) \vee \phi_2(\theta) \tag{C.4}$$

where,

$$\phi_1(\theta) = \sup_{v \in (-\infty, \theta]} \{f(v) \wedge g(\theta)\} \tag{C.5}$$

and

$$\phi_2(\theta) = \sup_{w \in (-\infty, \theta]} \{f(\theta) \wedge g(w)\} \tag{C.6}$$

In (C.5), $g(\theta)$ is a constant with respect to $v$, and in (C.6), $f(\theta)$ is a constant with respect to $w$; therefore,

$$\phi_1(\theta) = g(\theta) \wedge \sup_{v \in (-\infty, \theta]} f(v) \tag{C.7}$$

$$\phi_2(\theta) = f(\theta) \wedge \sup_{w \in (-\infty, \theta]} g(w) \tag{C.8}$$

We break $\theta$ into the following three ranges : $\theta < v_0$, $v_0 \leq \theta \leq v_1$ and $\theta > v_1$ (see Fig. 2.1). Recall that $f(v_0) = 1$ and $g(v_1) = 1$ and that $\tilde{\mathrm{F}}$ and $\tilde{\mathrm{G}}$ are both convex. Also, observe that *convexity of* $\tilde{F}$ *is equivalent to the condition that* $f$ *is monotonic non-decreasing in* $(-\infty, v_0]$ *and monotonic non-increasing in* $[v_0, \infty)$ (see Appendix C.1). Similarly, *convexity of* $\tilde{G}$ *is equivalent to the condition that* $g$ *is monotonic non-decreasing in* $(-\infty, v_1]$ *and monotonic non-increasing for* $[v_1, \infty)$.

$\underline{\theta = \theta_1 < v_0}$ : See Fig. 2.1 (a). Since $f$ and $g$ both are monotonic non-decreasing in $(-\infty, v_0]$,

$$\sup_{v \in (-\infty, \theta]} f(v) = f(\theta), \tag{C.9}$$

and

$$\sup_{w \in (-\infty, \theta]} g(w) = g(\theta) \; ; \tag{C.10}$$

therefore, from (C.7) and (C.8), we have

$$\phi_1(\theta) = \phi_2(\theta) = g(\theta) \wedge f(\theta) \tag{C.11}$$

Using (C.11) in (C.4), we get

$$\mu_{(\tilde{\mathrm{F}} \sqcup \tilde{\mathrm{G}})}(\theta) = g(\theta) \wedge f(\theta) \; ; \; \theta < v_0 \tag{C.12}$$

$\underline{v_0 \leq \theta = \theta_2 \leq v_1}$ : See Fig. 2.1 (a). Recall that $f(v_0) = 1$ and that $g$ is monotonic non-decreasing in $(-\infty, v_1]$; therefore, $\sup_{v \in (-\infty, \theta]} f(v) = 1$ and $\sup_{w \in (-\infty, \theta]} g(w) = g(\theta)$. Using these facts in (C.7) and (C.8), we have that in this range

$$\phi_1(\theta) = g(\theta) \wedge 1 = g(\theta) \tag{C.13}$$

and

$$\phi_2(\theta) = f(\theta) \wedge g(\theta) \tag{C.14}$$

Using (C.13) and (C.14) in (C.4), we have

$$\mu_{(\tilde{\mathrm{F}} \sqcup \tilde{\mathrm{G}})}(\theta) = g(\theta) \vee [f(\theta) \wedge g(\theta)] \tag{C.15}$$

Observe that, if $f(\theta) \leq g(\theta)$, the RHS of (C.15) simplifies to $g(\theta) \vee [f(\theta)] = g(\theta)$ and if $f(\theta) \geq g(\theta)$, the RHS gives $g(\theta) \vee [g(\theta)] = g(\theta)$. So, in either case

$$\mu_{(F \sqcup G)}(\theta) = g(\theta)\,;\; v_0 \leq \theta \leq v_1 \tag{C.16}$$

$\underline{\theta = \theta_3 > v_1}$ : For $\theta$ in this range [see Fig. 2.1 (a)], both $f$ and $g$ have already attained their maximum values; therefore,

$$\sup_{v \in (-\infty, \theta]} f(v) = 1 \tag{C.17}$$

$$\sup_{w \in (-\infty, \theta]} g(w) = 1 \tag{C.18}$$

Consequently,

$$\phi_1(\theta) = g(\theta) \tag{C.19}$$

$$\phi_2(\theta) = f(\theta)\,; \tag{C.20}$$

therefore, from (C.4),

$$\mu_{(\tilde{F} \sqcup \tilde{G})}(\theta) = f(\theta) \vee g(\theta)\,;\; \theta > v_1 \tag{C.21}$$

From (C.12), (C.16) and (C.21), we get (2.18).
**(a-II)** The *meet* operation between $\tilde{F}$ and $\tilde{G}$ can be expressed, as

$$\tilde{F} \sqcap \tilde{G} = \int_{v \in \Re} \int_{w \in \Re} [f(v) \wedge g(w)]/(v \wedge w) \tag{C.22}$$

This equation looks very similar to (C.3). The operations involved here are the same as for the *join* operation, except for the fact that every element of $\tilde{F} \sqcap \tilde{G}$ is obtained as a result of the *min* operation on one or more $\{v, w\}$ pairs, where $v \in \tilde{F}$ and $w \in \tilde{G}$. Consider $\theta \in \tilde{F} \sqcap \tilde{G}$. The possible pairs $\{v, w\}$ that can give us $\theta$ as a result of the *min* operation are $\{v, \theta\}$ where $v \in [\theta, \infty)$ and $\{\theta, w\}$ where $w \in [\theta, \infty)$. To find the membership grade of $\theta$, we find the minimum of the memberships for each of these $\{v, w\}$ pairs and then take the maximum of all these results. Again, we break this process into three steps : first we find the minima of the membership grades of all the pairs $\{v, \theta\}$ such that $v \in [\theta, \infty)$ and then find their supremum; then we do the same with all the pairs $\{\theta, w\}$ such that $w \in [\theta, \infty)$; and, finally, we find the maximum of the two suprema, i.e.,

$$\mu_{(\tilde{F}\sqcap\tilde{G})}(\theta) = \phi_3(\theta) \vee \phi_4(\theta) \tag{C.23}$$

where,

$$\phi_3(\theta) = \sup_{v\in[\theta,\infty)} \{f(v) \wedge g(\theta)\} \tag{C.24}$$

$$\phi_4(\theta) = \sup_{w\in[\theta,\infty)} \{f(\theta) \wedge g(w)\} \tag{C.25}$$

Again using similar reasoning as in part **(a-I)** of this proof, we have

$$\phi_3(\theta) = g(\theta) \wedge \sup_{v\in[\theta,\infty)} f(v) \tag{C.26}$$

$$\phi_4(\theta) = f(\theta) \wedge \sup_{w\in[\theta,\infty)} g(w) \tag{C.27}$$

We consider three ranges for $\theta : \theta > v_1$, $v_0 \leq \theta \leq v_1$ and $\theta < v_0$.

$\underline{\theta = \theta_3 > v_1}$ : See Fig. 2.1 (a). $f$ and $g$, both, are monotonic non-increasing in $(v_1, \infty)$; therefore,

$$\sup_{v\in[\theta,\infty)} f(v) = f(\theta) \tag{C.28}$$

$$\sup_{w\in[\theta,\infty)} g(w) = g(\theta) \tag{C.29}$$

Using (C.28) and (C.29) in (C.26) and (C.27), we get

$$\phi_3(\theta) = \phi_4(\theta) = f(\theta) \wedge g(\theta) \tag{C.30}$$

Therefore, from (C.23), we have

$$\mu_{(\tilde{F}\sqcap\tilde{G})}(\theta) = f(\theta) \wedge g(\theta) \tag{C.31}$$

$\underline{v_0 \leq \theta = \theta_2 \leq v_1}$ : See Fig. 2.1 (a). Recall that $f$ is monotonic non-increasing in $[v_0, \infty)$ and that $g(v_1) = 1$. This gives us

$$\sup_{v \in [\theta, \infty)} f(v) = f(\theta) \tag{C.32}$$

$$\sup_{w \in [\theta, \infty)} g(w) = 1 \tag{C.33}$$

Using (C.32) and (C.33) in (C.26) and (C.27), we have

$$\phi_3(\theta) = g(\theta) \wedge f(\theta) \tag{C.34}$$

$$\phi_4(\theta) = f(\theta) \tag{C.35}$$

Using (C.34) and (C.35) in (C.23), we have

$$\mu_{(\tilde{F} \sqcap \tilde{G})}(\theta) = [g(\theta) \wedge f(\theta)] \vee f(\theta) \tag{C.36}$$

Reasoning as in part **(a-I)** [see Eqs. (C.15) and (C.16)], we get

$$\mu_{(\tilde{F} \sqcap \tilde{G})}(\theta) = f(\theta)\,;\; v_0 \leq \theta \leq v_1 \tag{C.37}$$

$\underline{\theta = \theta_1 < v_0}$ : See Fig. 2.1 (a). We have that $f(v_0) = 1$ and $g(v_1) = 1$; therefore,

$$\sup_{v \in [\theta, \infty)} f(v) = 1 \tag{C.38}$$

$$\sup_{w \in [\theta, \infty)} g(w) = 1 \tag{C.39}$$

Using (C.38) and (C.39) in (C.26) and (C.27), we have

$$\phi_3(\theta) = g(\theta) \tag{C.40}$$

$$\phi_4(\theta) = f(\theta)\,; \tag{C.41}$$

therefore, from (C.23), we have

$$\mu_{(\tilde{F} \sqcap \tilde{G})}(\theta) = f(\theta) \vee g(\theta)\,;\; \theta < v_0 \tag{C.42}$$

From (C.31), (C.37) and (C.42), we get (2.19).

**(b-I)** In [8], Mizumoto and Tanaka show that results of *join* or *meet* operations, using *max* $t$-conorm and *min* $t$-norm, on convex and normal type-1 sets are also convex and normal. Using this fact, we generalize the result in part (a) of Theorem 2.1 to more than two sets.

Consider $n$ convex, normal, type-1 fuzzy sets $\tilde{F}_1, \ldots, \tilde{F}_n$ characterized by membership functions $f_1, \ldots, f_n$, respectively. Let $v_1, v_2, \ldots, v_n$ be real numbers such that $v_1 \leq v_2 \leq \cdots \leq v_n$ and $f_1(v_1) = f_2(v_2) = \cdots = f_n(v_n) = 1$.

Using (2.18), we have

$$\mu_{\tilde{F}_{n-1} \sqcup \tilde{F}_n}(\theta) = \begin{cases} f_{n-1}(\theta) \wedge f_n(\theta) & ; \quad \theta < v_{n-1} \\ f_n(\theta) & ; \quad v_{n-1} \leq \theta \leq v_n \\ f_{n-1}(\theta) \vee f_n(\theta) & ; \quad \theta > v_n \end{cases} \tag{C.43}$$

Using the associative property, we have (we are interested mainly in dealing with type-1 sets which are membership grades of type-2 sets; for more discussion on properties of type-1 fuzzy membership grades, see [4])

$$\tilde{F}_{n-2} \sqcup \tilde{F}_{n-1} \sqcup \tilde{F}_n = \tilde{F}_{n-2} \sqcup (\tilde{F}_{n-1} \sqcup \tilde{F}_n) \tag{C.44}$$

Let $f_{(n-1)n} = \mu_{\tilde{F}_{n-1} \sqcup \tilde{F}_n}$. Since $\tilde{F}_{n-1} \sqcup \tilde{F}_n$ is also a convex, normal, type-1 fuzzy set [$f_n(v_n) = 1$, and from (C.43) we see that $f_{(n-1)n}(v_n) = f_n(v_n)$], another application of (2.18) gives us

$$\mu_{\tilde{F}_{n-2} \sqcup \tilde{F}_{n-1} \sqcup \tilde{F}_n}(\theta) = \begin{cases} f_{(n-1)n}(\theta) \wedge f_{n-2}(\theta) & ; \quad \theta < v_{n-2} \\ f_{(n-1)n}(\theta) & ; \quad v_{n-2} \leq \theta \leq v_n \\ f_{(n-1)n}(\theta) \vee f_{n-2}(\theta) & ; \quad \theta > v_n \end{cases} \tag{C.45}$$

Since $v_{n-2} \leq v_{n-1}$, (C.43) and (C.45) can be rewritten as follows :

$$\mu_{\tilde{F}_{n-1} \sqcup \tilde{F}_n}(\theta) = \begin{cases} f_{n-1}(\theta) \wedge f_n(\theta) & ; \quad \theta < v_{n-2} \\ f_{n-1}(\theta) \wedge f_n(\theta) & ; \quad v_{n-2} \leq \theta < v_{n-1} \\ f_n(\theta) & ; \quad v_{n-1} \leq \theta \leq v_n \\ f_{n-1}(\theta) \vee f_n(\theta) & ; \quad \theta > v_n \end{cases} \tag{C.46}$$

$$\mu_{\tilde{F}_{n-2}\sqcup\tilde{F}_{n-1}\sqcup\tilde{F}_n}(\theta) = \begin{cases} f_{(n-1)n}(\theta) \wedge f_{n-2}(\theta) & ; \quad \theta < v_{n-2} \\ f_{(n-1)n}(\theta) & ; \quad v_{n-2} \leq \theta < v_{n-1} \\ f_{(n-1)n}(\theta) & ; \quad v_{n-1} \leq \theta \leq v_n \\ f_{(n-1)n}(\theta) \vee f_{n-2}(\theta) & ; \quad \theta > v_n \end{cases} \quad (C.47)$$

Substituting for $f_{(n-1)n}$ into (C.47) from (C.46), we obtain

$$\mu_{\tilde{F}_{n-2}\sqcup\tilde{F}_{n-1}\sqcup\tilde{F}_n}(\theta) = \begin{cases} f_{n-2}(\theta) \wedge f_{n-1}(\theta) \wedge f_n(\theta) & ; \quad \theta < v_{n-2} \\ f_{n-1}(\theta) \wedge f_n(\theta) & ; \quad v_{n-2} \leq \theta \leq v_{n-1} \\ f_n(\theta) & ; \quad v_{n-1} \leq \theta \leq v_n \\ f_{n-2}(\theta) \vee f_{n-1}(\theta) \vee f_n(\theta) & ; \quad \theta > v_n \end{cases} \quad (C.48)$$

Again, $\tilde{F}_{n-2} \sqcup \tilde{F}_{n-1} \sqcup \tilde{F}_n$ is also a convex and normal type-1 set, therefore (2.18) can be applied again. Continuing in this fashion, we get (2.20).

**(b-II)** The proof is very much similar to that of part (b) - I. Starting with $\tilde{F}_1 \sqcap \tilde{F}_2$ and using (2.19) repeatedly, we get (2.21).

## C.2 Proof of Assertion in the Proof of Theorem 2.1

The convexity of $\tilde{F} = \int f(\theta)/\theta$ is equivalent to the condition [8]

$$f(v_2) \geq \min\{f(v_1), f(v_3)\} \quad (C.49)$$

if $v_2$ is between $v_1$ and $v_3$, i.e., if $v_1 \leq v_2 \leq v_3$ or $v_3 \leq v_2 \leq v_1$. (See Figs. 2.1 and 2.3 for examples of arbitrarily shaped convex membership functions.) We first prove that convexity of $f$ implies the monotonicity conditions on $f$, and then prove that the monotonicity conditions on $f$ imply its convexity.

**(I)** Since $f$ is a membership function for a normalized type-1 fuzzy set (see Theorem 2.1), we know that $f(v) \leq 1$ for all $v$. Also, we know that $f(v_0) = 1$; therefore, letting $v_1 = v_0$ in (C.49), we get

$$f(v_2) \geq \min\{1, f(v_3)\}\,; \quad v_0 \leq v_2 \leq v_3 \text{ or } v_3 \leq v_2 \leq v_0 \quad (C.50)$$

i.e.,

$$f(v_2) \geq f(v_3)\,;\;\; v_0 \leq v_2 \leq v_3 \text{ or } v_3 \leq v_2 \leq v_0 \tag{C.51}$$

In other words, $f$ is monotonic non-decreasing in $(-\infty, v_0]$ and monotonic non-increasing in $[v_0, \infty)$.

**(II)** Now, assume that $f$ is monotonic non-decreasing in $(-\infty, v_0]$ and monotonic non-increasing in $[v_0, \infty)$, with $f(v_0) = 1$. Consider two points $v_1$ and $v_3$, such that $v_1 \leq v_3$. We will show that any point $v_2$ between $v_1$ and $v_3$, satisifes (C.49). There are three possibilities for $v_1$ and $v_3$ :

1. $\underline{v_1 \leq v_3 < v_0}$ : Since $f$ is monotonic non-decreasing in this range, for any point $v_2$ between $v_1$ and $v_3$, $f(v_2) \geq f(v_1)$; therefore, (C.49) is satisfied.

2. $\underline{v_1 \leq v_0 \leq v_3}$ : Since $f$ is monotonic non-decreasing in $(-\infty, v_0]$, if $v_1 \leq v_2 \leq v_0$, $f(v_2) \geq f(v_1)$. Also, since $f$ is monotonic non-increasing in $[v_0, \infty)$, if $v_0 \leq v_2 \leq v_3$, $f(v_2) \geq f(v_3)$. In either case, (C.49) is satisfied.

3. $\underline{v_0 < v_1 \leq v_3}$ : In this range, $f$ is monotonic non-increasing; therefore, for any $v_2$ between $v_1$ and $v_3$, $f(v_2) \geq f(v_3)$, which implies that (C.49) is satisfied.

Since (C.49) is satisfied in all the three cases, we conclude that $f$ is convex. □

## C.3 Proof of Corollary 2.1

**(a)** Let $f(v_0) = 1$. Using Theorem 2.1, we have

$$\mu_{\tilde{F}\sqcup\tilde{G}}(v) = \begin{cases} f(\theta) \wedge f(\theta - k) & ; \;\; \theta < v_0 \\ f(\theta - k) & ; \;\; v_0 \leq \theta \leq v_0 + k \\ f(\theta) \vee f(\theta - k) & ; \;\; \theta > v_0 + k \end{cases} \tag{C.52}$$

We have made use of the fact that the membership function of $\tilde{G}$ is a shifted version of $f$. The point $v_1$ in Theorem 2.1, now becomes $(v_0+k)$. As shown in Appendix C.1, the convexity of $\tilde{F}$ implies that $f$ is monotonic non-decreasing in $(-\infty, v_0]$ and monotonic non-increasing in $[v_0, \infty)$, which implies that $f(\theta) > f(\theta - k)$ for $\theta < v_0$ and $f(\theta) < f(\theta - k)$ for $\theta > v_0 + k$. Using these facts in (C.52), we have

$$\mu_{\tilde{F}\sqcup\tilde{G}}(\theta) = f(\theta - k) = \mu_{\tilde{G}}(\theta)\,;\; \forall \theta \in \Re$$

$$\Rightarrow \tilde{F} \sqcup \tilde{G} = \tilde{G} \tag{C.53}$$

A very similar proof can be used for the *meet* operation.
**(b)** A repeated application of part (a) yields part(b). □

## C.4 Proof of Theorem 2.2

From (2.15), we have

$$\neg\tilde{F} = \int_{\theta\in\Re} f(\theta)/(1-\theta) \tag{C.54}$$

Let $y = (1-\theta)$, then $\theta = (1-y)$ and $\theta \in \Re \Rightarrow y \in \Re$; therefore,

$$\neg\tilde{F} = \int_{y\in\Re} f(1-y)/y \tag{C.55}$$

$$= \int_{\theta\in\Re} f(1-\theta)/\theta \tag{C.56}$$

□

## C.5 Join under Product $t$-norm

**(a)** Consider the two convex normal type-1 fuzzy sets, $\tilde{F}$ and $\tilde{G}$ used in Theorem 2.1. The *join* operation between $\tilde{F}$ and $\tilde{G}$, using the product $t$-norm can be represented as

$$\tilde{F} \sqcup \tilde{G} = \int_{v\in\Re}\int_{w\in\Re} [f(v)g(w)]/(v \vee w) \tag{C.57}$$

where $\vee$ denotes the maximum. Equation (C.57) is the same as (C.3) with the *min* replaced by a product.

The following analysis is very similar to that in Theorem 2.1. If $\theta$ is an element of $\tilde{F} \sqcup \tilde{G}$, then the membership grade of $\theta$ can be determined by finding all the pairs $\{v, w\}$ such that $v \in \tilde{F}$, $w \in \tilde{G}$ and $v \vee w = \theta$; multiplying the membership grades of $v$ and $w$ in each pair; and then finding the maximum of these products of membership grades. The possible admissible $\{v, w\}$ pairs that can give us $\theta$ as the result of the *max* operation are $\{v, \theta\}$ where $v \in (-\infty, \theta]$ and $\{\theta, w\}$ where $w \in (-\infty, \theta]$. We find the products of membership grades of $v$ and $w$ from each such pair and take the maximum of all these products as the membership grade of $\theta$. We break this process into three steps : (1) find the product of the memberships of all the pairs $\{v, \theta\}$ where $v \in (-\infty, \theta]$ and then find their supremum; (2) do the same with all the pairs $\{\theta, w\}$ where $w \in (-\infty, \theta]$; and (3) find the maximum of the two suprema, i.e.,

$$\mu_{(\tilde{F} \sqcup \tilde{G})}(\theta) = \psi_1(\theta) \vee \psi_2(\theta) \tag{C.58}$$

where

$$\psi_1(\theta) = \sup_{v \in (-\infty, \theta]} \{f(v) g(\theta)\} \tag{C.59}$$

Since $g(\theta)$ is a constant for a given $\theta$,

$$\psi_1(\theta) = g(\theta) \sup_{v \in (-\infty, \theta]} f(v) \tag{C.60}$$

Similarly,

$$\psi_2(\theta) = \sup_{w \in (-\infty, \theta]} \{f(\theta) g(w)\} \tag{C.61}$$

$$= f(\theta) \sup_{w \in (-\infty, \theta]} g(w) \tag{C.62}$$

We break $\theta$ into the following three ranges : $\theta < v_0$, $v_0 \leq \theta \leq v_1$ and $\theta > v_1$.

$\underline{\theta = \theta_1 < v_0}$ : See Fig. 2.1 (a). Since, $f(v)$ and $g(w)$ both are monotonic non-decreasing in $(-\infty, v_0]$,

$$\sup_{v \in (-\infty, \theta]} f(v) = f(\theta), \tag{C.63}$$

and

$$\sup_{w \in (-\infty, \theta]} g(w) = g(\theta) \tag{C.64}$$

Consequently, from (C.60) and (C.62), we get

$$\psi_1(\theta) = \psi_2(\theta) = f(\theta) g(\theta) \tag{C.65}$$

which implies that [see (C.58)]

$$\mu_{(\tilde{F} \sqcup \tilde{G})}(\theta) = f(\theta) g(\theta)\ ;\ \theta < v_0 \tag{C.66}$$

$\underline{v_0 \leq \theta = \theta_2 \leq v_1}$ : See Fig. 2.1 (a). Recall that $f(v_0) = 1$ and that $g$ is monotonic non-decreasing in $(-\infty, v_1]$; therefore,

$$\sup_{v \in (-\infty, \theta]} f(v) = 1 \, , \tag{C.67}$$

and

$$\sup_{w \in (-\infty, \theta]} g(w) = g(\theta) \tag{C.68}$$

From (C.60) and (C.62), we get

$$\psi_1(\theta) \;\; = \;\; g(\theta) \tag{C.69}$$

$$\psi_2(\theta) \;\; = \;\; f(\theta) g(\theta) \tag{C.70}$$

Since $f(\theta) \leq 1$, $\psi_1(\theta) \geq \psi_2(\theta)$. Consequently, from (C.58)

$$\mu_{(\tilde{F} \sqcup \tilde{G})}(\theta) = g(\theta) \; ; \; v_0 \leq \theta \leq v_1 \tag{C.71}$$

$\underline{\theta = \theta_3 > v_1}$ : See Fig. 2.1 (a). For $\theta$ in this range, both $f$ and $g$ have already attained their maximum values, i.e.,

$$\sup_{v \in (-\infty, \theta]} f(v) = 1 \, , \tag{C.72}$$

and

$$\sup_{w \in (-\infty, \theta]} g(w) = 1 \, ; \tag{C.73}$$

therefore, from (C.60) and (C.62), we get

$$\psi_1(\theta) \;\; = \;\; g(\theta) \tag{C.74}$$

$$\psi_2(\theta) \;\; = \;\; f(\theta) \tag{C.75}$$

Consequently, from (C.58),

$$\mu_{(\tilde{F} \sqcup \tilde{G})}(\theta) = f(\theta) \vee g(\theta) \; ; \; \theta > v_1 \tag{C.76}$$

Combining (C.66), (C.71) and (C.76), we get (2.22).

**(b)** The proof of (2.23) is very similar to the proof of (2.20) in Appendix C.1. The only thing that we have to show is that using *max* $t$-conorm and *product* $t$-norm, the *meet* of two convex, normal type-1 fuzzy sets is also a convex, normal fuzzy type-1 set. Consider the convex, normal type-1 sets $\tilde{\mathrm{F}}$ and $\tilde{\mathrm{G}}$ described in part (a) of this proof. We must show that $\tilde{\mathrm{F}} \sqcup \tilde{\mathrm{G}}$ is also convex and normal under *max* $t$-conorm and product $t$-norm. To show convexity, we use the equivalent condition proved in Appendix C.1.

Since $\tilde{\mathrm{F}}$ and $\tilde{\mathrm{G}}$, both, are convex and normal, $f$ and $g$ both are monotonic non-decreasing in $(-\infty, v_0]$ (recall that $v_0 \le v_1$), which implies that $fg$ is also monotonic non-decreasing in $(-\infty, v_0]$. Also, $g(v_0) \ge f(v_0)g(v_0)$ and $g$ is monotonic non-decreasing in $[v_0, v_1]$. Consequently, $\mu_{\tilde{\mathrm{F}} \sqcup \tilde{\mathrm{G}}}(\theta)$ is monotonic non-decreasing in $(-\infty, v_1]$. Since, $f(\theta)$ and $g(\theta)$ are both monotonic non-increasing for $\theta > v_1$, $f(\theta) \vee g(\theta)$ is also monotonic non-increasing for $\theta > v_1$, which implies that $\mu_{\tilde{\mathrm{F}} \sqcup \tilde{\mathrm{G}}}(\theta)$ is monotonic non-increasing for $\theta > v_1$. Additionally, $\mu_{\tilde{\mathrm{F}} \sqcup \tilde{\mathrm{G}}}(v_1) = g(v_1) \vee f(v_1) = 1$; hence, $\tilde{\mathrm{F}} \sqcup \tilde{\mathrm{G}}$ is also convex and normal.

Suppose that we have $n$ convex, normal, type-1 fuzzy sets $\tilde{\mathrm{F}}_1, \ldots, \tilde{\mathrm{F}}_n$ characterized by membership functions $f_1, \ldots, f_n$, respectively. Let $v_1, v_2, \ldots, v_n$ be real numbers such that $v_1 \le v_2 \le \cdots \le v_n$ and $f_1(v_1) = f_2(v_2) = \cdots = f_n(v_n) = 1$, then proceeding exactly as in the case of the proof of part (b-I) in Theorem 2.1 (see Appendix C.1), by using the associative property of the *join* operation and by repeated application of (2.22), we get (2.23). □

## C.6 Meet of Gaussians under Product $t$-norm

Consider the case when $f(v)$ and $g(w)$ (as in Theorem 2.1) are Gaussians with support $[0, 1]$ with means $m_f$, $m_g$ and standard deviations $\sigma_f$, $\sigma_g$, respectively. Then,

$$\tilde{\mathrm{F}} \sqcap \tilde{\mathrm{G}} = \int_v \int_w e^{-\frac{1}{2}\left(\frac{v-m_f}{\sigma_f}\right)^2} e^{-\frac{1}{2}\left(\frac{w-m_g}{\sigma_g}\right)^2} / (vw) \tag{C.77}$$

Recall that the integral in the above equation denotes union in the continuum. If $\theta$ is an element of $\tilde{\mathrm{F}} \sqcap \tilde{\mathrm{G}}$, then the membership grade of $\theta$ can be found by : finding all the pairs $\{v, w\}$ such that $v \in \tilde{\mathrm{F}}$, $w \in \tilde{\mathrm{G}}$ and $vw = \theta$; multiplying the membership grades of $v$ and $w$ in each pair; and then finding the maximum of these products of membership grades, i.e.,

$$\mu_{\tilde{F}\sqcap\tilde{G}}(\theta) = \sup\{e^{-\frac{1}{2}(\frac{v-m_f}{\sigma_f})^2} e^{-\frac{1}{2}(\frac{w-m_g}{\sigma_g})^2}; vw = \theta; v \in \tilde{F}; w \in \tilde{G}\} \tag{C.78}$$

Given any $v$ (assuming $v \neq 0$), the constraint $vw = \theta$ gives us $w = \theta/v$. Further, since $w \in [0,1]$, it follows that $\theta/v \leq 1$ or $v \geq \theta$. So, given any $\theta \in [0,1]$, the acceptable $\{v, w\}$ pairs that can give $\theta$ as the result of the product operation are $\{(v, \frac{\theta}{v}); 0 \leq \theta \leq v \leq 1\}$; therefore, from (C.78), we have

$$\mu_{\tilde{F}\sqcap\tilde{G}}(\theta) = \sup_{v\in[\theta,1]} e^{-\frac{1}{2}[(\frac{v-m_f}{\sigma_f})^2 + (\frac{\frac{\theta}{v}-m_g}{\sigma_g})^2]} \tag{C.79}$$

Observe that, when $\theta = m_f m_g$, $v = m_f$ maximizes the above quantity, making the exponent 0. This implies that

$$\mu_{\tilde{F}\sqcap\tilde{G}}(m_f m_g) = 1, \tag{C.80}$$

which shows that our result is consistent with the type-1 case result, $m_f \star m_g = m_f m_g$, obtained by reducing type-1 sets $\tilde{F}$ and $\tilde{G}$ to singletons, having unity membership at $m_f$ and $m_g$ respectively and zero membership at all other points. The result of the meet operation is then a singleton also, with unity membership at $m_f m_g$ and zero membership at all other points.

For $\theta \neq m_f m_g$, the only thing that is easily observable is that the exponent in (C.79) does not reduce to zero, implying any $\theta$ other than $m_f m_g$ will have a membership grade less than unity. Now, let's see if we can determine an expression for $\theta$ in terms of $v$.

Let us call the quantity in the square bracket on the RHS of (C.79) $J(v)$. The $v$ that achieves the supremum in (C.79), minimizes $J(v)$. In order to find $\mu_{\tilde{F}\sqcap\tilde{G}}(\theta)$, we have to find an expression for $v$ that minimizes

$$J(v) = (\frac{v - m_f}{\sigma_f})^2 + (\frac{\frac{\theta}{v} - m_g}{\sigma_g})^2 \tag{C.81}$$

subject to the constraint $v \in [\theta, 1]$. Differentiating $J(v)$ and equating the derivative to 0, we see that the $v_*$ that achieves the minimum satisfies (with the constraint $v_* \in [\theta, 1]$)

$$\begin{aligned} J'(v_*) = 0 \quad &\Leftrightarrow \quad 2(\frac{v_* - m_f}{\sigma_f})(\frac{1}{\sigma_f}) + 2(\frac{\frac{\theta}{v_*} - m_g}{\sigma_g})(-\frac{\theta}{\sigma_g v_*^2}) = 0 \\ &\Leftrightarrow \quad \frac{v_*}{\sigma_f^2} - \frac{m_f}{\sigma_f^2} - \frac{\theta^2}{\sigma_g^2 v_*^3} + \frac{m_g \theta}{\sigma_g^2 v_*^2} = 0 \\ &\Leftrightarrow \quad v_*^4 - m_f v_*^3 + \theta \frac{\sigma_f^2}{\sigma_g^2} m_g v_* - \theta^2 \frac{\sigma_f^2}{\sigma_g^2} = 0 \end{aligned} \qquad \text{(C.82)}$$

Let us call the polynomial on the LHS of (C.82) $D(v)$. Observe that

$$\begin{aligned} D(m_f) &= m_f^4 - m_f^4 + \theta \frac{\sigma_f^2}{\sigma_g^2} m_g m_f - \theta^2 \frac{\sigma_f^2}{\sigma_g^2} \\ &= \theta \frac{\sigma_f^2}{\sigma_g^2}(m_f m_g - \theta) \end{aligned} \qquad \text{(C.83)}$$

$$\begin{aligned} D(\frac{\theta}{m_g}) &= (\frac{\theta}{m_g})^4 - m_f(\frac{\theta}{m_g})^3 + \theta \frac{\sigma_f^2}{\sigma_g^2} m_g \frac{\theta}{m_g} - \theta^2 \frac{\sigma_f^2}{\sigma_g^2} \\ &= (\frac{\theta}{m_g})^3 (\frac{\theta}{m_g} - m_f) \\ &= \frac{\theta^3}{m_g^4}(\theta - m_f m_g) \end{aligned} \qquad \text{(C.84)}$$

From (C.83) and (C.84), we observe that $D(m_f)$ and $D(\theta/m_g)$ are of opposite signs [since $\theta$, $m_f$ and $m_g$ are all in [0, 1], the quantity $(\theta - m_f m_g)$ decides the sign]. This implies that $D(v)$ always has a root between $m_f$ and $\theta/m_g$. As long as $\theta/m_g < 1$, $v_*$ always satisifes the constraint that $v_* \in [\theta, 1]$, because $m_f \leq 1$; however, after a critical value, say $\theta = \theta_c$, $v_* > 1$, and then $v = 1$ minimizes $J(v)$ while satisfying the constraint. The critical value $\theta_c$ can be found from (C.82) by expressing $\theta$ in terms of $v$. Rearranging (C.82) and solving for $\theta$, we get

$$\begin{aligned} &\theta^2(\frac{\sigma_f^2}{\sigma_g^2}) - \theta(\frac{\sigma_f^2}{\sigma_g^2} m_g v_*) + (m_f v_*^3 - v_*^4) = 0 \\ \Rightarrow \quad &\theta = \frac{v_*}{2}[m_g \pm \sqrt{m_g^2 + 4\frac{\sigma_g^2}{\sigma_f^2} v_*(v_* - m_f)}] \end{aligned} \qquad \text{(C.85)}$$

We are interested in finding those values of $\theta$ for which $v_* \geq 1$. Obviously, this implies that $v_* \geq m_f$, because $m_f \leq 1$ and therefore the second term in the bracket on the RHS of (C.85) is greater than or equal to $m_g$; hence, keeping the positive root of the above equation (recall that $\theta \geq 0$), we get

$$\theta = \frac{v_*}{2}[m_g + \sqrt{m_g^2 + 4\frac{\sigma_g^2}{\sigma_f^2}v_*(v_* - m_f)}] \tag{C.86}$$

The critical value of $\theta$ can be found by substituting $v_* = 1$ in (C.86) and is

$$\theta_c = \frac{1}{2}[m_g + \sqrt{m_g^2 + 4\frac{\sigma_g^2}{\sigma_f^2}(1 - m_f)}] \tag{C.87}$$

So, for $\theta < \theta_c$, $v_*$ can be obtained by solving Eq. (C.82) without using the constraint and then picking the root that satisfies the constraint and minimizes $J(v)$. If there is more than one root that satisfies the constraints, we check the value of $J(v)$ at each of the roots and pick the root at which $J(v)$ is minimum. For $\theta \geq \theta_c$, $v = 1$ minimizes $J(v)$.

In what follows, we attempt to solve Eq. (C.82). We rewrite (C.82) as (for notational simplicity, we drop the subscript "*")

$$v^4 - av^3 + bv - c = 0 \tag{C.88}$$

where

$$\begin{aligned} a &= m_f \\ b &= \frac{\theta m_g \sigma_f^2}{\sigma_g^2} \\ c &= \frac{\theta^2 \sigma_f^2}{\sigma_g^2} \end{aligned} \tag{C.89}$$

In the following, we use a standard procedure for solving quartic (4th order) equations [3]. Substituting $y = v - a/4$ (i.e., $v = y + a/4$) into (C.88), we get

$$(y + \frac{a}{4})^4 - a(y + \frac{a}{4})^3 + b(y + \frac{a}{4}) - c = 0 \tag{C.90}$$

which, upon simplification, gives

$$y^4 - (\frac{3}{8}a^2)y^2 + (b - \frac{a^3}{8})y + (\frac{ab}{4} - \frac{3}{256}a^4 - c) = 0 \qquad \text{(C.91)}$$

This equation has the following resolvent equation [i.e., if the roots of the following equation are found, the roots of (C.91) can be calculated from them]

$$z^3 - (\frac{3}{4}a^2)z^2 + (\frac{3}{16}a^4 + 4c - ab)z - (b - \frac{a^3}{8})^2 = 0 \qquad \text{(C.92)}$$

In order to find roots of (C.92), we simplify it further by substituting $z = t + a^2/4$ (i.e., $t = z - a^2/4$). Upon simplification, we get

$$t^3 + (4c - ab)t + (a^2c - b^2) = 0 \qquad \text{(C.93)}$$

Let $\alpha = (4c - ab)$, $\beta = (a^2c - b^2)$ and $D = (\beta/2)^2 + (\alpha/3)^3$; additionally, let

$$\begin{aligned} A &= (-\frac{\beta}{2} + \sqrt{D})^{1/3} \\ B &= -(\frac{\beta}{2} + \sqrt{D})^{1/3} \end{aligned} \qquad \text{(C.94)}$$

Then, the three roots of (C.93) are

$$\begin{aligned} t_1 &= A + B \\ t_2 &= -(\frac{A+B}{2}) + i(\frac{A-B}{2})\sqrt{3} \\ t_3 &= -(\frac{A+B}{2}) - i(\frac{A-B}{2})\sqrt{3} \end{aligned} \qquad \text{(C.95)}$$

If $D > 0$, one of these three roots is real and the other two are complex conjugates (which are discarded); if $D = 0$, all the three roots are real and at least two of them are equal; and if $D < 0$, all three roots are real and unequal.

