MODELING GUNSHOT BRUISES IN SOFT BODY ARMOR
WITH AN ADAPTIVE FUZZY SYSTEM

Ian Lee, Bart Kosko, Member, IEEE, W. French Anderson

Abstract

Gunshots produce bruise patterns on persons who wear soft body armor when shot even though the armor stops the bullets. An adaptive fuzzy system modeled these bruise patterns based on the depth and width of the deformed armor given a projectile’s mass and momentum. The fuzzy system used rules with sinc-shaped if-part fuzzy sets and was robust against random rule pruning: Median and mean test errors remained low even after removing up to one fifth of the rules. Handguns shot different caliber bullets at armor that had 10%-ordnance gelatin backing. The gelatin blocks were tissue simulants. The gunshot data tuned the additive fuzzy function approximator. The fuzzy system’s conditional variance $V[Y|X=x]$ described the second-order uncertainty of the function approximation. Handguns with different barrel lengths shot bullets over a fixed distance at armor-clad gelatin blocks that we made with Type 250A Ordnance Gelatin. The bullet-armor experiments found that a bullet’s weight and momentum correlated with the depth of its impact on armor-clad gelatin ($R^2 = 0.881$ and $p$-value < 0.001 for the null hypothesis that the regression line had zero slope). Related experiments on Plumber’s Putty showed that highspeed baseball impacts compared well to bullet-armor impacts for large-caliber handguns. A baseball’s momentum correlated with its impact depth in putty ($R^2 = 0.93$ and $p$-value < 0.001). A bullet’s momentum similarly correlated with its armor-impact in putty ($R^2 = 0.97$ and $p$-value < 0.001). A Gujarati-Chow test showed that the two putty-impact regression lines had statistically indistinguishable slopes for $p$-value = 0.396. Baseball impact depths were comparable to bullet-armor impact depths: Getting shot with a .22 caliber bullet when wearing soft body armor resembles getting hit in the chest with a 40-mph baseball. Getting shot with a .45 caliber bullet resembles getting hit with a 90-mph baseball.

I. MODELING BULLET-ARMOR BRUISE IMPACTS

How does it feel to get shot while wearing soft body armor? One police officer described it as a sting while another officer described it as a “hard blow” [1]. Fig. 1 shows the bruise beneath the armor after a .44 caliber bullet struck a police officer’s upper left chest. The armor stopped the bullet but the impact still injured soft tissue.
We examined the bruising effect with a fuzzy function approximator and a baseball analogy. Bullet impact experiments produced the bullet-armor bruise data that generated a quantitative bruise profile and a baseball-impact comparison. The bruise profile gave the depth and width of the deformation that a handgun bullet made on gelatin-backed armor for gelatin blocks that we made with Type 250A Ordnance Gelatin (from Kind & Knox Gelatin).

Few researchers have studied the relationship between the bruising effect and the so-called backface signature or the deformation in the armor’s backing material after a gunshot [2], [3]. Our bruise profile modeled the bullet-armor impact with the depth and width of the bruise as a blunt object that could injure soft tissue. We quantified the bullet-armor impacts and baseball impacts on a common backing material–Plumber’s Putty. The baseball analogy helped estimate gunshot impacts on armor. We found that a fast baseball could hit as hard as a large caliber handgun bullet on armor. The baseball and bullet impacts in putty had similar depths and statistically indistinguishable slopes.

A. Soft Body Armor

An adaptive fuzzy system learned to model the depth and width of bruise profiles from the bullet-armor impact experiments. Two armor-impact experimental setups used gelatin blocks to simulate human tissue. A third armor-impact experimental setup used Plumber’s Putty to record the bullet-armor deformation that compared with baseball impacts. The experiments confirmed the expectation that a bullet made a larger impact if it had a larger caliber or a larger momentum (see Table I). But a larger and slower handgun bullet hit harder than a smaller and faster one in the experiments. Impact depth correlated better with momentum than with kinetic energy. This corroborated the
Fig. 2. One of the authors holds a 14-ply Kevlar soft body armor panel (from a Superfeatherlite vest from Second Chance) and some sample cartridges (.22, .38, .357 magnum, .40, and .45 caliber). The right side of the image shows the pristine and armor-deformed bullets for the five calibers.

finding that kinetic energy was not a good predictor of bullet penetration wounds [4].

We picked the initial rules based on our ballistic judgement and experience. The experimental data tuned the rules of an adaptive standard-additive-model (SAM) fuzzy system [5]. The SAM system used two scalar subsystems to model the depth and width of a bullet-armor impact in parallel given the bullet’s weight and momentum. We tested the robustness of the fuzzy system by removing rules at random (random rule pruning). The median, mean, and maximal test errors resembled the initial system error for pruning that randomly removed up to 20% of the rules.

The next two sections review soft body armor and bullet-impact bruises. Section II reviews rule-based fuzzy function approximation. Section III reviews the standard additive model. Section IV provides the setup and results of the bullet-armor experiments. Section V provides the setup and results of the baseball impact experiments. Section VI provides the experimental setup and the results of the fuzzy function approximation.

Soft body armor prevents most handgun bullets from penetrating a user’s body [6]. Our armor experiments used a generic armor that combined many layers of fabric that wove together Kevlar fibers. Thinner armor is softer than thicker armor. Another type of armor material laminated together many layers of parallel fibers. Both types of armor deform under a bullet’s impact and spread the impact’s force over a wider area. Bullets penetrate by crushing [7]. So soft body armor arrests a handgun bullet by reducing its crushing force below a material-failure threshold [7].

Failure analysis does not consider the physiological effects as the armor stops a bullet. Some researchers define armor failure as material failure such as broken fibers or breached fabric layers [8], [9], [10], [11], [12]. Others require complete bullet passages [13]. Such definitions do not address the interactions that flexible armor permit with the underlying material.

These interactions have two effects. The first is that a bullet-armor impact can injure soft tissue even though the bullet does not penetrate the armor (see Fig. 1). The second effect is that soft body armor’s performance can differ
B. Bruising and the Bruise Profile

Bruising implies injury but need not indicate the severity of the injury. A bruise is escaped blood in the intercellular space after a blunt impact injures soft tissue [17]. The visible part of a bruise is the part of the escaped blood that is close to the skin surface. Scraping with a coin or a spoon can leave extensive but superficial bruises or welts that resemble bruises from abuse [18]. The visible bruise can change over time [19] at different rates based on sex, age, body fat [17], and medication [20]. So a bruise shows that a blunt impact occurred but need not show that internal injuries occurred [21], [22]. This can occur in sports injuries where soft tissue injuries can escape detection [23].

A bruise profile models the shape of the bullet-armor impact and can help guide the examination after an armor gunshot. This is similar to a wound profile [16] that can help the examination of a gunshot wound. The bruise profile can indicate the affected internal tissue beneath the visible armor bruise.

