Step 4: We must show that a circle which separates A and B defines A as a digital disk. This is clear, as well as the fact that if D is a digital disk, a defining circle will separate A and B. The time to compute all the $L(r, s)$ is $O(NN) = O(N^2)$.

For example, suppose where D is $\{(0, 0), (1, 0), (0, 1), (1, 1)\}$. In this case, $A = D$. Then $B = \{(0, -1), (1, -1), (2, 0), (2, 1), (1, 2), (0, 2), (-1, 1), (-1, 0)\}$.

Fig. 1 shows the Voronoi regions (solid lines) and S (the polygon centered at $(\frac{1}{2}, \frac{1}{2})$, bounded by dashed lines). The furthest point Voronoi regions are determined by the 2 lines $x = \frac{1}{2}$ and $y = \frac{1}{2}$.

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Counting with Fuzzy Sets

BART KOSKO

Abstract—**The notion of fuzzy set cardinality is examined. Zadeh's suggested measure of fuzzy cardinality, the sigma-count, is adopted and shown to generalize classical counting measure. This allows many combinatorial structures and counting techniques to be fuzzified, and hence used in knowledge representation and pattern recognition models. A fuzzy set review is found in the Appendix.**

Index Terms—**Combinatorics, counting measure, fuzzy cardinality, fuzzy sets, greedy algorithms, sigma-count.**

How big is a fuzzy set? Intuitively a fuzzy set has less stuffing than a nonfuzzy set. The fuzzy set of shiny red apples in a pile of apples seems less dense than the pile, even if all the apples shine. Similarly, the set of lightly gray pixels in a screen image seems to have less content, less measure, than the set of gray pixels.

The structure sought is a fuzzy set cardinality. Fuzzy cardinality can be cast in different ways; perhaps the most natural is Zadeh's sigma-count. Zadeh [1] has proposed generalizing the classical cardinality c of a subset $A \subset E$,

$$c(A) = \sum_{e \in E} I_A(e),$$

where

$$I_A(e) = \begin{cases} 1 & \text{if } e \in A, \\ 0 & \text{if } e \notin A, \end{cases}$$

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by extending the indicator function I_A to a [0, 1]-valued fuzzy-set membership function m_A. The new cardinality is called the *sigma-count*,

$$\Sigma\text{-Count}(A) = \sum_{e \in E} m_A(e).$$

Thus the fuzzy subset of shiny apples $\{(a_1, 0.3), (a_2, 0.4), (a_3, 0.2), (a_4, 0.7), (a_5, 1.0)\}$ has sigma-count 2.6. Clearly, on this interpretation of fuzzy cardinality, a fuzzy subset never has cardinality greater than its associated nonfuzzy superset (the "defuzzified" set whose indicator function is 1 if and only if the membership function is positive).

The sigma-count has the virtue of computability. It allows one to get a theoretical and practical grip on fuzzy combinatorics. For instance, the combinatorial structures known as greedy algorithms and greedy heuristics can be fuzzified by allowing the underlying set collection to contain fuzzy sets and by (carefully!) replacing nonfuzzy cardinality with the sigma-count in the requisite arguments. This allows fuzzy theory to be applied to heuristic search algorithms. Once combinatorial structures and techniques have been fuzzied, the sigma-count allows them to be applied to real problems by, ultimately, summing membership function estimates.

The main result of this correspondence is the proof that the sigma-count is a positive measure. Hence it generalizes counting measure. (Counting measure is the countably additive set function that yields $c(A)$ if A is finite, $+\infty$ if A is infinite.) That is, on nonfuzzy sets the sigma-count is finite when counting measure is finite and the two finite (integer) numbers are the same, and infinite when counting measure is infinite. In this sense Zadeh has identified the "right" cardinality structure, and so it seems reasonable to apply the sigma-count to the science of counting, combinatorics.

On fuzzy sets the sigma-count has special expressive power. Consider the positive integers $\{1, 2, 3, \cdots\}$, which has $+\infty$ counting measure. Define the fuzzy subset F with the membership function

$$m_F(i) = \frac{1}{2^i}.$$

Then $\Sigma\text{-Count}(F) = 1$, which in some sense means there is only one nonfuzzy integer. Intuitively the fuzzy subset F might correspond to a child's notion of numbers.