The three roots of the resolvent equation (C.92) can be obtained by adding $a^2/4$ to each of $t_1$, $t_2$ and $t_3$, as

$$z_1 = A + B + \frac{a^2}{4}$$

$$
\begin{aligned}
z_2 &= -(\frac{A+B}{2}) + i(\frac{A-B}{2})\sqrt{3} + \frac{a^2}{4} \\
z_3 &= -(\frac{A+B}{2}) - i(\frac{A-B}{2})\sqrt{3} + \frac{a^2}{4}
\end{aligned} \tag{C.96}
$$

From these, we can obtain the roots of (C.91) as

$$
\begin{aligned}
y_1 &= \frac{\sqrt{z_1} + \sqrt{z_2} + \sqrt{z_3}}{2} \\
y_2 &= \frac{\sqrt{z_1} - \sqrt{z_2} - \sqrt{z_3}}{2} \\
y_3 &= \frac{-\sqrt{z_1} + \sqrt{z_2} - \sqrt{z_3}}{2} \\
y_4 &= \frac{-\sqrt{z_1} - \sqrt{z_2} + \sqrt{z_3}}{2}
\end{aligned} \tag{C.97}
$$

Finally, these four roots of Eq. (C.91) give us the roots of (C.88), as

$$
\begin{aligned}
v_1 &= \frac{\sqrt{z_1} + \sqrt{z_2} + \sqrt{z_3}}{2} + \frac{a}{4} \\
v_2 &= \frac{\sqrt{z_1} - \sqrt{z_2} - \sqrt{z_3}}{2} + \frac{a}{4} \\
v_3 &= \frac{-\sqrt{z_1} + \sqrt{z_2} - \sqrt{z_3}}{2} + \frac{a}{4} \\
v_4 &= \frac{-\sqrt{z_1} - \sqrt{z_2} + \sqrt{z_3}}{2} + \frac{a}{4}
\end{aligned} \tag{C.98}
$$

Summarizing, we have four choices for $v_*$ :

$$
\begin{aligned}
v_1 &= \frac{\sqrt{z_1} + \sqrt{z_2} + \sqrt{z_3}}{2} + \frac{a}{4} \\
v_2 &= \frac{\sqrt{z_1} - \sqrt{z_2} - \sqrt{z_3}}{2} + \frac{a}{4} \\
v_3 &= \frac{-\sqrt{z_1} + \sqrt{z_2} - \sqrt{z_3}}{2} + \frac{a}{4} \\
v_4 &= \frac{-\sqrt{z_1} - \sqrt{z_2} + \sqrt{z_3}}{2} + \frac{a}{4}
\end{aligned} \tag{C.99}
$$

where

$$
z_1 = A + B + \frac{a^2}{4}
$$

$$
\begin{aligned}
z_2 &= -(\frac{A+B}{2}) + i(\frac{A-B}{2})\sqrt{3} + \frac{a^2}{4} \\
z_3 &= -(\frac{A+B}{2}) - i(\frac{A-B}{2})\sqrt{3} + \frac{a^2}{4}
\end{aligned}
\tag{C.100}
$$

and

$$
A = [\frac{b^2 - a^2c}{2} + \sqrt{(\frac{a^2 - b^2}{2})^2 + (\frac{4c - ab}{3})^3}]^{1/3} \tag{C.101}
$$

$$
B = [\frac{b^2 - a^2c}{2} - \sqrt{(\frac{a^2 - b^2}{2})^2 + (\frac{4c - ab}{3})^3}]^{1/3} \tag{C.102}
$$

with $a$, $b$, $c$ as in (C.89).

Of these four choices for $v_*$, we choose the one that is real, satisfies the constraint $v_* \in [\theta, 1]$, and minimizes $J(v)$. [This can be checked by examining $J(\theta)$, $J(1)$ and the value of the second drivative of $J$ at the root.] As mentioned earlier, if there is more than one root satisfying the constraints, we pick the one at which $J(v)$ attains the minimum value.

Summarizing, we have

$$
\mu_{\tilde{\mathrm{F}} \sqcap \tilde{\mathrm{G}}}(\theta) = e^{-\frac{1}{2}[(\frac{v_* - m_f}{\sigma_f})^2 + (\frac{\frac{\theta}{v_*} - m_g}{\sigma_g})^2]} \tag{C.103}
$$

where $v_*$ is obtained by solving (C.82) if $\theta < \theta_c$ and $v_* = 1$ if $\theta \geq \theta_c$, where $\theta_c$ is given in (C.87). Observe that computations begin by choosing a value for $\theta \in [0, 1]$ and must be repeated for every $\theta \in [0, 1]$, so that $\theta$ must, in practice, be discretized.

Figures 2.12 - 2.15 show some examples of Gaussian curves and the result of the *meet* operation between them. The *meet* curve in all the figures was obtained numerically. The order of the curves (i.e., the order $\{m_f, m_g\}$ or $\{\sigma_f, \sigma_g\}$) is not important, which means that the *meet* operation is commutative, as we would expect it to be.

## C.7 Solving for the Gaussian Meet Approximation

The problem of maximizing the RHS of (2.53) reduces to the problem of minimizing the objective function

$$H(v) = (\frac{v - m_f}{\sigma_f})^2 + (\frac{\theta - m_g v}{k\sigma_g})^2 \tag{C.104}$$

This minimization also needs to be performed with the constraint $v \in [\theta, 1]$; however, for simplicity, first we minimize $H$ unconstrained and then handle the constraint. Observe that $m_f, m_g, \sigma_f, \sigma_g$ are all positive (in addition, we will always have $m_f, m_g \in [0, 1]$). It can be easily seen that $H$ is convex ($H'' = 2/\sigma_f^2 + 2(m_g/k\sigma_g)^2 > 0$); therefore, equating the first derivative of $H$ to zero (and assuming that the infimum is obtained at $v = v^*$), we get

$$\begin{aligned} 2(\frac{v^* - m_f}{\sigma_f})(\frac{1}{\sigma_f}) + 2(\frac{\theta - m_g v^*}{k\sigma_g})(\frac{-m_g}{k\sigma_g}) &= 0 \\ v^*(\frac{1}{\sigma_f^2} + \frac{m_g^2}{k^2\sigma_g^2}) &= \theta\frac{m_g}{k^2\sigma_g^2} + \frac{m_f}{\sigma_f^2} \qquad \text{(C.105)} \\ v^* &= \frac{\theta m_g \sigma_f^2 + m_f k^2 \sigma_g^2}{m_g^2\sigma_f^2 + k^2\sigma_g^2} \qquad \text{(C.106)} \end{aligned}$$

Substituting (C.106) into (C.104), we get

$$\inf_v H(v) = (\frac{\theta - m_f m_g}{\sqrt{m_g^2\sigma_f^2 + k^2\sigma_g^2}})^2 \tag{C.107}$$

Now, let us handle the constraint. Since $H$ is convex in $v$, $H$ is monotonic increasing for $v > v^*$ and monotonic decreasing for $v < v^*$; therefore, if $v^* < \theta$, the minimum in the constraint set is fixed at $v = \theta$, and, if $v^* > 1$, the constrained minimum is fixed at $v = 1$. Otherwise, $v^*$ is as in (C.106).

Observe that the condition $v^* \in [\theta, 1]$ translates into conditions on $\theta$ and also depends on the parameters $m_f$, $\sigma_f$, $m_g$, $\sigma_g$ and $k$, since all of them appear on the RHS of (C.106). Next, we analyse these conditions.

From (C.106), we can see that $v^*$ is affine in $\theta$. Let

$$a = \frac{m_g \sigma_f^2}{m_g^2 \sigma_f^2 + k^2 \sigma_g^2} \quad (C.108)$$

$$b = \frac{m_f k^2 \sigma_g^2}{m_g^2 \sigma_f^2 + k^2 \sigma_g^2} \quad (C.109)$$

Then,

$$v^* = a\theta + b, \quad (C.110)$$

i.e., if we plot $v^*$ versus $\theta$, we get a straight line. Figures C.1 (a) - (d) show some examples.

Observe that $a \geq 0$ and $0 \leq b \leq 1$ always. For any $b$, if $a$ is such that the portion of the line in $[0,1]$ is contained completely in the area above the line $v^* = \theta$ and below the line $v^* = 1$ [Fig. C.1 (a)], then the constrained minimum is always equal to the unconstrained minimum. Let's call the critical value of $a$ that achieves this $a_c$. To find $a_c$, we use the condition that $a_c\theta + b = 1$ when $\theta = 1$ (this condition is required for the portion of the line in $[0,1]$ to be contained above $v^* = \theta$ and below $v^* = 1$). This gives us

$$a_c = 1 - b \quad (C.111)$$

If $a < a_c$, after some critical value of $\theta$, $\theta_{c1}$, in $[0,1]$ [Fig. C.1 (b)], $v^* < \theta$. $\theta_{c1}$ is the point of intersection of the lines $v^* = a\theta + b$ and $v^* = \theta$; therefore, $\theta_{c1}$ can be found as

$$a\theta_{c1} + b = \theta_{c1}$$
$$\theta_{c1} = \frac{b}{1-a} \quad (C.112)$$

where $a$ and $b$ are as in (C.108) and (C.109).

Similarly, we can see from Fig. C.1 (c) that when $a > a_c$, after some other critical value of $\theta$, $\theta_{c2}$, $v^* > 1$. $\theta_{c2}$ is the point of intersection of the lines $v^* = a\theta + b$ and $v^* = 1$; therefore,

$$a\theta_{c2} + b = 1$$
$$\theta_{c2} = \frac{1-b}{a} \quad (C.113)$$

where $a$ and $b$ are as in (C.108) and (C.109).

The above discussion can be summarized in terms of three cases, as follows : Let $\theta_{c1}$ and $\theta_{c2}$ be as in (C.112) and (C.113), respectively; then,

1. $\underline{a < a_c}$

$$v^* = \begin{cases} \dfrac{\theta m_g \sigma_f^2 + m_f k^2 \sigma_g^2}{m_g^2 \sigma_f^2 + k^2 \sigma_g^2} & ; \ \theta \le \theta_{c1} \\ \theta & ; \ \theta > \theta_{c1} \end{cases} \tag{C.114}$$

Substituting (C.114) into (C.104), we get

$$\inf_{v \in [\theta,1]} H(v) = \begin{cases} \left(\dfrac{\theta - m_f m_g}{\sqrt{m_g^2 \sigma_f^2 + k^2 \sigma_g^2}}\right)^2 & ; \ \theta \le \theta_{c1} \\ \left(\dfrac{\theta - m_f}{\sigma_f}\right)^2 + \theta^2 \left(\dfrac{1 - m_g}{k\sigma_g}\right)^2 & ; \ \theta > \theta_{c1} \end{cases} \tag{C.115}$$

Substituting (C.115) into (2.53), we obtain

$$\mu_{\tilde{E}}(\theta) = \begin{cases} e^{-\frac{1}{2}\left(\frac{\theta - m_f m_g}{\sqrt{m_g^2 \sigma_f^2 + k^2 \sigma_g^2}}\right)^2} & ; \ \theta \le \theta_{c1} \\ e^{-\frac{1}{2}\left(\frac{\theta - m_f}{\sigma_f}\right)^2} e^{-\frac{\theta^2}{2}\left(\frac{1 - m_g}{k\sigma_g}\right)^2} & ; \ \theta > \theta_{c1} \end{cases} \tag{C.116}$$

2. $\underline{a = a_c}$

$$v^* = \frac{\theta m_g \sigma_f^2 + m_f k^2 \sigma_g^2}{m_g^2 \sigma_f^2 + k^2 \sigma_g^2} \quad ; \quad \theta \in [0,1] \tag{C.117}$$

Substituting (C.117) into (C.104), we get (C.107), i.e.,

$$\inf_{v \in [\theta,1]} H(v) = \left(\frac{\theta - m_f m_g}{\sqrt{m_g^2 \sigma_f^2 + k^2 \sigma_g^2}}\right)^2 \quad ; \quad \theta \in [0,1] \tag{C.118}$$

Substituting (C.118) into (2.53), we obtain

$$\mu_{\tilde{E}}(\theta) = e^{-\frac{1}{2}\left(\frac{\theta - m_f m_g}{\sqrt{m_g^2 \sigma_f^2 + k^2 \sigma_g^2}}\right)^2} \quad ; \quad \theta \in [0,1] \tag{C.119}$$

3. $\underline{a > a_c}$

$$v^* = \begin{cases} \dfrac{\theta m_g \sigma_f^2 + m_f k^2 \sigma_g^2}{m_g^2 \sigma_f^2 + k^2 \sigma_g^2} & ; \ \theta \le \theta_{c2} \\ 1 & ; \ \theta > \theta_{c2} \end{cases} \tag{C.120}$$

Substituting (C.120) into (C.104), we get

$$\inf_{v\in[\theta,1]} H(v) = \begin{cases} \left(\dfrac{\theta - m_f m_g}{\sqrt{m_g^2\sigma_f^2 + k^2\sigma_g^2}}\right)^2 & ; \ \theta \le \theta_{c2} \\ \left(\dfrac{1-m_f}{\sigma_f}\right)^2 + \left(\dfrac{\theta - m_g}{k\sigma_g}\right)^2 & ; \ \theta > \theta_{c2} \end{cases} \tag{C.121}$$

Substituting (C.121) into (2.53), we obtain

$$\mu_{\tilde{\mathrm{E}}}(\theta) = \begin{cases} e^{-\frac{1}{2}\left(\frac{\theta - m_f m_g}{\sqrt{m_g^2\sigma_f^2 + k^2\sigma_g^2}}\right)^2} & ; \ \theta \le \theta_{c2} \\ e^{-\frac{1}{2}\left(\frac{1-m_f}{\sigma_f}\right)^2} e^{-\frac{1}{2}\left(\frac{\theta - m_g}{k\sigma_g}\right)^2} & ; \ \theta > \theta_{c2} \end{cases} \tag{C.122}$$

Now, we come back to the question of choosing an expression for $\mu_{\tilde{\mathrm{F}}\sqcap\tilde{\mathrm{G}}}$. Recall, that we solved this modified optimization problem because we wanted to find a simple expression for $\tilde{\mu}_{\tilde{\mathrm{F}}\sqcap\tilde{\mathrm{G}}}$, and it was for this reason that we simplified the actual optimization problem. Although (C.116), (C.119) and (C.122) give the exact solution to the simplified problem, the expressions are still too complicated. Even if we were to accept (C.116), (C.119) and (C.122) as they are, we are still going to have an approximate solution to the actual problem; therefore, it seems very reasonable to choose the simplest of the three possible expressions for $\mu_{\tilde{\mathrm{E}}}(\theta)$ and just use that as our approximation of $\mu_{\tilde{\mathrm{F}}\sqcap\tilde{\mathrm{G}}}(\theta)$. So, we choose the expression for the case $a = a_c$ as the required approximation (in effect, this is equivalent to simplifying the problem even further by disregarding the constraints $v^* \ge \theta$ and $v^* \le 1$); therefore,

$$\mu_{\tilde{\mathrm{F}}\sqcap\tilde{\mathrm{G}}}(\theta) \approx e^{-\frac{1}{2}\left(\frac{\theta - m_f m_g}{\sqrt{m_g^2\sigma_f^2 + k^2\sigma_g^2}}\right)^2} \tag{C.123}$$

As explained in Section 2.3.2, this expression is also consistent with the type-1 case.

Now, we have to choose some value for $k$. As explained in Section 2.3.2, we need some value in $[0,1]$. Since the *meet* operation is commutative, we want our approximation to also be commutative. This will make generalization to the case of more than two Gaussians easy. By observing (C.123), it is apparent that if we choose

$k = m_f$, the approximation becomes commutative (if we interchange $\{m_f, \sigma_f\}$ and $\{m_g, \sigma_g\}$, we still get the same result); so,

$$\mu_{\tilde{F} \sqcap \tilde{G}}(\theta) \approx e^{-\frac{1}{2}\left(\frac{\theta - m_f m_g}{\sqrt{m_g^2 \sigma_f^2 + m_f^2 \sigma_g^2}}\right)^2} \tag{C.124}$$

□

# C.8 Error Bounds for the Gaussian Meet Approximation

To obtain bounds on the Gaussian approximation error, we first find bounds on the result of the actual *meet* operation between two Gaussians. As explained in Section 2.3.2, just after (2.55), using $k = 1$ ($k = 0$) in (2.53) is equivalent to finding an upper (lower) bound on the result of the actual *meet* operation. Let's find an upper bound first.

## C.8.1 Upper Bound on the Meet between Gaussians

If we just substitute $k = 1$ in the expressions for $\mu_{\tilde{E}}(\theta)$, the resulting curves, generally, will not be Gaussian [see (C.116), (C.119) and (C.122)]. Here, we try to find a Gaussian upper bound for the *meet* operation, to facilitate generalization of this operation to more than two Gaussians. For this purpose, we consider $H(v)$ in (C.104), the objective function for the Gaussian approximation.

For any $k$, the unconstrained infimum of $H(v)$ should always be less than or equal to its constrained infimum; hence,

$$\begin{aligned} \inf_v H(v) &\leq \inf_{v \in [\theta, 1]} H(v) \\ \Rightarrow e^{-\frac{1}{2}[\inf_v H(v)]} &\geq e^{-\frac{1}{2}[\inf_{v \in [\theta,1]} H(v)]} \\ \Rightarrow e^{-\frac{1}{2}\left(\frac{\theta - m_f m_g}{\sqrt{m_g^2 \sigma_f^2 + k^2 \sigma_g^2}}\right)^2} &\geq \mu_{\tilde{E}}(\theta) \end{aligned} \tag{C.125}$$

where, we have made use of (C.107) and (2.53). A suitable upper bound on the result of the *meet* between two Gaussians can therefore be obtained by substituting $k = 1$ in (C.125), i.e.,

$$\mu^U_{\tilde{F} \sqcap \tilde{G}}(\theta) = e^{-\frac{1}{2}\left(\frac{\theta - m_f m_g}{\sqrt{m_g^2 \sigma_f^2 + \sigma_g^2}}\right)^2} \quad \text{(C.126)}$$

Observe that (C.126) is not symmetrical in $\{m_f, \sigma_f\}$ and $\{m_g, \sigma_g\}$, i.e., if we interchange the fuzzy sets $\tilde{F}$ and $\tilde{G}$, we get a different expression for the upper bound,

$$\mu^U_{\tilde{G} \sqcap \tilde{F}}(\theta) = e^{-\frac{1}{2}\left(\frac{\theta - m_g m_f}{\sqrt{m_f^2 \sigma_g^2 + \sigma_f^2}}\right)^2} \quad \text{(C.127)}$$

Both (C.126) and (C.127) give an upper bound for the *meet* between $\tilde{F}$ and $\tilde{G}$; therefore, to ensure that the upper bound is independent of the order of the two Gaussians, we choose the minimum of these two functions as the upper bound [because both (C.126) and (C.127) are upper bounds, the minimum of the two is also an upper bound], i.e.,

$$\mu^U_{\tilde{F} \sqcap \tilde{G}}(\theta) = e^{-\frac{1}{2}\left(\frac{\theta - m_f m_g}{\sigma_{u2}}\right)^2} \quad \text{(C.128)}$$

where

$$\sigma_{u2} = \min\left\{\sqrt{m_g^2 \sigma_f^2 + \sigma_g^2}, \sqrt{m_f^2 \sigma_g^2 + \sigma_f^2}\right\} \quad \text{(C.129)}$$

Let's see how this result generalizes to the *meet* of more than two Gaussians. Suppose that we have to find the *meet* between three type-1 Gaussian fuzzy sets $\tilde{F}_1$, $\tilde{F}_2$ and $\tilde{F}_3$, having means $m_1$, $m_2$ and $m_3$, respectively, and standard deviations $\sigma_1$, $\sigma_2$ and $\sigma_3$, respectively. If we perform the *meet* between $\tilde{F}_1$ and $\tilde{F}_2$ first, (C.128) gives us the following upper bound

$$\mu^U_{\tilde{F}_1 \sqcap \tilde{F}_2}(\theta) = e^{-\frac{1}{2}\left(\frac{\theta - m_1 m_2}{\sigma_{u12}}\right)^2} \quad \text{(C.130)}$$

where

$$\sigma_{u12} = \min\left\{\sqrt{m_1^2 \sigma_2^2 + \sigma_1^2}, \sqrt{m_2^2 \sigma_1^2 + \sigma_2^2}\right\} \quad \text{(C.131)}$$

which can also be rewritten as

$$\sigma_{u12} = \sqrt{\min\{m_1^2 \sigma_2^2 + \sigma_1^2, m_2^2 \sigma_1^2 + \sigma_2^2\}} \quad \text{(C.132)}$$

Now, an upper bound on the *meet* of $\tilde{F}_1$, $\tilde{F}_2$ and $\tilde{F}_3$ can be found by finding the upper bound on the *meet* of $\tilde{F}_3$ and the Gaussian in (C.130), i.e.,

$$\mu^U_{\tilde{F}_1 \sqcap \tilde{F}_2 \sqcap \tilde{F}_3}(\theta) = e^{-\frac{1}{2}(\frac{\theta - m_1 m_2 m_3}{\sigma_{u123}})^2} \tag{C.133}$$

where

$$\begin{aligned}
\sigma_{u123} &= \sqrt{\min\{m_1^2 m_2^2 \sigma_3^2 + \sigma_{u12}^2,\ m_3^2 \sigma_{u12}^2 + \sigma_3^2\}} \\
&= \Big[\min\Big\{m_1^2 m_2^2 \sigma_3^2 + \min\{m_1^2\sigma_2^2 + \sigma_1^2,\ m_2^2\sigma_1^2 + \sigma_2^2\}, \\
&\qquad m_3^2 \min\{m_1^2\sigma_2^2 + \sigma_1^2,\ m_2^2\sigma_1^2 + \sigma_2^2\} + \sigma_3^2\Big\}\Big]^{\frac{1}{2}} \\
&= \Big[\min\Big\{\min\{\sigma_1^2 + m_1^2\sigma_2^2 + m_1^2 m_2^2 \sigma_3^2,\ \sigma_2^2 + m_2^2\sigma_1^2 + m_1^2 m_2^2\sigma_3^2\}, \\
&\qquad \min\{\sigma_3^2 + m_3^2\sigma_2^2 + m_2^2 m_3^2\sigma_1^2,\ \sigma_3^2 + m_3^2\sigma_1^2 + m_1^2 m_3^2 \sigma_2^2\}\Big\}\Big]^{\frac{1}{2}} \\
&= \Big[\min\Big\{\sigma_1^2 + m_1^2\sigma_2^2 + m_1^2 m_2^2\sigma_3^2,\ \sigma_2^2 + m_2^2\sigma_1^2 + m_1^2 m_2^2 \sigma_3^2, \\
&\qquad \sigma_3^2 + m_3^2\sigma_2^2 + m_2^2 m_3^2 \sigma_1^2,\ \sigma_3^2 + m_3^2\sigma_1^2 + m_1^2 m_3^2\sigma_2^2\Big\}\Big]^{\frac{1}{2}}
\end{aligned} \tag{C.134}$$

This expression is also not symmetric in $\tilde{F}_1$, $\tilde{F}_2$ and $\tilde{F}_3$, i.e.,

$$\mu^U_{\tilde{F}_2 \sqcap \tilde{F}_3 \sqcap \tilde{F}_1}(\theta) = e^{-\frac{1}{2}(\frac{\theta - m_1 m_2 m_3}{\sigma_{u231}})^2} \tag{C.135}$$

where

$$\begin{aligned}
\sigma_{u231} &= \Big[\min\Big\{\sigma_1^2 + m_1^2\sigma_2^2 + m_1^2 m_2^2\sigma_3^2,\ \sigma_1^2 + m_1^2\sigma_3^2 + m_1^2 m_3^2\sigma_2^2, \\
&\qquad \sigma_2^2 + m_2^2\sigma_3^2 + m_2^2 m_3^2\sigma_1^2,\ \sigma_3^2 + m_3^2\sigma_2^2 + m_2^2 m_3^2\sigma_1^2\Big\}\Big]^{\frac{1}{2}},
\end{aligned} \tag{C.136}$$

and,

$$\mu^U_{\tilde{F}_1 \sqcap \tilde{F}_3 \sqcap \tilde{F}_2}(\theta) = e^{-\frac{1}{2}(\frac{\theta - m_1 m_2 m_3}{\sigma_{u132}})^2} \tag{C.137}$$

where

$$\sigma_{u132} = \Big[\min\Big\{\sigma_2^2 + m_2^2\sigma_1^2 + m_1^2 m_2^2\sigma_3^2,\ \sigma_2^2 + m_2^2\sigma_3^2 + m_2^2 m_3^2 \sigma_1^2,$$

$$\sigma_1^2 + m_1^2\sigma_3^2 + m_1^2m_3^2\sigma_2^2,\ \sigma_3^2 + m_3^2\sigma_1^2 + m_1^2m_3^2\sigma_2^2\}\Big]^{\frac{1}{2}} \quad \text{(C.138)}$$

Observe that we have considered all the possible orderings of $\tilde{F}_1$, $\tilde{F}_2$ and $\tilde{F}_3$ that would give us distinct results for $\mu^U(\theta)$, e.g., since the expression for the upper bound for the *meet* of two Gaussians is commutative, $\mu^U_{(\tilde{F}_1\sqcap\tilde{F}_2)\sqcap\tilde{F}_3} = \mu^U_{(\tilde{F}_2\sqcap\tilde{F}_1)\sqcap\tilde{F}_3}$.

We choose the minimum of the three Gaussians in (C.133), (C.135) and (C.137) as the final upper bound, i.e.,

$$\mu^U_{\tilde{F}_1\sqcap\tilde{F}_2\sqcap\tilde{F}_3} = e^{-\frac{1}{2}(\frac{\theta - m_1m_2m_3}{\sigma_{u3}})^2} \quad \text{(C.139)}$$

where (some terms are common to $\sigma_{u123}$, $\sigma_{u231}$ and $\sigma_{u132}$)

$$\begin{aligned} \sigma_{u3} &= \min\{\sigma_{u123},\ \sigma_{u231},\ \sigma_{132}\} \\ &= \Big[\min\big\{\sigma_1^2 + m_1^2\sigma_2^2 + m_1^2m_2^2\sigma_3^2,\ \sigma_1^2 + m_1^2\sigma_3^2 + m_1^2m_3^2\sigma_2^2, \\ &\qquad \sigma_2^2 + m_2^2\sigma_1^2 + m_1^2m_2^2\sigma_3^2,\ \sigma_2^2 + m_2^2\sigma_3^2 + m_2^2m_3^2\sigma_1^2, \\ &\qquad \sigma_3^2 + m_3^2\sigma_1^2 + m_1^2m_3^2\sigma_2^2,\ \sigma_3^2 + m_3^2\sigma_2^2 + m_2^2m_3^2\sigma_1^2\big\}\Big]^{\frac{1}{2}} \end{aligned} \quad \text{(C.140)}$$

Continuing in this fashion, the upper bound on the *meet* between $n$ Gaussians $\tilde{F}_1, \ldots, \tilde{F}_n$ having means $m_1, \ldots, m_n$ and standard deviations $\sigma_1, \ldots, \sigma_n$, respectively, is given as

$$\mu^U_{\sqcap_{i=1}^n \tilde{F}_i} = \exp\{-\frac{1}{2}(\frac{\theta - \prod_{i=1}^n m_i}{\bar{\sigma}_u})^2\} \quad \text{(C.141)}$$

where

$$\bar{\sigma}_u = \min\{\Sigma_1, \Sigma_2, \ldots, \Sigma_{n!}\} \quad \text{(C.142)}$$

where

$$\Sigma_j = \sqrt{c_{i_1}^2\sigma_{i_1}^2 + c_{i_2}^2\sigma_{i_2}^2 + \quad \cdots \quad + c_{i_n}^2\sigma_{i_n}^2} \quad ; \quad j = 1, \ldots, n! \quad \text{(C.143)}$$

where $\{i_1, \ldots, i_n\}$ indicates a permutation of $\{1, 2, \ldots, n\}$, and the $c_{i_k}$s ($k = 1, \ldots, n$) are calculated as follows :

$$\begin{aligned} c_{i_1} &= 1\,, \\ c_{i_k} &= m_{i_{(k-1)}} c_{i_{(k-1)}} \quad \text{for} \quad k = 2, \ldots, n\,. \end{aligned} \quad \text{(C.144)}$$

In order to illustrate the use and validity of (C.143) and (C.144), we consider the following example.

**Example C.1** For $n = 3$, $\bar{\sigma}_u$ can be calculated, as follows. There are $3! = 6$ possible permutations of $\{1, 2, 3\}$. For each one of these permutations, we use (C.144) to calcuate the $c_{i_j}$'s, and (C.143) to calculate the $\Sigma_j$'s :

$$\begin{array}{lll}
\{i_1, i_2, i_3\} = \{1, 2, 3\} & : \; c_1 = 1; & c_2 = m_1 c_1 = m_1; \quad c_3 = m_2 c_2 = m_1 m_2 \\
 & & \Sigma_1 = \sqrt{\sigma_1^2 + m_1^2 \sigma_2^2 + m_1^2 m_2^2 \sigma_3^2} \\
\{i_1, i_2, i_3\} = \{1, 3, 2\} & : \; c_1 = 1; & c_3 = m_1 c_1 = m_1; \quad c_2 = m_3 c_3 = m_1 m_3 \\
 & & \Sigma_2 = \sqrt{\sigma_1^2 + m_1^2 \sigma_3^2 + m_1^2 m_3^2 \sigma_2^2} \\
\{i_1, i_2, i_3\} = \{2, 1, 3\} & : \; c_2 = 1; & c_1 = m_2 c_2 = m_2; \quad c_3 = m_1 c_1 = m_1 m_2 \\
 & & \Sigma_3 = \sqrt{\sigma_2^2 + m_2^2 \sigma_1^2 + m_1^2 m_2^2 \sigma_3^2} \\
\{i_1, i_2, i_3\} = \{2, 3, 1\} & : \; c_2 = 1; & c_3 = m_2 c_2 = m_2; \quad c_1 = m_3 c_3 = m_2 m_3 \\
 & & \Sigma_4 = \sqrt{\sigma_2^2 + m_2^2 \sigma_3^2 + m_2^2 m_3^2 \sigma_1^2} \\
\{i_1, i_2, i_3\} = \{3, 1, 2\} & : \; c_3 = 1; & c_1 = m_3 c_3 = m_3; \quad c_2 = m_1 c_1 = m_1 m_3 \\
 & & \Sigma_5 = \sqrt{\sigma_3^2 + m_3^2 \sigma_1^2 + m_1^2 m_3^2 \sigma_2^2} \\
\{i_1, i_2, i_3\} = \{3, 2, 1\} & : \; c_3 = 1; & c_2 = m_3 c_3 = m_3; \quad c_1 = m_2 c_2 = m_2 m_3 \\
 & & \Sigma_6 = \sqrt{\sigma_3^2 + m_3^2 \sigma_2^2 + m_2^2 m_3^2 \sigma_1^2}
\end{array} \tag{C.145}$$

Using $\Sigma_1, \ldots, \Sigma_6$ from (C.145) in (C.142), we can verify that the $\bar{\sigma}_u$ we obtain, is the same as in (C.140). □

As is apparent from (C.142), the calculation of $\sigma_u$ is computationally intensive. To find an upper bound on the *meet* between 4 Gaussians, we need to find the minimum of $4! = 24$ terms; when 5 Gaussians are involved, the number of terms rises to 120, and so on. Observe, though, that the minimum in (C.142) gives the tightest of all bounds (i.e., tightest of the bounds that we have derived). If one just wants to find any upper bound, the minimization in (C.142) is not necessary. Each of the $\Sigma_j$s ($j = 1, 2, \ldots, n!$) in (C.142) is the standard deviation of one of the upper bounds of the *meet*; so, we can just choose one of the $n!$ terms and use it in (C.141) to get an upper bound on the *meet*.

### C.8.2 Effect of Clipping

So far, in our derivations, we have assumed that we have perfect Gaussians, i.e., we have neglected the fact that the actual curves are Gaussians contained in [0, 1] and may, therefore, be clipped (i.e., any portion of the Gaussians lying outside the interval [0, 1] is cut-off); however, this clipping does not change the upper bounds in (C.128) and (C.141). The reason can be explained as follows. Let $\tilde{F}_c$ and $\tilde{G}_c$ be clipped type-1 fuzzy sets having Gaussian membership functions contained in [0, 1]. Though the membership functions of $\tilde{F}_c$ and $\tilde{G}_c$ are defined only on [0, 1], in numerical calculations the membership functions are treated as if they are 0 before 0 and after 1. Therefore, $\mu_{\tilde{F}_c} \leq \mu_{\tilde{F}}$ and $\mu_{\tilde{G}_c} \leq \mu_{\tilde{G}}$, where $\tilde{F}$ and $\tilde{G}$ are type-1 sets whose membership functions are perfect (unclipped) Gaussians. (Figure C.2 shows an example of clipped and unclipped Gaussians.) Consequently, $\mu_{\tilde{F}_c \sqcap \tilde{G}_c} \leq \mu_{\tilde{F} \sqcap \tilde{G}}$ and therefore the upper bound derived above also holds in the case of clipped Gaussians. We will have to consider the explicit effects of clipping when deriving the lower bound for the *meet* in Section C.8.3.

### C.8.3 Lower Bound on the Meet between Gaussians

As we have seen earlier, substituting $k = 0$ into (2.53) is analogous to finding a lower bound on the *meet* between two Gaussians. Now, however, we also have to include the clipping effects mentioned in Section C.8.2, because a lower bound that assumes perfect (unclipped) Gaussians may not work for clipped Gaussians, since, as mentioned in Section C.8.2, $\mu_{\tilde{F}_c \sqcap \tilde{G}_c} \leq \mu_{\tilde{F} \sqcap \tilde{G}}$, where $\tilde{F}_c$ and $\tilde{G}_c$ are clipped versions of Gaussian type-1 sets $\tilde{F}$ and $\tilde{G}$. So, instead of just substituting $k = 0$ into (2.53), we go back to the beginning of the derivation for the Gaussian meet approximation.

Recall that using $k = 0$ is equivalent to assuming that one of the Gaussians has zero standard deviation (Section 2.3.2), i.e., it is equivalent to assuming that type-1 fuzzy set $\tilde{F}$ (or $\tilde{G}$) has a membership equal to 1 at $m_f$ (or $m_g$) and equal to 0 at all other points. Let's see what happens if this is really the case.

Assume that $\mu_{\tilde{F}}(m_f) = 1$ and $\mu_{\tilde{F}}(\theta) = 0$ for $\theta \neq m_f$, and $\mu_{\tilde{G}}(\theta)$ is a Gaussian contained in $[0, 1]$, i.e.,

$$\mu_{\tilde{G}}(\theta) = \begin{cases} e^{-\frac{1}{2}(\frac{\theta - m_g}{\sigma_g})^2} & ; \quad \theta \in [0, 1] \\ 0 & ; \quad \text{otherwise} \end{cases} \tag{C.146}$$

The result of the *meet* of $\tilde{F}$ and $\tilde{G}$ is just a scaled version of $\tilde{G}$, i.e.,

$$\begin{aligned} \mu_{\tilde{F}\sqcap\tilde{G}}(\theta) &= \mu_{\tilde{G}}(\theta/m_f) \\ &= \begin{cases} e^{-\frac{1}{2}(\frac{\frac{\theta}{m_f} - m_g}{\sigma_g})^2} & ; \quad \frac{\theta}{m_f} \in [0, 1] \\ 0 & ; \quad \text{otherwise} \end{cases} \\ &= \begin{cases} e^{-\frac{1}{2}(\frac{\theta - m_f m_g}{m_f \sigma_g})^2} & ; \quad \theta \in [0, m_f] \\ 0 & ; \quad \text{otherwise} \end{cases} \end{aligned} \tag{C.147}$$

Equation (C.147) follows from the definition of the *meet* operation under product $t$-norm [see Eq. (2.34) and also Section 2.3.1]. In Section 2.3.1, we ignored the clipping effects mentioned in Section C.8.2; but, here, we have to take them into account to make sure that the lower bound holds in all possible cases. Since the Gaussian $\mu_{\tilde{G}}(\theta)$ is contained in $[0, 1]$, the resulting function in (C.147) is nonzero only in $[0, m_f]$.