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<th>$\beta_1$</th>
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<td>0.340 (p &lt; 0.001)</td>
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TABLE I. Summary of linear regression statistics for the gelatin-backed bullet-armor impact experiments and for the simple linear model $y = \beta_0 + \beta_1 x$. Momentum $mv$ and weight $m$ correlated with an impact’s depth and width while kinetic energy $\frac{1}{2}mv^2$ correlated the least. Target distance $d$ correlated with an impact’s depth while speed $v$ and distance $d$ correlated little with the impact’s width. The $R^2$ measures the strength of the correlation between the input and output variables. The $t$ statistic tested the parameter significance of a single regression coefficient $\beta_1$ between deformation in millimeters and one of a bullet’s mean weight (grain), momentum (kg m/s), kinetic energy (kg m$^2$/s$^2$), target distance (yard), and speed (m/s). The $p$-value measured the credibility of the null hypothesis $H_0$ that the regression line had zero slope $\beta_1 = 0$ or that a dependent variable did not vary with the independent variable. A statistical test rejects the null hypothesis $H_0$ at a significance level $\alpha$ if the $p$-value is less than that significance level. So the regression rejected the null hypothesis $H_0$ for the customary significance levels $\alpha = 0.05$ and $\alpha = 0.01$ because $p$-value < 0.001.

for different backing material that supports the armor [7]. We found that a hammer strike breached several layers of concrete-backed armor fabric but a handgun bullet bounced off gelatin-backed armor fabric.

The backface signature is the deformation in the backing material after a bullet strikes armor [3]. Studies of backface signatures [2] give little information about the impact as a bruising force if the backing material differs from soft tissue. One industry standard [3] measures the backface signature on clay backing material. The clay records the impact in a plastic or permanent deformation but its properties differ from soft tissue.

Gelatin tissue simulant is elastic and responds to a bullet’s crushing force similar to how soft tissue responds in bullet penetration tests [14], [15], [16]. So testing gelatin-backed soft body armor can help study the performance of the armor on a user’s body. We performed the bullet-armor impact experiments on tissue simulant and defined a simple two-parameter bruise profile to describe the impact.
TABLE II. Initial fuzzy rules for the function approximation. Each entry represented one rule for the depth subsystem in (a) and width subsystem in (b). We based the 49 initial fuzzy rules for the armor-deformation depth and width on the experimenters’ ballistic judgement and experience. The rules had two-dimensional if-part sets and two parallel scalar then-part sets. A typical rule had the word form “If the bullet’s weight \( X_1 \) is medium small (MS) and the momentum \( X_2 \) is medium (MD) then the armor deformation depth \( Y_1 \) is small (SM) and the width \( Y_2 \) is very small (VS).” The then-part set functions \( b_j \) had center parameters \( c_j \) based on the entries in (a) and (b). The area parameters \( V_j \) were based on the entries in (c). Seven if-part fuzzy sets described the bullet’s weight and momentum \{Very Small (VS), Small (SM), Medium Small (MS), Medium (MD), Medium Large (ML), Large (LG), Very Large (VL)\}. Five then-part fuzzy sets described the armor deformation \{VS, SM, MD, LG, VL\}. Three volume or area values in (c) described the rule uncertainty \{Small, Medium, Large\} by fixing the dispersion or width of the then-part sets.

II. ADAPTIVE FUZZY SYSTEM

Bullet-impact experiments trained an adaptive fuzzy system to model the depth and width of the bullet-armor impact given a handgun bullet’s weight and momentum. We picked the fuzzy system’s initial rules in Table II based on the correlations in the experimental data (Table I) and based on our ballistic judgement and experience. Similar rules described the depth and width subsystems. A typical rule was “If a bullet’s weight is very small (VS) and its momentum is small (SM) then the armor deformation depth is SM and the width is VS.” A rule’s volume (or dispersion) reflected its uncertainty. The gunshot data tuned the rules in an adaptive standard-additive-model (SAM) function approximation.

We applied two scalar-valued additive fuzzy systems [5], [24] \( F: R^n \rightarrow R^p \) (for \( p = 1 \)) in parallel that used two-dimensional inputs \( n = 2 \) to model the depth and width of a bullet-armor impact. These systems approximated some unknown function \( f: R^n \rightarrow R \) by covering the graph of \( f \) with \( m \) fuzzy rule patches and averaging patches that overlap. An if-then rule of the form “If \( X \) is \( A \) then \( Y \) is \( B \)” defined a fuzzy Cartesian patch \( A \times B \) in the input-output space \( X \times Y \). The rules could use fuzzy sets of any shape for either their if-part sets \( A \) or then-part sets \( B \). This held for the feedforward standard additive model (SAM) fuzzy systems discussed below. Their generality...
Fig. 3. Sample if-part and then-part fuzzy sets. (a) Joint (product) sinc if-part set function for two-dimensional input case [24]. The joint set function has the factorable form $a_j(x) = a_j(x_1, x_2) = a_j^1(x_1) \times a_j^2(x_2)$. The shadows show the scalar sinc set functions $a_j^i : R \rightarrow [0, 1]$ for $i = 1, 2$ that generate $a_j : R^2 \rightarrow R$. (b) Scalar Gaussian then-part set function centered at $c_j$.

The if-part fuzzy sets $A_j \subset R^n$ and then-part fuzzy sets $B_j \subset R^p$ had set functions $a_j : R^2 \rightarrow [0, 1]$ and $b_j : R \rightarrow [0, 1]$ for $n = 2$ and $p = 1$. An additive fuzzy system [5], [24] summed the “fired” then-part sets $B'_j$

$$B(x) = \sum_{j=1}^{m} w_j B'_j = \sum_{j=1}^{m} w_j a_j(x)B_j.$$

The scaling choice $B'_j = a_j(x)B_j$ gave an SAM. The SAM output $F$ was the centroid of $B(x)$ in (1): $F(x) = \text{Centroid} (B(x))$.

The shape of if-part sets $A_j$ affected how well the feedforward SAM $F$ approximates a function $f$ and how quickly an adaptive SAM $F$ approximated it when learning based on input-output samples from $f$ tuned the parameters of $A_j$ and the centroids $c_j$ and volumes $V_j$ of the then-part set $B_j$. The shape of the then-part sets $B_j$ did not affect the first-order behavior of a feedforward SAM $F$ beyond the effect of the volume $V_j$ and centroid $c_j$. This held because the SAM output computed a convex-weighted sum of the then-part centroids $c_j$ for each vector input $x$

$$F(x) = \frac{\sum_{j=1}^{m} w_j a_j(x)V_j c_j}{\sum_{j=1}^{m} w_j a_j(x)V_j}$$

$$= \sum_{j=1}^{m} p_j(x)c_j = E[Y | X = x]$$
for if-part joint set function $a_j : R^n \rightarrow [0,1]$ that defined the if-part set $A_j \subset R^n$, rule weights $w_j \geq 0$, $p_j(x) \geq 0$, and $\sum_{j=1}^{m} p_j(x) = 1$ for each $x \in R^2$. Fig. 4 (a) and (c) are examples of the SAM output $F$ based on the rules in Table II with two-dimensional joint factorable sinc if-parts. The convex coefficient

$$p_j(x) = \frac{w_j a_j(x) V_j}{\sum_{i=1}^{m} w_i a_i(x) V_i}$$

(4)

depended on then-part set $B_j$ only through its volume or area $V_j$ (and perhaps through its rule weight $w_j$).

