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Operations on type-2 fuzzy sets

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Abstract

In this paper, we discuss set operations on type-2 fuzzy sets (including join and meet under minimum/product t-norm), algebraic operations, properties of membership grades of type-2 sets, and type-2 relations and their compositions. All this is needed to implement a type-2 fuzzy logic system (FLS).

Keywords: Fuzzy relations; Fuzzy numbers; Type-2 fuzzy sets

1. Introduction

The concept of a *type-2 fuzzy set* was introduced by Zadeh [30] as an extension of the concept of an ordinary fuzzy set (henceforth called a *type-1 fuzzy set*). Such sets are fuzzy sets whose membership grades themselves are type-1 fuzzy sets; they are very useful in circumstances where it is difficult to determine an exact membership function for a fuzzy set; hence, they are useful for incorporating linguistic uncertainties, e.g., the words that are used in linguistic knowledge can mean different things to different people [21].

A fuzzy relation of higher type (e.g., type-2) has been regarded as one way to increase the fuzziness of a relation, and, according to Hisdal, "increased fuzziness in a description means increased ability to handle inexact information in a logically correct manner [5]". According to John, "Type-2 fuzzy sets allow for linguistic grades of membership, thus assisting in knowledge representation, and they also offer improvement on inferencing with type-1 sets [6]".

Type-2 sets can be used to convey the uncertainties in membership functions of type-1 sets, due to the dependence of the membership functions on available linguistic and numerical information. Linguistic information (e.g., rules from experts), in general, does not give any information about the shapes of the membership functions. When membership functions are determined or tuned based on numerical data, the uncertainty in the numerical data, e.g., noise, translates into uncertainty in the membership functions. In all such cases, information about the linguistic/numerical uncertainty can be incorporated in the type-2 framework. In [18], Liang and Mendel demonstrated (using real data) that a type-2 fuzzy set, a Gaussian with fixed mean and

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uncertain standard deviation (*std*), is more appropriate to model the frame sizes of I/P/B frames in MPEG VBR video traffic than is a type-1 Gaussian membership function.

In this paper, A is a type-1 fuzzy set, and the membership grade (a synonym for the degree of membership) of $x \in X$ in A is $\mu_A(x)$, which is a crisp number in $[0,1]$. A type-2 fuzzy set in X is $\tilde{A}$, and the membership grade of $x \in X$ in $\tilde{A}$ is $\mu_{\tilde{A}}(x)$, which is a type-1 fuzzy set in $[0,1]$. The elements of the domain of $\mu_{\tilde{A}}(x)$ are called *primary memberships* of x in $\tilde{A}$ and the memberships of the primary memberships in $\mu_{\tilde{A}}(x)$ are called *secondary memberships* of x in $\tilde{A}$. The latter define the possibilities for the primary membership, e.g., $\mu_{\tilde{A}}(x)$ can be represented, for each $x \in X$ as

$$\mu_{\tilde{A}}(x) = f_x(u_1)/u_1 + f_x(u_2)/u_2 + \cdots + f_x(u_m)/u_m = \sum_i f_x(u_i)/u_i, \quad u_i \in J_x \subseteq [0,1].$$

When the secondary MFs are interval sets, we call them “interval type-2 fuzzy sets”. The operations of interval type-2 fuzzy sets are studied in [19,26].

Example 1. Fig. 1(a) shows a type-2 set in which the primary membership grade for each point in the set is a Gaussian type-1 set. The means of all these Gaussian type-1 sets, follow a Gaussian and the standard deviations of these Gaussians decrease as the mean decreases. Uncertainty in the primary membership grades of the type-2 MF consists of a shaded region (the union of all primary membership grades), that we call the *footprint of uncertainty* of the type-2 MF (e.g., see Fig. 1(a)). The *principal* membership function, i.e., the set of primary memberships having secondary membership equal to 1, is indicated with a thick line. This principal membership function is a Gaussian because of the way the set is constructed. Intensity of the shading is approximately proportional to secondary membership grades. Darker areas indicate higher secondary memberships. The flat portion from about 2.5 to 3.5 appears because primary memberships cannot be greater than 1 (since primary memberships, themselves, are possible membership values, they have to be in $[0,1]$) and so the Gaussians have to be “clipped”. The domain of the membership grade corresponding to $x = 4$ is also shown. Fig. 1(b) shows the secondary membership function for $x = 4$. Many other examples of type-2 sets can be found in [11].