To prove that the set function Σ-Count is a positive measure on sigma-algebras of fuzzy sets, and to motivate its measure-theoretic interpretation, some propositions are needed. Let E be the underlying nonfuzzy set and let $F(2^E)$ be the fuzzy power set of E—all fuzzy subsets of E (which includes, of course, those in 2^E). Then since $A \subset B$ implies $m_A \le m_B$, Proposition 1 immediately follows.

Proposition 1: For $A, B, \in F(2^E)$:

$$\text{If } A \subset B, \text{ then } \Sigma\text{-Count}(A) \le \Sigma\text{-Count}(B).$$

Proposition 1 and the four inclusions

$$A \cap B \subset A, \quad B \subset A \cup B$$

imply Proposition 2.

Proposition 2:

$$\begin{aligned} \Sigma\text{-Count}(A \cap B) &\le \min\{\Sigma\text{-Count}(A), \\ \Sigma\text{-Count}(B)\} &\le \max\{\Sigma\text{-Count}(A), \\ \Sigma\text{-Count}(B)\} &\le \Sigma\text{-Count}(A \cup B). \end{aligned}$$

As Zadeh has observed [2], the real-number identity

$$a + b = \min(a, b) + \max(a, b)$$

induces at once a like structure over fuzzy membership functions, and hence over sigma-counts. This gives Proposition 3.

Proposition 3: For $A, B \in F(2^E)$:

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$$\Sigma\text{-Count}(A) + \Sigma\text{-Count}(B) = \Sigma\text{-Count}(A \cap B) + \Sigma\text{-Count}(A \cup B).$$

Hence if A and B are disjoint ($m_{A \cap B} = 0$), the set function Σ-Count is finitely additive. This property is required to prove the Theorem.

Theorem: If $\theta \subset F(2^E)$ is a sigma-algebra of fuzzy sets (i.e., $E \in \theta$ and θ is closed under fuzzy complements and countable fuzzy unions), then Σ-Count: $\theta \to R^+$ is a positive measure.

Proof: It must be shown that Σ-Count is zero on the empty set and that it is countably additive, i.e., $\Sigma\text{-Count}(\cup_{i=1}^{\infty} A_i) = \Sigma_{i=1}^{\infty} \Sigma\text{-Count}(A_i)$ if the sets $\{A_i\}_{i=1}^{\infty}$ are pairwise disjoint. So, first, trivially $\Sigma\text{-Count}(\emptyset) = \Sigma_{e \in E} m_{\emptyset}(e) = 0$. Second, the finite additivity of Σ-Count extends to countable additivity if the following relationship holds: $\Sigma\text{-Count}(F_i)$ decreases to zero on every contracting sequence of sets $(F_i)_{i=1}^{\infty} \subset \theta$, i.e., on every sequence in θ such that $F_i \supset F_{i+1}$ and $\cap_{i=1}^{\infty} F_i = \emptyset$ and $\Sigma\text{-count}(F_j) < \infty$ for some j.

Now $F_i \supset F_{i+1}$ implies $m_{F_i} \geq m_{F_{i+1}}$. So Σ-Count is decreasing. And since $\Sigma\text{-Count}(F_i) \geq 0$ for all i,

$$\begin{aligned} 0 &\leq \liminf_{i \to \infty} \Sigma\text{-Count}(F_i) \leq \limsup_{i \to \infty} \Sigma\text{-Count}(F_i) \\ &= \limsup_{i \to \infty} \sum_{e \in E} m_{F_i}(e) \leq \sum_{e \in E} \limsup_{i \to \infty} m_{F_i}(e) \\ &= \sum_{e \in E} \inf_{j \geq 1} \sup_{k \geq j} m_{F_k}(e) \\ &= \sum_{e \in E} \inf_{j \geq 1} m_{F_j}(e) \qquad \text{since } m_{F_{k+1}} \leq m_{F_k} \text{ for all } k \geq j, \\ &= \sum_{e \in E} m_{\cap_{i=1}^{\infty} F_i}(e) = \sum_{e \in E} m_{\emptyset}(e) = 0. \end{aligned}$$