The fuzzy output $F(x)$ in (2) equals the conditional expectation $E[Y|X = x]$ in (3) because the then-part set functions $b_j$ are non-negative and have finite volume (area) and because $F(x)$ is the centroid of $B(x)$. Appendix II shows that these two conditions give a well-defined conditional probability density function $p(y|x)$ in $F(x) = \int_{-\infty}^{\infty} yp(y|x)dy$ [5]. So the SAM output describes the first-order behavior of the fuzzy system and does not depend on the shape of the then-part sets $B_j$. But the shape of $B_j$ did affect the second-order uncertainty or conditional variance $V[Y|X = x]$ of the SAM output $F(x)$ [5]:

$$V[Y|X = x] = \sum_{j=1}^{m} p_j(x) \sigma_{B_j}^2 + \sum_{j=1}^{m} p_j(x) [c_j - F(x)]^2$$

(5)

where $\sigma_{B_j}^2$ is the then-part set variance

$$\sigma_{B_j}^2 = \int_{-\infty}^{\infty} (y - c_j)^2 p_{B_j}(y)dy$$

(6)

where $p_{B_j}(y) = b_j(y)/V_j$ is an integrable probability density function and $b_j : R \rightarrow [0,1]$ is the integrable scalar set function of then-part set $B_j$. Fig. 4 (b) and (d) show examples of the conditional variance $V[Y|X = x]$ based on the rules in Table II with Gaussian then-parts. The first term on the right side of (5) gave an input-weighted sum of the then-part set uncertainties. The second term measured the interpolation penalty that resulted from computing the SAM output $F(x)$ in (2) as the weighted sum of centroids. The second-order structure of a fuzzy system’s output depended crucially on the size and shape of the then-part sets $B_j$.

Learning tuned the volumes $V_j$ and centroids $c_j$ of the then-part sets $B_j$ in our adaptive function-approximation. The adaptive SAM fuzzy system learned from data with learning laws that updated the fuzzy rules so as to reduce the squared error of the function approximation. The two-dimensional joint factorable sinc if-part (see Fig. 3 a) had learning laws that updated the two-dimensional vector center $m_j$ and dispersion (or width) $d_j$ parameters:

$$m_j^k(t + 1) = m_j^k(t) + \mu \varepsilon_t \frac{p_j(x) a_j(x) a_j^k(x_k)(c_j - F(x))(a_j(x) - \cos(\frac{x - m_j^k(t)}{d_j^k(t)})))}{1}$$

$$d_j^k(t + 1) = d_j^k(t) + \mu \varepsilon_t \frac{p_j(x) a_j(x) a_j^k(x_k)(c_j - F(x))(a_j(x) - \cos(\frac{x - m_j^k(t)}{d_j^k(t)})))}{1}$$

(7)

(8)

where $\mu_t$ was a learning rate at iteration $t$ and $\varepsilon_t$ was the difference between the fuzzy output $F(x)$ and the approximand $f$. The Gaussian then-part parameters had learning laws that updated the scalar centroid $c_j$ and
volume $V$:

$$c_j(t+1) = c_j(t) + \mu_t \varepsilon_t(x)p_j(x)$$  \hspace{1cm} (9)

$$V_j(t+1) = V_j(t) - \mu_t \varepsilon_t(x) \frac{p_j(x)}{V_j} [c_j - F(x)]$$  \hspace{1cm} (10)

where the volume or area of a scalar Gaussian then-part set was a function of the standard deviation: $V = \sqrt{2\pi}\sigma$.

This related the dispersion or width of a then-part set to its volume and so related the volume parameter to the set variance that contributed to the conditional variance.
Fuzzy systems $F : \mathbb{R}^n \rightarrow \mathbb{R}^p$ suffer from rule explosion in high dimensions. A fuzzy system $F$ needs on the order of $k^{n+p-1}$ rules to cover the graph and thus to approximate a vector function $f : \mathbb{R}^n \rightarrow \mathbb{R}^p$. Optimal rules can help deal with the exponential rule explosion. Lone or local mean-squared optimal rule patches cover the extrema of the approximand $f$: They patch the bumps [5]. Better learning schemes move rule patches to or near extrema and then fill in between extrema with extra rule patches if the rule budget allows.

III. BULLET-ARMOR IMPACT EXPERIMENTS

A. Experimental Setup

Two experimental setups measured bullet-armor impacts’ depths and widths against a gelatin block backing in the Orange County Indoor Shooting Range. The first setup used donated commercial armor to produce the deformation data. It fixed the bullet caliber (9-mm) and varied the distance between the target and the handgun. The distances were 5, 7, and 10 yards. These three short distances produced three measurably different bullet impacts because air friction quickly slows a bullet after it leaves the gun barrel.

The second setup used generic armor that we made from eight layers of Kevlar fabric to produce the deformation data. It varied both the bullet caliber and the handguns’ barrel lengths and fixed the distance (5 feet) between the target and the handguns. The setup used .22, .38, .40, and .45 caliber bullets. The choice of bullets gave a monotonic increase in bullet diameter and weight. We also used .357 magnum bullets that had weight similar to .38 caliber bullets and momentum similar to .45 caliber bullets due to higher speeds. A .44 magnum bullet penetrated the 8-layer generic armor. So we had no deformation data for the .44 magnum bullets. Hexcel Schwebel donated the Style 713 Aramid fabric that had 1000 deniers of Kevlar 29 fibers in plain weave.

The gel-backed experiments used 10% gelatin blocks to simulate tissue that backed the armor. Fig. 5 shows a sample gelatin block. The gelatin blocks consisted of Type 250A Ordnance Gelatin from Kind & Knox Gelatin.
and water at a ratio of 1:9 by weight. We weighed the water-filled mold to find the 10% weight for the gelatin powder and weighed the mold while we added water to achieve the 1:9 ratio. Slowly stirring water into the gelatin powder reduced bubble formation. The mixture required refrigeration for 24 hours to prevent spoilage while the powder hydrated. This further reduced air bubble formation. A warm water bath of about 40°C melted the mixture and prevented denaturing the gelatin. A cold water bath set the melted gelatin in molds that measured about 40 cm long by 30 cm wide by 12 cm thick.

Refrigeration hardened the gelatin blocks in 48–72 hours and ensured a uniform temperature of about 4°C in the blocks. The refrigerator ran for 24 hours before use to ensure that its temperature equilibrated in a range above freezing and below 4°C. This kept a gelatin block’s temperature low while avoiding damage from ice crystals. We used ice chests to transport the gelatin blocks to and from the shooting range.

A BB shot calibrated each gelatin block before use by giving BB penetration at known temperatures. A model 760B BB Repeater Air Rifle from Crossman Air Guns can shoot a 5.5-grain BB at between 530–590 feet per second (ft/s) with 10 pumps. An outdoor thermometer measured the gelatin temperature before using the blocks.

We measured the bullet velocity at 12 feet or 3 yards and separate from the impact experiments. The Prochrono Plus optical chronometer from Competition Electronics complicated shot placement in the indoor shooting range. It required a separation of at least 10–15 feet from handguns to prevent the propellant smoke from interfering. The indoor lighting dictated a 12-ft separation from the handgun and a 58-inch height to keep the chronometer close to the halogen lamp. This was because the optical sensors required some minimal amount of light to detect a bullet’s shadow as the bullet passed. The lack of a pistol holder allowed marksmanship to further complicate shot placement in the experiments.

The shooting range provided reload ammunition with full-copper-jacketed bullets. The 9-mm bullets weighed 115 grains on average. The mean velocity was 1216 feet per second (ft/s) for bullets from the same ammunition box. Table III shows the mean bullet speeds for each combination of bullet caliber and barrel length. The .22 caliber bullets had one mean weight (40 grains) and two different speeds (977 ft/s and 1059 ft/s on average). We used the data in Table III (except the .44 magnum bullets) for the weight and momentum that trained the fuzzy system.