Now, if we assume that $G$ is the singleton, i.e., if $\mu_{\tilde{G}}(m_g) = 0$ and $\mu_{\tilde{G}}(\theta) = 0$ for $\theta \neq m_g$, and $\mu_{\tilde{F}}(\theta)$ is a Gaussian contained in $[0, 1]$, then we get

$$\mu_{\tilde{F}\sqcap\tilde{G}}(\theta) = \begin{cases} e^{-\frac{1}{2}(\frac{\theta - m_f m_g}{m_g \sigma_f})^2} & ; \quad \theta \in [0, m_g] \\ 0 & ; \quad \text{otherwise} \end{cases} \tag{C.148}$$

The actual lower bound can be taken as the maximum (since each of them is a lower bound) of the results in (C.147) and (C.148). If we assume that $m_f \leq m_g$, then the lower bound is

$$\mu^L_{\tilde{F}\sqcap\tilde{G}}(\theta) = \begin{cases} \exp\left\{ -\frac{1}{2}(\frac{\theta - m_f m_g}{\max\{m_f\sigma_g, m_g\sigma_f\}})^2 \right\} & ; \quad \theta \in [0, m_f] \\ \exp\left\{ -\frac{1}{2}(\frac{\theta - m_f m_g}{m_g \sigma_f})^2 \right\} & ; \quad \theta \in [m_f, m_g] \\ 0 & ; \quad \text{otherwise} \end{cases} \tag{C.149}$$

If we assume that $m_g \leq m_f$, the lower bound is

$$\mu^L_{\tilde{F}\sqcap\tilde{G}}(\theta) = \begin{cases} \exp\left\{ -\frac{1}{2}\left(\frac{\theta - m_f m_g}{\max\{m_f\sigma_g, m_g\sigma_f\}}\right)^2 \right\} & ; \quad \theta \in [0, m_g] \\ \exp\left\{ -\frac{1}{2}\left(\frac{\theta - m_f m_g}{m_f\sigma_g}\right)^2 \right\} & ; \quad \theta \in [m_g, m_f] \\ 0 & ; \quad \text{otherwise} \end{cases} \tag{C.150}$$

To simplify (C.149) and (C.150) a little bit, we ignore the part of the curve lying outside $[0, \min\{m_f, m_g\}]$. Doing this will make the lower bound a little bit loose, but will let us generalize to the *meet* of more than two sets easily. Hence, without any assumptions on $m_f$ and $m_g$,

$$\mu^L_{\tilde{F}\sqcap\tilde{G}}(\theta) = \begin{cases} \exp\left\{ -\frac{1}{2}\left(\frac{\theta - m_f m_g}{\max\{m_f\sigma_g, m_g\sigma_f\}}\right)^2 \right\} & ; \quad \theta \in [0, \min\{m_f, m_g\}] \\ 0 & ; \quad \text{otherwise} \end{cases} \tag{C.151}$$

A generalization of (C.151) to the case of more than two Gaussians is a bit tedious; therefore, we take a different approach as illustrated by the following :

**Example C.2** Consider the meet between three Gaussians $\tilde{F}_1$, $\tilde{F}_2$, $\tilde{F}_3$, in $[0, 1]$, with means $m_1$, $m_2$, $m_3$ and standard deviations $\sigma_1$, $\sigma_2$, $\sigma_3$. We consider three cases to find a lower bound for the *meet*. First, we assume that $\tilde{F}_2$ and $\tilde{F}_3$ are singletons at $m_2$ and $m_3$, respectively (zero standard deviations), and $\tilde{F}_1$ is a Gaussian contained in $[0, 1]$ with mean $m_1$ and standard deviation $\sigma_1$. This is equivalent to finding the *meet* between $\tilde{F}_1$ and a singleton at $m_2 m_3$, because *meet* under product $t$-norm is multiplication under product $t$-norm. Let the result of the *meet* be $\tilde{F}^1_{23}$. Proceeding as in the derivation of (C.147) and (C.148), the membership function of $\tilde{F}^1_{23}$ can be obtained as

$$\mu_{\tilde{F}^1_{23}} = \begin{cases} \exp\left\{ -\frac{1}{2}\left(\frac{\theta - m_1 m_2 m_3}{m_2 m_3 \sigma_1}\right)^2 \right\} & ; \quad \theta \in [0, m_2 m_3] \\ 0 & ; \quad \text{otherwise} \end{cases} \tag{C.152}$$

Then, we find the *meet* between $\tilde{F}_1$, $\tilde{F}_2$ and $\tilde{F}_3$ by assuming that $\tilde{F}_1$ and $\tilde{F}_3$ are singletons at $m_1$ and $m_3$, respectively, and $\tilde{F}_2$ is not a singleton. Let us call the result of this *meet* operation $\tilde{F}^2_{13}$. Its membership function is

$$\mu_{\tilde{F}^2_{13}} = \begin{cases} \exp\left\{ -\frac{1}{2}\left(\frac{\theta - m_1 m_2 m_3}{m_1 m_3 \sigma_2}\right)^2 \right\} & ; \quad \theta \in [0, m_1 m_3] \\ 0 & ; \quad \text{otherwise} \end{cases} \tag{C.153}$$

Finally, we find the *meet* between $\tilde{F}_1$, $\tilde{F}_2$ and $\tilde{F}_3$ by assuming that $\tilde{F}_1$ and $\tilde{F}_2$ are singletons at $m_1$ and $m_2$, respectively, and $\tilde{F}_3$ is not a singleton. Let us call the result of this *meet* operation $\tilde{F}^3_{12}$. Its membership function is

$$\mu_{\tilde{F}^3_{12}} = \begin{cases} \exp\left\{ -\frac{1}{2}\left(\frac{\theta - m_1 m_2 m_3}{m_1 m_2 \sigma_3}\right)^2 \right\} & ; \quad \theta \in [0, m_1 m_2] \\ 0 & ; \quad \text{otherwise} \end{cases} \tag{C.154}$$

Since $\mu_{\tilde{F}^1_{23}}$, $\mu_{\tilde{F}^2_{13}}$ and $\mu_{\tilde{F}^3_{12}}$, each give a lower bound on the *meet* between $\tilde{F}_1$, $\tilde{F}_2$ and $\tilde{F}_3$, we choose the maximum of them [again, as in (C.151), we keep only that part of the maximum function which lies in $[0, \min\{m_1 m_2, m_2 m_3, m_1 m_3\}]$, for simplicity]. This gives us

$$\mu^L_{\tilde{F}_1 \sqcap \tilde{F}_2 \sqcap \tilde{F}_3}(\theta) = \begin{cases} \exp\left\{ -\frac{1}{2}\left(\frac{\theta - m_1 m_2 m_3}{\bar{\sigma}_3}\right)^2 \right\} & ; \quad \theta \in [0, l_3] \\ 0 & ; \quad \text{otherwise} \end{cases} \tag{C.155}$$

where

$$\bar{\sigma}_3 = \max\{m_1 m_2 \sigma_3, m_1 m_3 \sigma_2, m_2 m_3 \sigma_1\}\ , \tag{C.156}$$

and

$$l_3 = \min\{m_1 m_2, m_2 m_3, m_1 m_3\} \tag{C.157}$$

□

Now, suppose that we have $n$ Gaussian fuzzy sets, $\tilde{F}_1, \tilde{F}_2, \ldots, \tilde{F}_n$, in $[0,1]$ having means $m_1, m_2, \ldots, m_n$ and standard deviations $\sigma_1, \sigma_2, \ldots, \sigma_n$. A lower bound on their *meet* is obtained by generalizing (C.155) to the case of $n$ Gaussians, i.e.,

$$\mu^L_{\sqcap_{i=1}^n \tilde{F}_i}(\theta) = \begin{cases} \exp\left\{ -\frac{1}{2}\left(\frac{\theta - \prod_{i=1}^n m_i}{\bar{\sigma}_l}\right)^2 \right\} & ; \quad \theta \in [0, l_n] \\ 0 & ; \quad \text{otherwise} \end{cases} \tag{C.158}$$

where $(i = 1, 2, \cdots, n)$

$$\bar{\sigma}_l = \max\Big\{\sigma_1 \prod_{i;i\neq 1} m_i, \sigma_2 \prod_{i;i\neq 2} m_i, \cdots, \sigma_j \prod_{i;i\neq j} m_i, \cdots, \sigma_n \prod_{i;i\neq n} m_i\Big\} \tag{C.159}$$

and

$$l_n = \min\Big\{ \prod_{i;i\neq 1} m_i, \prod_{i;i\neq 2} m_i, \cdots, \prod_{i;i\neq j} m_i, \cdots, \prod_{i;i\neq n} m_i\Big\} \tag{C.160}$$

Figure C.3 shows some examples of the just-derived upper and lower bounds. Both, the upper and the lower bounds, that we have derived are quite conservative. It may be possible to derive tighter bounds; however, we will not pursue this issue any further.

### C.8.4 Bounds on the Gaussian Approximation Error

Before proceeding to find bounds on the error for the *meet* between Gaussians and the Gaussian approximation of the *meet*, we shall show that, just as the upper and lower bounds derived in this section enclose the actual *meet* curve between them, they also enclose the Gaussian approximation for the *meet*, i.e., we shall show that

$$\mu^L_{\sqcap_{i=1}^n \tilde{F}_i}(\theta) \leq \hat{\mu}_{\sqcap_{i=1}^n \tilde{F}_i}(\theta) \leq \mu^U_{\sqcap_{i=1}^n \tilde{F}_i}(\theta) \quad ; \quad \theta \in [0,1] \tag{C.161}$$

Recall that to find the Gaussian approximation for the *meet* between two Gaussian fuzzy sets, $\tilde{F}$ and $\tilde{G}$, we solved an optimization problem and arrived at the following expression for the Gaussian's standard deviation [see Eq. (C.123)] ,

$$\sigma(k) = \sqrt{m_g^2\sigma_f^2 + k^2\sigma_g^2} \quad , \quad k \in [0,1] \tag{C.162}$$

Obviously, $\sigma(0) \leq \sigma(k) \leq \sigma(1)$ for any $k \in [0,1]$. In particular, $\sigma(0) \leq \sigma(m_f) \leq \sigma(1)$, i.e.,

$$m_g\sigma_f \leq \sqrt{m_g^2\sigma_f^2 + m_f^2\sigma_g^2} \leq \sqrt{m_g^2\sigma_f^2 + \sigma_g^2} \tag{C.163}$$

These results are also true when we change the order of $\tilde{F}$ and $\tilde{G}$; hence,

$$m_f\sigma_g \leq \sqrt{m_g^2\sigma_f^2 + m_f^2\sigma_g^2} \leq \sqrt{m_f^2\sigma_g^2 + \sigma_f^2} \tag{C.164}$$

Combining (C.163) and (C.164), we get

$$\max\{m_f\sigma_g\,,\,m_g\sigma_f\} \le \sqrt{m_f^2\sigma_g^2 + m_g^2\sigma_f^2} \le \min\{\sqrt{m_f^2\sigma_g^2+\sigma_f^2}\,,\,\sqrt{m_g^2\sigma_f^2+\sigma_g^2}\} \tag{C.165}$$

Consequently,

$$e^{-\frac{1}{2}\left(\frac{\theta-m_fm_g}{\max\{m_f\sigma_g,m_g\sigma_f\}}\right)^2} \le e^{-\frac{1}{2}\left(\frac{\theta-m_fm_g}{\sqrt{m_f^2\sigma_g^2+m_g^2\sigma_f^2}}\right)^2} \le e^{-\frac{1}{2}\left(\frac{\theta-m_fm_g}{\min\left\{\sqrt{m_f^2\sigma_g^2+\sigma_f^2},\sqrt{m_g^2\sigma_f^2+\sigma_g^2}\right\}}\right)^2} \tag{C.166}$$

Also, from (C.151), it follows that

$$\mu^L_{\tilde{F}\sqcap\tilde{G}} \le e^{-\frac{1}{2}(\frac{\theta-m_fm_g}{\max\{m_f\sigma_g,m_g\sigma_f\}})^2} \tag{C.167}$$

If we denote the Gaussian approximation in (C.124) by $\hat{\mu}_{\tilde{F}\sqcap\tilde{G}}$, it follows from (C.128), (C.129), (C.166), and (C.167) that

$$\mu^L_{\tilde{F}\sqcap\tilde{G}}(\theta) \le \hat{\mu}_{\tilde{F}\sqcap\tilde{G}}(\theta) \le \mu^U_{\tilde{F}\sqcap\tilde{G}}(\theta) \quad ; \quad \theta \in [0,1] \tag{C.168}$$

This is in general true for the *meet* of any number of Gaussians, as can be verified from (2.59), (C.141), and (C.158) in a similar manner; hence,

$$\mu^L_{\sqcap_{i=1}^n\tilde{F}_i}(\theta) \le \hat{\mu}_{\sqcap_{i=1}^n\tilde{F}_i}(\theta) \le \mu^U_{\sqcap_{i=1}^n\tilde{F}_i}(\theta) \quad ; \quad \theta \in [0,1]$$

Suppose that we have $n$ Gaussian fuzzy sets, $\tilde{F}_1, \tilde{F}_2, \ldots, \tilde{F}_n$, in $[0,1]$ having means $m_1, m_2, \ldots, m_n$ and standard deviations $\sigma_1, \sigma_2, \ldots, \sigma_n$; then, we have shown earlier (in Sections C.8.1 and C.8.3) that

$$\mu^L_{\sqcap_{i=1}^n\tilde{F}_i}(\theta) \le \mu_{\sqcap_{i=1}^n\tilde{F}_i}(\theta) \le \mu^U_{\sqcap_{i=1}^n\tilde{F}_i}(\theta) \quad ; \quad \theta \in [0,1] \tag{C.169}$$

From (C.161) and (C.169), we can see that (dropping the subscript "$\sqcap_{i=1}^n\tilde{F}_i$" for notational convenience) if $\hat{\mu}(\theta) \le \mu(\theta)$,

$$\mu(\theta) - \hat{\mu}(\theta) \le \mu^U(\theta) - \hat{\mu}(\theta) \tag{C.170}$$

Similarly, if $\hat{\mu}(\theta) \geq \mu(\theta)$,

$$\hat{\mu}(\theta) - \mu(\theta) \leq \hat{\mu}(\theta) - \mu^L(\theta) \tag{C.171}$$

From (C.170) and (C.171), we can bound the approximation error as

$$\begin{aligned} \left|\hat{\mu}_{\sqcap_{i=1}^n \tilde{F}_i}(\theta) - \mu_{\sqcap_{i=1}^n \tilde{F}_i}(\theta)\right| \;\leq\; & \max\left\{\left[\mu^U_{\sqcap_{i=1}^n \tilde{F}_i}(\theta) - \hat{\mu}_{\sqcap_{i=1}^n \tilde{F}_i}(\theta)\right],\right. \\ & \left.\left[\hat{\mu}_{\sqcap_{i=1}^n \tilde{F}_i}(\theta) - \mu^L_{\sqcap_{i=1}^n \tilde{F}_i}(\theta)\right]\right\} \;;\; \theta \in [0,1] \end{aligned} \tag{C.172}$$

where $\hat{\mu}_{\sqcap_{i=1}^n \tilde{F}_i}(\theta)$ is given in (2.59), $\mu^U_{\sqcap_{i=1}^n \tilde{F}_i}(\theta)$ is given in (C.141) and $\mu^L_{\sqcap_{i=1}^n \tilde{F}_i}(\theta)$ is given in (C.158).

Equation (C.172) gives an upper bound on the approximation error at any point $\theta \in [0,1]$. To use (C.172), one needs to compute the upper and lower bounds $\left[\mu^U_{\sqcap_{i=1}^n \tilde{F}_i}(\theta) \text{ and } \mu^L_{\sqcap_{i=1}^n \tilde{F}_i}(\theta)\right]$ along with the Gaussian approximation $\left[\hat{\mu}_{\sqcap_{i=1}^n \tilde{F}_i}(\theta)\right]$. The approximation error at $\theta$ is less than or equal to the larger of $\left[\mu^U_{\sqcap_{i=1}^n \tilde{F}_i}(\theta) - \hat{\mu}_{\sqcap_{i=1}^n \tilde{F}_i}(\theta)\right]$ and $\left[\hat{\mu}_{\sqcap_{i=1}^n \tilde{F}_i}(\theta) - \mu^L_{\sqcap_{i=1}^n \tilde{F}_i}(\theta)\right]$.

Observe, from Fig. C.3, that the approximation error is less in the high membership regions than in the low membership regions, and is equal to zero at the point having unity membership (this point is equal to the product of the centers of all the participating Gaussians).

## C.9 Proof of Theorem 2.4

We prove the theorem in two parts : (a) we prove that $\alpha_i \tilde{F}_i + \beta$ is a Gaussian fuzzy number with mean $\alpha_i m_i + \beta$ and standard deviation $|\alpha_i \sigma_i|$; and (b) we prove that $\sum_{i=1}^n \tilde{F}_i$ is a Gaussian fuzzy number with mean $\sum_{i=1}^n m_i$ and standard deviation $\Sigma''$, where

$$\Sigma'' = \begin{cases} \sqrt{\sum_{i=1}^n \sigma_i^2} & , \text{ if product } t\text{-norm is used} \\ \sum_{i=1}^n \sigma_i & , \text{ if minimum } t\text{-norm is used} \end{cases} \tag{C.173}$$

(a) Consider

$$\tilde{F}_i = \int_v e^{-\frac{1}{2}\left(\frac{v-m_i}{\sigma_i}\right)^2} / v \tag{C.174}$$

Multiplying $\tilde{F}_i$ by a constant $\alpha_i(=1/\alpha_i)$ yields [see Section 2.4.1]

$$\begin{aligned}
\alpha_i\tilde{F}_i &= \int_v \left[e^{-\frac{1}{2}\left(\frac{v-m_i}{\sigma_i}\right)^2} \star 1\right]/(\alpha_i v) \\
&= \int_v e^{-\frac{1}{2}\left(\frac{v-m_i}{\sigma_i}\right)^2}/(\alpha_i v) \qquad \text{(C.175)}
\end{aligned}$$

Now, adding a crisp constant $\beta(=1/\beta)$ to $\alpha_i\tilde{F}_i$, we get [see Section 2.4.2]

$$\begin{aligned}
\alpha_i\tilde{F}_i + \beta &= \int_v \left[e^{-\frac{1}{2}\left(\frac{v-m_i}{\sigma_i}\right)^2} \star 1\right]/(\alpha_i v + \beta) \\
&= \int_v e^{-\frac{1}{2}\left(\frac{v-m_i}{\sigma_i}\right)^2}/(\alpha_i v + \beta) \qquad \text{(C.176)}
\end{aligned}$$

Let $\alpha_i v + \beta = v'$; this gives $v = (v' - \beta)/\alpha_i$, which when substituted into (C.176), leads to

$$\begin{aligned}
\alpha_i\tilde{F}_i + \beta &= \int_{v'} \exp\left\{-\frac{1}{2}\left[\frac{\left(\frac{v'-\beta}{\alpha_i}\right) - m_i}{\sigma_i}\right]^2\right\}\Big/ v' \\
&= \int_{v'} \exp\left\{-\frac{1}{2}\left[\frac{v' - (\alpha_i m_i + \beta)}{\alpha_i\sigma_i}\right]^2\right\}\Big/ v' \qquad \text{(C.177)}
\end{aligned}$$

which shows that $\alpha_i\tilde{F}_i + \beta$ is a Gaussian fuzzy number with mean $\alpha_i m_i + \beta$ and standard deviation $|\alpha_i\sigma_i|$. Note that this result does not depend on the kind of $t$-norm used, since $\alpha_i$ and $\beta$ are crisp numbers.

**(b)** Consider $\tilde{F}_1$ and $\tilde{F}_2$, with means $m_1$ and $m_2$ and standard deviations $\sigma_1$ and $\sigma_2$ respectively. The sum of these two fuzzy numbers can be expressed as [see Section 2.4.2]

$$\tilde{F}_1 + \tilde{F}_2 = \int_{v\in\tilde{F}_1}\int_{w\in\tilde{F}_2} e^{-\frac{1}{2}\left(\frac{v-m_1}{\sigma_1}\right)^2} \star e^{-\frac{1}{2}\left(\frac{w-m_2}{\sigma_2}\right)^2}/[v+w] \qquad \text{(C.178)}$$

where $\star$ indicates the chosen $t$-norm.

**(i)** Product $t$-norm : In this case, (C.178) reduces to

$$\tilde{F}_1 + \tilde{F}_2 = \int_{v\in\tilde{F}_1}\int_{w\in\tilde{F}_2} e^{-\frac{1}{2}\left(\frac{v-m_1}{\sigma_1}\right)^2} e^{-\frac{1}{2}\left(\frac{w-m_2}{\sigma_2}\right)^2}/[v+w] \qquad \text{(C.179)}$$

If $\theta$ is an element of $\tilde{F}_1 + \tilde{F}_2$, the membership grade of $\theta$ in $\tilde{F}_1 + \tilde{F}_2$ can be obtained by considering all the $\{v, w\}$ pairs such that $v \in \tilde{F}_1$ and $w \in \tilde{F}_2$ and $v + w = \theta$, multiplying the memberships of $v$ and $w$ in every pair, and, choosing the maximum of all these membership products. In other words,

$$\begin{aligned}\mu_{(\tilde{F}_1+\tilde{F}_2)}(\theta) &= \sup_v e^{-\frac{1}{2}\left(\frac{v-m_1}{\sigma_1}\right)^2} e^{-\frac{1}{2}\left[\frac{(\theta-v)-m_2}{\sigma_2}\right]^2} \\ &= \sup_v e^{-\frac{1}{2}\left[\left(\frac{v-m_1}{\sigma_1}\right)^2+\left(\frac{(\theta-v)-m_2}{\sigma_2}\right)^2\right]} \qquad \text{(C.180)}\end{aligned}$$

Let us call the expression in the square bracket in the exponent of (C.180) $J(v)$, i.e.,

$$J(v) = \left(\frac{v - m_1}{\sigma_1}\right)^2 + \left[\frac{(\theta - v) - m_2}{\sigma_2}\right]^2 \qquad \text{(C.181)}$$

The value of $v$ that maximizes the exponent on the RHS of (C.180) can be obtained by minimizing $J(v)$. Note that $J$ is convex ($J'' = \frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2} > 0$), so equating the first derivative of $J$ to zero (assuming the minimum is reached at $v^*$), we get

$$\begin{aligned}2\left(\frac{v^* - m_1}{\sigma_1}\right)\left(\frac{1}{\sigma_1}\right) + 2\left[\frac{(\theta - v^*) - m_2}{\sigma_2}\right]\left(-\frac{1}{\sigma_2}\right) &= 0 \\ \frac{v^* - m_1}{\sigma_1^2} + \frac{v^* - (\theta - m_2)}{\sigma_2^2} &= 0 \\ v^*\left(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}\right) &= \frac{m_1}{\sigma_1^2} + \frac{\theta - m_2}{\sigma_2^2} \\ v^* &= \frac{m_1\sigma_2^2 + (\theta - m_2)\sigma_1^2}{\sigma_1^2 + \sigma_2^2} \qquad \text{(C.182)}\end{aligned}$$

Substituting (C.182) into (C.181), we get

$$\inf_v J(v) = \left[\frac{\theta - (m_1 + m_2)}{\sqrt{\sigma_1^2 + \sigma_2^2}}\right]^2 \qquad \text{(C.183)}$$

Substituting (C.183) into (C.180), we get

$$\mu_{(\tilde{F}_1+\tilde{F}_2)}(\theta) = e^{-\frac{1}{2}\left[\frac{\theta-(m_1+m_2)}{\sqrt{\sigma_1^2+\sigma_2^2}}\right]^2} \qquad \text{(C.184)}$$

This result generalizes easily to the case of more than two Gaussians. Let $m_{12} = m_1 + m_2$ and $\sigma_{12} = \sqrt{\sigma_1^2 + \sigma_2^2}$, so that $\tilde{F}_1 + \tilde{F}_2$ is a Gaussian with mean $m_{12}$ and standard deviation $\sigma_{12}$. Now, if a third Gaussian fuzzy set $\tilde{F}_3$, with mean $m_3$ and standard deviation $\sigma_3$, adds to this sum, the mean and standard deviation of the resulting Gaussian are

$$m_{123} = m_{12} + m_3 = m_1 + m_2 + m_3 \tag{C.185}$$

$$\sigma_{123} = \sqrt{\sigma_{12}^2 + \sigma_3^2} = \sqrt{\sigma_1^2 + \sigma_2^2 + \sigma_3^2} \tag{C.186}$$

Generalizing the result to the case of $n$ Gaussians, we see that $\sum_{i=1}^n \tilde{F}_i$ is a Gaussian fuzzy number with mean $\sum_{i=1}^n m_i$ and standard deviation $\sqrt{\sum_{i=1}^n \sigma_i^2}$.

(ii) Minimum $t$-norm : In this case, (C.178) reduces to

$$\tilde{F}_1 + \tilde{F}_2 = \int_{v\in\tilde{F}_1} \int_{w\in\tilde{F}_2} e^{-\frac{1}{2}\left(\frac{v-m_1}{\sigma_1}\right)^2} \wedge e^{-\frac{1}{2}\left(\frac{w-m_2}{\sigma_2}\right)^2} /[v+w] \tag{C.187}$$

If $\theta$ is an element of $\tilde{F}_1 + \tilde{F}_2$, the membership grade of $\theta$ in $\tilde{F}_1 + \tilde{F}_2$ can be obtained by considering all the $\{v, w\}$ pairs such that $v \in \tilde{F}_1$ and $w \in \tilde{F}_2$ and $v + w = \theta$, finding the minimum of the memberships of $v$ and $w$ in every pair, and, choosing the maximum of all these minimums. In other words,

$$\mu_{(\tilde{F}_1+\tilde{F}_2)}(\theta) = \sup_v \left[e^{-\frac{1}{2}\left(\frac{v-m_1}{\sigma_1}\right)^2} \wedge e^{-\frac{1}{2}\left[\frac{(\theta-v)-m_2}{\sigma_2}\right]^2}\right] \tag{C.188}$$

We make use of the fact that the supremum of the minimum of two Gaussians is reached at their point of intersection lying between their means. To solve for the point of intersection, we equate the equations of the two Gaussians.

$$\begin{aligned} e^{-\frac{1}{2}\left(\frac{v_*-m_1}{\sigma_1}\right)^2} &= e^{-\frac{1}{2}\left[\frac{(\theta-v_*)-m_2}{\sigma_2}\right]^2} \\ \Rightarrow \quad \left(\frac{v_*-m_1}{\sigma_1}\right)^2 &= \left[\frac{(\theta-v_*)-m_2}{\sigma_2}\right]^2 \\ \Rightarrow \quad \left(\frac{v_*-m_1}{\sigma_1}\right)^2 &= \left[\frac{(\theta-m_2)-v_*}{\sigma_2}\right]^2 \\ \Rightarrow \quad \frac{v_*-m_1}{\sigma_1} &= \pm\frac{(\theta-m_2)-v_*}{\sigma_2} \end{aligned} \tag{C.189}$$

The positive square-root on the RHS of (C.189) gives us the point of intersection lying between the means [$m_1$ and $(\theta - m_2)$]. Solving further, we find

$$\begin{aligned} v_*\Big(\frac{1}{\sigma_1} + \frac{1}{\sigma_2}\Big) &= \frac{m_1}{\sigma_1} + \frac{(\theta - m_2)}{\sigma_2} \\ \Rightarrow \quad v_* &= \frac{m_1\sigma_2 + (\theta - m_2)\sigma_1}{\sigma_1 + \sigma_2} \end{aligned} \tag{C.190}$$

Since $v_*$ is the point of intersection of the two Gaussians, it has the same membership in $\tilde{F}_1$ and $\tilde{F}_2$; therefore, membership grade of $\theta$ in $\tilde{F}_1 + \tilde{F}_2$ is

$$\begin{aligned} \mu_{(\tilde{F}_1+\tilde{F}_2)}(\theta) &= \exp\Big\{-\frac{1}{2}\Big(\frac{v_* - m_1}{\sigma_1}\Big)^2\Big\} \\ &= \exp\left\{-\frac{1}{2}\left[\frac{\frac{m_1\sigma_2+(\theta-m_2)\sigma_1}{\sigma_1+\sigma_2} - m_1}{\sigma_1}\right]^2\right\} \\ &= \exp\Big\{-\frac{1}{2}\Big(\frac{\theta - (m_1 + m_2)}{\sigma_1 + \sigma_2}\Big)^2\Big\} \end{aligned} \tag{C.191}$$

This result generalizes easily to the case of more than two Gaussians. Let $m_{12} = m_1 + m_2$ and $\sigma_{12} = \sigma_1 + \sigma_2$, so that $\tilde{F}_1 + \tilde{F}_2$ is a Gaussian with mean $m_{12}$ and standard deviation $\sigma_{12}$. Now, if a third Gaussian fuzzy set $\tilde{F}_3$, with mean $m_3$ and standard deviation $\sigma_3$, adds to this sum, the mean and standard deviation of the resulting Gaussian are

$$m_{123} = m_{12} + m_3 = m_1 + m_2 + m_3 \tag{C.192}$$

$$\sigma_{123} = \sigma_{12} + \sigma_3 = \sigma_1 + \sigma_2 + \sigma_3 \tag{C.193}$$

Generalizing to the case of $n$ Gaussians, we see that $\sum_{i=1}^{n} \tilde{F}_i$ is a Gaussian fuzzy number with mean $\sum_{i=1}^{n} m_i$ and standard deviation $\sum_{i=1}^{n} \sigma_i$.

Combining parts (a) and (b), we get the desired result. □

## C.10 Proof of the Claim in Example 2.4

Consider an interval type-2 set resulting from a Gaussian type-1 set, whose mean is uncertain in the interval $[m_1, m_2]$, and whose standard deviation is $\sigma$. We show that, if $(m_2 - m_1)$ is small compared to $\sigma$, the centroid of this type-2 set is approximately an interval type-1 set with domain $[m_1, m_2]$.

Figure C.4 (a) depicts an example of such a set, $\tilde{\tilde{\text{A}}}$. The type-1 Gaussians with centers $m_1$ and $m_2$ and standard deviation $\sigma$ are also shown. From the discussion in Example 2.4, we know that : (1) the centroid of $\tilde{\tilde{\text{A}}}$ is some interval, $[c_l, c_r]$, which contains $[m_1, m_2]$, and (2) $c_l$ is the centroid of an embedded type-1 set, $\tilde{\text{A}}_l$ [see Fig. C.4 (b)] whose membership function assigns the highest possible memberships to all the points to the left of $c_l$, and the lowest possible memberships to all the points to the right of $c_l$. We now focus on the left end-point, $c_l$, of this interval. The discussion about $c_r$ is similar.

We show that $c_l > m_1 - \Delta$, where

$$\Delta = \frac{m_2 - m_1}{2} \tag{C.194}$$

Consider the embedded type-1 set, $\tilde{\text{A}}_1$, shown in Fig. C.4 (c). The membership function of this embedded set assigns the highest possible memberships to the points less than $(m_1 - \Delta)$, and the lowest possible membership to the points greater than $(m_1 - \Delta)$. It is easy to verify that the area under this curve, to the left of $(m_1 - \Delta)$, is the same as the area under it to the right of $(m_1 + \Delta)$ [see Fig. C.4 (c)]; therefore, the centroid of this curve, $c'$, lies between $(m_1 - \Delta)$ and $(m_1 + \Delta)$. Now consider any other embedded set, $\tilde{\text{A}}_2$, which assigns the highest possible memberships to all the points to the left of some point $m'$ and the lowest possible memberships to all the points to the right of $m'$, where $m' < m_1 - \Delta$. It is easy to see that the centroid of $\tilde{\text{A}}_2$ will lie to the right of $c'$. We, therefore, conclude that $c_l > m_1 - \Delta$. Using this result with the fact that $[c_l, c_r]$ includes $[m_1, m_2]$, we get

$$m_1 - \Delta < c_l \leq m_1 \tag{C.195}$$

Obviously, as $\Delta \to 0$ [i.e., as $(m_2 - m_1) \to 0$], $c_l \to m_1$. We will make use of (C.195) in the sequel, to show the dependence of $c_l$ on $\sigma$.

To show the effect of $\sigma$ on $c_l$, we write the expression for the centroid of $\tilde{\text{A}}_l$ [see Fig. C.4 (b)]. Let $G(x; m, \sigma) = \exp\{-\frac{1}{2}(\frac{x-m}{\sigma})^2\}$; then

$$c_l = \tilde{C}_{\tilde{\tilde{\text{A}}}_l} = \frac{\int_{-\infty}^{c_l} xG(x; m_1, \sigma)dx + \int_{c_l}^{m_1+\Delta} xG(x; m_2, \sigma)dx + \int_{m_1+\Delta}^{\infty} xG(x; m_1, \sigma)dx}{\int_{-\infty}^{c_l} G(x; m_1, \sigma)dx + \int_{c_l}^{m_1+\Delta} G(x; m_2, \sigma)dx + \int_{m_1+\Delta}^{\infty} G(x; m_1, \sigma)dx}$$

$$= \frac{\int_{-\infty}^{\infty} xG(x;m_1,\sigma)dx - \int_{c_l}^{m_1+\Delta} x[G(x;m_1,\sigma) - G(x;m_2,\sigma)]dx}{\int_{-\infty}^{\infty} G(x;m_1,\sigma)dx - \int_{c_l}^{m_1+\Delta}[G(x;m_1,\sigma) - G(x;m_2,\sigma)]dx}$$

$$= \frac{\sqrt{2\pi}\sigma m_1 - I_1}{\sqrt{2\pi}\sigma - I_2} \qquad (C.196)$$

where

$$I_1 = \int_{c_l}^{m_1+\Delta} x[G(x;m_1,\sigma) - G(x;m_2,\sigma)]dx, \qquad (C.197)$$

$$I_2 = \int_{c_l}^{m_1+\Delta} [G(x;m_1,\sigma) - G(x;m_2,\sigma)]dx \qquad (C.198)$$

and, we have made use of the facts that : (1) the area under a Gaussian having standard deviation $\sigma$ is $\sqrt{2\pi}\sigma$, and (2) since the centroid of a Gaussian is equal to its mean,

$$\int_{-\infty}^{\infty} xG(x;m_1,\sigma)dx = m_1 \int_{-\infty}^{\infty} G(x;m_1,\sigma)dx = \sqrt{2\pi}\sigma m_1 \qquad (C.199)$$

From (C.195) and (C.196), we have

$$0 \leq m_1 - c_l = \frac{I_1 - m_1 I_2}{\sqrt{2\pi}\sigma - I_2} \qquad (C.200)$$

Observe, from (C.197) and (C.198), that

$$I_1 \leq \int_{c_l}^{m_1+\Delta} (m_1+\Delta)[G(x;m_1,\sigma) - G(x;m_2,\sigma)]dx = (m_1+\Delta)I_2 \qquad (C.201)$$

Using (C.201) with (C.200), we get

$$0 \leq m_1 - c_l \leq \frac{\Delta I_2}{\sqrt{2\pi}\sigma - I_2} \qquad (C.202)$$

Now, observe, from (C.198), (C.195) and the fact that $G(x;m_1,\sigma) > G(x;m_2,\sigma)$ for $x < m_1 + \Delta$, that

$$I_2 < \int_{m_1-\Delta}^{m_1+\Delta} [G(x;m_1,\sigma) - G(x;m_2,\sigma)]dx \qquad (C.203)$$

Observe also that, for $x \in [m_1 - \Delta, m_1 + \Delta]$,

$$\begin{aligned} 0 &\leq G(x; m_1, \sigma) - G(x; m_2, \sigma) \\ &= G(x; m_1, \sigma)\Big[1 - \exp\Big\{-\frac{1}{2}\Big[\Big(\frac{x - m_2}{\sigma}\Big)^2 - \Big(\frac{x - m_1}{\sigma}\Big)^2\Big]\Big\}\Big] \\ &= G(x; m_1, \sigma)\Big[1 - \exp\Big\{-\frac{2\Delta}{\sigma^2}(m_1 + \Delta - x)\Big\}\Big] \\ &\leq G(x; m_1, \sigma)\Big[1 - \exp\Big\{-\Big(\frac{2\Delta}{\sigma}\Big)^2\Big\}\Big] \end{aligned} \tag{C.204}$$

In the last step of (C.204), we have made use of the fact that $x \geq m_1 - \Delta$.

Using (C.204) with (C.203), we have

$$\begin{aligned} I_2 &< \Big[1 - \exp\Big\{-\Big(\frac{2\Delta}{\sigma}\Big)^2\Big\}\Big] \int_{m_1 - \Delta}^{m_1 + \Delta} G(x; m_1, \sigma)dx \\ &\leq \Big[1 - \exp\Big\{-\Big(\frac{2\Delta}{\sigma}\Big)^2\Big\}\Big] \int_{-\infty}^{\infty} G(x; m_1, \sigma)dx \\ &= \Big[1 - \exp\Big\{-\Big(\frac{2\Delta}{\sigma}\Big)^2\Big\}\Big]\sqrt{2\pi}\sigma \end{aligned} \tag{C.205}$$

Observe that

$$\begin{aligned} \sqrt{2\pi}\sigma\Big[1 - \exp\Big\{-\Big(\frac{2\Delta}{\sigma}\Big)^2\Big\}\Big] &= \sqrt{2\pi}\sigma\Big[1 - \Big(1 + \sum_{i=1}^{\infty}(-1)^i\frac{1}{i!}\Big(\frac{2\Delta}{\sigma}\Big)^{2i}\Big)\Big] \\ &= \sqrt{2\pi}\sigma\Big[\sum_{i=1}^{\infty}(-1)^{i+1}\frac{1}{i!}\Big(\frac{2\Delta}{\sigma}\Big)^{2i}\Big)\Big] \\ &= 2\sqrt{2\pi}\Delta\Big[\sum_{i=1}^{\infty}(-1)^{i+1}\frac{1}{i!}\Big(\frac{2\Delta}{\sigma}\Big)^{2i-1}\Big)\Big] \end{aligned} \tag{C.206}$$

It is clear, from (C.205) and (C.206), that, as $2\Delta/\sigma = (m_2 - m_1)/\sigma \to 0$, $I_2 \to 0$; hence, from (C.202), $c_l \to m_1$.

In a similar manner, it can be shown that $m_2 \leq c_r < m_2 + \Delta$, and as $(m_2 - m_1)/\sigma \to 0$, $c_r \to m_2$ [see Fig. C.4 (d)]. □

## C.11 Proof of Theorem 2.5

Let $G(x; m, \sigma) \triangleq \exp\left\{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2\right\}$. Equation (2.73) can now be rewritten as

$$\tilde{Y} = \int_{z_1} \cdots \int_{z_M} \int_{w_1} \cdots \int_{w_M} \mathcal{T}_{i=1}^{M} G(z_l; m_l, \sigma_l) \star \mathcal{T}_{l=1}^{M} G(w_l; h_l, \Delta_l) \Big/ \frac{\sum_{l=1}^{M} w_l z_l}{\sum_{l=1}^{M} w_l} \quad \text{(C.207)}$$

where $\tilde{Y} \triangleq \tilde{Y}(\tilde{Z}_1, \ldots, \tilde{Z}_M, \tilde{W}_1, \ldots, \tilde{W}_M)$.

If we let $\gamma_l = z_l - m_l$ and $\delta_l = w_l - h_l$ for $l = 1, \ldots, M$, (C.207) becomes

$$\tilde{Y} = \int_{\gamma_1} \cdots \int_{\gamma_M} \int_{\delta_1} \cdots \int_{\delta_M} \mathcal{T}_{l=1}^{M} G(\gamma_l; 0, \sigma_l) \star \mathcal{T}_{l=1}^{M} G(\delta_l; 0, \Delta_l) \Big/ \frac{\sum_{l=1}^{M} (h_l + \delta_l)(m_l + \gamma_l)}{\sum_{l=1}^{M} (h_l + \delta_l)} \quad \text{(C.208)}$$

Theoretically, each $\delta_l$ can take any value in the interval $[0, 1]$; however, only those values which lie within 2 or 3 standard deviations of $h_l$ contribute significantly to the union in (C.208); therefore, we assume that each $\delta_l$ takes values between $\pm k\Delta_l$, where $k = 2$ or 3. Similarly, we assume that each $\gamma_l$ takes values between $\pm k\sigma_l$.

The term to the right of the slash in (C.208) can be rewritten as

$$\begin{aligned} \frac{\sum_l w_l z_l}{\sum_l w_l} &= \frac{\sum_l (h_l + \delta_l)(m_l + \gamma_l)}{\sum_l (h_l + \delta_l)} \\ &= \frac{\sum_l h_l m_l + \sum_l h_l \gamma_l + \sum_l \delta_l m_l + \sum_l \delta_l \gamma_l}{\sum_l h_l + \sum_l \delta_l} \end{aligned} \quad \text{(C.209)}$$

where the limits on each sum are from 1 to $M$.