Therefore, the required limit exists: $\lim_{i \to \infty} \Sigma\text{-Count}(F_i) = 0$. Therefore, the set function Σ-Count is countably additive, and thus a positive measure on θ. Q.E.D

Appendix
Fuzzy Set Review

Let $E = \{e_1, e_2, \cdots\}$ be a standard set. A *fuzzy subset A* of E is the collection of element/membership-degree pairs $\{(e, m_A(e))\}_{e \in E}$, where the *membership function* m_A is a generalized indicator function, $m_A: E \to [0, 1]$. Let $F(2^E)$ denote the *fuzzy power set* of E, all fuzzy subsets of E.

Fuzzy set operations are defined in terms of membership functions. Let $A, B \in F(2^E)$.

$$A \cup B = \{(e \max\{m_A(e), m_B(e)\})\}_{e \in E}.$$

$$A \cap B = \{(e, \min\{m_A(e), m_B(e)\})\}_{e \in E}.$$

$$A^c = \{(e, 1 - m_A(e))\}_{e \in E}.$$

$$A \subset B \text{ iff } m_A(e) \leq m_B(e)$$

$$\text{for all } e \in E, \text{ or, more compactly, } m_A \leq m_B.$$

The empty set $\emptyset$ has zero membership function, $m_{\emptyset} = 0$.

More generally, if I is an arbitrary index set and $A_i \in F(2^E)$ for all $i \in I$, arbitrary union and intersection membership functions are defined with suprema and infima:

$$m_{\cup_{i \in I} A_i} = \sup_{i \in I} m_{A_i} \quad \text{and} \quad m_{\cap_{i \in I} A_i} = \inf_{i \in I} m_{A_i}.$$

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A Significant Plane for Two-Class Discrimination Problems

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***Abstract*—In solving classification problems, the Fisher linear discriminant is often used for discriminating between two pattern classes. In addition to this discriminant direction, this correspondence proposes the use of a second direction, orthogonal to the first, which maximizes the projected scatter. An example is provided to illustrate the method.**

***Index Terms*—Discriminants, multivariate data projection, pattern recognition.**

For discrimination problems between two classes of vector samples, the Fisher linear discriminant [1] is often used as the optimal linear method, providing a linear combination of the original parameters in the sense of the best discrimination between these classes.

More generally, the solution of such a problem of discrimination for L classes is given by computing the eigenvectors of the $T^{-1} \cdot B$ matrix, with T as the total covariance matrix and B the between-class covariance matrix. For L classes, the rank of the between-class matrix is $(L - 1)$ in the best case, and therefore only one nonzero eigenvalue exists for a discrimination problem between two classes. Then the subspace obtained by this method is a one-dimensional space.

In order to represent the original vector samples onto a plane including the previous discriminant vector, two different methods can be proposed: the first way is to determine a second (or more) discriminant vector, orthogonal to the first [2], [3] (the optimal discriminant plane). The other way is to combine the discriminant analysis and a principal components analysis (so-called Karhunen–Loeve expansion) [4]: in that case, the objective is to obtain a plane that could simultaneously give information on discrimination (first vector) and scatter (second vector).

The expression of the first vector is well known, if the problem is solved by the optimization of the Fisher criterion [2]

$$d = \alpha \cdot W^{-1} \cdot \delta$$

where α is chosen so that $d^t \cdot \text{d} = 1$, with δ as the difference between the two class-centers and W the within-class covariance matrix.

We propose to obtain the second vector u by computing a vector which maximizes the projected scatter and which is orthogonal to the first one, i.e.,

$$V = u^t T u \text{ maximum}$$

with the two constraints

$$u^t d = 0 \text{ and } u^t u = 1.$$

These expressions can be written using the Lagrange multipliers:

$$C = u^t T u - \lambda_1 (u^t u - 1) - \lambda_2 u^t d.$$

Setting the partial of C with respect to u equal to zero:

$$\frac{\partial C}{\partial u} = 2Tu - 2\lambda_1 u - \lambda_2 d = 0 \tag{1}$$

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