<table>
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<th>.38</th>
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TABLE III. Average bullet speeds for the available combinations of bullet weight (caliber) and barrel length.

B. Experimental Results

The gelatin-backed bullet-armor experiments found that a bullet’s impact depth correlated with it’s distance to target \( d \) with the first setup and found that depth correlated with bullet properties that included the mean weight \( m \), momentum \( p = mv \), and kinetic energy \( \frac{1}{2} mv^2 \) in decreasing order with the second setup. Depth correlation
with kinetic energy was much less than the correlations with weight and momentum. Impact depth also correlated
with combinations of weight, momentum, and kinetic energy. The impact width similarly correlated with the bullet
properties except distance. The impact depth and width correlated very little with a bullet’s mean speed \( v \). This
held because the .45 caliber bullet-armor impacts were deeper and wider than both the .22 and .38 caliber impacts
when the .22 caliber bullets were faster and the .38 caliber bullets were slower. Table IV shows the regression
statistics. The statistics were based on two subsets of the experimental data: Dataset (1) was complete and Dataset
(2) excluded the .357 magnum data. The key results held for both datasets. Dataset (2) had stronger correlations
(larger \( R^2 \)) than Dataset (1). Momentum correlated with deformation more than weight in (2).

Regression analysis fit the experimental data to a straight line. The simple linear model used functions of \( x \) and
had smaller \( p \)-values than quadratic and cubic models while the quadratic and cubic models had marginally larger
\( R^2 \) values. The \( R^2 \) measured the strength of the correlation between the dependent and the independent variables.
The first gel-backed experiments used a regression equation of the form

\[
y = \beta_0 + \beta_1 x
\]

(11)

where \( y \) was the deformation depth or width in millimeters (mm) and \( x \) was the distance between the target and
the handgun in yards.

The second gel-backed experiments used equations of the form in (11) for the correlations between deformation
and one of the bullet properties that included mean weight, speed, momentum, and kinetic energy. The correlation
between deformation and the combination of weight and momentum used a multiple regression equation of the
form

\[
y = \beta_0 + \beta_1 x_1 + \beta_2 x_2
\]

(12)

where \( y \) was the deformation depth or width in mm and \( x_1 \) and \( x_2 \) were the weight in grains, momentum in
kilograms meter per second (kg m/s), or kinetic energy in kilograms meter squared per second squared (kg m^2/s^2).

We applied the simple linear model in (11) to test whether bullet deformation correlated with each of a bullet’s
properties. The null hypothesis \( H_0 : \beta_1 = 0 \) stated that the slope \( \beta_1 \) of the regression line in (11) was zero and thus
the impact deformation’s depth and width (dependent variables) did not vary with a bullet’s weight, momentum,
kinetic energy, speed, or distance to target (independent variables). The \( p \)-value measures the credibility of \( H_0 \). A
statistical test rejects the null hypothesis \( H_0 \) at a significance level \( \alpha \) if the \( p \)-value is less than that significance
level: Reject \( H_0 \) if \( p \)-value < \( \alpha \). The test rejected the null hypothesis \( H_0 \) at the standard significance levels \( \alpha = 0.05 \)
and \( \alpha = 0.01 \) because \( p \)-value < 0.001.

We applied multiple regression for combinations of a bullet’s property such as weight and momentum in (12)
and tested the null hypothesis \( H_0 : \beta_i = 0 \) for \( i = 0, 1, 2 \) that all the parameters were statistically insignificant. This
tested whether the deformation’s depth and width varied with the combination of a bullet’s weight and momentum.
The test found that the regression coefficient for weight was statistically more significant than the coefficients for
momentum or kinetic energy (see Table IV). This helped guide our choice of weight and momentum as the inputs
of the fuzzy system.
The impact depth correlates with distance to target. The correlation was \( R^2 = 0.415 \) and linear regression gave \( y_1 = 29.328 - 1.203x \) with \( p \)-value < 0.001 where \( x \) was the distance in yards and \( y_1 \) was the deformation depth in mm. (b) The impact width correlates poorly with distance: \( R^2 = 0.004 \). The regression equation was \( y_2 = 58.162 + 0.1082x \). Note that the regression line is nearly horizontal.

Fig. 6 shows the depth and the width of the bullet-armor deformation after each impact in the first gel-backed experiments. The deformation depth decreased as target distance increased. Linear regression (see Table IV) gave \( R^2 = 0.415 \), regression equation \( y_1 = 29.328 - 1.203x \), and \( p \)-value < 0.001 where \( x \) was the distance in yards and \( y_1 \) was the deformation depth in mm. The width \( y_2 \) did not appear to vary with target distance for \( R^2 = 0.004 \), \( y_2 = 58.162 + 0.1082x \), and \( p \)-value < 0.001. These statistics suggested a non-zero slope but almost no correlation for \( R^2 = 0.004 \).

The correlations between deformation and target distance also held between deformation and momentum because bullet weight was approximately constant and velocity was a monotonic function of distance. The limited number of data points was due to the availability of the donated armor. The shots should overlap as little as possible and avoid the armor panel edges for consistency. Marksmanship further decreased the number of shots as distance increased.

Fig. 7 shows the complete deformation depth data in (a) and the complete width data in (b) of the bullet-armor deformation as a function of bullet weight and momentum in the second gel-backed experiments. The regression planes suggested that depth correlated with bullet momentum more than weight and that width correlated with weight more than momentum. Excluding the magnum data in (c) and (d) changed the depth regression plane: It suggested that deformation depth correlated with both a bullet’s weight and momentum. The correlation strengths were \( R^2 = 0.881 \) for the depth and \( R^2 = 0.865 \) for the width using the complete set of experimental data. The depth equation was \( y_1 = 5.550 + 0.304x_1 - 1.361x_2 \) where \( x_1 \) was bullet weight (grain) and \( x_2 \) was momentum (kg m/s). The width equation was \( y_2 = 84.846 + 0.425x_1 - 0.240x_2 \). The test statistics for depth correlation were \( F = 196.020 \) and \( p \)-value < 0.001. The test statistics for width correlation were \( F = 169.430 \) and \( p \)-value < 0.001. The test rejected the null hypothesis \( H_0 \) at the standard significance levels \( \alpha = 0.05 \) and \( \alpha = 0.01 \).
Fig. 7. The deformation data from the second gelatin-backed bullet-armor experiments. The figures show the complete set of experimental depth data in (a) and width data in (b) and their fit to regression planes as functions of weight ($X_1$) and momentum ($X_2$). (c) and (d) show a subset of the data that excluded the .357 magnum experiments. The regression planes consist of those points that satisfy the regression equations: $y_1 = 5.550 + 0.304x_1 - 1.361x_2$ for depth in (a), $y_2 = 84.846 + 0.425x_1 - 0.240x_2$ for width in (b), $y_1 = 4.913 + 0.207x_1 + 4.625x_2$ for depth in (c), and $y_2 = 82.646 + 0.093x_1 + 20.438x_2$ for width in (d).

The experiments used five bullet calibers (.22, .38, .40, and .45 caliber and .357 magnum, see Fig. 2) and two different speeds (such as on average 808 ft/s and 897 ft/s for the .45, see Table III) per caliber to produce 56 sets of input-output data. This gave the sparse sampling of the input space in Fig. 7.