In what follows, we express the term on the RHS of (C.209) as an affine combination of Gaussian $\gamma_l$'s and $\delta_l$'s so that we can make use of Theorem 2.4 to find an (approximate) expression for $\tilde{Y}$ in (C.207). We expand the denominator of (C.209) by first rewriting it as

$$\frac{1}{\sum_l h_l + \sum_l \delta_l} = \frac{1}{\sum_l h_l} \left( \frac{1}{1 + \frac{\sum_l \delta_l}{\sum_l h_l}} \right) \quad \text{(C.210)}$$

If $|\sum_l \delta_l| < \sum_l h_l$ (since $\delta_l$ varies between $-k\Delta_l$ and $k\Delta_l$, this is equivalent to assuming that $k\sum_l \Delta_l < \sum_l h_l$), we can express the parenthetical term on the RHS of (C.210), as

$$\frac{1}{1+\frac{\sum_l \delta_l}{\sum_l h_l}} = 1 - \Big(\frac{\sum_l \delta_l}{\sum_l h_l}\Big) + \Big(\frac{\sum_l \delta_l}{\sum_l h_l}\Big)^2 - \Big(\frac{\sum_l \delta_l}{\sum_l h_l}\Big)^3 + \cdots \tag{C.211}$$

where we have made use of the identity

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots \quad \text{if} \quad |x| < 1 \tag{C.212}$$

If

$$k\frac{\sum_l \Delta_l}{\sum_l h_l} \ll 1\,, \tag{C.213}$$

which means that

$$\frac{|\sum_l \delta_l|}{\sum_l h_l} \ll 1\,, \tag{C.214}$$

we can ignore powers of $\sum_l \delta_l \big/ \sum_l h_l$ greater than 1 in (C.211). This gives us

$$\frac{1}{1+\frac{\sum_l \delta_l}{\sum_l h_l}} \approx 1 - \Big(\frac{\sum_l \delta_l}{\sum_l h_l}\Big) \tag{C.215}$$

Substituting (C.215) into (C.210), we get

$$\frac{1}{\sum_l h_l + \sum_l \delta_l} \approx \frac{1}{\sum_l h_l}\Big(1 - \frac{\sum_l \delta_l}{\sum_l h_l}\Big) \tag{C.216}$$

Using (C.216) in (C.209), we get

$$\frac{\sum_l w_l z_l}{\sum_l w_l} \approx \frac{\sum_l h_l m_l + \sum_l h_l \gamma_l + \sum_l \delta_l m_l + \sum_l \delta_l \gamma_l}{\sum_l h_l}\Big(1 - \frac{\sum_l \delta_l}{\sum_l h_l}\Big) \tag{C.217}$$

Ignoring all the terms containing powers of $\sum_l \delta_l / \sum_l h_l$ higher than 1, we get

$$\begin{aligned} \frac{\sum_l w_l z_l}{\sum_l w_l} &\approx \frac{\sum_l h_l m_l}{\sum_l h_l}\Big(1 - \frac{\sum_l \delta_l}{\sum_l h_l}\Big) + \frac{\sum_l h_l \gamma_l}{\sum_l h_l}\Big(1 - \frac{\sum_l \delta_l}{\sum_l h_l}\Big) + \frac{\sum_l \delta_l m_l}{\sum_l h_l} + \frac{\sum_l \delta_l \gamma_l}{\sum_l h_l} \\ &= \frac{\sum_l h_l m_l}{\sum_l h_l} - \frac{\sum_l \delta_l}{\sum_l h_l}\Big(\frac{\sum_l h_l m_l}{\sum_l h_l}\Big) + \frac{\sum_l h_l \gamma_l}{\sum_l h_l} \end{aligned}$$

$$-\frac{\sum_l \delta_l}{\sum_l h_l}\Big(\frac{\sum_l h_l\gamma_l}{\sum_l h_l}\Big)+\frac{\sum_l \delta_l m_l}{\sum_l h_l}+\frac{\sum_l \delta_l\gamma_l}{\sum_l h_l} \tag{C.218}$$

Let

$$\mathcal{M}=\frac{\sum_l h_l m_l}{\sum_l h_l}; \tag{C.219}$$

then (C.218) can be rewritten as

$$\frac{\sum_l w_l z_l}{\sum_l w_l}\approx\mathcal{M}-\mathcal{M}\frac{\sum_l \delta_l}{\sum_l h_l}+\frac{\sum_l \delta_l m_l}{\sum_l h_l}+\frac{\sum_l h_l\gamma_l}{\sum_l h_l}-\frac{\sum_l \delta_l}{\sum_l h_l}\Big(\frac{\sum_l h_l\gamma_l}{\sum_l h_l}\Big)+\frac{\sum_l \delta_l\gamma_l}{\sum_l h_l} \tag{C.220}$$

Next we focus on the last two terms in (C.220). Observe that

$$\Big|\frac{\sum_l \delta_l\gamma_l}{\sum_l h_l}\Big|\le\Big|\max_l \gamma_l\Big(\frac{\sum_l \delta_l}{\sum_l h_l}\Big)\Big| \tag{C.221}$$

Since $\gamma_l$ takes values between $\pm k\sigma_l$, (C.221) is equivalent to

$$\Big|\frac{\sum_l \delta_l\gamma_l}{\sum_l h_l}\Big|\le k\max_l \sigma_l\Big|\frac{\sum_l \delta_l}{\sum_l h_l}\Big| \tag{C.222}$$

Similarly,

$$\Big|\frac{\sum_l h_l\gamma_l}{\sum_l h_l}\Big|\le k\max_l \sigma_l \tag{C.223}$$

Consequently,

$$\Big|-\frac{\sum_l \delta_l}{\sum_l h_l}\Big(\frac{\sum_l h_l\gamma_l}{\sum_l h_l}\Big)+\frac{\sum_l \delta_l\gamma_l}{\sum_l h_l}\Big|\le\Big|\frac{\sum_l \delta_l}{\sum_l h_l}\Big(\frac{\sum_l h_l\gamma_l}{\sum_l h_l}\Big)\Big|+\Big|\frac{\sum_l \delta_l\gamma_l}{\sum_l h_l}\Big|\le 2k\max_l \sigma_l\Big|\frac{\sum_l \delta_l}{\sum_l h_l}\Big| \tag{C.224}$$

Observe, from (C.220) and (C.224), that if condition (C.214) is satisfied, we can ignore the last two terms on the RHS of (C.220) in comparison with $\sum_l h_l\gamma_l / \sum_l h_l$, which, according to (C.223), takes values in $\pm k\max_l \sigma_l$. Doing this gives us

$$\begin{aligned}\frac{\sum_l w_l z_l}{\sum_l w_l} &\approx \mathcal{M}-\mathcal{M}\frac{\sum_l \delta_l}{\sum_l h_l}+\frac{\sum_l \delta_l m_l}{\sum_l h_l}+\frac{\sum_l h_l\gamma_l}{\sum_l h_l}\\ &= \sum_{l=1}^{M}\Big[\gamma_l\Big(\frac{h_l}{\sum_l h_l}\Big)+\delta_l\Big(\frac{m_l-\mathcal{M}}{\sum_l h_l}\Big)\Big]+\mathcal{M}\end{aligned} \tag{C.225}$$

Using (C.225), (C.208) can be rewritten as

$$\tilde{Y} \approx \int_{\gamma_1} \cdots \int_{\gamma_M} \int_{\delta_1} \cdots \int_{\delta_M} \mathcal{T}_{l=1}^{M} G(\gamma_l; 0, \sigma_l) \star \mathcal{T}_{l=1}^{M} G(\delta_l; 0, \Delta_l) \Big/ \sum_{l=1}^{M} \Big[ \gamma_l \Big( \frac{h_l}{\sum_l h_l} \Big) + \delta_l \Big( \frac{m_l - \mathcal{M}}{\sum_l h_l} \Big) \Big] + \mathcal{M} \qquad \text{(C.226)}$$

Recall that $\gamma_l = z_l - m_l$ and $\delta_l = w_l - h_l$. Let

$$\underline{\tilde{Z}}_l = \tilde{Z}_l - m_l \quad \text{for} \quad l = 1, \ldots, M \qquad \text{(C.227)}$$

and

$$\underline{\tilde{W}}_l = \tilde{W}_l - h_l \quad \text{for} \quad l = 1, \ldots, M\,; \qquad \text{(C.228)}$$

so that each $\underline{\tilde{Z}}_l$ is a type-1 Gaussian fuzzy number with zero mean and standard deviation equal to $\sigma_l$, and each $\underline{\tilde{W}}_l$ is also a type-1 Gaussian fuzzy number with zero mean and standard deviation $\Delta_l$. Observe that the RHS of (C.226) is equal to $\sum_{l=1}^{M} \big[ \underline{\tilde{Z}}_l \big( \frac{h_l}{\sum_l h_l} \big) + \underline{\tilde{W}}_l \big( \frac{m_l - \mathcal{M}}{\sum_l h_l} \big) \big] + \mathcal{M}$ (see Section 2.4.2), i.e.,

$$\tilde{Y} \approx \sum_{l=1}^{M} \Big[ \underline{\tilde{Z}}_l \Big( \frac{h_l}{\sum_l h_l} \Big) + \underline{\tilde{W}}_l \Big( \frac{m_l - \mathcal{M}}{\sum_l h_l} \Big) \Big] + \mathcal{M} \qquad \text{(C.229)}$$

The result in Theorem 2.5 follows by applying Theorem 2.4 to (C.229), using the fact that all $\underline{\tilde{Z}}_l$'s and $\underline{\tilde{W}}_l$'s have zero means. □

**Comment 1** : When all $\Delta_l = 0$, there is only one source of fuzziness in $\sum_l w_l z_l / \sum_l w_l$, namely, the $\tilde{Z}_l$'s. In this case, (2.73) reduces to

$$\tilde{Y}(\tilde{Z}_1, \ldots, \tilde{Z}_M, h_1, \ldots, h_M) = \int_{z_1} \cdots \int_{z_M} \mathcal{T}_{l=1}^{M} G(z_l; m_l, \sigma_l) \Big/ \frac{\sum_{l=1}^{M} h_l z_l}{\sum_{l=1}^{M} h_l} \qquad \text{(C.230)}$$

Again, letting $\gamma_l = z_l - m_l$ for $l = 1, \ldots, M$, we have

$$\begin{aligned} \tilde{Y}(\tilde{Z}_1, \ldots, \tilde{Z}_M, h_1, \ldots, h_M) &= \int_{\gamma_1} \cdots \int_{\gamma_M} \mathcal{T}_{l=1}^{M} G(\gamma_l; 0, \sigma_l) \Big/ \frac{\sum_{l=1}^{M} h_l (m_l + \gamma_l)}{\sum_{l=1}^{M} h_l} \\ &= \int_{\gamma_1} \cdots \int_{\gamma_M} \mathcal{T}_{l=1}^{M} G(\gamma_l; 0, \sigma_l) \Big/ \end{aligned}$$

$$
\begin{aligned}
&\Big[\frac{\sum_{l=1}^{M} h_l m_l}{\sum_{l=1}^{M} h_l} + \sum_{l=1}^{M} \gamma_l \Big(\frac{h_l}{\sum_{l=1}^{M} h_l}\Big)\Big] \\
&= \int_{\gamma_1} \cdots \int_{\gamma_M} \mathcal{T}_{l=1}^{M} G(\gamma_l; 0, \sigma_l) \Big/ \Big[\mathcal{M} + \sum_{l=1}^{M} \gamma_l \Big(\frac{h_l}{\sum_{l=1}^{M} h_l}\Big)\Big] \\
&= \sum_{l=1}^{M} \underline{\tilde{\mathrm{W}}}_l \Big(\frac{h_l}{\sum_{l=1}^{M} h_l}\Big) + \mathcal{M} \qquad \text{(C.231)}
\end{aligned}
$$

where $\underline{\tilde{\mathrm{W}}}_l$ is as defined in (C.228) and $\mathcal{M}$ is as in (C.219). Applying Theorem 2.4 to (C.231), it follows that $\tilde{\mathrm{Y}}(\tilde{\mathrm{Z}}_1, \ldots, \tilde{\mathrm{Z}}_M, h_1, \ldots, h_M)$ is a Gaussian type-1 set with mean $\mathcal{M}$ and standard deviation $\Sigma$, where

$$
\Sigma = \begin{cases} \dfrac{\sqrt{\sum_{l=1}^{M} (h_l \sigma_l)^2}}{\sum_{l=1}^{M} h_l} & , \text{ if product } t\text{-norm is used} \\ \dfrac{\sum_{l=1}^{M} (h_l \sigma_l)}{\sum_{l=1}^{M} h_l} & , \text{ if minimum } t\text{-norm is used} \end{cases} \qquad \text{(C.232)}
$$

Observe that this result is exact and it can also be obtained by substituting $\Delta_l = 0$ $(l = 1, \ldots, M)$ in (2.75).

**Comment 2** : It is not very easy to find an expression for the error between the approximation in Theorem 2.5 and the true set $\tilde{\mathrm{Y}}$; however, we can find bounds on the domain of $\tilde{\mathrm{Y}}$ easily.

Recall Eq. (C.208). If each $\delta_l$ varies between $\pm k\Delta_l$ and each $\gamma_l$ varies between $\pm k\sigma_l$, the term to the right of the slash can be bounded as

$$
\frac{\sum_l (h_l - k\Delta_l)(m_l - k\sigma_l)}{\sum_l (h_l + k\Delta_l)} \le \frac{\sum_l (h_l + \delta_l)(m_l + \gamma_l)}{\sum_l (h_l + \delta_l)} \le \frac{\sum_l (h_l + k\Delta_l)(m_l + k\sigma_l)}{\sum_l (h_l - k\Delta_l)} \qquad \text{(C.233)}
$$

Let $k_1 = \max_l[\Delta_l / h_l]$ for $h_l \neq 0$ and let $k_2 = \max_l[\sigma_l / m_l]$, assuming $m_l > 0$ $(l = 1, \ldots, M)$. From (C.233), we have

$$
\begin{aligned}
&\frac{\sum_l (h_l - kk_1 h_l)(m_l - kk_2 m_l)}{\sum_l (h_l + kk_1 h_l)} \le \frac{\sum_l (h_l + \delta_l)(m_l + \gamma_l)}{\sum_l (h_l + \delta_l)} \le \\
&\qquad \frac{\sum_l (h_l + kk_1 h_l)(m_l + kk_2 m_l)}{\sum_l (h_l - kk_1 h_l)} \\
\Rightarrow\ & \mathcal{M}\Big[\frac{(1 - kk_1)(1 - kk_2)}{(1 + kk_1)}\Big] \le \frac{\sum_l (h_l + \delta_l)(m_l + \gamma_l)}{\sum_l (h_l + \delta_l)} \le
\end{aligned}
$$

$$\mathcal{M}\left[\frac{(1+kk_1)(1+kk_2)}{(1-kk_1)}\right] \tag{C.234}$$

where $\mathcal{M} = \sum_l h_l m_l / \sum_l h_l$. Since the term to the right of the slash in (C.208) indicates a general point in $\tilde{\mathrm{Y}}$, it is clear that the entire domain of $\tilde{\mathrm{Y}}$ lies between the bounds in (C.234).

Though the bounds in (C.234) are generally very conservative, they illustrate how $\tilde{\mathrm{Y}}(\tilde{\mathrm{Z}}_1,\ldots,\tilde{\mathrm{Z}}_M,\tilde{\mathrm{W}}_1,\ldots,\tilde{\mathrm{W}}_M)$ collapses to $y(z_1,\ldots,z_M,w_1,\ldots,l_M)$ [see (2.72) and (2.73)], when all the type-2 uncertainties collapse to type-1 uncertainties. When all the type-2 uncertainties disappear, $k_1 = k_2 = 0$ and the upper and lower bounds on $\tilde{\mathrm{Y}}$ both equal $\mathcal{M}$, implying that the type-1 set $\tilde{\mathrm{Y}}$ collapses to a crisp point equal to $\mathcal{M}$, i.e., $1/\mathcal{M}$ (since $\mu_{\tilde{\mathrm{Y}}}(\mathcal{M}) = 1$).

Similar bounds can be obtained when all or some of the $m_l < 0$.

**Comment 3** : Since the bounds in (C.234) enclose the entire domain of the type-reduced set between them, $\mathcal{M}$, the unity membership point in $\tilde{\mathrm{Y}}$ and $C_{\tilde{\mathrm{Y}}}$, the centroid of $\tilde{\mathrm{Y}}$, also lie between these two bounds. The difference between $\mathcal{M}$ and $\tilde{\mathrm{Y}}$ can, therefore, be loosely bounded as (assuming $m_l > 0$ for $l = 1,\ldots,M$).

$$\begin{aligned}
|\mathcal{M} - C_{\tilde{\mathrm{Y}}}| &\le \mathcal{M}\left[\frac{(1+kk_1)(1+kk_2)}{(1-kk_1)} - \frac{(1-kk_1)(1-kk_2)}{(1+kk_1)}\right] \\
\Rightarrow\quad |\mathcal{M} - C_{\tilde{\mathrm{Y}}}| &\le \mathcal{M}\left[\frac{(1+kk_1)^2(1+kk_2) - (1-kk_1)^2(1-kk_2)}{(1-k^2k_1^2)}\right] \\
\Rightarrow\quad |\mathcal{M} - C_{\tilde{\mathrm{Y}}}| &\le \mathcal{M}\left[\frac{4kk_1 + 2kk_2(1+k^2k_1^2)}{(1-k^2k_1^2)}\right] \qquad \text{(C.235)}
\end{aligned}$$

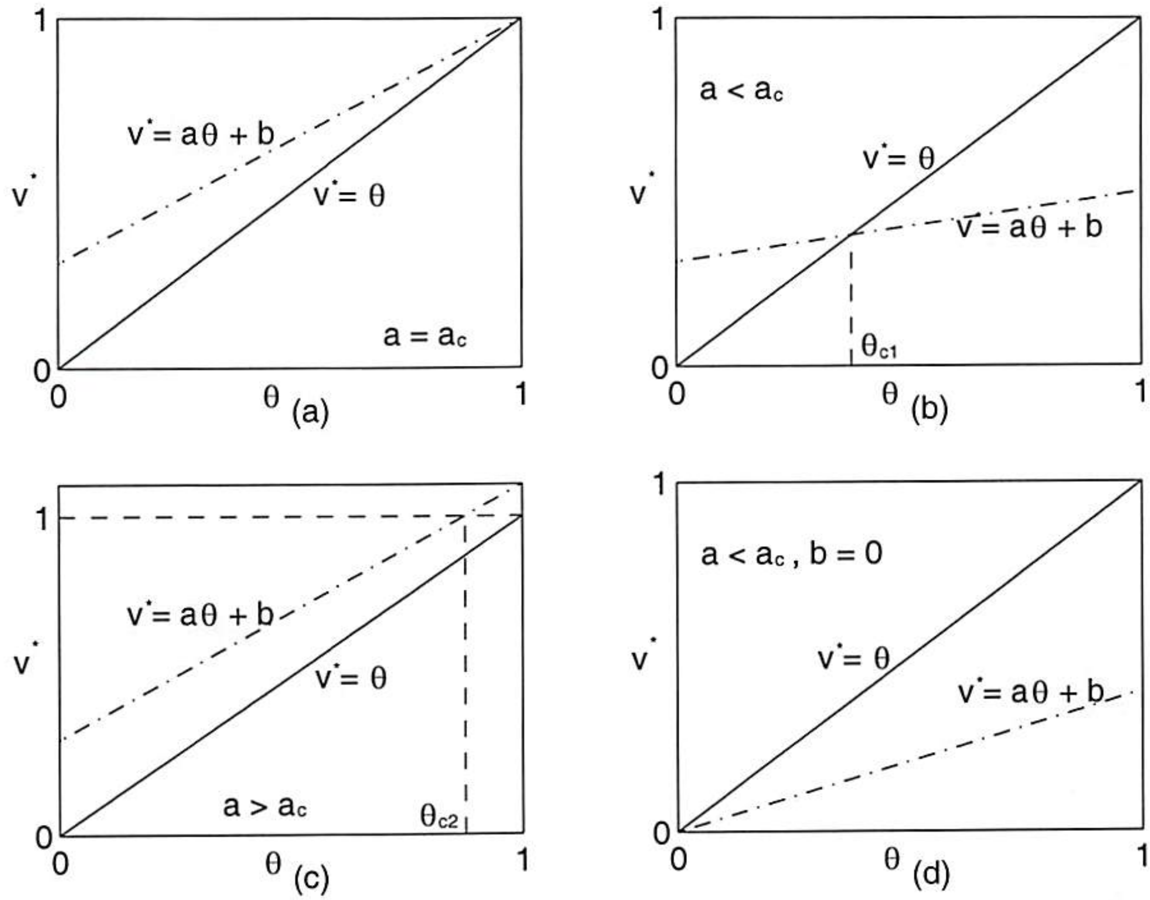



Figure C.1: Plots of $v^* = a\theta + b$ versus $\theta$ and different possibilities that can arise depending on the value of $a$ and $b$. The critical value of $a$ is $a_c = 1 - b$. (a) $a = a_c$. In this case, the constrained minimum is always equal to the true minimum. (b) $a < a_c$. (c) $a > a_c$. (d) The special case, when $b = 0$ and $a > a_c$. In this case, the constrained minimum is equal to the true minimum only at $\theta = 0$.

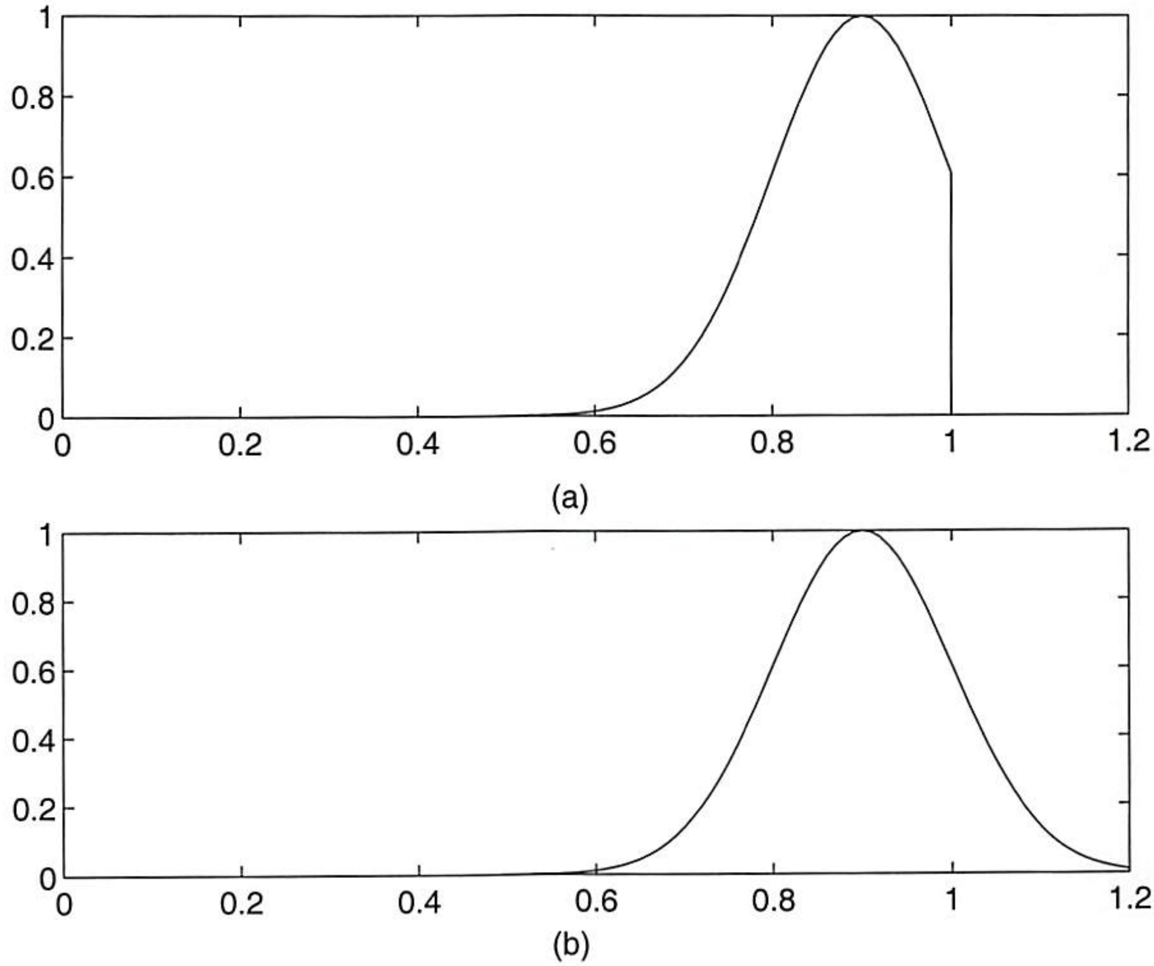



Figure C.2: A Gaussian contained in $[0, 1]$ may be clipped as shown in (a). Figure (b) shows the unclipped version of the same Gaussian.

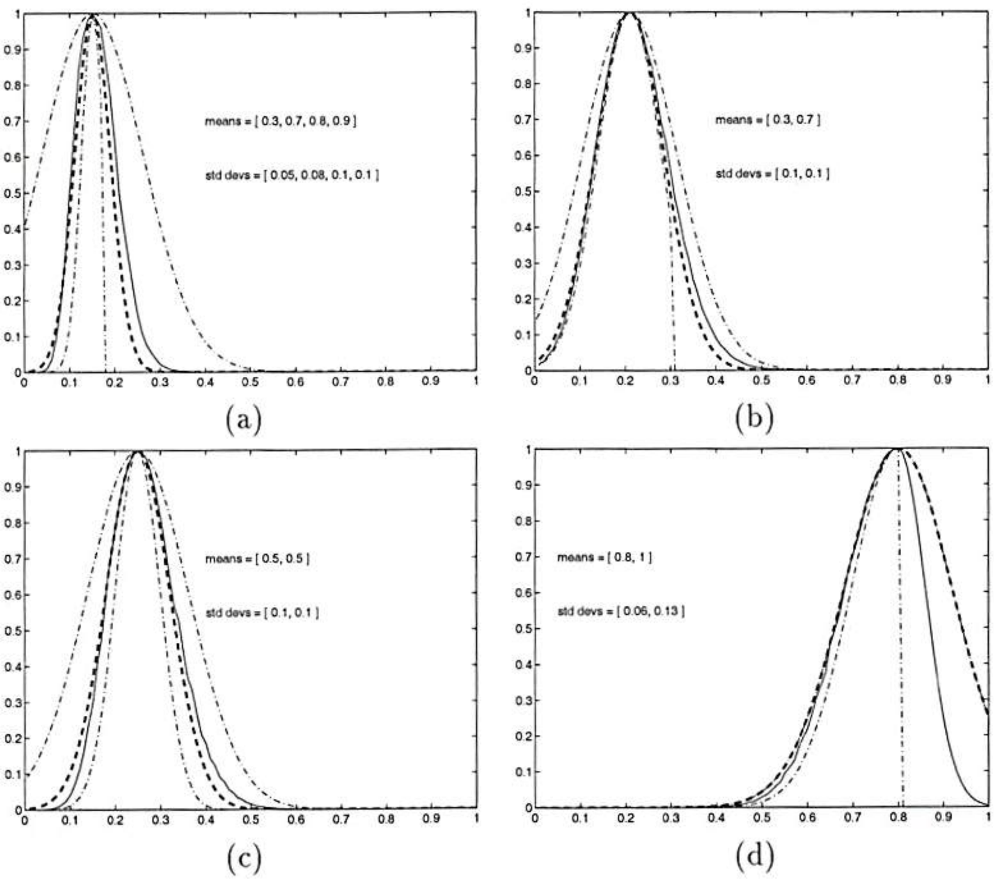



Figure C.3: Some examples of upper and lower bounds given in (C.141) and (C.158). In each figure, the solid line shows the actual result of the *meet* (computed numerically), the thick dashed line shows the Gaussian approximation and the dash-dotted lines show upper and lower bounds. The Gaussians in (c) are coincident (same means and standard deviations). The Gaussians in (d) are the same as in Fig. 2.15 (f). Since one of the Gaussians is centered at 1 (half of it is clipped), the approximation does not work as well as in the other cases; however, the upper and lower bounds still hold. In this case, the upper bound coincides with the approximation.

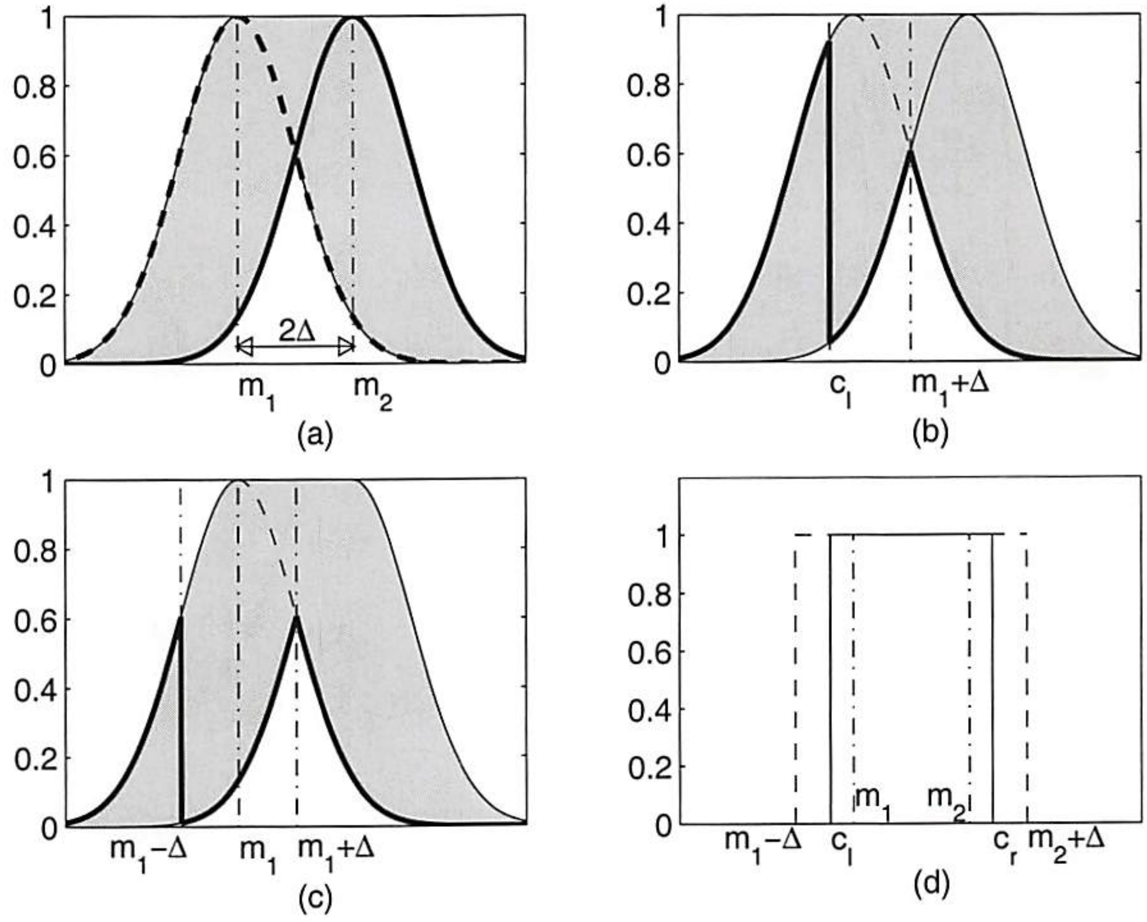



Figure C.4: Figures for Appendix C.10. (a) An interval type-2 set $\tilde{\tilde{A}}$ resulting from a Gaussian type-1 set with standard deviation equal to $\sigma$ and mean uniformly uncertain in the interval $[m_1, m_2]$. The Gaussian type-1 sets having standard deviations $\sigma$ and means $m_1$ (thick dashed line) and $m_2$ (thick solid line) are also shown. (b) The embedded type-1 set, $\tilde{A}_l$, whose centroid equals $c_l$ is shown with a thick solid line. (c) The embedded type-1 set, $\tilde{A}_1$, whose membership function assigns the highest possible memberships to the points to the left of $m_1 - \Delta$ and the lowest possible memberships to the points to the right of $m_1 - \Delta$, where $\Delta = (m_2 - m_1)/2$, is shown with a thick solid line. The Gaussian with center $m_1$ is shown with a thin dashed line. (d) The centroid of $\tilde{\tilde{A}}$ is a crisp set with domain $[c_l, c_r]$, where $m_1 - \Delta < c_l \leq m_1$ and $m_2 \leq c_r < m_2 + \Delta$.

# Appendix D

# Weighted Average of Interval Type-1 Sets

Here, we develop a computational procedure for computing the exact result of a weighted average of interval type-1 sets. This procedure can be used to compute the centroid of an interval type-2 set (Section 2.5).

Since every point in the domain of an interval type-1 set has a unity membership, we can describe an interval type-1 set just by its domain, e.g., an interval type-1 set with domain $[l, r]$ can be indicated as just $[l, r]$. If we let $m = (l + r)/2$ (mean) and $s = (r - l)/2$ (spread), we can also indicate an interval type-1 set as $[m - s, m + s]$.

## D.1 Exact Result : Computational Procedure

Consider the weighted average of type-1 fuzzy sets [see (2.73)], which we reproduce here for convenience

$$\tilde{Y}(\tilde{Z}_1, \ldots, \tilde{Z}_M, \tilde{W}_1, \ldots, \tilde{W}_M) = \int_{z_1} \cdots \int_{z_M} \int_{w_1} \cdots \int_{w_M} \mathcal{T}_{l=1}^{M} \mu_{\tilde{Z}_l}(z_l) \star \mathcal{T}_{l=1}^{M} \mu_{\tilde{W}_l}(w_l) \Big/ \frac{\sum_{l=1}^{M} w_l z_l}{\sum_{l=1}^{M} w_l} \tag{D.1}$$

If each $\tilde{Z}_l$ and $\tilde{W}_l$ $(l = 1, \ldots, M)$ is an interval type-1 set, then, using the fact that $\mu_{\tilde{Z}_l}(z_l) = \mu_{\tilde{W}_l}(w_l) = 1$, (D.1) can be rewritten as

$$Y(Z_1, \ldots, Z_M, W_1, \ldots, W_M) = \int_{z_1} \cdots \int_{z_M} \int_{w_1} \cdots \int_{w_M} 1 \Big/ \frac{\sum_{l=1}^{M} w_l z_l}{\sum_{l=1}^{M} w_l} \tag{D.2}$$

where we have omitted the tilde, since all the sets involved are crisp. We present an iterative procedure to compute the actual weighted average $Y$, when each $Z_l$ in (D.2) is an interval type-1 set, having center $c_l$ and spread $s_l$ ($s_l \geq 0$), and when each $W_l$ is also an interval type-1 set with center $h_l$ and spread $\Delta_l$ ($\Delta_l \geq 0$) [we assume that $h_l \geq \Delta_l$, so that $w_l \geq 0$ for $l = 1, \ldots, M$].

We make the following observations :

1. Since each set in the weighted average on the RHS of (D.2) is an interval type-1 set, $Y(Z_1, \ldots, Z_M, W_1, \ldots, W_M)$ will also be an interval type-1 set, i.e., it will be a crisp set having an interval on the real line as its domain. So, to find $Y(Z_1, \ldots, Z_M, W_1, \ldots, W_M)$, we need to compute just the two end-points of this interval.

2. Since $w_l \geq 0$ for all $l$, the parital derivative $\partial Y / \partial z_k = w_k / \sum_l w_l \geq 0$; therefore, $Y$ always increases with increasing $z_k$; and, for any combination of $\{w_1, \ldots, w_M\}$ chosen so that $w_l \in W_l$, $Y(Z_1, \ldots, Z_M, W_1, \ldots, W_M)$ is maximized when $z_l = c_l + s_l$ for $l = 1, \ldots, M$; and $Y(Z_1, \ldots, Z_M, W_1, \ldots, W_M)$ is minimized when $z_l = c_l - s_l$. The right end-point of the domain of $Y(Z_1, \ldots, Z_M, W_1, \ldots, W_M)$ is, therefore, obtained by maximizing $\left[\sum_l w_l(c_l + s_l)\right] / \left[\sum_l w_l\right]$ subject to the constraints $w_l \in W_l$ for $l = 1, \ldots, M$; and, the left end-point of the domain of $Y(Z_1, \ldots, Z_M, W_1, \ldots, W_M)$ is obtained by minimizing $\left[\sum_l w_l(c_l - s_l)\right] / \left[\sum_l w_l\right]$ subject to the constraints $w_l \in W_l$ for $l = 1, \ldots, M$.

From these two observations, it is clear that in order to compute $Y(Z_1, \ldots, Z_M, W_1, \ldots, W_M)$, we only need to consider the problem of optimizing (maximizing / minimizing) the weighted average

$$S(w_1, \ldots, w_M) = \frac{\sum_{l=1}^{M} z_l w_l}{\sum_{l=1}^{M} w_l} \tag{D.3}$$

subject to the constraints $w_l \in [h_l - \Delta_l, h_l + \Delta_l]$ for $l = 1, \ldots, M$, where, $h_l \geq \Delta_l$, so that $w_l \geq 0$, for $l = 1, \ldots, M$. As explained in observation (2) above, we set $z_l = c_l + s_l$ ($l = 1, \ldots, M$), when maximizing $S$, and $z_l = c_l - s_l$ ($l = 1, \ldots, M$), when minimizing $S$.