IV. BASEBALL IMPACT EXPERIMENTS

A. Experimental Setup

Two putty-backed experimental setups compared the baseball impacts to bullet-armor impacts. The baseball setup measured baseball impacts on tubs of Oatey’s Plumber’s Putty at the Homerun Park in Anaheim, California. The outdoor batting cage used pitching machines that threw baseballs at 40, 50, 60, 70, 80, and 90 miles per hour (mph). The chronometer used sunlight and a baseball’s passing shadow to measure its speed. We placed the chronometer
approximately a half foot from the putty target. The putty-filled tubs were approximately three to five feet from the pitching machines. Each tub consisted of at least 15 pounds of the putty. Fig. 8 shows one sample deformation that a baseball made in putty.

![Fig. 8. A regulation baseball and a record of its impact. Pitching machines threw baseballs at tubs of Plumber's putty. A chronograph measured the speed of each baseball. The baseball speeds were approximately 40, 50, 60, 70, 80, and 90 miles per hour.](image)

The bullet-armor setup shot bullets at putty-backed soft body armor. The experiments shot .22, .38, .40, and .45 caliber bullets and .357 magnum and .44 magnum bullets. The armor was the Superfeatherlite model from Second Chance. The target was 5 feet from the handguns. Fig. 9 shows the depths of the putty deformations from baseballs next to the depths of the putty deformations from bullet-armor impacts. The deformations deepened as baseball speeds increased and as bullet caliber increased. The baseball impact experiments used regulation baseballs (Fig. 8) to produce at least 10 data points for each of the six different speeds. The putty deformed and recorded each impact. The baseballs’ weight was constant so plotting the deformation depth against momentum preserved the proportion of plotting against speed. The gelatin-backed bullet-armor experiments found that momentum correlated with deformation depth better than weight or speed alone. So we plotted depth against momentum in Fig. 9.

### B. Experimental Results

The baseball experiments compared bullet-armor impacts to baseball impacts in two ways: Deformation depth in putty and the slopes in the fitted regression lines. The first way compared how the two types of projectile deformations differed. Baseball impacts and bullet-armor impacts had similar depths in Oatey’s Plumber’s Putty (Table V). The similarity of impact depths suggested that handgun shots on soft body armor would feel like baseball impacts without armor. Fast-baseball impact depths were comparable to bullet-armor impact depths: Getting shot with a .22 caliber bullet when wearing soft body armor resembles getting hit on the chest with a 40-mph baseball. Getting shot with a .45 caliber bullet resembles getting hit with a 90-mph baseball.

Fast baseballs were similar to large caliber bullets (on armor) in denting putty (see Fig. 9) except for the .44 magnum. The .44 magnum dented the putty to a 40mm-depth or about twice the depth of the 90-mph baseballs. The 90-mph baseballs made 21.6-mm-deep dents in putty on average. This is similar to the depths of the dents from a .40 caliber (19 mm), a .45 caliber (22 mm), and a .357 magnum (21 mm) bullets. A .38 caliber bullet made
Second was that kinetic energy had a weaker correlation with deformation than either weight or momentum. A bullet’s weight monotonically increased with its caliber in Dataset (2) and could represent caliber by excluding the magnitude data. Dataset (2) had stronger correlations (larger $R^2$) than Dataset (1). Momentum correlated with deformation more than weight in (2).

---

**TABLE IV.** Linear regression statistics for the gelatin-backed bullet-armor impact experiments. The statistics were based on two subsets of the experimental data: Dataset (1) was complete and Dataset (2) excluded the .357 magnum data. The key results held for both datasets. First was that deformation depth and width correlated with a bullet’s weight, momentum, and kinetic energy while speed correlated little. Second was that kinetic energy had a weaker correlation with deformation than either weight or momentum.

Bullet Caliber | .22 | .38 | .40 | .357 | .45 | .44
---|---|---|---|---|---|---
Depth (mm) | 5 | 15 | 19 | 21 | 22 | 40
Baseball Speed (mph) | 40 | 70 | 80 | 90 | n/a | n/a
Depth (mm) | 6.5 | 13.6 | 17.0 | 21.6 | n/a | n/a
a 15-mm-deep dent in armor-clad putty. This depth is between the mean depth values of the 70-mph (13.6 mm) and the 80-mph (17 mm) baseballs. Table V summarizes these results.

The second way compared the correlation and regression slope of the two types of impacts. The experiments found that the mean depth of a baseball’s impact and the depth of a bullet’s armor-impact both correlated with projectile momentum (see Fig. 9). The baseball impacts had correlation $R^2 = 0.93$, regression equation $y = -6.155 + 5.188x$, and $p$-value $< 0.001$ where $x$ was a baseball’s momentum in kilograms meter per second (kg m/s) and $y$ was the putty deformation depth in millimeters (mm). The bullet-armor impacts had similar correlation $R^2 = 0.97$, regression equation $y = -2.12 + 4.76x$, and $p$-value $< 0.001$.

Fig. 9. Baseball and bullet impact depth in plumber’s putty versus momentum. The baseball impact depth correlated with baseball momentum $R^2 = 0.93$ and $p$-value $< 0.001$ for the null hypothesis: $\beta_1 = 0$. The solid line on the right is the regression line for the baseball impacts (blue dots) $y = -6.155 + 5.188x$ where $x$ is baseball momentum and $y$ is putty deformation depth. Only two data points fell outside of the 95% confidence bounds. Bullet-armor impact depths correlated with bullet momentum $R^2 = 0.97$. The green dashed line on the left is the regression line for the bullet-armor impacts (green circles) $y = 2.124 + 4.766x$ where $x$ is bullet momentum and $y$ is depth. The two regression lines have the similar slope $\beta_1 \approx 5$. A multiple regression analysis with dummy variables (Gujarati-Chow test) could not reject the null hypothesis $H_0 : \beta_1($baseball$) = \beta_1($armor$)$ for the test statistics $t = 0.855$ and $p$-value $= 0.396$. So the test retained the null hypothesis that the two types of impacts had the same slope.

The putty-impact regression lines had similar slopes $\beta_1 \approx 5$ for the baseball impacts ($\beta_1 = 5.188$) and the bullet-armor impacts ($\beta_1 = 4.766$). Fig. 9 suggests that the two lines are parallel: Same slope with different intercepts. But the multiple-regression-based Chow test [25], [26] gave only that the two regression lines differed for $F = 61.826$, degrees of freedom $(2, 62)$, and $p$-value $< 0.001$. So we applied a modified Chow test (Gujarati-Chow test [27]) to test whether the slope terms $\beta_1$ differed, the intercept terms $\beta_0$ differed or both.

The Gujarati-Chow test was a multiple regression analysis with a dummy variable approach [25], [27] that compared two groups of data. It used a binary dummy variable $D$ (see Appendix) to test separately whether the slope terms $\beta_1$ differed and whether the intercept terms $\beta_0$ differed. The intercept-term ($\beta_0$) test statistics were $t = -4.995$ and $p$-value $< 0.001$. The putty-impact regression lines had statistically distinct intercepts because we rejected the null hypothesis that the intercept terms were identical for the standard significance levels $\alpha = 0.05$ and $\alpha = 0.01$. This confirmed the Chow-test result that the two regression lines were distinct.
The slope-term ($\beta_1$) test statistics were $t = 0.855$ and $p$-value $= 0.396$. The two putty-impact regression lines had statistically indistinguishable slopes because we retained the null hypothesis that the slope terms were identical for the standard significance levels $\alpha = 0.05$ and $\alpha = 0.01$. This confirmed that the putty-backed baseball impacts and the bullet-armor impacts had the same slope $\beta_1$ in their regression lines as Fig. 9 suggested.

We note that the regulation baseball’s constant weight implied that the correlation between deformation and speed was the same as the correlation between deformation and momentum. This corroborated the results from the first gelatin-backed bullet-armor experiments.

V. FUZZY EXPERT AND GUNSHOT ARMOR BRUISE

A. Fuzzy System Setup

The adaptive SAM fuzzy system used the learning laws (7), (8), (9), (10) to learn from the gelatin-backed bullet-armor experimental data. We used the two-dimensional joint factorable sinc function (see Fig. 3) for the if-part fuzzy sets $A_j$. Sinc sets often converge faster and with greater accuracy than do triangles, Gaussian bell curves, Cauchy bell curves, and other familiar set shapes [24]. We used scalar Gaussian set functions for the one-dimensional then-part fuzzy sets $B_j$. This gave the set variance $\sigma_{B_j}^2$ from the then-part set volume $V_j$:

$$\sigma_{B_j}^2 = \frac{V_j^2}{2\pi}.$$  

A larger then-part rule volume $V_j$ produced more uncertainty in the $j^{th}$ rule and so should result in less weight. So we weighted each rule with the inverse of its squared volume [5]: $w_j = 1/V_j^2$. This gave the final form of the SAM output $F$ from (2):

$$F(x) = \frac{\sum_{j=1}^{m} \frac{a_j(x)}{V_j} c_j}{\sum_{j=1}^{m} \frac{a_j(x)}{V_j}} = \sum_{j=1}^{m} \frac{p_j(x)}{c_j}.$$  

A larger volume $V_j$ also gave a larger conditional variance.

We picked the fuzzy system’s initial rules according to the observed correlations in Table I: Same-weight bullets hit harder if they were faster. Same-speed bullets hit harder if they were heavier. But heavier and slower handgun bullets can hit harder than some lighter and faster ones. The adaptive SAM system used 49 rules according to Table II. The if-part set functions $a_j$ used center and width parameters to uniformly cover the input space. The then-part set functions $b_j$ used center parameters or centroids $c_j$ that gave an output according to Table II (a) and (b) and used volume or dispersion parameters in (c) that reflected the uncertainty of the rules. The fuzzy sets in Table II listed the initial rules that we created based on our experience with ballistics and soft body armor. The volume $V_j$ was a function of the then-part’s dispersion or width parameter. A rule was less certain if its if-part covered untested combinations of bullet weight and momentum so its then-part had a larger set variance. Fig. 4 shows the fuzzy system’s initial first-order output $F(x) = E[Y|X = x]$ and second-order uncertainty $V[Y|X = x]$.

A random resampling scheme selected half of the sparse experimental data as the bootstrapped training set and the remaining half as the test set. A bootstrap scheme [28] sampled the training data with replacement at random to generate 300 sets of input-output data to tune the fuzzy system.
B. Fuzzy System Results

We used two sets of the experimental data to train the fuzzy system: Dataset (1) was complete and Dataset (2) excluded the .357-magnum data. Tuning with Dataset (1) reduced the fuzzy system’s error function that summed the squared approximation error $f - F$ (SSE) for 3000 epochs of learning. Fig. 10 shows the tuned system output $F(x) = E[Y|X = x]$ and its conditional variance $V[Y|X = x]$. The tuning was effective because the depth subsystem had an initial error of SSE= 45, final error of SSE= 17, and low test error of SSE= 16 and the width subsystem had initial SSE= 126, final SSE= 44, and low test SSE= 46 (see Fig. 11 (a)). Tuning with Dataset (2) produced system output $F(x)$ and conditional variance $V[Y|X = x]$ similar to the initial state in Fig. 4. The fuzzy system’s final SSE resembled its initial SSE after tuning even though the low test SSE suggested that the tuning was effective: The depth subsystem had an initial error of SSE= 43, final error of SSE= 15, and test error of SSE= 14 and the width subsystem had initial SSE= 45, final SSE= 31, and test SSE= 25 (see Fig. 11 (b)).

We tested the robustness of the tuned fuzzy system by randomly pruning its rules (see Fig. 12). The fuzzy system proved robust both when the learning included the magnum data and when it did not. Pruning randomly removed a fraction of the rules over 100 trials. The depth and the width subsystems gave similar results. The maximal test error remained low (SSE < 100) for up to 20 percent using Dataset (1) and up to 10 percent using Dataset (2) of
Fig. 11. Learning curve. Tuning reduced the fuzzy system’s sum squared error (SSE) of the function approximation for two subsets of data: Dataset (a) was complete and Dataset (b) excluded the .357 magnum data. The depth subsystem in (a) reduced its SSE from 45 to 17 over 3000 epochs of learning and had a test SSE of 16. The width subsystem in (a) reduced its SSE from 126 to 44 and had test SSE = 46. The depth subsystem in (b) reduced its SSE from 43 to 15 and had test SSE = 14. The width subsystem in (b) reduced its SSE from 45 to 31 and had a test SSE of 25.

randomly removed rules. This was comparable to the approximation errors in data tuning. Both the mean and the median of the test error remained low for random pruning that removed up to 30 percent of the rules using Dataset (1) and 20 percent using Dataset (2).

VI. Conclusion

The adaptive SAM system modeled the bruise profile of a bullet impact based on bullet-armor experiments. The fuzzy system’s output conditional variance measured the inherent uncertainty in the rules. A baseball analogy gave further insight into armor gunshots based on baseball-impact experiments. The bullet-armor experiments found correlations between a bullet impact’s depth and width and its weight and momentum. These correlations were stronger than the correlation between a bullet’s impact and its kinetic energy. The baseball correlations corroborated the bullet-armor results.

The findings from the bruise analysis should help engineers design lighter and thinner armor that reduces a bullet’s bruising effect while improving the armor users mobility. These results would benefit from further testing with more than one weight per caliber and with higher-speed projectiles. Measurements of the deformed armor would also be more accurate if they came from highspeed photographs of the bullet-armor experiments rather than from the post-impact deformation. The post-impact measurements tended to give smaller values because the elastic gelatin blocks often reduced the armor deformation by springing back after bullet impact.

The bruising effect of flexible armor will change as armor technology evolves. New armor designs can reduce bullet-armor deformation. One such example is the use of colloidal shear-thickening fluid in fluid-impregnated...
Fig. 12. Rule pruning. The fuzzy system was robust against random pruning. The figure plots the system’s test error in log scale versus the percent of pruned depth rules in (a). Similar results hold for random pruning of width rules in (b) and for pruning rules that trained without magnum data in (c) and (d). The vertical bars show the maximal and minimal range of 100 trials. The solid polygonal line interpolates the median of those trials. The dashed line interpolates the mean. The maximal error remained below 100 sum squared error (SSE) for up to 20% of randomly pruned rules. Both the mean and median error remained low for rule losses of up to 35%.