Differentiating $S(w_1, \ldots, w_M)$ w.r.t. $w_k$ gives us

$$\begin{aligned}
\frac{\partial}{\partial w_k} S(w_1, \ldots, w_M) &= \frac{\partial}{\partial w_k}\left[\frac{\sum_{l=1}^{M} z_l w_l}{\sum_{l=1}^{M} w_l}\right] \\
&= \frac{\partial}{\partial w_k}\left[\frac{z_k w_k + \sum_{l \neq k} z_l w_l}{w_k + \sum_{l \neq k} w_l}\right] \\
&= \left[\frac{1}{w_k + \sum_{l \neq k} w_l}\right](z_k) \\
&\qquad + \left(z_k w_k + \sum_{l \neq k} z_l w_l\right)\left[\frac{-1}{\left(w_k + \sum_{l \neq k} w_l\right)^2}\right] \\
&= \frac{z_k}{\sum_{l=1}^{M} w_l} - \frac{\sum_{l=1}^{M} z_l w_l}{\left(\sum_{l=1}^{M} w_l\right)^2} \\
&= \frac{z_k}{\sum_{l=1}^{M} w_l} - \left[\frac{\sum_{l=1}^{M} z_l w_l}{\sum_{l=1}^{M} w_l}\right]\frac{1}{\sum_{l=1}^{M} w_l} \\
&= \frac{z_k - S(w_1, \ldots, w_M)}{\sum_{l=1}^{M} w_l} \qquad \text{(D.4)}
\end{aligned}$$

Since $\sum_{l=1}^{M} w_l > 0$, it is easy to see from (D.4) that

$$\frac{\partial}{\partial w_k} S(w_1, \ldots, w_M) \gtreqless 0 \qquad \text{if} \qquad z_k \gtreqless S(w_1, \ldots, w_M) \qquad \text{(D.5)}$$

As shown below, equating $\partial S / \partial w_k$ to zero does not give us any information about the value of $w_k$ when $S$ is maximized or minimized.

$$\begin{aligned}
\frac{\sum_{l=1}^{M} z_l w_l}{\sum_{l=1}^{M} w_l} &= z_k \\
\Rightarrow \quad \sum_{l=1}^{M} z_l w_l &= z_k \sum_{l=1}^{M} w_l \\
\Rightarrow \quad z_k w_k + \sum_{\substack{l=1 \\ l \neq k}}^{M} z_l w_l &= z_k w_k + z_k \sum_{\substack{l=1 \\ l \neq k}}^{M} w_l \\
\Rightarrow \quad \frac{\sum_{l \neq k} z_l w_l}{\sum_{l \neq k} w_l} &= z_k \qquad \text{(D.6)}
\end{aligned}$$

Observe that $w_k$ no longer appears in (D.6). Equation (D.5), however, gives us the direction in which $w_k$ should be changed to increase or decrease $S$. Observe, from

(D.5), that if $z_k > S$, $S$ increases as $w_k$ increases; and, if $z_k < S$, $S$ increases as $w_k$ decreases.

Recall that the maximum value that $w_k$ can attain is $h_k + \Delta_k$ and the minimum value that it can attain is $h_k - \Delta_k$. The discussion in the previous paragraph, therefore, implies that $S(w_1, \ldots, w_M)$ attains its maximum value if: (1) $w_k = h_k + \Delta_k$ for those values of $k$ for which $z_k > S$, and, (2) $w_k = h_k - \Delta_k$ for those values of $k$ for which $z_k < S$. Similarly, $S(w_1, \ldots, w_M)$ attains its minimum value, if : (1) $w_k = h_k - \Delta_k$ for those values of $k$ for which $z_k > S$, and, (2) $w_k = h_k + \Delta_k$ for those values of $k$ for which $z_k < S$.

The maximum of $S$ can be obtained by following the iterative procedure given next. We set $z_l = c_l + s_l$ $(l = 1, \ldots, M)$; and, without loss of generality, assume that the $z_l$'s are arranged in ascending order, i.e., $z_1 \leq z_2 \leq \ldots \leq z_M$.

1. Set $w_l = h_l$ for $l = 1, \ldots, M$, and compute $S' = S(h_1, \ldots, h_M)$ using (D.3).

2. Find $k$ $(1 \leq k \leq M - 1)$ such that $z_k \leq S' \leq z_{k+1}$.

3. Set $w_l = h_l - \Delta_l$ for $l \leq k$ and $w_l = h_l + \Delta_l$ for $l \geq k + 1$, and compute $S'' = S(h_1 - \Delta_1, \ldots, h_k - \Delta_k, h_{k+1} + \Delta_{k+1}, \ldots, h_M + \Delta_M)$ using (D.3).

   [Since the $z_l$'s are arranged in ascending order, observe, from (D.5) - see also the sentences after Eq. (D.6) - and the fact that $z_k \leq S' \leq z_{k+1}$, that, because we are decreasing the $w_l$'s for $l \leq k$ and increasing the $w_l$'s for $l \geq k + 1$, $S'' \geq S'$.]

4. Check if $S'' = S'$. If yes, stop. $S''$ is the maximum value of $S(w_1, \ldots, w_M)$. If no, go to step 5.

5. Set $S'$ equal to $S''$. Go to step 2.

It can easily be shown that *this iterative procedure converges in at most $M$ iterations*, where one iteration consists of one pass through steps 1 to 5. At any iteration, let $k'$ be such that $z_{k'} \leq S'' \leq z_{k'+1}$. Since $S'' \geq S'$, $k' \geq k$. If $k'$ is the same as $k$, the algorithm converges at the end of the next iteration. (Note that it is possible to have $S'' \neq S'$ even when $k' = k$. This happens when both $S'$ and $S''$ are in $[z_k, z_{k+1}]$; however, if this happens, at the end of the next iteration, $S'' = S'$.) Since $k$ can take at most $M - 1$ values, the algorithm converges in at most $M$ iterations.

The minimum of $S(w_1, \ldots, w_M)$ can be obtained by using a procedure similar to the one described above. Only two changes need to be made : (1) we must set $z_l = c_l - s_l$ for $l = 1, \ldots, M$; and, (2) in Step 3, we must set $w_l = h_l + \Delta_l$ for $l \leq k$ and $w_l = h_l - \Delta_l$ for $l \geq k+1$, to compute the weighted average $S'' = S(h_1 + \Delta_1, \ldots, h_k + \Delta_k, h_{k+1} - \Delta_{k+1}, \ldots, h_M - \Delta_M)$.

### D.1.1 Centroid of an Interval Type-2 Set

Observe, from (2.67), that the centroid of an interval type-2 set $\tilde{\tilde{A}}$, whose domain is discretized into $N$ points, is given as

$$\tilde{C}_{\tilde{\tilde{A}}} = \int_{\theta_1} \cdots \int_{\theta_N} 1 \Bigg/ \frac{\sum_{l=1}^{N} x_l \theta_l}{\sum_{l=1}^{N} \theta_l} \tag{D.7}$$

where $\theta_l$ belongs to some interval in $[0, 1]$. Equation (D.7) has the same form as (D.2), except for the fact that $x_l$'s in (D.7) are crisp numbers unlike $Z_l$'s in (D.2); therefore, the same computational procedure described above can be used to compute $\tilde{C}_{\tilde{\tilde{A}}}$, with the $x_l$'s and $\theta_l$'s in (D.7) corresponding to $z_l$'s and $w_l$'s in (D.3), respectively. Note that in this case, $s_l = 0$ for all $l$, because the $x_l$'s are crisp. If $N$ is very large, in Step (4), we can check if $|S'' - S'| < \epsilon$ instead of $S'' = S'$, for some predecided $\epsilon$.

## D.2 Approximate Result

In this section, we give a result similar to Theorems 2.5 and E.4 for interval type-2 sets. Before giving the approximate result, we obtain a result similar to Theorem 2.4 for interval type-1 sets.

**Theorem D.1** *Given $n$ interval type-1 numbers $F_1, \ldots, F_n$, with means $m_1, m_2, \ldots, m_n$ and spreads $s_1, s_2, \ldots, s_n$, their affine combination $\sum_{i=1}^{n} \alpha_i F_i + \beta$, where $\alpha_i$ ($i = 1, \ldots, n$) and $\beta$ are crisp constants, is also an interval type-1 number with mean $\sum_{i=1}^{n} \alpha_i m_i + \beta$, and spread $\sum_{i=1}^{n} |\alpha_i| s_i$.* □

**Proof :** Consider $F_i = [m_i - s_i, m_i + s_i]$. Multiplying $F_i$ by a crisp constant $\alpha_i$ $(= 1/\alpha_i)$ yields (see 2.61)

$$\alpha_i F_i = \int_v 1/(\alpha_i v) \quad ; \quad v \in [m_i - s_i, m_i + s_i] \tag{D.8}$$

Adding a crisp constant $\beta$ $(= 1/\beta)$ to $\alpha_i F_i$ yields [see (2.63)]

$$\alpha_i F_i + \beta = \int_v 1/(\alpha_i v + \beta) \quad ; \quad v \in [m_i - s_i, m_i + s_i] \tag{D.9}$$

Substituting $w = \alpha_i v + \beta$, (D.9) gives us

$$\alpha_i F_i + \beta = \int_w 1/w \quad ; \quad w \in [\alpha_i m_i + \beta - |\alpha_i| s_i \,,\, \alpha_i m_i + \beta + |\alpha_i| s_i] \tag{D.10}$$

Recall that $F_i$ can be represented as $[l_i, r_i]$, where $l_i = m_i - s_i$ and $r_i = m_i + s_i$. Observe, therefore, from (2.64) [see, also, the discussion after (2.64)], that

$$\sum_{i=1}^{n} F_i = \left[\sum_{i=1}^{n} m_i - \sum_{i=1}^{n} s_i \,,\, \sum_{i=1}^{n} m_i + \sum_{i=1}^{n} s_i\right] \tag{D.11}$$

Using (D.10) and (D.11), we get the result in Theorem D.1. □

We now give an approximation to the weighted average of interval type-1 sets.

**Theorem D.2** *If each $Z_l$ in (D.2) is an interval type-1 set, having center $c_l$ and spread $s_l$, and if each $W_l$ is also an interval type-1 set with center $h_l$ and spread $\Delta_l$, then $Y$ is approximately an interval type-1 set, with center $\mathcal{C}$ and spread $\mathcal{S}$, where*

$$\mathcal{C} = \frac{\sum_{l=1}^{M} h_l c_l}{\sum_{l=1}^{M} h_l} \tag{D.12}$$

*and*

$$\mathcal{S} = \frac{\sum_{l=1}^{M}[(h_l s_l) + |c_l - \mathcal{C}|\Delta_l]}{\sum_{l=1}^{M} h_l} \tag{D.13}$$

*provided that*

$$\frac{\sum_{l=1}^{M} \Delta_l}{\sum_{l=1}^{M} h_l} \ll 1\,. \tag{D.14}$$

*The approximation improves as $\left(\sum_{l=1}^{M} \Delta_l \big/ \sum_{l=1}^{M} h_l\right)$ grows smaller. The result is exact when $\sum_{l=1}^{M} \Delta_l = 0$, i.e., when $\Delta_l = 0$ for $l = 1, \ldots, M$.* □

**Proof :** The proof proceeds exactly like the proof of Theorem 2.5 in Appendix C.11, the only difference being that now the condition required for a good approximation is $[\sum_{l=1}^{M} \Delta_l]/[\sum_{l=1}^{M} h_l] \ll 1$ instead of $k[\sum_{l=1}^{M} \Delta_l]/[\sum_{l=1}^{M} h_l] \ll 1$ [see (2.76)]. The factor of $k$ appeared in the Gaussian case, because the membership of a point in a Gaussian type-1 set is never exactly equal to 0; we just neglected the memberships outside $\pm k\Delta_l$, since they were too small. In the interval case, however, the memberships of points outside $\pm\Delta_l$ are equal to 0; and therefore, the factor of $k$ disappears.

We get a result similar to (C.229) :

$$Y \approx \sum_{l=1}^{M} \left[\underline{Z}_l\left(\frac{h_l}{\sum_l h_l}\right) + \underline{W}_l\left(\frac{c_l - \mathcal{C}}{\sum_l h_l}\right)\right] + \mathcal{C} \tag{D.15}$$

where $\underline{Z}_l$'s are zero-mean interval type-1 sets with spreads $s_l$'s, $\underline{W}_l$'s are zero-mean interval type-1 sets with spreads $\Delta_l$'s, and all the summations and "+" signs denote algebraic sum. The result in Theorem D.2 follows by applying Theorem D.1 to (D.15). When applying Theorem D.1, we set $n = 2M$; $F_i = \underline{Z}_i$, and $\alpha_i = h_i/[\sum_{l=1}^{M} h_l]$ for $i = 1, \ldots, M$; and, $F_i = \underline{W}_i$, and $\alpha_i = (c_i - \mathcal{C})/[\sum_{l=1}^{M} h_l]$ for $i = M+1, \ldots, 2M$. □

**Comment 1 :** In this case, the true weighted average [i.e., the LHS of (D.2)], $Y$, will also be an interval type-1 set (since all the sets involved are interval type-1 sets); however, the approximation is useful because the actual end-points of the domain of $Y$ can only be obtained computationally.

**Comment 2 :** Comments 2 and 3 at the end of Appendix C.11 apply in this case as well.

**Comment 3 :** Though Theorem D.2 is very much similar to Theorems 2.5 and E.4, there is one difference. In case of Gaussians and triangular sets, the secondary membership functions may be clipped because they have to be contained in $[0, 1]$; and therefore, may not remain true Gaussians or triangles. We ignored these clipping effects for simplicity; therefore, the results in Theorems 2.5 and E.4 contained a "clipping effect" approximation, in addition to the approximations introduced subject to conditions (2.76) and (E.184), respectively. In the case of interval sets, however, no clipping effects need to be considered, because any clipped version of an interval is again an interval; so, the only approximation that is introduced in the result in Theorem D.2 is the one subject to condition (D.14).

# Appendix E

## Triangular Membership Functions

Most of the results we have developed so far are quite generally applicable to any convex normal membership functions, with the exception of two important results : (a) the Gaussian *meet* approximation (Section 2.3.2) and (b) the weighted average of Gaussian fuzzy numbers (Theorem 2.5). Here, we develop results similar to (a) and (b) for triangular membership functions, so that one can use a type-2 FLS with triangular type-2 sets. By a triangular type-2 set, we mean a type-2 set which assigns a triangular type-1 membership grade to every point in its domain.

For simplicity, we deal only with symmetrical triangles (i.e., triangles with left spread equal to right spread). Such triangles can be described by two parameters : the center $c$ and the spread $s$, so that the domain of the triangular fuzzy set is the interval $[c - s, c + s]$. In the sequel, a "triangle" will always mean a "symmetrical triangle" unless explicitly stated otherwise.

In Section E.1, we first derive the actual result of the *meet* of two triangular fuzzy sets. The result, however, is not triangular and therefore does not easily generalize to the *meet* of more than two sets; therefore, we find a triangular approximation to the actual *meet* and also find upper and lower bounds on the approximation. In Section E.1.2, we provide MATLAB files for verification of the results stated in Theorem E.1. In Section E.2, we develop results for the algebraic sum of triangular type-1 fuzzy numbers (similar to the result for the sum of Gaussian type-1 fuzzy numbers in Section 2.4.2.2). This result is used in Section E.2.2.3, where we develop a result similar to Theorem 2.5 for triangular fuzzy numbers.

## E.1 Meet Approximation under Product $t$-Norm

The derivation of the actual *meet* of two triangular fuzzy sets gives a closed form solution, unlike the Gaussian case (see Appendix C.6 for derivation of the actual *meet* between two Gaussians). So, we first derive the actual result and then see if an approximation is needed.

In type-2 applications, the triangular sets involved in the *meet* operation will be membership grades of type-2 sets; therefore, we restrict our attention to triangles contained in $[0, 1]$.

### E.1.1 Meet of Triangles under Product $t$-norm

**Theorem E.1** *Suppose that we have two triangular type-1 sets $\tilde{F}$ and $\tilde{G}$, characterized by membership functions $f$ and $g$, such that*

$$f(v) = \begin{cases} 1 - \left|\frac{v-c_f}{s_f}\right| & ; \quad v \in [\underline{c_f - s_f}, \overline{c_f + s_f}] \\ 0 & ; \quad otherwise \end{cases} \tag{E.1}$$

*and*

$$g(w) = \begin{cases} 1 - \left|\frac{w-c_g}{s_g}\right| & ; \quad w \in [\underline{c_g - s_g}, \overline{c_g + s_g}] \\ 0 & ; \quad otherwise \end{cases} \tag{E.2}$$

*where $\underline{x} = \max\{x, 0\}$, $\overline{x} = \min\{x, 1\}$, $c_f, c_g \in [0, 1]$ and $s_f, s_g > 0$.*

*The meet of $\tilde{F}$ and $\tilde{G}$ under product t-norm is characterized by the following membership function :*

*<u>For $\theta \leq c_f c_g$</u>*

*If $c_f > s_f$ and $c_g > s_g$,*

$$\mu_{\tilde{F}\sqcap\tilde{G}}(\theta) = \begin{cases} \frac{1}{s_f s_g}\left[\sqrt{\theta} - \sqrt{(c_f - s_f)(c_g - s_g)}\right]^2 & ; \quad (c_f - s_f)(c_g - s_g) \leq \theta \leq cs_1 \\ 1 + \frac{\theta - c_f c_g}{cs} & ; \quad cs_1 \leq \theta \leq c_f c_g \end{cases} \tag{E.3}$$

*If $c_f \leq s_f$ and/or $c_g \leq s_g$,*

$$\mu_{\tilde{F}\sqcap\tilde{G}}(\theta) = 1 + \left(\frac{\theta - c_f c_g}{cs}\right) \quad ; \quad 0 \leq \theta \leq c_f c_g \tag{E.4}$$

*For $\theta \geq c_f c_g$*

*If $c_f + s_f < 1$ and $c_g + s_g < 1$,*

$$\mu_{\tilde{F}\sqcap\tilde{G}}(\theta) = \begin{cases} 1 - \frac{\theta - c_f c_g}{cs} & ; \quad c_f c_g \leq \theta \leq cs_2 \\ \frac{1}{s_f s_g}\left[\sqrt{\theta} - \sqrt{(c_f + s_f)(c_g + s_g)}\right]^2 & ; \quad cs_2 \leq \theta \leq (c_f + s_f)(c_g + s_g) \end{cases} \tag{E.5}$$

*If $c_f + s_f \geq 1$ and/or $c_g + s_g \geq 1$,*

$$\mu_{\tilde{F}\sqcap\tilde{G}}(\theta) = \begin{cases} 1 - \frac{\theta - c_f c_g}{cs} & ; \quad c_f c_g \leq \theta \leq cs_2 \\ \frac{1}{s_f s_g}\left[\sqrt{\theta} - \sqrt{(c_f + s_f)(c_g + s_g)}\right]^2 & ; \quad cs_2 \leq \theta \leq cs_3 \\ \max\left\{\left[1 - \left(\frac{\theta - c_g}{s_g}\right)\right]\left[1 - \left(\frac{1 - c_f}{s_f}\right)\right], \right. \\ \qquad \left. \left[1 - \left(\frac{1 - c_g}{s_g}\right)\right]\left[1 - \left(\frac{\theta - c_f}{s_f}\right)\right]\right\} & ; \quad cs_3 \leq \theta \leq \overline{(c_f + s_f)}\overline{(c_g + s_g)} \end{cases} \tag{E.6}$$

*where*

$$cs = \max\{c_f s_g, c_g s_f\} \tag{E.7}$$

$$cs_1 = \min\left\{c_f^2\left(\frac{c_g - s_g}{c_f - s_f}\right), c_g^2\left(\frac{c_f - s_f}{c_g - s_g}\right)\right\} \tag{E.8}$$

$$cs_2 = \max\left\{c_f^2\left(\frac{c_g + s_g}{c_f + s_f}\right), c_g^2\left(\frac{c_f + s_f}{c_g + s_g}\right)\right\} \tag{E.9}$$

$$cs_3 = \min\left\{\frac{c_f + s_f}{c_g + s_g}, \frac{c_g + s_g}{c_f + s_f}\right\} \tag{E.10}$$

**Proof** :

Equation (2.28) describes the *meet* operation between two type-1 sets under product $t$-norm. We reproduce it here for convenience :

$$\tilde{F} \sqcap \tilde{G} = \int_{v \in \tilde{F}} \int_{w \in \tilde{G}} [f(v)g(w)]/(vw), \tag{E.11}$$

where the integrals denote logical union. Equation (E.11) can be interpreted as follows. Each element $v$ of set $\tilde{F}$ multiplies every element $w$ of set $\tilde{G}$, and, at the same time, the membership grade of $v$ in $\tilde{F}$ multiplies the membership grade of $w$ in $\tilde{G}$. So, given a particular element $v_1$ of $\tilde{F}$, what we get as a result of these multiplications is a scaled version of the membership function of $\tilde{G}$ [scaled along both the axes : along the independent axis by $v_1$ and along the dependent axis by $f(v_1)$]. This process is repeated for every element of $\tilde{F}$ and finally, the meet of $\tilde{F}$ and $\tilde{G}$ is given by the envelope of all the above scaled triangles. Figure E.1 shows this interpretation pictorially. Observe that if we interchange $\tilde{F}$ and $\tilde{G}$, the above interpretaion still holds. [Recall that we used the same interpretation of the *meet* operation while deriving the Gaussian *meet* approximation (Section 2.3.2).]

To find the membership function of $\tilde{F} \sqcap \tilde{G}$, we start by considering that $\tilde{F} \sqcap \tilde{G}$ is made up of a collection of scaled sets $\tilde{G}$, one scaled set corresponding to each element of $\tilde{F}$. One obvious thing that can be observed from (E.11) is that only one point in the domain of $\tilde{F} \sqcap \tilde{G}$ will have a membership grade equal to unity, and this point is equal to $c_f c_g$. Observe, from Fig. E.1 (b), that to the right of the unity membership point $c_f c_g$, $\mu_{\tilde{F} \sqcap \tilde{G}}$ is a curve that touches the right side of each scaled triangle. Similarly, to the left of $c_f c_g$, $\mu_{\tilde{F} \sqcap \tilde{G}}$ touches the left side of each scaled triangle. From this, we infer that to the left of $c_f c_g$, the left half of $\tilde{G}$ (i.e., $v < c_g$) contributes to $\mu_{\tilde{F} \sqcap \tilde{G}}$, and, to the right of $c_f c_g$, the right half of $\tilde{G}$ (i.e., $v > c_g$) contributes to $\mu_{\tilde{F} \sqcap \tilde{G}}$. Also, from (E.1) and (E.2), it is clear that the domain of $\tilde{F} \sqcap \tilde{G}$ is $[\underline{(c_f - s_f)(c_g - s_g)}\,,\, \overline{(c_f + s_f)(c_g + s_g)}]$.

To begin with, we assume that the triangles are completely contained in $[0, 1]$, i.e., $c_f - s_f > 0$, $c_g - s_g > 0$, $c_f + s_f < 1$ and $c_g + s_g < 1$ (they may or may not overlap). Later on, we consider the other possibilities. Observe that, as argued in the preceding paragraph, to the left of $c_f c_g$, only the left half of $\tilde{G}$ contributes and to the right of $c_f c_g$, only the right half of $\tilde{G}$ contributes; therefore, the conditions $c_f - s_f > 0$ and $c_g - s_g > 0$ matter only for the part of the *meet* to the left of $c_f c_g$; and the conditions $c_f + s_f < 1$ and $c_g + s_g < 1$ matter only for the part of the *meet* to the right of $c_f c_g$.

**Triangles completely contained in $[0, 1]$ :**

We can write the membership function of $\tilde{F} \sqcap \tilde{G}$ as follows : For $(c_f - s_f)(c_g - s_g) \leq \theta \leq c_f c_g$,

$$\mu_{\tilde{F} \sqcap \tilde{G}}(\theta) = \sup_{v \in V_l} \Big[1 + \Big(\frac{\frac{\theta}{v} - c_g}{s_g}\Big)\Big]\Big[1 + \Big(\frac{v - c_f}{s_f}\Big)\Big] \tag{E.12}$$

where the interval $V_l$ is found as follows. Since $v \in \tilde{F}$, $v \geq (c_f - s_f)$ and here, we are considering points to the left of $c_f$ [see (E.1) and discussion in the preceding paragraph]. This gives us $v \in [c_f - s_f, c_f]$. Similarly, since $\frac{\theta}{v} \in \tilde{G}$ and we are considering points to the left of $c_g$, we have $\frac{\theta}{v} \in [c_g - s_g, c_g]$, i.e., $v \in [\frac{\theta}{c_g}, \frac{\theta}{c_g - s_g}]$. Combining these two constraints, we have

$$V_l = [c_f - s_f, c_f] \cap \Big[\frac{\theta}{c_g}, \frac{\theta}{c_g - s_g}\Big] \tag{E.13}$$

Observe that $(c_f - s_f)(c_g - s_g) \le \theta \Rightarrow \frac{\theta}{c_g - s_g} \ge (c_f - s_f)$ and $\theta \le c_f c_g \Rightarrow \frac{\theta}{c_g} \le c_f$; therefore, $V_l$ is a non-empty interval.

Similarly, for $c_f c_g \le \theta \le (c_f + s_f)(c_g + s_g)$,

$$\mu_{\tilde{F} \sqcap \tilde{G}}(\theta) = \sup_{v \in V_r} \Big[1 - \Big(\frac{\frac{\theta}{v} - c_g}{s_g}\Big)\Big]\Big[1 - \Big(\frac{v - c_f}{s_f}\Big)\Big] \tag{E.14}$$

where, arguing as in case of (E.12), we find that

$$V_r = [c_f, c_f + s_f] \cap \Big[\frac{\theta}{c_g + s_g}, \frac{\theta}{c_g}\Big] \tag{E.15}$$

It is easy to see that $V_r$ is also non-empty for $c_f c_g \le \theta \le (c_f + s_f)(c_g + s_g)$.

Combining (E.12) and (E.14), we have

$$\mu_{\tilde{F} \sqcap \tilde{G}}(\theta) = \begin{cases} \sup_{v \in V_l} \Big[1 + \Big(\frac{\frac{\theta}{v} - c_g}{s_g}\Big)\Big]\Big[1 + \Big(\frac{v - c_f}{s_f}\Big)\Big] & ; \ (c_f - s_f)(c_g - s_g) \le \theta \le c_f c_g \\ \sup_{v \in V_r} \Big[1 - \Big(\frac{\frac{\theta}{v} - c_g}{s_g}\Big)\Big]\Big[1 - \Big(\frac{v - c_f}{s_f}\Big)\Big] & ; \ c_f c_g \le \theta \le (c_f + s_f)(c_g + s_g) \end{cases} \tag{E.16}$$

where $V_l$ and $V_r$ are as in (E.13) and (E.15), respectively.

For $(c_f - s_f)(c_g - s_g) \le \theta \le c_f c_g$, the objective function to be maximized is

$$\begin{aligned} D1(v) &= \Big[1 + \Big(\frac{\frac{\theta}{v} - c_g}{s_g}\Big)\Big]\Big[1 + \Big(\frac{v - c_f}{s_f}\Big)\Big] \\ &= \Big[\Big(1 - \frac{c_g}{s_g}\Big) + \frac{\theta}{s_g}\frac{1}{v}\Big]\Big[\Big(1 - \frac{c_f}{s_f}\Big) + \frac{v}{s_f}\Big] \\ &= \Big(1 - \frac{c_g}{s_g}\Big)\Big(1 - \frac{c_f}{s_f}\Big) + \Big(1 - \frac{c_g}{s_g}\Big)\frac{v}{s_f} + \Big(1 - \frac{c_f}{s_f}\Big)\frac{\theta}{s_g}\frac{1}{v} + \frac{\theta}{s_f s_g} \end{aligned} \tag{E.17}$$

The first and second derivatives of $D1$ are

$$D1'(v) = \Big(1 - \frac{c_g}{s_g}\Big)\frac{1}{s_f} - \Big(1 - \frac{c_f}{s_f}\Big)\frac{\theta}{s_g}\frac{1}{v^2} \tag{E.18}$$

$$D1''(v) = \left(1 - \frac{c_f}{s_f}\right)\frac{\theta}{s_g}\frac{2}{v^3} \tag{E.19}$$

Since $c_f > s_f$, $D1'' < 0$, which implies that $D1$ is concave. To find its supremum, we equate $D1'$ to zero. If the supremum is achieved at $v = v^*$, we have

$$\left(1 - \frac{c_g}{s_g}\right)\frac{1}{s_f} - \left(1 - \frac{c_f}{s_f}\right)\frac{\theta}{s_g}\frac{1}{(v^*)^2} = 0$$

$$\Rightarrow \quad v^* = \sqrt{\theta\left(\frac{c_f - s_f}{c_g - s_g}\right)} \tag{E.20}$$

Let us see if $v^*$ satisfies the constraint $v^* \in V_l$. For $v^*$ to belong to $V_l$, it has to satisfy the following four conditions [see Eq. (E.13)] : $v^* \geq c_f - s_f$, $v^* \leq c_f$, $v^* \geq \frac{\theta}{c_g}$ and $v^* \leq \frac{\theta}{c_g - s_g}$. Since $\theta \geq (c_f - s_f)(c_g - s_g)$, $(c_f - s_f) \leq v^* \leq \frac{\theta}{c_g - s_g}$. To satisfy the other two conditions,

$$\begin{aligned}
v^* \leq c_f \quad &\Leftrightarrow \quad \theta \leq c_f^2\left(\frac{c_g - s_g}{c_f - s_f}\right) \\
v^* \geq \frac{\theta}{c_g} \quad &\Leftrightarrow \quad \theta \leq c_g^2\left(\frac{c_f - s_f}{c_g - s_g}\right) \\
\Rightarrow v^* \in V_l \quad &\Leftrightarrow \quad \theta \leq \min\{c_f^2\left(\frac{c_g - s_g}{c_f - s_f}\right), c_g^2\left(\frac{c_f - s_f}{c_g - s_g}\right)\}
\end{aligned} \tag{E.21}$$

Substituting (E.20) into (E.17), we get

$$\mu_{\tilde{F}\sqcap\tilde{G}}(\theta) = \frac{1}{s_f s_g}\left[\sqrt{\theta} - \sqrt{(c_f - s_f)(c_g - s_g)}\right]^2 \quad ; \quad (c_f - s_f)(c_g - s_g) \leq \theta \leq cs_1 \tag{E.22}$$

where

$$cs_1 = \min\left\{c_f^2\left(\frac{c_g - s_g}{c_f - s_f}\right), c_g^2\left(\frac{c_f - s_f}{c_g - s_g}\right)\right\} \tag{E.23}$$

If $\theta > cs_1$, the supremum of $D1$ is reached at a point $v^*$ which is either greater than $c_f$ or less than $\frac{\theta}{c_g}$ [(E.21)]. If $cs_1 = c_f^2\left(\frac{c_g - s_g}{c_f - s_f}\right)$, $\theta > cs_1 \Leftrightarrow v^* > c_f$. Since, $D1$ is concave, the supremum in $V_l$ is attained at $v = c_f$; therefore, substituting $v = c_f$ in (E.17), we get

$$\mu_{\tilde{F}\sqcap\tilde{G}}(\theta) = 1 + \left(\frac{\theta - c_f c_g}{c_f s_g}\right) \quad ; \quad c_f^2\left(\frac{c_g - s_g}{c_f - s_f}\right) < \theta \leq c_f c_g \tag{E.24}$$

If $cs_1 = c_g^2\left(\frac{c_f - s_f}{c_g - s_g}\right)$, $\theta > cs_1 \Leftrightarrow v^* < \frac{\theta}{c_g}$. Since, $D1$ is concave, the supremum in $V_l$ is attained at $v = \frac{\theta}{c_g}$; therefore, substituting $v = \frac{\theta}{c_g}$ in (E.17), we get

$$\mu_{\tilde{F}\sqcap\tilde{G}}(\theta) = 1 + \left(\frac{\theta - c_f c_g}{c_g s_f}\right) \quad ; \quad c_g^2\left(\frac{c_f - s_f}{c_g - s_g}\right) < \theta \leq c_f c_g \tag{E.25}$$

Note that

$$c_f s_g \gtrless c_g s_f \quad \Leftrightarrow \quad c_f^2\left(\frac{c_g - s_g}{c_f - s_f}\right) \lessgtr c_g^2\left(\frac{c_f - s_f}{c_g - s_g}\right) \tag{E.26}$$

Combining (E.24), (E.25) and (E.26), we get

$$\mu_{\tilde{F}\sqcap\tilde{G}}(\theta) = 1 + \left(\frac{\theta - c_f c_g}{cs}\right) \quad ; \quad cs_1 < \theta \leq c_f c_g \tag{E.27}$$

where

$$cs = \max\{c_f s_g, c_g s_f\} \tag{E.28}$$

For $c_f c_g \leq \theta \leq (c_f + s_f)(c_g + s_g)$, the objective function to be maximized is [(E.16)]

$$\begin{aligned} D2(v) &= \left[1 - \left(\frac{\frac{\theta}{v} - c_g}{s_g}\right)\right]\left[1 - \left(\frac{v - c_f}{s_f}\right)\right] \\ &= \left[\left(1 + \frac{c_g}{s_g}\right) - \frac{\theta}{s_g}\frac{1}{v}\right]\left[\left(1 + \frac{c_f}{s_f}\right) - \frac{v}{s_f}\right] \\ &= \left(1 + \frac{c_g}{s_g}\right)\left(1 + \frac{c_f}{s_f}\right) - \left(1 + \frac{c_g}{s_g}\right)\frac{v}{s_f} - \left(1 + \frac{c_f}{s_f}\right)\frac{\theta}{s_g}\frac{1}{v} + \frac{\theta}{s_f s_g} \end{aligned} \tag{E.29}$$

The first and second derivatives of $D2$ are

$$D2'(v) = -\left(1 + \frac{c_g}{s_g}\right)\frac{1}{s_f} + \left(1 + \frac{c_f}{s_f}\right)\frac{\theta}{s_g}\frac{1}{v^2} \tag{E.30}$$

$$D2''(v) = -\left(1 + \frac{c_f}{s_f}\right)\frac{\theta}{s_g}\frac{2}{v^3} \tag{E.31}$$

From (E.31), we can see that $D2''$ is always negative (for $v \in [0, 1]$), implying that $D2$ is concave. If the supremum of $D2$ is reached at $v_*$, equating $D2'$ to zero, we get

$$-\left(1 + \frac{c_g}{s_g}\right)\frac{1}{s_f} + \left(1 + \frac{c_f}{s_f}\right)\frac{\theta}{s_g}\frac{1}{v_*^2} = 0$$

$$\Rightarrow \quad v_* \;=\; \sqrt{\theta\Big(\frac{c_f + s_f}{c_g + s_g}\Big)} \tag{E.32}$$

Let us see if $v^*$ satisfies the constraint $v^* \in V_r$. For $v^*$ to belong to $V_r$, it has to satisfy the following four conditions [see Eq. (E.15)] : $v^* \geq c_f$, $v^* \leq c_f + s_f$, $v^* \geq \frac{\theta}{c_g+s_g}$ and $v^* \leq \frac{\theta}{c_g}$. Since $\theta \leq (c_f+s_f)(c_g+s_g)$, $\frac{\theta}{c_g+s_g} \leq v^* \leq (c_f+s_f)$. To satisfy the other two conditions,

$$\begin{aligned}
v^* \geq c_f \quad &\Leftrightarrow \quad \theta \geq c_f^2\Big(\frac{c_g+s_g}{c_f+s_f}\Big)\\
v^* \leq \frac{\theta}{c_g} \quad &\Leftrightarrow \quad \theta \geq c_g^2\Big(\frac{c_f+s_f}{c_g+s_g}\Big)\\
\Rightarrow v^* \in V_r \quad &\Leftrightarrow \quad \theta \geq \max\{c_f^2\Big(\frac{c_g+s_g}{c_f+s_f}\Big)\,,\, c_g^2\Big(\frac{c_f+s_f}{c_g+s_g}\Big)\}
\end{aligned} \tag{E.33}$$

Substituting (E.32) into (E.29), we get

$$\mu_{\tilde{F}\sqcap\tilde{G}}(\theta) = \frac{1}{s_f s_g}\Big[\sqrt{\theta} - \sqrt{(c_f+s_f)(c_g+s_g)}\Big]^2 \quad ; \quad cs_2 \leq \theta \leq (c_f+s_f)(c_g+s_g) \tag{E.34}$$

where

$$cs_2 = \max\Big\{c_f^2\Big(\frac{c_g+s_g}{c_f+s_f}\Big)\,,\, c_g^2\Big(\frac{c_f+s_f}{c_g+s_g}\Big)\Big\} \tag{E.35}$$

If $\theta < cs_2$, the supremum of $D2$ is reached at a point $v^*$ which is either less than $c_f$ or greater than $\frac{\theta}{c_g}$ [(E.33)]. If $cs_2 = c_f^2\Big(\frac{c_g+s_g}{c_f+s_f}\Big)$, $\theta < cs_2 \Leftrightarrow v^* < c_f$. Since, $D2$ is concave, the supremum in $V_r$ is attained at $v = c_f$; therefore, substituting $v = c_f$ in (E.29), we get

$$\mu_{\tilde{F}\sqcap\tilde{G}}(\theta) = 1 - \Big(\frac{\theta - c_f c_g}{c_f s_g}\Big) \quad ; \quad c_f c_g \leq \theta < c_f^2\Big(\frac{c_g+s_g}{c_f+s_f}\Big) \tag{E.36}$$

If $cs_2 = c_g^2\Big(\frac{c_f+s_f}{c_g+s_g}\Big)$, $\theta < cs_2 \Leftrightarrow v^* > \frac{\theta}{c_g}$. Since, $D2$ is concave, the supremum in $V_r$ is attained at $v = \frac{\theta}{c_g}$; therefore, substituting $v = \frac{\theta}{c_g}$ in (E.29), we get

$$\mu_{\tilde{F}\sqcap\tilde{G}}(\theta) = 1 - \Big(\frac{\theta - c_f c_g}{c_g s_f}\Big) \quad ; \quad c_f c_g \leq \theta < c_g^2\Big(\frac{c_f+s_f}{c_g+s_g}\Big) \tag{E.37}$$

Observe that

$$c_f s_g \gtrless c_g s_f \quad \Leftrightarrow \quad c_f^2\Big(\frac{c_g+s_g}{c_f+s_f}\Big) \gtrless c_g^2\Big(\frac{c_f+s_f}{c_g+s_g}\Big) \tag{E.38}$$

Combining (E.36), (E.37), and (E.38), we get

$$\mu_{\tilde{F}\sqcap\tilde{G}}(\theta) = 1 - \Big(\frac{\theta - c_f c_g}{cs}\Big) \quad ; \quad c_f c_g < \theta \le cs_2 \tag{E.39}$$

where $cs$ is as in (E.7).

From (E.22), (E.27), (E.34) and (E.39), if $c_f - s_f > 0$, $c_g - s_g > 0$, $c_f + s_f < 1$ and $c_g + s_g < 1$, then

$$\mu_{\tilde{F}\sqcap\tilde{G}}(\theta) = \begin{cases} \frac{1}{s_f s_g}\Big[\sqrt{\theta} - \sqrt{(c_f - s_f)(c_g - s_g)}\Big]^2 & ; \quad (c_f - s_f)(c_g - s_g) \le \theta \le cs_1 \\ 1 + \frac{\theta - c_f c_g}{cs} & ; \quad cs_1 \le \theta \le c_f c_g \\ 1 - \frac{\theta - c_f c_g}{cs} & ; \quad c_f c_g \le \theta \le cs_2 \\ \frac{1}{s_f s_g}\Big[\sqrt{\theta} - \sqrt{(c_f + s_f)(c_g + s_g)}\Big]^2 & ; \quad cs_2 \le \theta \le (c_f + s_f)(c_g + s_g) \end{cases} \tag{E.40}$$

where $cs$, $cs_1$ and $cs_2$ are as in (E.7), (E.8) and (E.9), respectively.

Now, let's consider the case when the triangles are not fully contained in $[0, 1]$, i.e., when the triangles may be clipped by the lines $\theta = 0$ and/or $\theta = 1$. As explained earlier, the conditions $c_f - s_f > 0$ and $c_g - s_g > 0$ matter only for $\theta \le c_f c_g$ and the conditions $c_f + s_f < 1$ and $c_g + s_g < 1$ matter only for $\theta \ge c_f c_g$.

**Triangles clipped on the left**

Consider $c_f \le s_f$ and $c_g \le s_g$. Now, $\underline{c_f - s_f} = \underline{c_g - s_g} = 0$. We can write the membership function of $\tilde{F} \sqcap \tilde{G}$ for $0 \le \theta \le c_f c_g$ as follows :

$$\mu_{\tilde{F}\sqcap\tilde{G}}(\theta) = \sup_{v \in V_{l1}} \Big[1 + \Big(\frac{\frac{\theta}{v} - c_g}{s_g}\Big)\Big]\Big[1 + \Big(\frac{v - c_f}{s_f}\Big)\Big] \tag{E.41}$$

where the interval $V_{l1}$ is found as follows. Since $v \in \tilde{F}$, and is to the left of $c_f$, $v \in [\underline{c_f - s_f} = 0, c_f]$. Similarly, since $\frac{\theta}{v} \in \tilde{G}$ and we are considering points to the left of $c_g$, we have $\frac{\theta}{v} \in [\underline{c_g - s_g} = 0, c_g]$, i.e., $v \ge \frac{\theta}{c_g}$. Combining these two constraints, we have

$$V_{l1} = \Big[\frac{\theta}{c_g}, c_f\Big] \tag{E.42}$$

Observe that $\theta \ge c_f c_g \Leftrightarrow \frac{\theta}{c_g} \le c_f$; therefore, $V_{l1}$ is a non-empty interval.

Now $D1''$ [see (E.19)] is positive, which implies that $D1$ is convex; therefore, the supremum of $D1$ is reached at one of the end-points of $V_{l1}$, i.e.,

$$\begin{aligned}\mu_{\tilde{F}\sqcap\tilde{G}}(\theta) &= D1\Big(\frac{\theta}{c_g}\Big) \vee D1(c_f) \\ &= 1 + \Big(\frac{\theta - c_f c_g}{cs}\Big) \quad ; \quad 0 \leq \theta \leq c_f c_g \end{aligned} \tag{E.43}$$

where $cs$ is as in (E.7).

Consider $c_f \leq s_f$ and $c_g > s_g$. Now, $\underline{c_f - s_f} = 0$ and $\underline{c_g - s_g} = c_g - s_g$. We can write the membership function of $\tilde{F} \sqcap \tilde{G}$ for $0 \leq \theta \leq c_f c_g$ as follows :

$$\mu_{\tilde{F}\sqcap\tilde{G}}(\theta) = \sup_{v \in V_{l2}} \Big[1 + \Big(\frac{\frac{\theta}{v} - c_g}{s_g}\Big)\Big]\Big[1 + \Big(\frac{v - c_f}{s_f}\Big)\Big] \tag{E.44}$$

where the interval $V_{l2}$ can be found as

$$V_{l2} = [0, c_f] \cap \Big[\frac{\theta}{c_g}, \frac{\theta}{c_g - s_g}\Big] \tag{E.45}$$

Since $0 \leq \frac{\theta}{c_g} \leq c_f$, the expression for $V_{l2}$ simplifies to

$$V_{l2} = \Big[\frac{\theta}{c_g}, \min\Big\{c_f, \frac{\theta}{c_g - s_g}\Big\}\Big] \tag{E.46}$$

Again $D1$ is convex. But, now $D1' < 0$, which implies that $D1$ is monotonic decreasing for $v \in [0, 1]$; therefore, the supremum is reached at $v = \frac{\theta}{c_g}$, i.e.,

$$\begin{aligned}\mu_{\tilde{F}\sqcap\tilde{G}}(\theta) &= D1\Big(\frac{\theta}{c_g}\Big) \\ &= 1 + \Big(\frac{\theta - c_f c_g}{c_g s_f}\Big) \end{aligned} \tag{E.47}$$

If we consider $c_f > s_f$ and $c_g \leq s_g$, $\underline{c_f - s_f} = c_f - s_f$ and $\underline{c_g - s_g} = 0$. The membership function of $\tilde{F} \sqcap \tilde{G}$ for $0 \leq \theta \leq c_f c_g$ is

$$\mu_{\tilde{F}\sqcap\tilde{G}}(\theta) = \sup_{v \in V_{l3}} \Big[1 + \Big(\frac{\frac{\theta}{v} - c_g}{s_g}\Big)\Big]\Big[1 + \Big(\frac{v - c_f}{s_f}\Big)\Big] \tag{E.48}$$

where the interval $V_{l3}$ can be found as follows : since $v$ belongs to the left half of $f$, $v \in [c_f - s_f, c_f]$ and since $\frac{\theta}{v}$ belongs to the left half of $g$, $\frac{\theta}{v} \in [0, c_g]$, i.e., $v \geq \frac{\theta}{c_g}$. Combining these two conditions, we get

$$V_{l3} = \left[\max\left\{c_f - s_f, \frac{\theta}{c_g}\right\}, c_f\right] \tag{E.49}$$

In this case $D1'' < 0$, which implies that $D1$ is concave. Observe, however, that now $D1' > 0$, implying that $D1$ is monotonic increasing for $v \in [0, 1]$. Therefore, the supremum is reached at $v = c_f$, i.e.,

$$\mu_{\tilde{F}\sqcap\tilde{G}}(\theta) = 1 + \left(\frac{\theta - c_f c_g}{c_f s_g}\right) \quad ; \quad 0 \leq \theta \leq c_f c_g \tag{E.50}$$

Combining (E.47) and (E.50), we see that if $c_f \leq s_f$ or $c_g \leq s_g$, then

$$\mu_{\tilde{F}\sqcap\tilde{G}}(\theta) = 1 + \left(\frac{\theta - c_f c_g}{cs}\right) \quad ; \quad 0 \leq \theta \leq c_f c_g \tag{E.51}$$

From (E.43) and (E.51), we see that if $c_f \leq s_f$ and/or $c_g \leq s_g$, the membership function of $\tilde{F} \sqcap \tilde{G}$ for $0 \leq \theta c_f c_g$ is

$$\mu_{\tilde{F}\sqcap\tilde{G}}(\theta) = 1 + \left(\frac{\theta - c_f c_g}{cs}\right) \quad ; \quad 0 \leq \theta \leq c_f c_g \tag{E.52}$$

**Triangles Clipped on the Right**

Let $c_f + s_f \geq 1$ and $c_g + s_g \geq 1$, so that $\overline{c_f + s_f} = \overline{c_g + s_g} = 1$. The membership function of $\tilde{F} \sqcap \tilde{G}$ for $c_f c_g \leq \theta \leq 1$ is as follows :

$$\mu_{\tilde{F}\sqcap\tilde{G}}(\theta) = \sup_{v \in V_{r1}} \left[1 - \left(\frac{\frac{\theta}{v} - c_g}{s_g}\right)\right]\left[1 - \left(\frac{v - c_f}{s_f}\right)\right] \tag{E.53}$$

where the interval $V_{r1}$ is found as follows. Since $v \in \tilde{F}$, and is to the right of $c_f$, $v \in [c_f\,, \overline{c_f + s_f} = 1]$. Similarly, since $\frac{\theta}{v} \in \tilde{G}$ and we are considering points to the right of $c_g$, we have $\frac{\theta}{v} \in [c_g\,, \overline{c_g + s_g} = 1]$, i.e., $v \in \left[\theta, \frac{\theta}{c_g}\right]$. Combining these two constraints, we have

$$V_{r1} = [c_f, 1] \cap \left[\theta, \frac{\theta}{c_g}\right] \tag{E.54}$$

Observe that $\theta \geq c_f c_g \Leftrightarrow c_f \leq \frac{\theta}{c_g}$ and obviously $\theta \leq 1$; therefore, $V_{r1}$ is non-empty.

From (E.31), we see that $D2$ is concave; therefore, equating $D2'$ to zero, we find the point at which the supremum is achieved, as [see(E.32)]

$$v^* = \sqrt{\theta\left(\frac{c_f + s_f}{c_g + s_g}\right)} \tag{E.55}$$

Let's see under what conditions $v^*$ satisfies the constraint $v^* \in V_{r1}$. For $v^*$ to belong to $V_{r1}$, it has to satisfy the following four conditions [see Eq. (E.54)] : $v^* \geq c_f$, $v^* \leq 1$, $v^* \geq \theta$ and $v^* \leq \frac{\theta}{c_g}$. From (E.33), $c_f \leq v^* \leq \frac{\theta}{c_g} \Leftrightarrow v^* \geq cs_2$, where $cs_2$ is as in (E.9). To satisfy the remaining two conditions,

$$v^* \geq \theta \quad \Leftrightarrow \quad \theta \leq \frac{c_f + s_f}{c_g + s_g} \tag{E.56}$$

$$v^* \leq 1 \quad \Leftrightarrow \quad \theta \leq \frac{c_g + s_g}{c_f + s_f} \tag{E.57}$$

From (E.33), (E.56), and (E.57), we have

$$v^* \in V_{r1} \quad \Leftrightarrow \quad cs_2 \leq \theta \leq cs_3 \tag{E.58}$$

where

$$cs_3 = \min\left\{\frac{c_f + s_f}{c_g + s_g}, \frac{c_g + s_g}{c_f + s_f}\right\} \tag{E.59}$$

For $\theta \leq cs_3$, the expressions for $\mu_{\tilde{F}\sqcap\tilde{G}}(\theta)$ are similar to those in (E.40). If $(c_f + s_f) < (c_g + s_g)$, we have $cs_3 = \frac{c_f+s_f}{c_g+s_g}$. In this case, $\theta > cs_3 \Leftrightarrow v^* < \theta$; therefore, the supremum of $D2$ is reached at $v = \theta$, i.e.,

$$\begin{aligned} \mu_{\tilde{F}\sqcap\tilde{G}}(\theta) &= D2(\theta) \\ &= \left[1 - \left(\frac{1 - c_g}{s_g}\right)\right]\left[1 - \left(\frac{\theta - c_f}{s_f}\right)\right] \end{aligned} \tag{E.60}$$

If $(c_g + s_g) < (c_f + s_f)$, we have $cs_3 = \frac{c_g+s_g}{c_f+s_f}$. In this case, $\theta > cs_3 \Leftrightarrow v^* > 1$; therefore, the supremum of $D2$ is reached at $v = 1$, i.e.,

$$\begin{aligned} \mu_{\tilde{F}\sqcap\tilde{G}}(\theta) &= D2(1) \\ &= \left[1 - \left(\frac{\theta - c_g}{s_g}\right)\right]\left[1 - \left(\frac{1 - c_f}{s_f}\right)\right] \end{aligned} \tag{E.61}$$

Also observe that

$$c_f+s_f \gtrless c_g+s_g \quad \Leftrightarrow \quad \Big[1-\big(\frac{\theta-c_g}{s_g}\big)\Big]\Big[1-\big(\frac{1-c_f}{s_f}\big)\Big] \gtrless \Big[1-\big(\frac{1-c_g}{s_g}\big)\Big]\Big[1-\big(\frac{\theta-c_f}{s_f}\big)\Big] \tag{E.62}$$

From (E.60), (E.61) and (E.62), we have that

$$\begin{aligned}\mu_{\tilde{F}\sqcap\tilde{G}}(\theta) &= \max\Big\{\Big[1-\big(\frac{\theta-c_g}{s_g}\big)\Big]\Big[1-\big(\frac{1-c_f}{s_f}\big)\Big], \\ &\qquad \Big[1-\big(\frac{1-c_g}{s_g}\big)\Big]\Big[1-\big(\frac{\theta-c_f}{s_f}\big)\Big]\Big\} \quad ; \quad cs_3 \le \theta \le 1 \end{aligned} \tag{E.63}$$

Now, we consider the case where only one of the triangles is clipped by the line $\theta = 1$. Let $c_f + s_f \ge 1$ and $c_g + s_g < 1$. Now, $\overline{c_f + s_f} = 1$ and $\overline{c_g + s_g} = c_g + s_g$; consequently, $(\overline{c_f + s_f})(\overline{c_g + s_g}) = c_g + s_g$. The membership function for $\tilde{F} \sqcap \tilde{G}$ for $c_f c_g \le \theta \le (c_g + s_g)$ is

$$\mu_{\tilde{F}\sqcap\tilde{G}}(\theta) = \sup_{v \in V_{r2}} \Big[1-\big(\frac{\frac{\theta}{v}-c_g}{s_g}\big)\Big]\Big[1-\big(\frac{v-c_f}{s_f}\big)\Big] \tag{E.64}$$

where the interval $V_{r2}$ is found as follows. Since $v \in \tilde{F}$, and is to the right of $c_f$, $v \in [c_f\,,\,\overline{c_f+s_f} = 1]$. Similarly, since $\frac{\theta}{v} \in \tilde{G}$ and we are considering points to the right of $c_g$, we have $\frac{\theta}{v} \in [c_g\,,\,c_g+s_g]$, i.e., $v \in \Big[\frac{\theta}{c_g+s_g}\,,\,\frac{\theta}{c_g}\Big]$. Combining these two constraints, we have

$$V_{r2} = [c_f, 1] \cap \Big[\frac{\theta}{c_g+s_g}, \frac{\theta}{c_g}\Big] \tag{E.65}$$

Observe that $\theta \ge c_f c_g \Leftrightarrow c_f \le \frac{\theta}{c_g}$ and $\theta \le (c_g+s_g) \Leftrightarrow \frac{\theta}{c_g+s_g} \le 1$; therefore, $V_{r2}$ is non-empty. The supremum of $D2$ is attained at $v^*$ which is as in (E.55).

For $v^*$ to belong to $V_{r2}$, it has to satisfy the following four conditions [see Eq. (E.54)] : $v^* \ge c_f$, $v^* \le 1$, $v^* \ge \frac{\theta}{c_g+s_g}$ and $v^* \le \frac{\theta}{c_g}$. From (E.33), $c_f \le v^* \le \frac{\theta}{c_g} \Leftrightarrow v^* \ge cs_2$, where $cs_2$ is as in (E.9). Also, $\theta \le (c_g+s_g) \le (c_f+s_f)(c_g+s_g) \Leftrightarrow v^* \ge \frac{\theta}{c_g+s_g}$, and

$$v^* \le 1 \quad \Leftrightarrow \quad \theta \le \frac{c_g+s_g}{c_f+s_f} \tag{E.66}$$

Hence,

$$v^* \in V_{r2} \quad \Leftrightarrow \quad cs_2 \le \theta \le \frac{c_g+s_g}{c_f+s_f} \tag{E.67}$$

For $\theta \leq cs_3$, $\mu_{\tilde{F}\sqcap\tilde{G}}(\theta)$ is as in (E.40). If $\theta > \frac{c_g+s_g}{c_f+s_f}$ (observe that $c_g + s_g < c_f + s_f$ since we have assumed that $c_f + s_f > 1$ and $c_g + s_g < 1$), $v^* > 1$. In this case, the supremum within $V_{r2}$ is achieved at $v = 1$, i.e.,

$$\begin{aligned} \mu_{\tilde{F}\sqcap\tilde{G}}(\theta) &= D2(1) \\ &= \left[1 - \left(\frac{\theta - c_g}{s_g}\right)\right]\left[1 - \left(\frac{1 - c_f}{s_f}\right)\right] \quad ; \quad \frac{c_g + s_g}{c_f + s_f} \leq \theta \leq 1 \end{aligned} \qquad \text{(E.68)}$$

If we interchange the order of $\tilde{F}$ and $\tilde{G}$, i.e., if we assume that $c_f + s_f < 1$ and $c_g + s_g \geq 1$, we get

$$\begin{aligned} \mu_{\tilde{F}\sqcap\tilde{G}}(\theta) &= D2(\theta) \\ &= \left[1 - \left(\frac{1 - c_g}{s_g}\right)\right]\left[1 - \left(\frac{\theta - c_f}{s_f}\right)\right] \quad ; \quad \frac{c_f + s_f}{c_g + s_g} \leq \theta \leq 1 \end{aligned} \qquad \text{(E.69)}$$

For $c_f c_g \leq \theta \leq \frac{c_g+s_g}{c_f+s_f}$, the membership function is as in (E.40).

Observe that (E.51) and (E.68), both consider the situation when only one of the triangles is clipped by the line $\theta = 1$. Since, the result should be independent of the order of $\tilde{F}$ and $\tilde{G}$, the actual membership function in this case is the maximum of (E.51) and (E.68); therefore, combining (E.68) and (E.69), we find that if $c_f + s_f \geq 1$ and/or $c_g + s_g \geq 1$, the membership function of $\tilde{F}\sqcap\tilde{G}$ for $c_f c_g \leq \theta \leq (\overline{c_f + s_f})(\overline{c_g + s_g})$ is

$$\mu_{\tilde{F}\sqcap\tilde{G}}(\theta) = \begin{cases} 1 - \frac{\theta - c_f c_g}{cs} & ; \quad c_f c_g \leq \theta \leq cs_2 \\ \frac{1}{s_f s_g}\left[\sqrt{\theta} - \sqrt{(c_f + s_f)(c_g + s_g)}\right]^2 & ; \quad cs_2 \leq \theta \leq cs_3 \\ \max\Big\{\left[1 - \left(\frac{\theta - c_g}{s_g}\right)\right]\left[1 - \left(\frac{1 - c_f}{s_f}\right)\right], & \\ \qquad \left[1 - \left(\frac{1 - c_g}{s_g}\right)\right]\left[1 - \left(\frac{\theta - c_f}{s_f}\right)\right]\Big\} & ; \quad cs_3 \leq \theta \leq (\overline{c_f + s_f})(\overline{c_g + s_g}) \end{cases} \qquad \text{(E.70)}$$

where $cs$, $cs_1$ and $cs_2$ are as in (E.7), (E.8), and (E.9), respectively.

From (E.40), (E.52), and (E.70), we get the results stated in Theorem E.1. Observe that the result of the *meet* of two triangles under product $t$-norm does not remain triangular. Figure E.1 (b) shows the *meet* of the two triangular fuzzy sets shown in Fig. E.1 (a).

From the length and complexity of the proof of Theorem E.1, it is apparent that the easiest way to verify its correctness is to compare Theorem E.1 with numerically computed results. In Section E.1.2, we provide the MATLAB programs used for the verification of Theorem E.1.

### E.1.2 M-Files[1] for Verification of Theorem E.1

```
% Function ''tri_meet.m'' :  Computes theoretical and numerical
results for the meet operation under product t-norm between two
triangular type-1 sets.

% Inputs :  Centers (c1,c2) and spreads (s1,s2) of the two
triangular type-1 sets.  The centers should be in [0,1] and the
spreads should be positive.  ''step'' is an optional stepsize
parameter.  A smaller value should be used for higher resolution.
If not specified, the default is 0.01.

% Outputs :  ''x'' is the x-axis (in [0,1]); ''y_num'' is the
numerically calculates result of the meet operation; and ''y_thm''
is the result stated in Theorem E.1.

% Uses functions ''triangle.m'' and ''extend.m''.

function[x,y_num,y_thm] = tri_meet(c1,s1,c2,s2,step)

if nargin == 4,
   step = 0.01 ;
end % if

x = [step :  step :  1]' ;
lx = length(x) ;

          % Numerically computed result
y1 = triangle(c1,s1,step) ;
y2 = triangle(c2,s2,step) ;
tol = step/2 ;
y_num = extend(x,y1,y2,tol) ;

          % Theoretical result :  Theorem E.1
cc = c1*c2 ;
cs_m1 = c1-s1 ;
```

[1] For use with MATLAB version 4.2c or higher

```
cs_m2 = c2-s2 ;
cs_p1 = c1+s1 ;
cs_p2 = c2+s2 ;
cs = max(c1*s2,c2*s1) ;
bcs_m1 = max(cs_m1,0) ;
bcs_m2 = max(cs_m2,0) ;
bcs_p1 = min(cs_p1,1) ;
bcs_p2 = min(cs_p2,1) ;

left_end = bcs_m1*bcs_m2 ;
[min1,in1] = min(abs(x - left_end)) ;
[min2,in2] = min(abs(x - cc)) ;
x_left = [x(in1) :  step :  x(in2)] ;
lx_l = length(x_left) ;

if (cs_m1 > 0) & (cs_m2 > 0),
   cs1 = min(c1^2*cs_m2/cs_m1, c2^2*cs_m1/cs_m2) ;
   [min1x,in1x] = min(abs(x_left - cs1)) ;
   x_l1 = x_left(1:in1x) ;
   x_l2 = x_left(in1x+1:lx_l) ;
   y_l1 = 1/(s1*s2)*(sqrt(x_l1) - sqrt(cs_m1*cs_m2)).^2 ;
   y_l2 = 1 + ((x_l2 - cc)./cs) ;
   y_left = [y_l1,y_l2] ;
else
   y_left = 1 + ((x_left - cc)./cs) ;
end % if

right_end = bcs_p1*bcs_p2 ;
[min3,in3] = min(abs(x - right_end)) ;
x_right = [x(in2) :  step :  x(in3)] ;
lx_r = length(x_right) ;

cs2 = max(c1^2*cs_p2/cs_p1, c2^2*cs_p1/cs_p2) ;

[min2x,in2x] = min(abs(x_right-cs2)) ;
x_r1 = x_right(1:in2x) ;
y_r1 = 1 - ((x_r1 - cc)./cs) ;

if (cs_p1 < 1)&(cs_p2 < 1),
   x_r2 = x_right(in2x+1:lx_r) ;
   y_r2 = 1/(s1*s2)*(sqrt(x_r2) - sqrt(cs_p1*cs_p2)).^2 ;
else
   cs3 = min(cs_p1/cs_p2 , cs_p2/cs_p1) ;
   [min3x,in3x] = min(abs(x_right-cs3)) ;
```

```
   x_r21 = x_right(in2x+1:in3x) ;
   y_r21 = 1/(s1*s2)*(sqrt(x_r21) - sqrt(cs_p1*cs_p2)).^2 ;
   x_r22 = x_right(in3x+1:lx_r) ;
   yt221 = 1/(s1*s2) * ((cs_p1 - x_r22)*(cs_p2 - 1)) ;
   yt222 = 1/(s1*s2) * ((cs_p1 - 1)*(cs_p2 - x_r22)) ;
   y_r22 = max(yt221,yt222) ;
   x_r2 = [x_r21 x_r22] ;
   y_r2 = [y_r21 y_r22] ;
end % if
y_right = [y_r1 y_r2] ;

if in1 > 1,
   z1 = zeros(1,in1-1) ;
else
   z1 = [] ;
end % if

if in3 < lx,
   z2 = zeros(1,lx-in3) ;
else
   z2 = [] ;
end % if

y_thm = [z1 y_left y_right(2:lx_r) z2]' ;

return ;
```

```
% Function ''triangle.m'' : Computes the ordinates for a
symmetrical triangle.

% Inputs : Center(''c'') and spread (''s'') of the triangle;
and optional stepsize parameter : "step". Smaller value for
''step'' gives higher resolution. The domain of the triangle
is always assumed to be [0,1]. The center ''c'' should always
be in [0,1] and spread ''s'' should be positive.

% Outputs : The ordinates ''y'' and abscissae ''x'' of the
triangle. If c+s>1 and/or c-s<0, the triangle appears clipped,
i.e., only the part of the triangle contained in [0,1] is produced.

function [y,x] = triangle(c,s,step)
```

```
if nargin == 2,
   step = 0.01 ;
end % if

x = [step :  step :  1]' ;
lnx = length(x) ;
lx = c - s ;
rx = c + s ;
minx = min(x) ;
maxx = max(x) ;
mx1 = x(max(1,round(c/step))) ;

if lx < minx ,
   if rx > maxx,
     x1 = [minx :  step :  mx1]' ;
     x2 = [mx1+step :  step :  maxx]' ;
     y1 = max(1 + (x1-c)/s,0) ;
     y2 = max(1 - (x2-c)/s,0) ;
     y = [y1' y2']' ;
   else
     inrx1 = (min(lnx,round(rx/step))) ;
     if inrx1 == 0,
       inrx1 = 1;
     end
     rx1 = x(inrx1) ;
     x1 = [minx :  step :  mx1]' ;
     x2 = [mx1+step :  step :  rx1]' ;
     x3 = [rx1+step :  step :  maxx]' ;
     y1 = max(1 + (x1-c)/s,0) ;
     y2 = max(1 - (x2-c)/s,0) ;
     y3 = zeros(size(x3)) ;
     y = [y1' y2' y3']' ;
   end % if rx
else
   if rx > maxx,
     lx1 = x(max(1,round(lx/step))) ;
     x1 = [minx :  step :  lx1-step]' ;
     x2 = [lx1 :  step :  mx1]' ;
     x3 = [mx1+step :  step :  maxx]' ;
     y1 = zeros(size(x1)) ;
     y2 = max(1 + (x2-c)/s,0) ;
     y3 = max(1 - (x3-c)/s,0) ;
     y = [y1' y2' y3']' ;
```

```
    else
      lx1 = x(max(1,round(lx/step))) ;
      inrx1 = (min(lnx,round(rx/step))) ;
      if inrx1 == 0,
        inrx1 = 1;
      end
      rx1 = x(inrx1) ;
      x1 = [minx :  step :  lx1-step]' ;
      x2 = [lx1 :  step :  mx1]' ;
      x3 = [mx1+step :  step :  rx1]' ;
      x4 = [rx1+step :  step :  maxx]' ;
      y1 = zeros(size(x1)) ;
      y2 = max(1 + (x2-c)/s,0) ;
      y3 = max(1 - (x3-c)/s,0) ;
      y4 = zeros(size(x4)) ;
      y = [y1' y2' y3' y4']' ;
    end % if rx
end % if lx

return ;
```

```
% Function ''extend.m'' :  Computes the meet under product t-norm
between two type-1 sets.  Both the type-1 sets are required to
have domain [0,1].

% Inputs :  All column vectors - domain of the type-1 sets ("x");
and their membership functions ("y1" and "y2").  Default value for
the optional scalar argument ''tol'' is 0.005.  While applying the
Extension Principle, if the difference between to domain points is
less than ''tol'', they are assumed to be equal.

function [y] = extend(x,y1,y2,tol)

if nargin == 3,
    tol = 0.005 ;
end % if

lx = length(x) ;
y = zeros(size(x)) ;
```

```
for i1 = 1 :  lx,
   if i1 > 1,
     clear xn yn1 yn2
   end % if
   k = x(i1) ;
   xn = x(i1 :  lx) ;
   yn1 = y1(i1 :  lx) ;
   yn2 = y2(i1 :  lx) ;
   x_mat = xn*xn' ;
   y_mat = yn1*yn2' ;
   [in1, in2] = find( abs(x_mat-k) <= tol) ;
   y(i1) = max(diag(y_mat(in1,in2))) ;
end % for

return ;
```

### E.1.3 Triangular Meet Approximation

Although we found an expression for the result of the actual *meet* operation for two triangular membership functions, it is too complicated, and moreover, it is difficult to extend it to the *meet* of more than two triangular membership functions. Hence, we seek a triangular approximation, because, if we can approximate the result of the *meet* of two symmetrical triangles with another symmetrical triangle, we will be able to generalize the results of the *meet* of two symmetrical triangles to the *meet* of more than two triangles and this will a save a lot of computations while implementing a triangular type-2 fuzzy logic system. In order to find such an approximation, we first find upper and lower bounds on the result of the actual *meet* operation.

#### E.1.3.1 Lower Bound

To find a lower bound on the *meet* between two triangular fuzzy sets, we proceed exactly in the same manner as we did in the Gaussian case [see Appendix C.8.3 -

Eq.(C.146) onwards]. Consider the two triangular type-1 sets considered in Theorem E.1. We rewrite their membership functions here for convenience,

$$f(v) = \begin{cases} 1 - \left|\frac{v-c_f}{s_f}\right| & ; \quad v \in [\underline{c_f - s_f}, \overline{c_f + s_f}] \\ 0 & ; \quad \text{otherwise} \end{cases} \tag{E.71}$$

$$g(w) = \begin{cases} 1 - \left|\frac{w-c_g}{s_g}\right| & ; \quad w \in [\underline{c_g - s_g}, \overline{c_g + s_g}] \\ 0 & ; \quad \text{otherwise} \end{cases} \tag{E.72}$$

where $\underline{x} = \max\{x, 0\}$, $\overline{x} = \min\{x, 1\}$.

In order to find the lower bound, we first assume that $\tilde{\mathrm{F}}$ is a singleton (let's call it $\tilde{\mathrm{F}}_s$) with $\mu_{\tilde{\mathrm{F}}_s}(c_f) = 1$ and $\mu_{\tilde{\mathrm{F}}_s}(v) = 0$ for $v \neq c_f$, and that $\tilde{\mathrm{G}}$ has the membership function $g$ described in (E.72). The resulting membership function is a scaled version of $g$, i.e.,

$$\begin{aligned} \mu_{\tilde{\mathrm{F}}_s \sqcap \tilde{\mathrm{G}}}(\theta) &= \mu_{\tilde{\mathrm{G}}}(\theta / c_f) \\ &= \begin{cases} 1 - \left|\frac{\frac{\theta}{c_f} - c_g}{s_g}\right| & ; \quad \frac{\theta}{c_f} \in [\underline{c_g - s_g}, \overline{c_g + s_g}] \\ 0 & ; \quad \text{otherwise} \end{cases} \\ &= \begin{cases} 1 - \left|\frac{\theta - c_f c_g}{c_f s_g}\right| & ; \quad \theta \in [c_f(\underline{c_g - s_g}), c_f(\overline{c_g + s_g})] \\ 0 & ; \quad \text{otherwise} \end{cases} \end{aligned} \tag{E.73}$$

Now, we assume that $\tilde{\mathrm{G}}$ (let's call it $\tilde{\mathrm{G}}_s$) is a singleton at $c_g$ and $\tilde{\mathrm{F}}$ has the membership function $f$ described in (E.71). The resulting membership function is a scaled version of $f$, i.e.,

$$\begin{aligned} \mu_{\tilde{\mathrm{F}} \sqcap \tilde{\mathrm{G}}_s}(\theta) &= \mu_{\tilde{\mathrm{F}}}(\theta / c_g) \\ &= \begin{cases} 1 - \left|\frac{\frac{\theta}{c_g} - c_f}{s_f}\right| & ; \quad \frac{\theta}{c_g} \in [\underline{c_f - s_f}, \overline{c_f + s_f}] \\ 0 & ; \quad \text{otherwise} \end{cases} \\ &= \begin{cases} 1 - \left|\frac{\theta - c_f c_g}{c_g s_f}\right| & ; \quad \theta \in [c_g(\underline{c_f - s_f}), c_g(\overline{c_f + s_f})] \\ 0 & ; \quad \text{otherwise} \end{cases} \end{aligned} \tag{E.74}$$

Observe that $\mu_{\tilde{F}_s} \leq \mu_{\tilde{F}}$; therefore $\mu_{\tilde{F}_s \sqcap \tilde{G}} \leq \mu_{\tilde{F} \sqcap \tilde{G}}$. Similarly, $\mu_{\tilde{F} \sqcap \tilde{G}_s} \leq \mu_{\tilde{F} \sqcap \tilde{G}}$. This shows that each of (E.73) and (E.74) is a lower bound on the result of the *meet*. We select the maximum of them as the actual lower bound. If we assume that $c_f(\overline{c_g + s_g}) \leq c_g(\overline{c_f + s_f})$, the lower bound is

$$\mu^L_{\tilde{F} \sqcap \tilde{G}}(\theta) = \begin{cases} 1 - \left|\frac{\theta - c_f c_g}{cs}\right| & ; \ \theta \in [cs', c_f(\overline{c_g + s_g})] \\ 1 - \left|\frac{\theta - c_f c_g}{c_g s_f}\right| & ; \ \theta \in [c_f(\overline{c_g + s_g}), c_g(\overline{c_f + s_f})] \end{cases} \tag{E.75}$$

where $cs = \max\{c_f s_g, c_g s_f\}$ [recall Eq. (E.7)] and

$$cs' = \min\{c_f(\underline{c_g - s_g}), \ c_g(\underline{c_f - s_f})\} \tag{E.76}$$

If we assume that $c_g(\overline{c_f + s_f}) \leq c_f(\overline{c_g + s_g})$, then the lower bound is

$$\mu^L_{\tilde{F} \sqcap \tilde{G}}(\theta) = \begin{cases} 1 - \left|\frac{\theta - c_f c_g}{cs}\right| & ; \ \theta \in [cs', c_g(\overline{c_f + s_f})] \\ 1 - \left|\frac{\theta - c_f c_g}{c_f s_g}\right| & ; \ \theta \in [c_g(\overline{c_f + s_f}), c_f(\overline{c_g + s_g})] \end{cases} \tag{E.77}$$

where $cs$ is as in (E.7) and $cs'$ is given in (E.76). Observe, from (E.75) and (E.77), that both the expressions for the lower bound are the same in the interval $[cs', \min\{c_f(\overline{c_g + s_g}), \ c_g(\overline{c_f + s_f})\}]$. They differ outside this interval; therefore, to simplify (E.75) and (E.77) a little bit, we ignore the part of the curve lying outside $[cs', \min\{c_f(\overline{c_g + s_g}), \ c_g(\overline{c_f + s_f})\}]$. This will let us generalize the bound to the case of more than two sets easily; hence, a lower bound on the *meet* of $\tilde{F}$ and $\tilde{G}$ is given as

$$\mu^L_{\tilde{F} \sqcap \tilde{G}}(\theta) = \begin{cases} 1 - \left|\frac{\theta - c_f c_g}{cs}\right| & ; \ \theta \in [cs', \min\{c_f(\overline{c_g + s_g}), c_g(\overline{c_f + s_f})\}] \\ 0 & ; \ \text{otherwise} \end{cases} \tag{E.78}$$

where $cs'$ is as in (E.76).

Generalization to the case of more than two triangular sets, proceeds in a manner exactly similar to the generalization of the lower bound on the *meet* between Gaussian type-1 sets. (See Example C.2). We state the result here.

If we have $n$ triangular fuzzy sets $\tilde{F}_1, \ldots, \tilde{F}_n$ having centers $c_1, c_2, \ldots, c_n$ and spreads $s_1, s_2, \ldots, s_n$, then a lower bound on the result of their *meet* is given by

$$\mu^L_{\sqcap_{i=1}^n \tilde{F}_i}(\theta) = \begin{cases} 1 - \left|\frac{\theta - \prod_{i=1}^n c_i}{s_l}\right| & ; \ \theta \in [l_n, r_n] \\ 0 & ; \ \text{otherwise} \end{cases} \tag{E.79}$$

where

$$l_n = \min\left\{\overline{c_1 - s_1}\prod_{i;i\neq 1} c_i, \cdots, \overline{c_j - s_j}\prod_{i;i\neq j} c_i, \cdots, \overline{c_n - s_n}\prod_{i;i\neq n} c_i\right\};\ i = 1, 2, \cdots, n \tag{E.80}$$

and

$$r_n = \min\left\{\overline{c_1 + s_1}\prod_{i;i\neq 1} c_i, \cdots, \overline{c_j + s_j}\prod_{i;i\neq j} c_i, \cdots, \overline{c_n + s_n}\prod_{i;i\neq n} c_i\right\};\ i = 1, 2, \cdots, n \tag{E.81}$$

### E.1.3.2 Upper Bound

$\underline{\theta \leq c_f c_g}$ :

First we consider the case where $c_f > s_f$ and $c_g > s_g$. We show that, in this case, the straight line, $l_1(\theta)$ [see Fig. E.1 (c)], joining the point $(c_f c_g, 1)$ and the point $[(c_f - s_f)(c_g - s_g), 0]$ forms an upper bound on the result of the *meet.* The equation of this line is

$$l_1(\theta) = \begin{cases} 1 + \frac{\theta - c_f c_g}{c_f c_g - (c_f - s_f)(c_g - s_g)} & ; \ \theta \in [(c_f - s_f)(c_g - s_g), c_f c_g] \\ 0 & ; \ \text{otherwise} \end{cases} \tag{E.82}$$

From (E.3), we have to show that

$$l_1(\theta) \geq \frac{1}{s_f s_g}\left[\sqrt{\theta} - \sqrt{(c_f - s_f)(c_g - s_g)}\right]^2; \quad (c_f - s_f)(c_g - s_g) \leq \theta \leq cs_1 \tag{E.83}$$

Let us assume that for $(c_f - s_f)(c_g - s_g) \leq \theta \leq cs_1$,

$$l_1(\theta) < \frac{1}{s_f s_g}\left[\sqrt{\theta} - \sqrt{(c_f - s_f)(c_g - s_g)}\right]^2 \tag{E.84}$$

Equation (E.84) implies that for $(c_f - s_f)(c_g - s_g) \leq \theta \leq cs_1$,

$$\begin{aligned}
& 1 + \frac{\theta - c_f c_g}{c_f c_g - (c_f - s_f)(c_g - s_g)} < \\
& \qquad \frac{1}{s_f s_g}\left[\sqrt{\theta} - \sqrt{(c_f - s_f)(c_g - s_g)}\right]^2 \\
\Rightarrow \quad & \frac{\theta - (c_f - s_f)(c_g - s_g)}{c_f c_g - (c_f - s_f)(c_g - s_g)} < \\
& \qquad \frac{1}{s_f s_g}\left[\sqrt{\theta} - \sqrt{(c_f - s_f)(c_g - s_g)}\right]^2 \qquad \text{(E.85)} \\
\Rightarrow \quad & \frac{\left[\sqrt{\theta} + \sqrt{(c_f - s_f)(c_g - s_g)}\right]}{c_f c_g - (c_f - s_f)(c_g - s_g)} < \\
& \qquad \frac{1}{s_f s_g}\left[\sqrt{\theta} - \sqrt{(c_f - s_f)(c_g - s_g)}\right] \qquad \text{(E.86)}
\end{aligned}$$

While going from (E.85) to (E.86), we have made use of the fact that $\theta \geq (c_f - s_f)(c_g - s_g)$, and

$$\theta - (c_f - s_f)(c_g - s_g) = \left[\sqrt{\theta} + \sqrt{(c_f - s_f)(c_g - s_g)}\right]\left[\sqrt{\theta} - \sqrt{(c_f - s_f)(c_g - s_g)}\right]. \tag{E.87}$$

Equation (E.86) implies that

$$\begin{aligned}
& \sqrt{(c_f - s_f)(c_g - s_g)}\left[\frac{1}{c_f c_g - (c_f - s_f)(c_g - s_g)} + \frac{1}{s_f s_g}\right] < \\
& \qquad \sqrt{\theta}\left[\frac{1}{s_f s_g} - \frac{1}{c_f c_g - (c_f - s_f)(c_g - s_g)}\right]
\end{aligned}$$

$$\Rightarrow \sqrt{(c_f - s_f)(c_g - s_g)} \frac{c_f s_g + c_g s_f}{[c_f c_g - (c_f - s_f)(c_g - s_g)] s_f s_g} <$$

$$\sqrt{\theta} \frac{c_f s_g + c_g s_f - 2 s_f s_g}{[c_f c_g - (c_f - s_f)(c_g - s_g)] s_f s_g}$$

$$\Rightarrow \sqrt{(c_f - s_f)(c_g - s_g)}(c_f s_g + c_g s_f) < \sqrt{\theta}(c_f s_g + c_g s_f - 2 s_f s_g) \quad \text{(E.88)}$$

Now, we use the fact that $\theta \leq cs_1$. Let us assume that $cs_1 = c_f^2\left(\frac{c_g - s_g}{c_f - s_f}\right)$, which implies that $c_f s_g > c_g s_f$ [Eq. (E.26)]. Continuing with (E.88), this implies

$$\begin{aligned}
\sqrt{(c_f - s_f)(c_g - s_g)}(c_f s_g + c_g s_f) &< c_f \sqrt{\left(\frac{c_g - s_g}{c_f - s_f}\right)}(c_f s_g + c_g s_f - 2 s_f s_g) \\
\Rightarrow \quad (c_f - s_f)(c_f s_g + c_g s_f) &< c_f(c_f s_g + c_g s_f - 2 s_f s_g) \\
\Rightarrow \quad -s_f(c_f s_g + c_g s_f) &< -2 c_f s_f s_g \\
\Rightarrow \quad c_f s_g + c_g s_f &> 2 c_f s_g \\
\Rightarrow \quad c_g s_f &> c_f s_g \qquad \text{(E.89)}
\end{aligned}$$

which contradicts our assumption that $c_f s_g > c_g s_f$ and consequently contradicts (E.84); implying that (E.83) is true. If we assume that $cs_1 = c_g^2\left(\frac{c_f - s_f}{c_g - s_g}\right)$, we get a similar contradiction.