Kevlar fabric. The shear-thickening fluid enhances the penetration resistance of flexible armor [30]. The United States Army is developing armor that uses nanomaterials and nanodevices [31] such as molecular chain mail that uses a mixture of nanoscopic materials [32]. These new materials change how armor deforms on bullet impact and so will change the bruising effect. Test data from the new armor designs can tune the fuzzy rules and so adapt the fuzzy function approximator.

APPENDIX I

THE STANDARD ADDITIVE MODEL (SAM) THEOREM

This Appendix derives the basic ratio structure (1) of a standard additive model fuzzy system [5], [24].
**SAM Theorem:** Suppose the fuzzy system $F : R^n \to R^p$ is a standard additive model: $F(x) = \text{Centroid}(B(x)) = \text{Centroid}(\sum_{j=1}^{m} w_j a_j(x) B_j)$ for if-part joint set function $a_j : R^n \to [0,1]$, rule weights $w_j \geq 0$, and then-part fuzzy set $B_j \subset R^p$. Then $F(x)$ is a convex sum of the $m$ then-part set centroids

$$F(x) = \sum_{j=1}^{m} \frac{w_j a_j(x) V_j c_j}{\sum_{j=1}^{m} w_j a_j(x) V_j} = \sum_{j=1}^{m} p_j(x)c_j$$

(14)

The convex coefficients or discrete probability weights $p_1(x), \ldots, p_m(x)$ depend on the input $x$ through

$$p_j(x) = \frac{w_j a_j(x) V_j}{\sum_{i=1}^{m} w_i a_i(x) V_i}$$

(15)

$V_j$ is the finite positive volume (or area if $p = 1$) and $c_j$ is the centroid of then-part set $B_j$

$$V_j = \int_{R^p} b_j(y_1, \ldots, y_p) \, dy_1 \ldots dy_p > 0$$

(16)

$$c_j = \frac{\int_{R^p} b_j(y_1, \ldots, y_p) \, dy_1 \ldots dy_p}{\int_{R^p} b_j(y_1, \ldots, y_p) \, dy_1 \ldots dy_p}.$$ 

(17)

**APPENDIX II**

**ADDITIVE STATISTICS Theorem:** [5]

Suppose $F : R^n \to R$ is an additive fuzzy system with scalar output such that $F(x) = \text{Centroid}(B(x))$ and $B(x) = \sum_{j=1}^{m} w_j B_j'(x)$. Then

$$F(x) = E[Y|X = x] = \sum_{j=1}^{m} p_j(x)E_{B_j'}[Y|X = x]$$

(18)

$$V[Y|X = x] = \sum_{j=1}^{m} p_j(x)V[Y|X = x, B_j']$$

(19)

the convex coefficients $p_j(x)$ are weighted volume ratios of the “fired” sets $B_j'$:

$$p_j(x) = \frac{w_j V_j'(x)}{\sum_{k=1}^{m} w_k V_k'(x)}$$

(20)

where

$$V_j'(x) = \int b_j'(x, y) \, dy$$

(21)

**Proof:**

The proof that centroidal fuzzy systems are conditional means follows from the ratio structure of the centroid and the boundedness and non-negativity of the set values $b(x, y) \geq 0$ of the combined set $B$ of “fired” then-part
sets \( B_j \). Each input \( x \) gives its own \( B(x) \) and thus its own output \( F(x) \):

\[
F(x) = \text{Centroid}(B(x)) \quad (22)
\]

\[
= \frac{\int_{\mathbb{R}^p} yb(x, y)dy}{\int_{\mathbb{R}^p} b(x, y)dy} \quad (23)
\]

\[
= \int_{\mathbb{R}^p} yp(y|x)dy \quad (24)
\]

\[
= E[Y|X = x] \quad (25)
\]

for each \( x \in \mathbb{R}^n \). This holds because the ratio of the joint distribution to the marginal defines a proper conditional probability density

\[
p(y|x) = \frac{b(x, y)}{\int_{\mathbb{R}^p} b(x, y)dy} \quad (26)
\]

even though \( b(x,y) > 1 \) may hold.

Then we continue the same chain of equalities to show that \( F(x) \) is a convex sum of local conditional mean realizations or centroids:

\[
F(x) = \frac{\sum_{j=1}^{m} w_j \int_{-\infty}^{\infty} yb'_j(x, y)dy}{\sum_{j=1}^{m} w_j \int_{-\infty}^{\infty} yb'_j(x, y)dy} \quad (27)
\]

\[
= \frac{\sum_{j=1}^{m} w_j V'_j \int_{-\infty}^{\infty} yb'_j(y|x)dy}{\sum_{j=1}^{m} w_j V'_j} \quad (28)
\]

\[
= \sum_{j=1}^{m} p_j(x)E_{B'_j}[Y|X = x] \quad (29)
\]

\[
= \sum_{j=1}^{m} p_j(x)c'_j \quad (30)
\]

The variance (covariance) result follows in the same way:

\[
V = [Y|X = x] = \int_{-\infty}^{\infty} (y - E[Y|X = x])^2b(x, y)dy \int_{-\infty}^{\infty} b(x, y)dy \quad (31)
\]

\[
= \sum_{j=1}^{m} p_j(x) \int_{-\infty}^{\infty} (y - E[Y|X = x])^2p_{B'_j}(y|x)dy \quad (32)
\]
\[ = \sum_{j=1}^{m} p_j(x) V[Y|X = x, B'_j] \] (33)

The SAM structure \( B'_j(x) = a_j(x)B_j \) simplifies the conditional variance \( V_{B'_j}[Y|X = x] \) because the conditional probability density \( p_{B'_j}(y|x) \) does not depend on the input \( x \) for \( a_j(x) > 0 \):

\[
p_{B'_j}(y|x) = \frac{a_j(x)b_j(y)}{a_j(x) \int_{R^p} b_j(y)dy} = \frac{b_j(y)}{\int_{R^p} b_j(y)dy} = \frac{b_j(y)}{V_j} = p_{B_j}(y) \] (34)

Then we use these facts to simplify the first-order and second-order conditional statistics of a SAM system. It gives the SAM conditional variance as a convex sum of the local then-part variances plus a global dispersion term.

**Corollary (SAM Statistics)**

The scalar-valued SAM system

\[ F(x) = \text{Centroid} \left( \sum_{j=1}^{m} w_j a_j(x) B_j \right) \] (36)

has conditional mean and variance

\[ E[Y|X = x] = F(x) = \sum_{j=1}^{m} p_j(x)c_j \] (37)

and

\[ V[Y|X = x] = \sum_{j=1}^{m} p_j(x) \sigma_{B_j}^2 + \sum_{j=1}^{m} p_j(x)(c_j - F(x))^2 \] (38)

with convex coefficients

\[ p_j(x) = \frac{w_j a_j(x) V_j}{\sum_{k=1}^{m} w_k a_k(x) V_j} \] (39)

The second term in (38) acts as a penalty term. It is positive if and only if some \( j^{th} \) rule fires \( a_j(x) > 0 \) and thus \( p_j(x) > 0 \) and \( c_j \neq F(x) \). So it is positive iff the fuzzy system \( F \) interpolates to reach the output \( F(x) \). Each rule comes with its own output uncertainty \( \sigma_{B_j}^2 \). Interpolated outputs have more variance than just the sum of these weighted rule variances. Combining two uncertain structures gives a still less certain third structure.