For $cs_1 < \theta \leq c_f c_g$, we have to show that [see (E.3)]

$$l_1(\theta) \geq 1 + \frac{\theta - c_f c_g}{cs} \quad \text{(E.90)}$$

i.e., (after some algebra) we have to show that

$$c_f c_g - (c_f - s_f)(c_g - s_g) \geq cs \quad \text{(E.91)}$$

where $cs$ is given in (E.7). Observe that

$$\begin{aligned}
c_f c_g - (c_f - s_f)(c_g - s_g) &= c_f s_g + c_g s_f - s_f s_g \\
&= c_f s_g + s_f(c_g - s_g) \quad > \quad c_f s_g
\end{aligned}$$

$$= c_g s_f + s_g(c_f - s_f) \quad > \quad c_g s_f \qquad \text{(E.92)}$$

which shows that (E.91) is true. Observe that we have made use of the fact that $c_f > s_f$ and $c_g > s_g$. From (E.83) and (E.91), we see that $l_1(\theta)$ is, indeed, an upper bound on $\mu_{\tilde{F}\sqcap\tilde{G}}(\theta)$ when $c_f > s_f$ and $c_g > s_g$.

Next, we consider the case $c_f \leq s_f$ and/or $c_g \leq s_g$. From (E.4), we see that we again have to show (E.91) for $0 \leq \theta \leq c_f c_g$. Observe, however, that in this case, the inequalitites shown in (E.92) may not hold true; therefore, for this case $l_1(\theta)$ may not remain an upper bound. For this reason, we choose the following straight line as an upper bound for $\theta \leq c_f c_g$,

$$l_2(\theta) = \begin{cases} 1 + \frac{\theta - c_f c_g}{c_f s_g + c_g s_f} & ; \quad \theta \in \left[\max\{0, c_f c_g - (c_f s_g + c_g s_f)\}, c_f c_g\right] \\ 0 & ; \quad \text{otherwise} \end{cases} \qquad \text{(E.93)}$$

Since $c_f s_g + c_g s_f > c_f c_g - (c_f - s_f)(c_g - s_g)$, $l_2(\theta) \geq l_1(\theta)$; therefore, $l_2(\theta)$ forms an upper bound when $c_f > s_f$ and $c_g > s_g$. Also, since $c_f s_g + c_g s_f > \max\{c_f s_g, c_g s_f\}$, $l_2(\theta)$ forms an upper bound when $c_f \leq s_f$ and/or $c_g \leq s_g$; therefore,

$$\mu^U_{\tilde{F}\sqcap\tilde{G}}(\theta) = l_2(\theta) \quad ; \quad (\underline{c_f - s_f})(\underline{c_g - s_g}) \leq \theta \leq c_f c_g \qquad \text{(E.94)}$$

where $l_2(\theta)$ is as in (E.93).

$\underline{\theta \geq c_f c_g}$ :

We show that the straight line joining the points $(c_f c_g, 1)$ and $[(c_f + s_f)(c_g + s_g), 0]$ [see Fig. E.1 (c)] forms an upper bound on $\mu_{\tilde{F}\sqcap\tilde{G}}(\theta)$, i.e., we want to show that the line

$$l_3(\theta) = \begin{cases} 1 - \frac{\theta - c_f c_g}{(c_f + s_f)(c_g + s_g) - c_f c_g} & ; \quad \theta \in [c_f c_g, (\overline{c_f + s_f})(\overline{c_g + s_g})] \\ 0 & ; \quad \text{otherwise} \end{cases} \qquad \text{(E.95)}$$

forms an upper bound for $\theta \geq c_f c_g$.

Consider the case when $c_f + s_f < 1$ and $c_g + s_g < 1$. For $c_f c_g \leq \theta \leq cs_2$, we have to show that [see (E.5)]

$$l_3(\theta) \geq 1 - \frac{\theta - c_f c_g}{cs} \tag{E.96}$$

Observe that

$$\begin{aligned} (c_f + s_f)(c_g + s_g) - c_f c_g &= c_f s_g + c_g s_f + s_f s_g \\ &> \max\{c_f s_g, c_g s_f\} \quad = \quad cs \end{aligned} \tag{E.97}$$

which shows that (E.96), is true. For $cs_2 \leq \theta \leq (c_f + s_f)(c_g + s_g)$, we have to show that [see (E.5)]

$$1 - \frac{\theta - c_f c_g}{(c_f + s_f)(c_g + s_g) - c_f c_g} \geq \frac{1}{s_f s_g}\Big[\sqrt{\theta} - \sqrt{(c_f + s_f)(c_g + s_g)}\Big]^2 \tag{E.98}$$

Let us assume that

$$1 - \frac{\theta - c_f c_g}{(c_f + s_f)(c_g + s_g) - c_f c_g} < \frac{1}{s_f s_g}\Big[\sqrt{\theta} - \sqrt{(c_f + s_f)(c_g + s_g)}\Big]^2 \tag{E.99}$$

$$\Rightarrow \frac{-\theta + (c_f + s_f)(c_g + s_g)}{(c_f + s_f)(c_g + s_g) - c_f c_g} < \frac{1}{s_f s_g}\Big[\sqrt{\theta} - \sqrt{(c_f + s_f)(c_g + s_g)}\Big]^2 \tag{E.100}$$

$$\Rightarrow \frac{\Big[\sqrt{(c_f + s_f)(c_g + s_g)} + \sqrt{\theta}\Big]}{(c_f + s_f)(c_g + s_g) - c_f c_g} < \frac{1}{s_f s_g}\Big[\sqrt{(c_f + s_f)(c_g + s_g)} - \sqrt{\theta}\Big] \tag{E.101}$$

While going from (E.100) to (E.101), we have made use of the facts that $\theta \leq (c_f + s_f)(c_g + s_g)$, and

$$(c_f + s_f)(c_g + s_g) - \theta = \Big[\sqrt{(c_f + s_f)(c_g + s_g)} + \sqrt{\theta}\Big]\Big[\sqrt{(c_f + s_f)(c_g + s_g)} - \sqrt{\theta}\Big]. \tag{E.102}$$

Equation (E.101) implies that

$$\sqrt{\theta}\Big[\frac{1}{s_f s_g} + \frac{1}{(c_f + s_f)(c_g + s_g) - c_f c_g}\Big] <$$

$$\sqrt{(c_f+s_f)(c_g+s_g)}\Big[\frac{1}{s_fs_g}-\frac{1}{(c_f+s_f)(c_g+s_g)-c_fc_g}\Big]$$

$$\Rightarrow \quad \sqrt{\theta}(c_fs_g+c_gs_f+2s_fs_g) < \sqrt{(c_f+s_f)(c_g+s_g)}(c_fs_g+c_gs_f) \quad \text{(E.103)}$$

Now, we use the fact that $\theta \geq cs_2$. Let us assume that $cs_2 = c_f^2\left(\frac{c_g+s_g}{c_f+s_f}\right)$, which implies that $c_fs_g > c_gs_f$ [Eq. (E.38)]. Therefore, continuing with (E.103), we have

$$\begin{aligned}
c_f\sqrt{\Big(\frac{c_g+s_g}{c_f+s_f}\Big)}(c_fs_g+c_gs_f+2s_fs_g) &< \sqrt{(c_f+s_f)(c_g+s_g)}(c_fs_g+c_gs_f) \\
\Rightarrow \quad c_f(c_fs_g+c_gs_f+2s_fs_g) &< (c_f+s_f)(c_fs_g+c_gs_f) \\
\Rightarrow \quad 2c_fs_fs_g &< s_f(c_fs_g+c_gs_f) \\
\Rightarrow \quad 2c_fs_g &< c_fs_g+c_gs_f \\
\Rightarrow \quad c_fs_g &< c_gs_f \qquad \text{(E.104)}
\end{aligned}$$

which contradicts our assumption that $c_fs_g > c_gs_f$ and consequently contradicts (E.99), implying that (E.98) is true. If we assume that $cs_2 = c_g^2\left(\frac{c_f+s_f}{c_g+s_g}\right)$, we get a similar contradiction. From (E.96) and (E.98), we see that $l_3(\theta)$ forms an upper bound on $\mu_{\tilde{F}\sqcap\tilde{G}}(\theta)$ for $c_fc_g \leq \theta \leq (c_f+s_f)(c_g+s_g)$, when $c_f+s_f<1$ and $c_g+s_g<1$.

Next, we consider the case when $c_f+s_f \geq 1$ and/or $c_g+s_g \geq 1$. From (E.6), we see that for $c_fc_g \leq \theta \leq cs_3$, we have to show (E.96) and (E.98) again, and this can be done in exactly the same manner as we did above for the case $c_f+s_f<1$ and $c_g+s_g<1$. For $\theta > cs_3$, we take an indirect approach to prove that $l_3(\theta)$ is an upper bound. We show that for $cs_3 \leq \theta \leq (\overline{c_f+s_f})(\overline{c_g+s_g})$,

$$\begin{aligned}
\frac{1}{s_fs_g}\Big[\sqrt{\theta}-\sqrt{(c_f+s_f)(c_g+s_g)}\Big]^2 \ \geq \ \max\Big\{&\Big[1-\Big(\frac{\theta-c_g}{s_g}\Big)\Big]\Big[1-\Big(\frac{1-c_f}{s_f}\Big)\Big], \\
&\Big[1-\Big(\frac{1-c_g}{s_g}\Big)\Big]\Big[1-\Big(\frac{\theta-c_f}{s_f}\Big)\Big]\Big\} \quad \text{(E.105)}
\end{aligned}$$

Observe that the RHS of (E.105) can be rewritten as

$$\begin{aligned}
&\max\left\{\left[1-\left(\frac{\theta-c_g}{s_g}\right)\right]\left[1-\left(\frac{1-c_f}{s_f}\right)\right],\ \left[1-\left(\frac{1-c_g}{s_g}\right)\right]\left[1-\left(\frac{\theta-c_f}{s_f}\right)\right]\right\}\\
=\ &\max\left\{\frac{1}{s_f s_g}[(c_g+s_g)-\theta][(c_f+s_f)-1]\,,\right.\\
&\qquad\left.\frac{1}{s_f s_g}[(c_f+s_f)-\theta][(c_g+s_g)-1]\right\} \qquad \text{(E.106)}
\end{aligned}$$

Let us assume that $(c_f + s_f) < (c_g + s_g)$, so that from (E.62), we have

$$\begin{aligned}
&\max\left\{\frac{1}{s_f s_g}[(c_g+s_g)-\theta][(c_f+s_f)-1]\,,\ \frac{1}{s_f s_g}[(c_f+s_f)-\theta][(c_g+s_g)-1]\right\}\\
=\ &\frac{1}{s_f s_g}[(c_f+s_f)-\theta][(c_g+s_g)-1] \qquad \text{(E.107)}
\end{aligned}$$

Now assume that

$$\begin{aligned}
&\frac{1}{s_f s_g}\left[\sqrt{\theta}-\sqrt{(c_f+s_f)(c_g+s_g)}\right]^2 <\\
&\qquad\frac{1}{s_f s_g}[(c_f+s_f)-\theta][(c_g+s_g)-1] \qquad \text{(E.108)}\\
\Rightarrow\ &\theta+(c_f+s_f)(c_g+s_g)-2\sqrt{\theta(c_f+s_f)(c_g+s_g)} <\\
&\qquad(c_f+s_f)(c_g+s_g)-(c_f+s_f)-\theta(c_g+s_g)+\theta\\
\Rightarrow\ &-2\sqrt{\theta(c_f+s_f)(c_g+s_g)} < -(c_f+s_f)-\theta(c_g+s_g)\\
\Rightarrow\ &2\sqrt{\theta(c_f+s_f)(c_g+s_g)} > (c_f+s_f)+\theta(c_g+s_g)\\
\Rightarrow\ &4\theta(c_f+s_f)(c_g+s_g) > [(c_f+s_f)+\theta(c_g+s_g)]^2\\
\Rightarrow\ &4\theta(c_f+s_f)(c_g+s_g) > (c_f+s_f)^2+\theta^2(c_g+s_g)^2+2\theta(c_f+s_f)(c_g+s_g)\\
\Rightarrow\ &0 > (c_f+s_f)^2+\theta^2(c_g+s_g)^2-2\theta(c_f+s_f)(c_g+s_g)\\
\Rightarrow\ &0 > [(c_f+s_f)-\theta(c_g+s_g)]^2 \qquad \text{(E.109)}
\end{aligned}$$

which is, of course, false and therefore, shows that (E.108) does not hold; implying that (E.105) is true. Observe that our assumption $(c_f + s_f) < (c_g + s_g)$ does not contribute to the result. Even if we assume otherwise, we arrive at the same conclusion. Equation (E.105) along with (E.98) proves that $l_3(\theta)$ is also an upper bound on $\mu_{\tilde{F}\sqcap\tilde{G}}(\theta)$, when $c_f + s_f \geq 1$ and/or $c_g + s_g \geq 1$.

From (E.83) , (E.91) , (E.96), (E.98) and (E.105), we see that

$$\mu^U_{\tilde{F}\sqcap\tilde{G}}(\theta) = l_3(\theta) \quad ; \quad c_f c_g \leq \theta \leq (\overline{c_f + s_f})(\overline{c_g + s_g}) \tag{E.110}$$

where $l_3(\theta)$ is as in (E.95).

Combining (E.94) and (E.110), we get the upper bound for the *meet* as

$$\mu^U_{\tilde{F}\sqcap\tilde{G}}(\theta) = \begin{cases} 1 + \frac{\theta - c_f c_g}{c_f s_g + c_g s_f} & ; \ \theta \in \left[(\underline{c_f - s_f})(\underline{c_g - s_g}), c_f c_g\right] \\ 1 - \frac{\theta - c_f c_g}{(c_f + s_f)(c_g + s_g) - c_f c_g} & ; \ \theta \in \left[c_f c_g, (\overline{c_f + s_f})(\overline{c_g + s_g})\right] \\ 0 & ; \quad \text{otherwise} \end{cases} \tag{E.111}$$

For $\theta \leq c_f c_g$, generalization of (E.111) to the case of more than two triangular type-1 sets proceeds exactly in the same manner as in the case of the first approximation for *meet* between Gaussian type-1 sets (see Section 2.3.1). For $\theta > c_f c_g$, we find the upper bound on the *meet* of more than two triangular type-1 sets using the same principle as in the case of two triangular sets : the straight line joining the unity membership point $(\prod_{i=1}^n c_i, 1)$ and the rightmost point of the domain $[\prod_{i=1}^n (c_i + s_i), 0]$ forms an upper bound on the result of the *meet*.

If we have $n$ triangular fuzzy sets $\tilde{F}_1, \ldots, \tilde{F}_n$ having centers $c_1, c_2, \ldots, c_n$ and spreads $s_1, s_2, \ldots, s_n$, then an upper bound on the result of their *meet* is given by

$$\mu^U_{\sqcap_{i=1}^n \tilde{F}_i}(\theta) = \begin{cases} 1 + \left[\frac{\theta - \prod_{i=1}^n c_i}{s_{u1}}\right] & ; \ \theta \in \left[\prod_{i=1}^n (\underline{c_i - s_i}), \prod_{i=1}^n c_i\right] \\ 1 - \left[\frac{\theta - \prod_{i=1}^n c_i}{s_{u2}}\right] & ; \ \theta \in \left[\prod_{i=1}^n c_i, \prod_{i=1}^n (\overline{c_i + s_i})\right] \\ 0 & ; \quad \text{otherwise} \end{cases} \tag{E.112}$$

where

$$s_{u1} = \left[s_1 \prod_{i;i\neq 1} c_i + s_2 \prod_{i;i\neq 2} c_i + \cdots + s_j \prod_{i;i\neq j} c_i + \cdots + s_n \prod_{i;i\neq n} c_i\right]; \quad i = 1, 2, \cdots, n \qquad \text{(E.113)}$$

$$s_{u2} = \prod_{i=1}^{n}(c_i + s_i) - \prod_{i=1}^{n} c_i \qquad \text{(E.114)}$$

### E.1.3.3 Meet Approximation

We want an approximation of the *meet* between triangular fuzzy sets that is a symmetrical triangle lying between the lower and upper bounds derived in Sections E.1.3.1 and E.1.3.2, respectively.

Consider the case of two triangles. Observe from (E.78) and (E.111), that for the approximation to lie between the two bounds for $\theta < c_f c_g$, we need the spread of the approximating triangle to lie between $cs = \max\{c_f s_g, c_g s_f\}$ and $c_f s_g + c_g s_f$. One simple choice for the spread that satisfies this condition is $\sqrt{c_f^2 s_g^2 + c_g^2 s_f^2}$. It is easy to see that this choice lies between the bounds for $\theta > c_f c_g$ also; therefore, we choose the following approximation for the *meet* of $\tilde{F}$ and $\tilde{G}$.

$$\hat{\mu}_{\tilde{F}\sqcap\tilde{G}} = \begin{cases} 1 - \left|\frac{\theta - c_f c_g}{\sqrt{c_f^2 s_g^2 + c_g^2 s_f^2}}\right| & ; \quad \theta \in \left[\underline{(c_f c_g - \sqrt{c_f^2 s_g^2 + c_g^2 s_f^2})}, \overline{(c_f c_g + \sqrt{c_f^2 s_g^2 + c_g^2 s_f^2})}\right] \\ 0 & ; \quad \text{otherwise} \end{cases} \qquad \text{(E.115)}$$

Generalization of this result can be done exactly in the same manner as the generalization of the Gaussian approximation (see Section 2.3.2). We state the result here.

If we have $n$ triangular fuzzy sets $\tilde{F}_1, \ldots, \tilde{F}_n$ having centers $c_1, c_2, \ldots, c_n$ and spreads $s_1, s_2, \ldots, s_n$, then the triangular approximation to their *meet* is given by

$$\hat{\mu}_{\sqcap_{i=1}^{n}\tilde{F}_i}(\theta) = \begin{cases} 1 - \left|\frac{\theta - \prod_{i=1}^{n} c_i}{\bar{s}}\right| & ; \quad \theta \in [\underline{(\prod_{i=1}^{n} c_i - \bar{s})}, \overline{(\prod_{i=1}^{n} c_i + \bar{s})}] \\ 0 & ; \quad \text{otherwise} \end{cases} \qquad \text{(E.116)}$$

where

$$\bar{s} = \left[s_1^2 \prod_{i;i\neq 1} c_i^2 + \cdots + s_j^2 \prod_{i;i\neq j} c_i^2 + \cdots + s_n^2 \prod_{i;i\neq n} c_i^2\right]^{\frac{1}{2}} ; \; i = 1, 2, \cdots, n \qquad \text{(E.117)}$$

Observe that this approximation is very much similar to the Gaussian approximation for the *meet* between Gaussians [see (2.59)].

### E.1.4 Bounds on the Approximation Error

The procedure for finding bounds on the approximation error for the *meet* between triangular fuzzy sets is also exactly the same as in the Gaussian case (see Appendix C.8.4). Though we were able to find an exact expression for the actual *meet* curve for the case of two triangles, we do not have expressions for the case of more than two triangles; therefore, we have to rely on the upper and lower bounds to find bounds on the approximation error. From the discussion in Section E.1.3, it is clear that both our approximation as well as the actual result of the *meet* lie between the derived upper and lower bounds; therefore, the difference between the approximation and the error is always less than the larger of the difference between the approximation and the upper bound, and the difference between the approximation and the lower bound, i.e.,

$$\begin{aligned} \left|\hat{\mu}_{\sqcap_{i=1}^n \tilde{F}_i}(\theta) - \mu_{\sqcap_{i=1}^n \tilde{F}_i}(\theta)\right| \;\; \leq \;\; & \max\left\{\left[\mu^U_{\sqcap_{i=1}^n \tilde{F}_i}(\theta) - \hat{\mu}_{\sqcap_{i=1}^n \tilde{F}_i}(\theta)\right],\right. \\ & \left.\left[\hat{\mu}_{\sqcap_{i=1}^n \tilde{F}_i}(\theta) - \mu^L_{\sqcap_{i=1}^n \tilde{F}_i}(\theta)\right]\right\} ; \; \theta \in [0,1] \quad \text{(E.118)} \end{aligned}$$

where $\hat{\mu}_{\sqcap_{i=1}^n \tilde{F}_i}(\theta)$, $\mu^U_{\sqcap_{i=1}^n \tilde{F}_i}(\theta)$ and $\mu^L_{\sqcap_{i=1}^n \tilde{F}_i}(\theta)$ are given in (E.116), (E.112) and (E.79), respectively.

Figure E.2 shows some examples of the triangular *meet* approximation and the upper and lower bounds. Observe that the actual result is not symmetrical and the approximation seems to be closer to the actual result on the left-hand side of the unity membership point.

## E.2 Algebraic Sum of Triangular Type-1 Fuzzy Numbers

### E.2.1 Algebraic Sum of Triangular Fuzzy Numbers under Minimum $t$-norm

**Theorem E.2** *Given $n$ type-1 triangular fuzzy numbers $\tilde{F}_1, \ldots, \tilde{F}_n$, with centers $c_1, c_2, \ldots, c_n$ and spreads $s_1, s_2, \ldots, s_n$, their affine combination $\sum_{i=1}^{n} \alpha_i \tilde{F}_i + \beta$, where $\alpha_i$ ($i = 1, \ldots, n$) and $\beta$ are crisp constants, is also a triangular fuzzy number with center $\sum_{i=1}^{n} \alpha_i c_i + \beta$, and spread $\sum_{i=1}^{n} |\alpha_i s_i|$.*

**Proof :** We prove the theorem in two parts : (a) we prove that $\alpha_i \tilde{\mathrm{F}}_i + \beta$ is a triangular fuzzy number with center $\alpha_i c_i + \beta$ and spread $|\alpha_i s_i|$; and (b) we prove that $\sum_{i=1}^{n} \tilde{\mathrm{F}}_i$ is a triangular fuzzy number with center $\sum_{i=1}^{n} c_i$ and spread $\sum_{i=1}^{n} \sigma_i$.
(a) Consider

$$\tilde{\mathrm{F}}_i = \int_{v \in V_i} \left(1 - \left|\frac{v - c_i}{s_i}\right|\right) / v \tag{E.119}$$

where $V_i = [c_i - s_i, c_i + s_i]$. Multiplying $\tilde{\mathrm{F}}_i$ by a constant $\alpha_i (= 1/\alpha_i)$ yields [using the Extension Principle (Chapter 2)]

$$\begin{aligned} \alpha_i \tilde{\mathrm{F}}_i &= \int_{v \in \alpha_i V_i} \left[\left(1 - \left|\frac{v - c_i}{s_i}\right|\right) \star 1\right] \Big/ (\alpha_i v) \\ &= \int_{v \in \alpha_i V_i} \left(1 - \left|\frac{v - c_i}{s_i}\right|\right) \Big/ (\alpha_i v) \end{aligned} \tag{E.120}$$

where $\alpha_i V_i = [\alpha_i(c_i - s_i), \alpha_i(c_i + s_i)]$. Now, adding a crisp constant $\beta(= 1/\beta)$ to $\alpha_i \tilde{\mathrm{F}}_i$, we get

$$\begin{aligned}\alpha_i \tilde{\mathrm{F}}_i + \beta &= \int_{v \in \alpha_i V_i + \beta} \left[\left(1 - \left|\frac{v - c_i}{s_i}\right|\right) \star 1\right] \Big/ (\alpha_i v + \beta) \\ &= \int_{v \in \alpha_i V_i + \beta} \left(1 - \left|\frac{v - c_i}{s_i}\right|\right) \Big/ (\alpha_i v + \beta) \end{aligned} \tag{E.121}$$

where $\alpha_i V_i + \beta = [\alpha_i(c_i - s_i) + \beta, \alpha_i(c_i + s_i) + \beta]$. Let $\alpha_i v + \beta = v'$; this gives $v = (v' - \beta)/\alpha_i$, which when substituted into (E.121), leads to

$$\begin{aligned}\alpha_i \tilde{\mathrm{F}}_i + \beta &= \int_{v' \in \alpha_i V_i + \beta} \left[1 - \left|\frac{\left(\frac{v' - \beta}{\alpha_i}\right) - c_i}{s_i}\right|\right] \Big/ v' \\ &= \int_{v' \in \alpha_i V_i + \beta} \left(1 - \left|\frac{v' - \alpha_i c_i}{\alpha_i s_i}\right|\right) \Big/ v' \end{aligned} \tag{E.122}$$

which shows that $\alpha_i \tilde{\mathrm{F}}_i + \beta$ is a triangular fuzzy number with center $\alpha_i c_i + \beta$ and spread $|\alpha_i s_i|$. Note that this result does not depend on the kind of $t$-norm used, since $\alpha_i$ and $\beta$ are crisp numbers.

**(b)** Suppose that we have two triangular type-1 fuzzy numbers $\tilde{\mathrm{F}}$ and $\tilde{\mathrm{G}}$, characterized by membership functions $f$ and $g$, such that

$$f(v) = \begin{cases} 1 - \left|\frac{v - c_f}{s_f}\right| & ; \quad v \in [c_f - s_f, c_f + s_f] \\ 0 & ; \quad \text{otherwise} \end{cases} \tag{E.123}$$

and

$$g(w) = \begin{cases} 1 - \left|\frac{w - c_g}{s_g}\right| & ; \quad w \in [c_g - s_g, c_g + s_g] \\ 0 & ; \quad \text{otherwise} \end{cases} \tag{E.124}$$

The algebraic sum of $\tilde{\mathrm{F}}$ and $\tilde{\mathrm{G}}$ under minimum $t$-norm is given as

$$\tilde{\mathrm{F}} + \tilde{\mathrm{G}} = \int_{v \in \tilde{\mathrm{F}}} \int_{w \in \tilde{\mathrm{G}}} [f(v) \wedge g(w)] / (v + w) \tag{E.125}$$

If $\theta \in \tilde{\mathrm{F}} + \tilde{\mathrm{G}}$, the membership grade of $\theta$ can be expressed as

$$\mu_{\tilde{\mathrm{F}}+\tilde{\mathrm{G}}}(\theta) = \sup_{v \in \tilde{\mathrm{F}}} [f(v) \wedge g(\theta - v)] \tag{E.126}$$

Observe, from (E.123) and (E.124), that $f(v)$ is a triangle with center $c_f$ and spread $s_f$, and $g(\theta - v)$ is a triangle with center $(\theta - c_g)$ and spread $s_g$. It is easy to see that the supremum of the minimum of these two triangles is reached at their point of intersection between their means. To find the point of intersection, we equate $f(v)$ and $g(\theta - v)$. Observe that, since we want to find the point of intersection lying between the means, if $c_f > (\theta - c_g)$ [i.e., if $\theta < (c_f + c_g)$], we have to use the equation for $v < c_f$ for $f(v)$ and the equation for $v > (\theta - c_g)$ for $g(\theta - v)$. Similarly, if $c_f < (\theta - c_g)$ [i.e., if $\theta > (c_f + c_g)$], we have to use the equation for $v > c_f$ for $f(v)$ and the equation for $v < (\theta - c_g)$ for $g(\theta - v)$. If $c_f = (\theta - c_g)$, $\theta = c_f + c_g$ and $\mu_{\tilde{\mathrm{F}}+\tilde{\mathrm{G}}}(\theta) = 1$.

$\underline{\theta < c_f + c_g}$ : Equating $f(v)$ and $g(\theta - v)$, if $v^*$ is the point of intersection, we have

$$\begin{aligned} 1 + \frac{v^* - c_f}{s_f} &= 1 + \frac{(\theta - c_g) - v^*}{s_g} \\ \Rightarrow \quad (v^* - c_f)s_g &= [(\theta - c_g) - v^*]s_f \\ \Rightarrow \quad v^* &= \frac{(\theta - c_g)s_f + c_f s_g}{s_f + s_g} \end{aligned} \tag{E.127}$$

Using (E.127) in (E.126), we have

$$\mu_{\tilde{\mathrm{F}}+\tilde{\mathrm{G}}}(\theta) = f(v^*) = g(\theta - v^*) = 1 + \frac{\theta - (c_f + c_g)}{s_f + s_g} \quad ; \quad \theta < c_f + c_g \tag{E.128}$$

For $\theta > c_f + c_g$, equating $f(v)$ and $g(\theta - v)$, we have

$$1 - \frac{v^* - c_f}{s_f} = 1 - \frac{(\theta - c_g) - v^*}{s_g} \tag{E.129}$$

which gives us the same $v^*$ as in (E.127). From (E.126), we get

$$\mu_{\tilde{\mathrm{F}}+\tilde{\mathrm{G}}}(\theta) = f(v^*) = g(\theta - v^*) = 1 - \frac{\theta - (c_f + c_g)}{s_f + s_g} \quad ; \quad \theta > c_f + c_g \tag{E.130}$$

Observe also, from (E.123) and (E.124), that $(c_f - s_f) + (c_g - s_g) \leq \theta \leq (c_f + s_f) + (c_g + s_g)$; therefore, from (E.128) and (E.130), we get

$$\mu_{\tilde{\mathrm{F}}+\tilde{\mathrm{G}}}(\theta) = \begin{cases} 1 + \left|\frac{\theta-(c_f+c_g)}{s_f+s_g}\right| & ; \quad (c_f - s_f) + (c_g - s_g) \leq \theta \leq (c_f + s_f) + (c_g + s_g) \\ 0 & ; \quad \text{otherwise} \end{cases} \tag{E.131}$$

Generalizing to the sum of $n$ triangular type-1 fuzzy numbers, we get that if we have $n$ triangular type-1 fuzzy numbers, $\tilde{\mathrm{F}}_1, \ldots, \tilde{\mathrm{F}}_n$ with centers $c_1, \ldots, c_n$ and spreads $s_1, \ldots, s_n$, thier algebraic sum is given as

$$\mu_{\sum_{i=1}^n \tilde{\mathrm{F}}_i}(\theta) = \begin{cases} 1 - \left|\frac{\theta-\sum_{i=1}^n c_i}{\sum_{i=1}^n s_i}\right| & ; \quad \theta \in [\sum_{i=1}^n c_i - \sum_{i=1}^n s_i \, , \, \sum_{i=1}^n c_i + \sum_{i=1}^n s_i] \\ 0 & ; \quad \text{otherwise} \end{cases} \tag{E.132}$$

Combining parts (**a**) and (**b**), we get the result in Theorem E.2. □

### E.2.2 Algebraic Sum of Triangular Fuzzy Numbers under Product $t$-norm

Under the product $t$-norm, the result of the algebraic sum of two type-1 triangular numbers is not a triangle. We first give the exact result and then approximate it with a symmetrical triangle.

**Theorem E.3** *Suppose that we have two triangular type-1 fuzzy numbers $\tilde{F}$ and $\tilde{G}$, characterized by membership functions $f$ and $g$, where $f$ and $g$ are as given in*

*(E.123) and (E.124). The algebraic sum of $\tilde{F}$ and $\tilde{G}$ under product t-norm is given as*

$$\mu_{\tilde{F}+\tilde{G}}(\theta) = \begin{cases} \frac{1}{4s_f s_g}\Big[\theta - [(c_f - s_f) + (c_g - s_g)]\Big]^2 & ; \quad (c_f - s_f) + (c_g - s_g) \leq \theta \leq cs_4 \\ 1 + \frac{\theta-(c_f+c_g)}{\max\{s_f, s_g\}} & ; \quad cs_4 < \theta \leq c_f + c_g \\ 1 - \frac{\theta-(c_f+c_g)}{\max\{s_f, s_g\}} & ; \quad c_f + c_g \leq \theta < cs_5 \\ \frac{1}{4s_f s_g}\Big[\theta - [(c_f + s_f) + (c_g + s_g)]\Big]^2 & ; \quad cs_5 \leq \theta \leq (c_f + s_f) + (c_g + s_g) \end{cases} \tag{E.133}$$

*where*

$$cs_4 = \min\{(c_f + s_f) + (c_g - s_g)\, , (c_f - s_f) + (c_g + s_g)\} \tag{E.134}$$

$$cs_5 = \max\{(c_f + s_f) + (c_g - s_g)\, , (c_f - s_f) + (c_g + s_g)\} \tag{E.135}$$

**Proof :** The algebraic sum of $\tilde{\mathrm{F}}$ and $\tilde{\mathrm{G}}$ under product $t$-norm is given as

$$\tilde{\mathrm{F}} + \tilde{\mathrm{G}} = \int_{v\in\tilde{\mathrm{F}}} \int_{w\in\tilde{\mathrm{G}}} [f(v)g(w)]/(v+w) \tag{E.136}$$

Equation (E.11) can be interpreted as follows. Each element $v$ of set $\tilde{\mathrm{F}}$ adds to every element $w$ of set $\tilde{\mathrm{G}}$, and, at the same time, the membership grade of $v$ in $\tilde{\mathrm{F}}$ multiplies the membership grade of $w$ in $\tilde{\mathrm{G}}$. So, given a particular element $v_1$ of $\tilde{\mathrm{F}}$, what we get as a result of these multiplications is a shifted and scaled version of the membership function of $\tilde{\mathrm{G}}$ [shifted along the independent axis by $v_1$ and scaled along the dependent axis by $f(v_1)$]. This process is repeated for every element of $\tilde{\mathrm{F}}$ and finally, the meet of $\tilde{\mathrm{F}}$ and $\tilde{\mathrm{G}}$ is given by the envelope of all the above scaled triangles.

Figure E.3 shows this interpretation pictorially. Observe that if we interchange $\tilde{\mathrm{F}}$ and $\tilde{\mathrm{G}}$, the above interpretaion still holds.

One obvious thing that can be observed from (E.136) is that only one point in the domain of $\tilde{\mathrm{F}} + \tilde{\mathrm{G}}$ will have a membership grade equal to unity, and this point is equal to $c_f + c_g$. Observe, from Fig. E.1 (b), that to the right of the unity membership point $c_f + c_g$, $\mu_{\tilde{F}+\tilde{G}}$ is a curve that touches the right side of each scaled triangle. Similarly, to the left of $c_f + c_g$, $\mu_{\tilde{F}+\tilde{G}}$ touches the left side of each scaled triangle. From this, we infer that to the left of $c_f + c_g$, the left half of $\tilde{G}$ (i.e., $v < c_g$) contributes to $\mu_{\tilde{F}+\tilde{G}}$, and, to the right of $c_f + c_g$, the right half of $\tilde{G}$ (i.e., $v > c_g$) contributes to $\mu_{\tilde{F}+\tilde{G}}$. Also, from (E.1) and (E.2), it is clear that the domain of $\tilde{F} + \tilde{G}$ is $[(c_f - s_f) + (c_g - s_g), (c_f + s_f) + (c_g + s_g)]$.

We can write the membership function of $\tilde{F} + \tilde{G}$ as follows : for $(c_f - s_f) + (c_g - s_g) \leq \theta \leq c_f + c_g$,

$$\mu_{\tilde{F}+\tilde{G}}(\theta) = \sup_{v \in V_l} \Big[1 + \frac{(\theta - v) - c_g}{s_g}\Big]\Big[1 + \frac{v - c_f}{s_f}\Big] \tag{E.137}$$

where the interval $V_l$ is found as follows. Since $v \in \tilde{F}$ and here, we are considering points to the left of $c_f$ [see (E.123) and the discussion in the preceding paragraph]. This gives us $v \in [c_f - s_f, c_f]$. Similarly, since $(\theta - v) \in \tilde{G}$ and we are considering points to the left of $c_g$, we have $(\theta - v) \in [c_g - s_g, c_g]$, i.e., $v \in [\theta - c_g, \theta - (c_g - s_g)]$. Combining these two constraints, we have

$$V_l = [c_f - s_f, c_f] \cap [\theta - c_g, \theta - (c_g - s_g)] \tag{E.138}$$

Observe that $(c_f - s_f) + (c_g - s_g) \leq \theta \Rightarrow \theta - (c_g - s_g) \geq c_f - s_f$ and $\theta \leq c_f + c_g \Rightarrow \theta - c_g \leq c_f$; therefore, $V_l$ is a non-empty interval.