**APPENDIX III**

**LEARNING LAWS FOR SCALAR AND JOINT FACTORABLE IF-PART SETS**

This appendix derives the learning laws for scalar and joint factorable if-part sets [24]. Supervised gradient descent can tune all the parameters in the SAM (2). A gradient descent learning law for a SAM parameter \( \xi \) has the form

\[ \xi(t + 1) = \xi(t) - \mu_t \frac{\partial E}{\partial \xi} \] (40)

where \( \mu_t \) is a learning rate at iteration \( t \). We seek to minimize the squared error \( E(x) = \frac{1}{2}(f(x) - F(x))^2 \) of the function approximation. The vector function \( f : R^n \to R^p \) has components \( f(x) = (f_1(x), \ldots, f_p(x))^T \) and so
The learning laws for the if-part set parameters follow in like manner for both scalar and joint sets as we show. 

Expansions of the error function with respect to the parameter of the if-part set function $\xi_j^k$, with respect to the then-part set centroid $c_j = (c_j^1, \ldots, c_j^p)^T$, and with respect to the then-part set volume $V_j$

The partial derivative of the joint factorable sinc set function is

Then the chain rule gives the gradient of the error function with respect to the if-part set parameter $\xi_j^k$, with respect to the then-part set centroid $c_j = (c_j^1, \ldots, c_j^p)^T$, and with respect to the then-part set volume $V_j$

The SAM ratios (2) with inverse-volume-ratio weights $w_j = 1/V_j^2$ give

Then the learning laws for the then-part set centroids $c_j$ and volumes $V_j$ have the final form

The learning laws for the if-part set parameters follow in like manner for both scalar and joint sets as we show below. Chain rule gives for scalar sinc set function

A joint factorable set function $a_j(x) = a_j^1(x) \ldots a_j^p(x)$ leads to a new form for the error gradient. The gradient with respect to the parameter of the $j^{th}$ set function $a_j$ has the form

The partial derivative of the joint factorable sinc set function is

Combining (40), (41), (47), (48), and (49) gives the if-part learning laws:

$$m_j^k(t + 1) = m_j^k(t) + \mu_t \varepsilon_t \frac{p_j(x)}{a_j(x)} \frac{a_j(x)}{a_j^k(x_k)} (c_j - F(x)) (a_j(x) - \cos(\frac{x - m_j^k(t)}{d_j^k(t)})) \frac{1}{x - m_j^k(t)}$$

$$d_j^k(t + 1) = d_j^k(t) + \mu_t \varepsilon_t \frac{p_j(x)}{a_j(x)} \frac{a_j(x)}{a_j^k(x_k)} (c_j - F(x)) (a_j(x) - \cos(\frac{x - m_j^k(t)}{d_j^k(t)})) \frac{1}{d_j^k(t)}$$

where $a_j(x) = \prod_{i \neq k} a_j^i(x_i)$
APPENDIX IV
HYPOTHESIS TEST FOR $\beta_1 = c$ FOR SIMPLE LINEAR REGRESSION

This section reviews the hypothesis test whether the slope term of a simple linear regression equation has a certain value $c$: $H_0 : \beta_1 = c$ versus $H_a : \beta_1 \neq c$ [29]. The simple linear regression model for $n$ observations has the form

$$y = \beta_0 + \beta_1 x + \epsilon$$  \hspace{1cm} (52)

where the independent variable $x$, dependent variable $y$, and error term $\epsilon$ are $n \times 1$ vector representations of the $n$ observations. The prediction equation

$$y = \hat{\beta}_0 + \hat{\beta}_1 x$$  \hspace{1cm} (53)

uses the least squares estimators

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n} (x_i - \bar{x})^2} \text{ and } \hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}.\hspace{1cm} (54)$$

The test statistic is

$$t = \frac{\hat{\beta}_1 - c}{\sqrt{s^2 \sum_{i=1}^{n} (x_i - \bar{x})^2 / (n - 2)}}$$  \hspace{1cm} (55)

where $s^2 = \frac{SSE}{n - 2}$ and $SSE = \sum_{i=1}^{n} (y_i - \hat{y}_i)^2$.

The test rejects the null hypothesis $H_0$ if the test statistic $|t| \geq t_{\alpha/2,n-2}$. We note that $t$ has a noncentral distribution $t(n - 2, \delta)$ with noncentrality parameter $\delta$ that has the form

$$\delta = \frac{E(\hat{\beta}_1)}{\sqrt{\text{var}(\hat{\beta}_1)}} = \frac{\beta_1}{\sigma / \sum_{i=1}^{n} (x_i - \bar{x})^2}.\hspace{1cm} (56)$$

The hypothesis test for $H_0 : \beta_1 = 0$ versus $H_a : \beta_1 \neq 0$ can use (55)–(56) for $c = 0$ and $\delta = 0$.

APPENDIX V
MULTIPLE REGRESSION ANALYSIS AND THE DUMMY VARIABLE APPROACH (GUJARATI-CHOW TEST)

Multiple regression [25] can compare two groups with unequal numbers of samples. The dummy variable approach [27] can compare the two groups’ linear regression coefficients to see whether they have different intercepts, different slopes, or both. This modifies the Chow test [26] that tests whether both the intercepts and slopes differ between the two groups.

The Gujarati-Chow test is a multiple regression analysis that uses a dummy variable $D$ [27], [25] to create interaction terms. The test concatenates the data from the two groups and uses the binary dummy variable $D$ to
determine whether a data sample belongs to Group A or Group B:

\[
D = \begin{cases} 
1 & \text{if the observation lies in the Group A} \\
& \text{(of } N_1 \text{ observations).} \\
0 & \text{if the observation lies in the Group B} \\
& \text{(of } N_2 \text{ observations).}
\end{cases}
\]

The dummy-variable regression equation has the form

\[
y = a_0 + a_1 D + a_2 x + a_3 (Dx)
\]

where \(x\) is the independent variable and \(y\) is the dependent variable for both groups of data. The dummy variable \(D\) is in both an additive and a multiplicative forms. The constants \(a_1\) and \(a_3\) are the differential intercept and differential slope coefficients. The Gujarati-Chow test performs a multiple linear regression and can test two null hypothesis: \(H_0^{a_1} : a_1 = 0\) and \(H_0^{a_3} : a_3 = 0\).

The intercept value of Group A’s data is \(a_1 + a_0\) if \(a_1\) is statistically significant then \(a_0\) is the intercept value of Group B’s data in this case. Else \(a_0\) estimates the intercept term of both groups of data if \(a_1\) is statistically insignificant (large \(p\)-value).

The slope of Group A’s data is \(a_3 + a_2\) if \(a_3\) is statistically significant then \(a_2\) is the slope of Group B’s data in this case. Else \(a_2\) estimates the slope term of both groups of data if \(a_3\) is statistically insignificant (large \(p\)-value). The additive and multiplicative dummies give whether two linear regressions differ either in the intercept or the slope or both.

We want to test the null hypothesis \(H_0\) that the two regression slopes are identical for the bullet-armor impact data (Group A) and the baseball impact data (Group B). This tests the hypothesis \(H_0 : a_3 = 0\) versus \(H_a : a_3 \neq 0\). The hypothesis test retains the null hypothesis for a high \(p\)-value: \(a_3\) is not statistically significant and \(a_2\) estimates the slope term of both the bullet-armor data and the baseball data.

REFERENCES