Similarly, for $c_f + c_g \leq \theta \leq (c_f + s_f) + (c_g + s_g)$,

$$\mu_{\tilde{F}+\tilde{G}}(\theta) = \sup_{v \in V_r} \Big[1 + \Big(\frac{(\theta - v) - c_g}{s_g}\Big)\Big]\Big[1 + \Big(\frac{v - c_f}{s_f}\Big)\Big] \tag{E.139}$$

where the interval $V_r$ is found as follows. Since $v \in \tilde{F}$ and here, we are considering points to the right of $c_f$ [see (E.123) and the discussion in the preceding paragraph]. This gives us $v \in [c_f, c_f + s_f]$. Similarly, since $(\theta - v) \in \tilde{G}$ and we are considering points to the right of $c_g$, we have $(\theta - v) \in [c_g, c_g + s_g]$, i.e., $v \in [\theta - (c_g + s_g), \theta - c_g]$. Combining these two constraints, we have

$$V_r = [c_f, c_f + s_f] \cap [\theta - (c_g + s_g), \theta - c_g] \tag{E.140}$$

It is easy to see that $V_r$ is non-empty for $c_f + c_g \leq \theta \leq (c_f + s_f) + (c_g + s_g)$.

Combining (E.137) and (E.139), we have

$$\mu_{\tilde{F}+\tilde{G}}(\theta) = \begin{cases} \sup_{v \in V_l} \left[1 + \left(\frac{(\theta - v) - c_g}{s_g}\right)\right]\left[1 + \left(\frac{v - c_f}{s_f}\right)\right] & ; \quad (c_f - s_f) + (c_g - s_g) \leq \\ & \qquad \theta \leq c_f + c_g \\ \sup_{v \in V_r} \left[1 - \left(\frac{(\theta - v) - c_g}{s_g}\right)\right]\left[1 - \left(\frac{v - c_f}{s_f}\right)\right] & ; \quad c_f + c_g \leq \\ & \qquad \theta \leq (c_f + s_f) + (c_g + s_g) \end{cases} \tag{E.141}$$

where $V_l$ and $V_r$ are as in (E.138) and (E.140), respectively.

For $(c_f - s_f) + (c_g - s_g) \leq \theta \leq c_f + c_g$, the objective function to be maximized is

$$\begin{aligned} S1(v) &= \left[1 + \left(\frac{(\theta - v) - c_g}{s_g}\right)\right]\left[1 + \left(\frac{v - c_f}{s_f}\right)\right] \\ &= \left[\left(1 - \frac{c_g}{s_g}\right) + \frac{(\theta - v)}{s_g}\right]\left[\left(1 - \frac{c_f}{s_f}\right) + \frac{v}{s_f}\right] \\ &= \left(1 - \frac{c_g}{s_g}\right)\left(1 - \frac{c_f}{s_f}\right) + \left(1 - \frac{c_f}{s_f}\right)\frac{(\theta - v)}{s_g} \\ &\quad + \left(1 - \frac{c_g}{s_g}\right)\frac{v}{s_f} + \frac{v(\theta - v)}{s_f s_g} \end{aligned} \tag{E.142}$$

The first and second derivatives of $S1$ are

$$S1'(v) = -\left(1 - \frac{c_f}{s_f}\right)\frac{1}{s_g} + \left(1 - \frac{c_g}{s_g}\right)\frac{1}{s_f} + \frac{(\theta - 2v)}{s_f s_g} \tag{E.143}$$

$$S1''(v) = \frac{-2}{s_f s_g} \tag{E.144}$$

Since $S1'' < 0$, $S1$ is concave. To find its supremum, we equate $S1'$ to zero. If the supremum is achieved at $v = v^*$, we have

$$\begin{aligned} & -\Big(1 - \frac{c_f}{s_f}\Big)\frac{1}{s_g} + \Big(1 - \frac{c_g}{s_g}\Big)\frac{1}{s_f} + \frac{(\theta - 2v^*)}{s_f s_g} = 0 \\ \Rightarrow \quad & -(s_f - c_f) + (s_g - c_g) + \theta - 2v^* = 0 \\ \Rightarrow \quad & v^* = \frac{\theta + (c_f - s_f) - (c_g - s_g)}{2} \end{aligned} \tag{E.145}$$

Let us see if $v^*$ satisifes the constraint $v^* \in V_l$. For $v^*$ to belong to $V_l$, it has to satisfy four conditions [see (E.138)] : $v^* \geq c_f - s_f$, $v^* \leq c_f$, $v^* \geq \theta - c_g$ and $v^* \leq \theta - (c_g - s_g)$. Since $\theta \geq (c_f - s_f) + (c_g - s_g)$, $(c_f - s_f) \leq v^* \leq \theta - (c_g - s_g)$. To satisfy the other two conditions,

$$\begin{aligned} v^* \leq c_f \quad & \Leftrightarrow \quad \theta \leq (c_f + s_f) + (c_g - s_g) \\ v^* \geq \theta - c_g \quad & \Leftrightarrow \quad \theta \leq (c_f - s_f) + (c_g + s_g) \\ \Rightarrow v^* \in V_l \quad & \Leftrightarrow \quad \theta \leq \min\{(c_f + s_f) + (c_g - s_g), (c_f - s_f) + (c_g + s_g)\} \end{aligned} \tag{E.146}$$

Substituting (E.145) into (E.142), we get

$$\mu_{\tilde{F}+\tilde{G}}(\theta) = \frac{1}{4 s_f s_g}\Big[\theta - [(c_f - s_f) + (c_g - s_g)]\Big]^2 \quad ; \quad (c_f - s_f) + (c_g - s_g) \leq \theta \leq cs_4 \tag{E.147}$$

where

$$cs_4 = \min\{(c_f + s_f) + (c_g - s_g), (c_f - s_f) + (c_g + s_g)\} \tag{E.148}$$

If $\theta > cs_4$, the supremum of $S1$ is reached at a point greater than $c_f$ or less than $(\theta - c_g)$ [(E.146)]. If $cs_4 = (c_f + s_f) + (c_g - s_g)$, then $\theta > cs_4 \Leftrightarrow v^* > c_f$. Since $S1$ is concave, the supremum in $V_l$ is attained at $v = c_f$; therefore, substituting $v = c_f$ in (E.142), we get

$$\mu_{\tilde{F}+\tilde{G}}(\theta) = 1 + \frac{\theta - (c_f + c_g)}{s_g} \quad ; \quad (c_f + s_f) + (c_g - s_g) < \theta \leq c_f + c_g \tag{E.149}$$

If $cs_4 = (c_f - s_f) + (c_g + s_g)$, then $\theta > cs_4 \Leftrightarrow v^* < (\theta - c_g)$ [see (E.146)]. Since $S1$ is concave, the supremum in $V_l$ is attained at $v = (\theta - c_g)$; therefore, substituting $v = (\theta - c_g)$ in (E.142), we get

$$\mu_{\tilde{F}+\tilde{G}}(\theta) = 1 + \frac{\theta - (c_f + c_g)}{s_f} \quad ; \quad (c_f - s_f) + (c_g + s_g) < \theta \leq c_f + c_g \tag{E.150}$$

Note that

$$s_f \gtreqless s_g \quad \Leftrightarrow \quad (c_f + s_f) + (c_g - s_g) \gtreqless (c_f - s_f) + (c_g + s_g) \tag{E.151}$$

Combining (E.149), (E.150) and (E.151), we get

$$\mu_{\tilde{F}+\tilde{G}}(\theta) = 1 + \frac{\theta - (c_f + c_g)}{\max\{s_f, s_g\}} \quad ; \quad cs_4 < \theta \leq c_f + c_g \tag{E.152}$$

For $c_f + c_g \leq \theta \leq (c_f + s_f) + (c_g + s_g)$, the objective function to be maximized is

$$\begin{aligned} S2(v) &= \Big[1 - \Big(\frac{(\theta - v) - c_g}{s_g}\Big)\Big]\Big[1 - \Big(\frac{v - c_f}{s_f}\Big)\Big] \\ &= \Big[\Big(1 + \frac{c_g}{s_g}\Big) - \frac{(\theta - v)}{s_g}\Big]\Big[\Big(1 + \frac{c_f}{s_f}\Big) - \frac{v}{s_f}\Big] \\ &= \Big(1 + \frac{c_g}{s_g}\Big)\Big(1 + \frac{c_f}{s_f}\Big) - \Big(1 + \frac{c_f}{s_f}\Big)\frac{(\theta - v)}{s_g} \\ &\quad - \Big(1 + \frac{c_g}{s_g}\Big)\frac{v}{s_f} + \frac{v(\theta - v)}{s_f s_g} \end{aligned} \tag{E.153}$$

The first and second derivatives of $S2$ are

$$S2'(v) = \Big(1 + \frac{c_f}{s_f}\Big)\frac{1}{s_g} - \Big(1 + \frac{c_g}{s_g}\Big)\frac{1}{s_f} + \frac{(\theta - 2v)}{s_f s_g} \tag{E.154}$$

$$S2''(v) = \frac{-2}{s_f s_g} \tag{E.155}$$

Since $S2'' < 0$, $S2$ is concave. To find its supremum, we equate $S2'$ to zero. If the supremum is achieved at $v = v^*$, we have

$$\begin{aligned}
&\Big(1+\frac{c_f}{s_f}\Big)\frac{1}{s_g} - \Big(1+\frac{c_g}{s_g}\Big)\frac{1}{s_f} + \frac{(\theta - 2v^*)}{s_f s_g} = 0 \\
\Rightarrow\ \ & (c_f + s_f) - (c_g + s_g) + \theta - 2v^* = 0 \\
\Rightarrow\ \ & v^* = \frac{\theta + (c_f + s_f) - (c_g + s_g)}{2}
\end{aligned} \tag{E.156}$$

Let us see if $v^*$ satisifes the constraint $v^* \in V_r$. For $v^*$ to belong to $V_r$, it has to satisfy four conditions [see (E.140)] : $v^* \geq c_f$, $v^* \leq c_f + s_f$, $v^* \geq \theta - (c_g + s_g)$ and $v^* \leq (\theta - c_g)$. Since $\theta \leq (c_f + s_f) + (c_g + s_g)$, $\theta - (c_g + s_g) \leq v^* \leq (c_f + s_f)$. To satisfy the other two conditions,

$$\begin{aligned}
v^* \geq c_f \ \ &\Leftrightarrow\ \ \theta \geq (c_f - s_f) + (c_g + s_g) \\
v^* \leq \theta - c_g \ \ &\Leftrightarrow\ \ \theta \geq (c_f + s_f) + (c_g - s_g) \\
\Rightarrow v^* \in V_r \ \ &\Leftrightarrow\ \ \theta \geq \max\{(c_f - s_f) + (c_g + s_g), (c_f + s_f) + (c_g - s_g)\}
\end{aligned} \tag{E.157}$$

Substituting (E.156) into (E.153), we get

$$\mu_{\tilde{\mathrm{F}}+\tilde{\mathrm{G}}}(\theta) = \frac{1}{4 s_f s_g}\Big[\theta - [(c_f + s_f) + (c_g + s_g)]\Big]^2 \quad ; \quad cs_5 \leq \theta \leq (c_f + s_f) + (c_g + s_g) \tag{E.158}$$

where

$$cs_5 = \max\{(c_f - s_f) + (c_g + s_g), (c_f + s_f) + (c_g - s_g)\} \tag{E.159}$$

If $\theta < cs_5$, the supremum of $S2$ is reached at a point $v^*$, which is either less than $c_f$ or greater than $(\theta - c_g)$ [(E.157)]. If $cs_5 = (c_f - s_f) + (c_g + s_g)$, then $\theta < cs_5 \Leftrightarrow v^* < c_f$. Since $S2$ is concave, its supremum in $V_r$ is attained at $v = c_f$. Substituting $v = c_f$ in (E.153), we get

$$\mu_{\tilde{F}+\tilde{G}}(\theta) = 1 - \frac{\theta - (c_f + c_g)}{s_g} \quad ; \quad c_f + c_g \leq \theta < (c_f - s_f) + (c_g + s_g) \tag{E.160}$$

If $cs_5 = (c_f + s_f) + (c_g - s_g)$, then $\theta < cs_5 \Leftrightarrow v^* > (\theta - c_g)$ [see (E.157)]. Since $S2$ is concave, the supremum in $V_r$ is attained at $v = (\theta - c_g)$; therefore, substituting $v = (\theta - c_g)$ in (E.153), we get

$$\mu_{\tilde{F}+\tilde{G}}(\theta) = 1 - \frac{\theta - (c_f + c_g)}{s_f} \quad ; \quad c_f + c_g \leq \theta < (c_f + s_f) + (c_g - s_g) \tag{E.161}$$

Note that

$$s_f \gtrless s_g \quad \Leftrightarrow \quad (c_f + s_f) + (c_g - s_g) \gtrless (c_f - s_f) + (c_g + s_g) \tag{E.162}$$

Combining (E.160), (E.161) and (E.162), we get

$$\mu_{\tilde{F}+\tilde{G}}(\theta) = 1 - \frac{\theta - (c_f + c_g)}{\max\{s_f, s_g\}} \quad ; \quad c_f + c_g \leq \theta < cs_5 \tag{E.163}$$

Combining (E.147), (E.152), (E.158) and (E.163), we get

$$\mu_{\tilde{F}+\tilde{G}}(\theta) = \begin{cases} \frac{1}{4s_f s_g}\left[\theta - [(c_f - s_f) + (c_g - s_g)]\right]^2 & ; \quad (c_f - s_f) + (c_g - s_g) \leq \theta \leq cs_4 \\ 1 + \frac{\theta - (c_f + c_g)}{\max\{s_f, s_g\}} & ; \quad cs_4 < \theta \leq c_f + c_g \\ 1 - \frac{\theta - (c_f + c_g)}{\max\{s_f, s_g\}} & ; \quad c_f + c_g \leq \theta < cs_5 \\ \frac{1}{4s_f s_g}\left[\theta - [(c_f + s_f) + (c_g + s_g)]\right]^2 & ; \quad cs_5 \leq \theta \leq (c_f + s_f) + (c_g + s_g) \end{cases} \tag{E.164}$$

where $cs_4$ and $cs_5$ are as in (E.148) and (E.159), respectively. □

In order to find a triangular approximation to the algebraic sum of triangular type-1 fuzzy numbers under product $t$-norm, we first find upper and lower bounds on the actual result of the algebraic sum in (E.164).

#### E.2.2.1 Lower Bound

We, next, show that the symmetrical triangle with center $c_f + c_g$ and spread $\max\{s_f, s_g\}$ forms a lower bound on the actual result of the algebraic sum in (E.164). The equation of this triangle is

$$l(\theta) = \begin{cases} 1 - \left|\frac{\theta-(c_f+c_g)}{s_l}\right| & ; \quad \theta \in [c_f + c_g - s_l, c_f + c_g + s_l] \\ 0 & ; \quad \text{otherwise} \end{cases} \tag{E.165}$$

where

$$s_l = \max\{s_f, s_g\} \tag{E.166}$$

Observe, from (E.164), that for $cs_4 < \theta < cs_5$, $\mu_{\tilde{F}+\tilde{G}}(\theta) = l(\theta)$. For $(c_f - s_f) + (c_g - s_g) \leq \theta \leq cs_4$, let us assume that $\mu_{\tilde{F}+\tilde{G}}(\theta) < l(\theta)$, i.e., assume that, for $(c_f - s_f) + (c_g - s_g) \leq \theta \leq cs_4$,

$$\frac{1}{4s_f s_g}\left[\theta - [(c_f - s_f) + (c_g - s_g)]\right]^2 < 1 + \frac{\theta - (c_f + c_g)}{s_l} \tag{E.167}$$

Let $s_l = s_f$ (i.e., let $s_f \geq s_g$). Then (E.167) implies

$$\begin{aligned} & \frac{1}{4s_f s_g}\left[\theta - [(c_f - s_f) + (c_g - s_g)]\right]^2 < 1 + \frac{\theta - (c_f + c_g)}{s_f} \\ \Rightarrow \quad & \frac{1}{4s_f s_g}\left[\theta - [(c_f - s_f) + (c_g - s_g)]\right]^2 < \frac{\theta - (c_f - s_f) - c_g}{s_f} \\ \Rightarrow \quad & \theta^2 - 2[(c_f - s_f) + (c_g - s_g)]\theta + [(c_f - s_f) + (c_g - s_g)]^2 < \\ & \qquad 4s_g[\theta - (c_f - s_f) - c_g] \\ \Rightarrow \quad & \theta^2 - 2[(c_f - s_f) + (c_g + s_g)]\theta + [(c_f - s_f) + (c_g - s_g)]^2 < \\ & \qquad -4s_g[(c_f - s_f) + c_g] \end{aligned} \tag{E.168}$$

Observe that

$$[(c_f - s_f) + (c_g - s_g)]^2 + 4s_g[(c_f - s_f) + c_g] = [(c_f - s_f) + (c_g + s_g)]^2 \tag{E.169}$$

From (E.168) and (E.169), we have

$$\left[\theta - [(c_f - s_f) + (c_g + s_g)]\right]^2 < 0 \tag{E.170}$$

which shows that our assumption (E.167) must be false; therefore,

$$l(\theta) \leq \mu_{\tilde{\mathrm{F}}+\tilde{\mathrm{G}}}(\theta) \quad ; \quad (c_f - s_f) + (c_g - s_g) \leq \theta < cs_4 \tag{E.171}$$

If we assume that $s_g \geq s_f$, we get a similar contradiction.

By proceeding in a similar manner, we can show that $l(\theta)$ also forms a lower bound on $\mu_{\tilde{\mathrm{F}}+\tilde{\mathrm{G}}}(\theta)$ for $cs_5 < \theta \leq (c_f + s_f) + (c_g + s_g)$; therefore, we can say that

$$\mu^L_{\tilde{\mathrm{F}}+\tilde{\mathrm{G}}}(\theta) = l(\theta) \tag{E.172}$$

where $l(\theta)$ is as given in (E.165).

#### E.2.2.2 Upper Bound

We show that the symmetrical triangle with center $(c_f + c_g)$ and spread $(s_f + s_g)$ forms a upper bound on the actual result of the algebraic sum in (E.164). The equation of this triangle is

$$u(\theta) = \begin{cases} 1 - \left|\frac{\theta - (c_f + c_g)}{s_f + s_g}\right| & ; \quad \theta \in [(c_f - s_f) + (c_g - s_g), (c_f + s_f) + (c_g + s_g)] \\ 0 & ; \quad \text{otherwise} \end{cases} \tag{E.173}$$

Since the spreads $s_f$ and $s_g$ are positive, $(s_f + s_g) > \max\{s_f, s_g\}$; and therefore, $u(\theta) \geq \mu_{\tilde{\mathrm{F}}+\tilde{\mathrm{G}}}$ for $cs_4 < \theta < cs_5$. It is obvious that at $\theta = [(c_f - s_f) + (c_g - s_g)]$, $\mu_{\tilde{\mathrm{F}}+\tilde{\mathrm{G}}} = u(\theta)$. For $(c_f - s_f) + (c_g - s_g) < \theta \leq cs_4$, let us assume that $u(\theta) < \mu_{\tilde{\mathrm{F}}+\tilde{\mathrm{G}}}$, i.e., let us assume that

$$1 + \frac{\theta - (c_f + c_g)}{s_f + s_g} \; < \; \frac{1}{4 s_f s_g}\left[\theta - [(c_f - s_f) + (c_g - s_g)]\right]^2$$

$$\Rightarrow \quad \frac{\theta - [(c_f - s_f) + (c_g - s_g)]}{s_f + s_g} < \frac{1}{4s_f s_g}\left[\theta - [(c_f - s_f) + (c_g - s_g)]\right]^2$$

$$\Rightarrow \quad \frac{4s_f s_g}{s_f + s_g} < \theta - [(c_f - s_f) + (c_g - s_g)]$$

$$\Rightarrow \quad \theta > (c_f - s_f) + (c_g - s_g) + \frac{4s_f s_g}{s_f + s_g} \qquad \text{(E.174)}$$

Let $s_f \geq s_g$. This implies that $[4s_f s_g]/[s_f + s_g] = [4s_g]/[1 + s_g/s_f] \geq 2s_g$, which, together with (E.174) implies that

$$\theta > (c_f - s_f) + (c_g + s_g) \qquad \text{(E.175)}$$

Our assumption that $s_f \geq s_g$, together with (E.151) and (E.148), implies that $cs_4 = (c_f - s_f) + (c_g + s_g)$ and since we have considered that $\theta < cs_4$, (E.175) can not be true, which means that our assumption that $u(\theta) < \mu_{\tilde{F}+\tilde{G}}(\theta)$ must be false. A similar conrtradiction can be obtained by assuming that $s_g \geq s_f$. This shows that $u(\theta)$ forms an upper bound on $\mu_{\tilde{F}+\tilde{G}}(\theta)$ for $(c_f - s_f) + (c_g - s_g) \leq \theta \leq cs_4$.

By proceeding in a similar manner, we can show that $u(\theta)$ also forms an upper bound on $\mu_{\tilde{F}+\tilde{G}}(\theta)$ for $cs_5 \leq \theta \leq (c_f + s_f) + (c_g + s_g)$; therefore, we can say that

$$\mu^U_{\tilde{F}+\tilde{G}}(\theta) = u(\theta) \qquad \text{(E.176)}$$

where $u(\theta)$ is as given in (E.173).

#### E.2.2.3 Triangular Approximation

To find a triangular approximation, we choose a symmetrical triangle that lies between the upper and lower bounds on the actual result of the algebraic sum. By observing (E.165) and (E.173), we can see that a triangle with center $(c_f + c_g)$ and spread $\sqrt{s_f^2 + s_g^2}$ gives such an approximation, i.e., we propose the following approximation for the algebraic sum of $\tilde{F}$ and $\tilde{G}$ :

$$\mu_{\tilde{F}+\tilde{G}}(\theta) \approx \begin{cases} 1 - \left|\frac{\theta-(c_f+c_g)}{s_{fg}}\right| & ; \quad \theta \in [c_f + c_g - s_{fg}, c_f + c_g + s_{fg}] \\ 0 & ; \quad \text{otherwise} \end{cases} \tag{E.177}$$

where

$$s_{fg} = \sqrt{s_f^2 + s_g^2} \tag{E.178}$$

Generalizing to the sum of $n$ triangular type-1 fuzzy numbers, we find that the algebraic sum of $n$ triangular type-1 fuzzy numbers, $\tilde{F}_1, \ldots, \tilde{F}_n$ with centers $c_1, \ldots, c_n$ and spreads $s_1, \ldots, s_n$, under product $t$-norm can be approximated as

$$\mu_{\sum_{i=1}^n \tilde{F}_i}(\theta) \approx \begin{cases} 1 - \left|\frac{\theta-\sum_{i=1}^n c_i}{\sqrt{\sum_{i=1}^n s_i^2}}\right| & ; \quad \theta \in [\sum_{i=1}^n c_i - \sqrt{\sum_{i=1}^n s_i^2}, \sum_{i=1}^n c_i + \sqrt{\sum_{i=1}^n s_i^2}] \\ 0 & ; \quad \text{otherwise} \end{cases} \tag{E.179}$$

Using part **(a)** of Theorem E.2 with (E.179), we have the following result : Given $n$ type-1 triangular fuzzy numbers $\tilde{F}_1, \ldots, \tilde{F}_n$, with centers $c_1, c_2, \ldots, c_n$ and spreads $s_1, s_2, \ldots, s_n$, their affine combination $\sum_{i=1}^n \alpha_i \tilde{F}_i + \beta$, where $\alpha_i$ $(i = 1, \ldots, n)$ and $\beta$ are crisp constants, is approximately a triangular fuzzy number with center $\sum_{i=1}^n \alpha_i c_i + \beta$, and spread $\sqrt{\sum_{i=1}^n \alpha_i^2 s_i^2}$.

## E.3 Weighted Average of Triangular Type-1 Fuzzy Numbers

Consider the weighted average

$$y(z_1, \ldots, z_M, w_1, \ldots, w_M) = \frac{\sum_{l=1}^M w_l z_l}{\sum_{l=1}^M w_l} \tag{E.180}$$

where $z_l \in \Re$ and $w_l \in [0, 1]$ for $l = 1, \ldots, M$. If each $z_l$ is replaced by a type-1 fuzzy set $\tilde{Z}_l$ and each $w_l$ is replaced by a type-1 fuzzy set $\tilde{W}_l$, then the extension of (E.180) gives

$$\tilde{Y}(\tilde{Z}_1, \ldots, \tilde{Z}_M, \tilde{W}_1, \ldots, \tilde{W}_M) = \int_{z_1} \cdots \int_{z_M} \int_{w_1} \cdots \int_{w_M} \mathcal{T}_{l=1}^{M} \mu_{\tilde{Z}_l}(z_l) \star \mathcal{T}_{l=1}^{M} \mu_{\tilde{W}_l}(w_l) \Big/ \frac{\sum_{l=1}^{M} w_l z_l}{\sum_{l=1}^{M} w_l} \tag{E.181}$$

where $\mathcal{T}$ and $\star$ both indicate the $t$-norm used ... product or minimum.

**Theorem E.4** *If each $\tilde{Z}_l$ is a triangular type-1 fuzzy number, with center $c_l$ and spread $s_l$, and if each $\tilde{W}_l$ is also a triangular type-1 fuzzy number with center $h_l$ and spread $\Delta_l$, then $\tilde{Y}$ is approximately a triangular type-1 fuzzy number, with center $\mathcal{C}$ and spread $\mathcal{S}$, where*

$$\mathcal{C} = \frac{\sum_{l=1}^{M} h_l c_l}{\sum_{l=1}^{M} h_l} \tag{E.182}$$

*and*

$$\mathcal{S} = \begin{cases} \dfrac{\sqrt{\sum_{l=1}^{M} [(h_l s_l)^2 + (c_l - \mathcal{C})^2 \Delta_l^2]}}{\sum_{l=1}^{M} h_l} , & \textit{if product t-norm is used} \\ \dfrac{\sum_{l=1}^{M} [(h_l s_l) + |c_l - \mathcal{C}| \Delta_l]}{\sum_{l=1}^{M} h_l} , & \textit{if minimum t-norm is used} \end{cases} \tag{E.183}$$

*provided that*

$$\frac{\sum_{l=1}^{M} \Delta_l}{\sum_{l=1}^{M} h_l} \ll 1 , \tag{E.184}$$

*The triangular approximation improves as $\left(\sum_{l=1}^{M} \Delta_l \big/ \sum_{l=1}^{M} h_l\right)$ grows smaller. Under minimum t-norm, the result is exact when $\sum_{l=1}^{M} \Delta_l = 0$, i.e., when $\Delta_l = 0$ for $l = 1, \ldots, M$.*

**Proof :** The proof proceeds exactly like the proof of Theorem 2.5 in Appendix C.11, the only difference being that now the condition required for a good approximation is $[\sum_{l=1}^{M} \Delta_l]/[\sum_{l=1}^{M} h_l] \ll 1$ instead of $k[\sum_{l=1}^{M} \Delta_l]/[\sum_{l=1}^{M} h_l] \ll 1$ [see (2.76)]. The factor of $k$ appeared in the Gaussian case, because the membership of a point in a Gaussian type-1 set is never exactly equal to 0; we just neglected the memberships outside $\pm k\Delta_l$, since they were too small. In the triangular case, however, the memberships of points outside $\pm\Delta_l$ are equal to 0; and therefore, the factor of $k$ disappears.

We get a result similar to (C.229) :

$$\tilde{\mathrm{Y}} \approx \sum_{l=1}^{M} \left[\underline{\tilde{\mathrm{Z}}}_l\left(\frac{h_l}{\sum_l h_l}\right) + \underline{\tilde{\mathrm{W}}}_l\left(\frac{c_l - \mathcal{C}}{\sum_l h_l}\right)\right] + \mathcal{C} \tag{E.185}$$

where $\underline{\tilde{\mathrm{Z}}}_l$'s are zero-mean triangular type-1 fuzzy numbers with spreads $s_l$'s and $\underline{\tilde{\mathrm{W}}}_l$'s are zero-mean triangular type-1 fuzzy numbers with spreads $\Delta_l$'s. The result in Theorem E.4 follows by applying results in Theorem E.2 and Section E.2.2.3 to (E.185). □

**Comment 1** : When product $t$-norm is used, the result in Theorem E.4 is not exact even when $\Delta_l = 0$ for $l = 1, \ldots, M$, because the algebraic sum of triangular type-1 sets under product $t$-norm does not remain triangular (see Section E.2).

**Comment 2** : Comments 2 and 3 at the end of Appendix C.11 apply in this case as well.

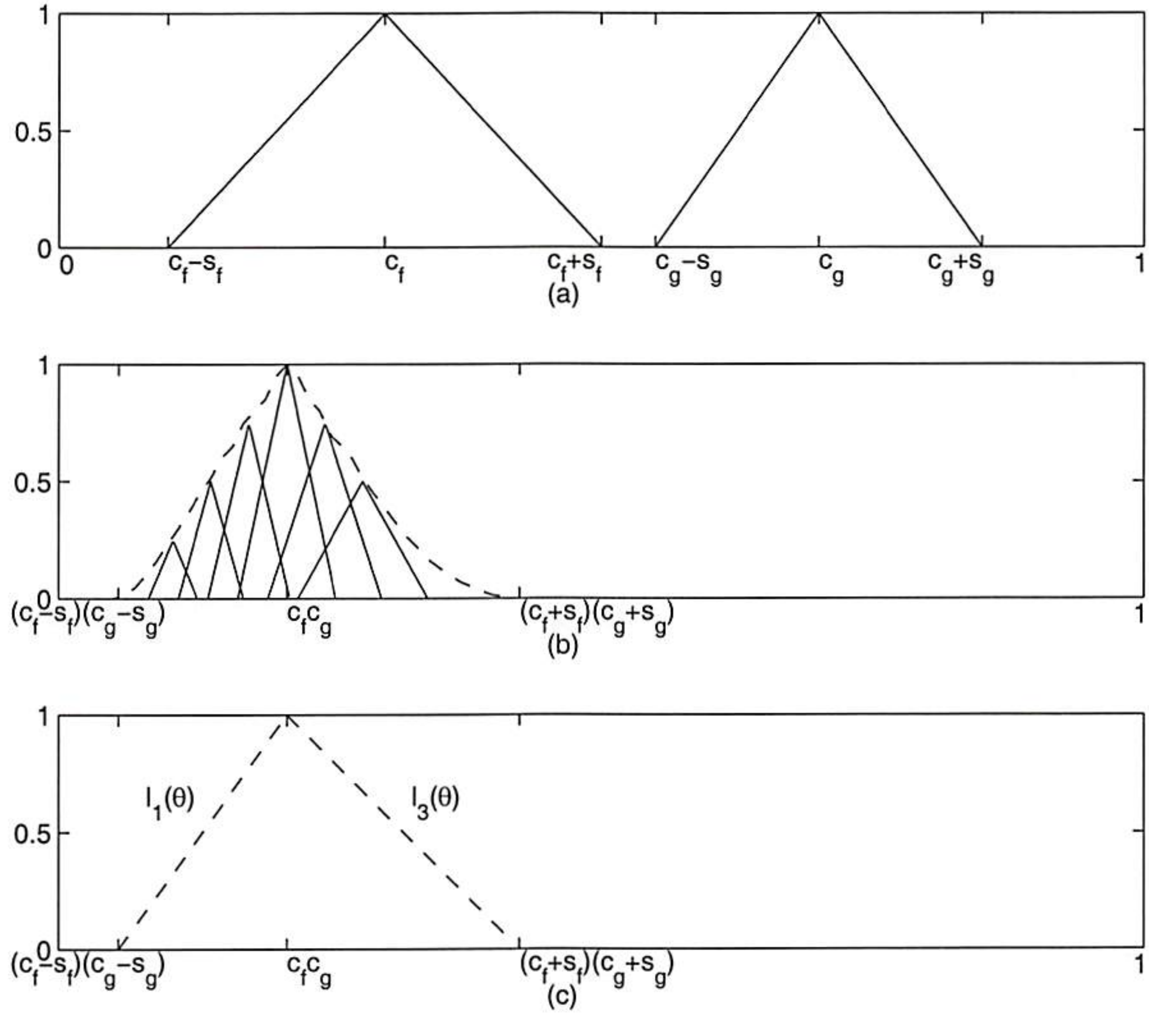



Figure E.1: (a) Two triangular fuzzy sets. (b) Interpretation of the *meet* as the envelope of a collection of scaled triangles. The dashed line shows the actual result (Theorem E.1). (c) The domain of the actual *meet* result and the lines $l_1(\theta)$ and $l_3(\theta)$ used in the calculation of the upper bound.

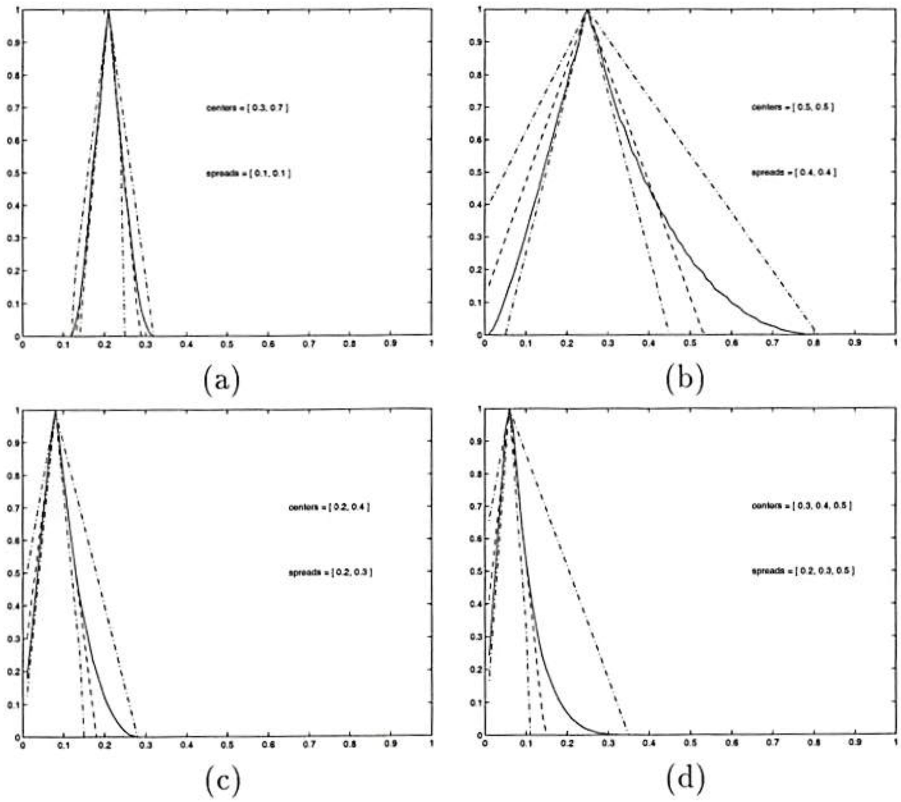



Figure E.2: Some examples of upper and lower bounds in (E.112) and (E.79), and the approximation in (E.116). In each figure, the solid line shows the actual result of the *meet* (computed numerically), the dashed line shows the triangular approximation and the dash-dotted lines show upper and lower bounds. In (b), the two triangles are coincident.

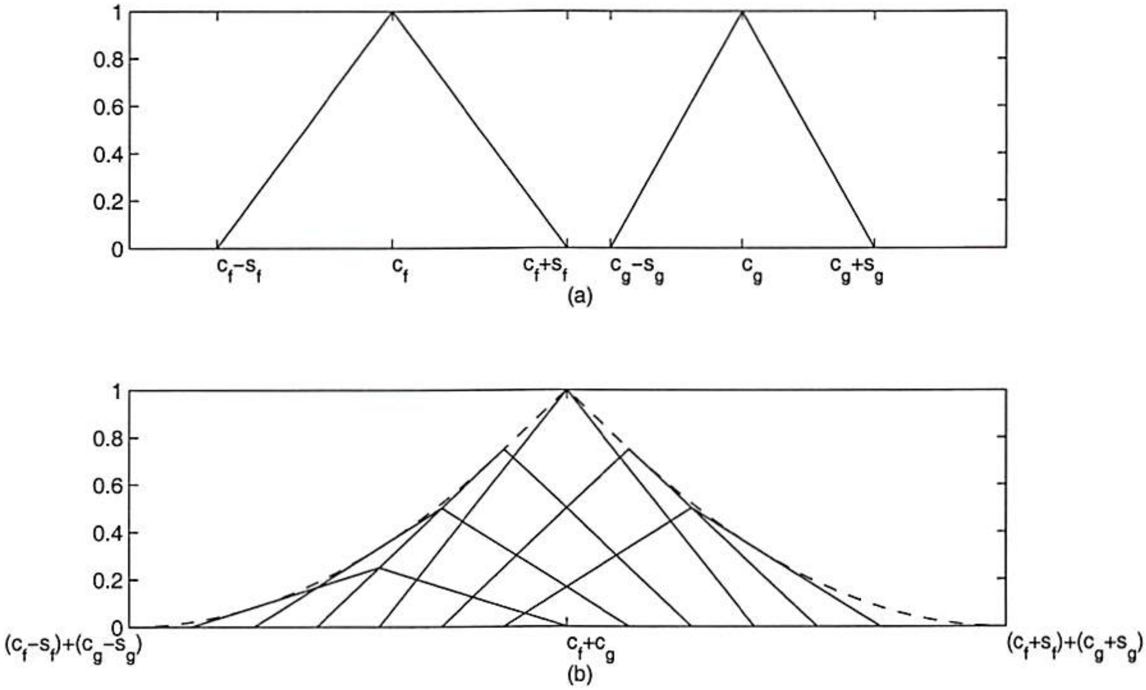



Figure E.3: (a) Two triangular fuzzy sets. (b) Interpretation of their algebraic sum as the envelope of a collection of shifted and scaled triangles. The dashed line shows the actual result (Theorem E.3).

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