The concept of a principal membership function also illustrates the fact that a type-1 fuzzy set can be thought of as a special case of a type-2 fuzzy set. We can think of a type-1 fuzzy set as a type-2 fuzzy set whose membership grades are type-1 fuzzy singletons, having secondary membership equal to unity for only one primary membership and zero for all others, i.e., we can think of the principal membership function of a type-2 set as an *embedded type-2 set*. A type-2 fuzzy set can also be thought of as a fuzzy valued function, which assigns to every $x \in X$ a type-1 fuzzy membership grade. In this sense, we will call X the *domain* of the type-2 fuzzy set.

Mizumoto and Tanaka [23] have studied the set-theoretic operations of type-2 sets, properties of membership grades of such sets, and, have examined the operations of algebraic product and algebraic sum for them [24]. Nieminen [25] has provided more details about algebraic structure of type-2 sets. Dubois and Prade [4] and Kaufmann and Gupta [14] have discussed the *join* and *meet* operations between fuzzy numbers under minimum t-norm. Dubois and Prade [2–4] have also discussed fuzzy valued logic and have given a formula for the composition of type-2 relations as an extension of the type-1 sup-star composition, but this formula is only for minimum t-norm. Karnik and Mendel [11,13] have provided a general formula for the extended sup-star composition of type-2 relations.

Type-2 fuzzy sets have been used in decision making [1,29], solving fuzzy relation equations [27], survey processing [12], pre-processing of data [8], and modeling uncertain channel states [22,13]. In engineering applications of fuzzy logic, minimum and product t-norm are usually used, so we focus on product and minimum t-norm in this paper, and leave the extensions of our results to other t-norm to others. Recently, a type-2 FLS [11,13] has been developed which uses either t-norm; however, in order to implement such a

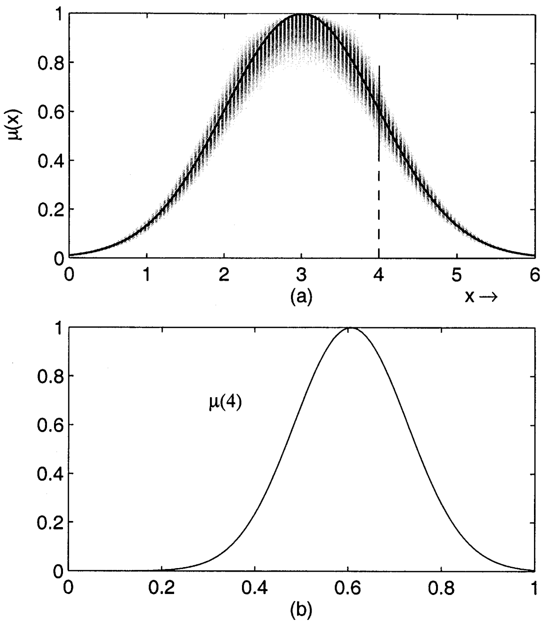

Fig. 1. (a) Pictorial representation of a Gaussian type-2 set. The standard deviations of the secondary Gaussians decrease by design, as x moves away from 3. The secondary memberships in this type-1 fuzzy set are shown in (b), and are also Gaussian.

type-2 FLS, one needs algorithms for set-theoretical operations on type-2 sets (which are given in Section 2), algebraic operations for type-2 sets (which are given in Section 3), an understanding of set properties of membership grades (which is provided in Section 4) and algorithms for type-2 fuzzy relations and their compositions (which are given in Section 5). So, the main purpose of this paper is to provide the *foundation* for type-2 FLSs.

2. Set-theoretic operations on type-2 sets

Consider two fuzzy sets of type-2, $\tilde{A}$ and $\tilde{B}$, in a universe X. Let $\mu_{\tilde{A}}(x)$ and $\mu_{\tilde{B}}(x)$ be the membership grades (fuzzy sets in $J_x \subseteq [0,1]$) of these two sets, represented, for each x, as $\mu_{\tilde{A}}(x) = \int_u f_x(u)/u$ and $\mu_{\tilde{B}}(x) = \int_w g_x(w)/w$, respectively, where $u, w \in J_x$ indicate the primary memberships of x and $f_x(u)$, $g_x(w) \in [0,1]$ indicate the secondary memberships (grades) of x. Using Zadeh's Extension Principle [30,4,11], the membership grades for union, intersection and complement of type-2 fuzzy sets $\tilde{A}$ and $\tilde{B}$ have been defined as follows [23]:

$$\text{Union:} \quad \tilde{A} \cup \tilde{B} \Leftrightarrow \mu_{\tilde{A}\cup\tilde{B}}(x) = \mu_{\tilde{A}}(x) \sqcup \mu_{\tilde{B}}(x) = \int_u \int_w (f_x(u) \star g_x(w))/(u \vee w), \tag{1}$$

$$\text{Intersection:} \quad \tilde{A} \cap \tilde{B} \Leftrightarrow \mu_{\tilde{A}\cap\tilde{B}}(x) = \mu_{\tilde{A}}(x) \sqcap \mu_{\tilde{B}}(x) = \int_u \int_w (f_x(u) \star g_x(w))/(u \star w), \tag{2}$$

$$\text{Complement:} \; \bar{\tilde{A}} \Leftrightarrow \mu_{\bar{\tilde{A}}}(x) = \neg\mu_{\tilde{A}}(x) = \int_u f_x(u)/(1-u), \tag{3}$$

where $\vee$ represents the *max* t-conorm and $\star$ represents a t-norm. The integrals indicate logical union. In the sequel, we adhere to these definitions, and, as in [23], we refer to the operations $\sqcup$, $\sqcap$ and $\neg$ as *join*, *meet* and *negation*, respectively. If $\mu_A(x)$ and $\mu_B(x)$ have discrete domains, the integrals in (1)–(3) are replaced by summations.

In (1), if more than one calculation of u and w gives the same point $u \vee w$, then in the union we keep the one with the largest membership grade. We do the same in (2). Suppose, for example, $u_1 \vee w_1 = \theta^*$ and $u_2 \vee w_2 = \theta^*$. Then within the computation of (1) we would have

$$f_x(u_1) \star g_x(w_1)/\theta^* + f_x(u_2) \star g_x(w_2)/\theta^* \tag{4}$$

where $+$ denotes union. Combining these two terms for the common θ^* is a type-1 computation in which we can use a t-conorm. We choose to use the maximum as suggested by Zadeh [30] and Mizumoto and Tanaka [23].

Using the definitions in (1)–(3), we examine the operations of *join*, *meet* and *negation* in more detail, both under minimum ($\wedge$) and product t-norms (which are the most widely used t-norms in engineering applications of fuzzy sets and logic). We consider only *convex*, *normal* membership grades. Our goal is to obtain algorithms that let us compute the *join*, *meet* and *negation*. These algorithms will be needed later to compute type-2 relations. Since these operations are performed on membership grades of type-2 sets, which themselves are type-1 sets, the results discussed next are in terms of type-1 sets. The type-1 sets, F_i, in Sections 2.1–2.4, can be thought of as membership grades of some type-2 sets, $\tilde{A}_i$, for some arbitrary input x_0. For example, $\mu(4)$ in Fig. 1(b) is a type-1 set associated with $x = 4$.

2.1. Join and meet under minimum t-norm

Our major result for join and meet under minimum t-norm is given in:

Theorem 1. *Suppose that we have n convex, normal, type-1 real fuzzy sets $F_1, \ldots, F_n$ characterized by membership functions $f_1, \ldots, f_n$, respectively. Let $v_1, v_2, \ldots, v_n$ be real numbers such that $v_1 \leqslant v_2 \leqslant \cdots \leqslant v_n$ and $f_1(v_1) = f_2(v_2) = \cdots = f_n(v_n) = 1$. Then, using max t-conorm and min t-norm,*

$$\mu_{\sqcup_{i=1}^n F_i}(\theta) = \begin{cases} \bigwedge_{i=1}^n f_i(\theta), & \theta < v_1, \\ \bigwedge_{i=k+1}^n f_i(\theta), & v_k \leqslant \theta < v_{k+1},\ 1 \leqslant k \leqslant n-1, \\ \bigvee_{i=1}^n f_i(\theta), & \theta \geqslant v_n, \end{cases} \tag{5}$$

and

$$\mu_{\sqcup_{i=1}^n F_i}(\theta) = \begin{cases} \bigvee_{i=1}^n f_i(\theta), & \theta < v_1, \\ \bigwedge_{i=1}^k f_i(\theta), & v_k \leqslant \theta < v_{k+1},\ 1 \leqslant k \leqslant n-1, \\ \bigwedge_{i=1}^n f_i(\theta), & \theta \geqslant v_n. \end{cases} \tag{6}$$

See Appendix A for the proof of Theorem 1. Fig. 2 shows an example of an application of Theorem 1 for the case $n = 4$. Observe that Eqs. (5) and (6) are very easy to program.

Dubois and Prade present the result in Theorem 1 just for the case $n = 2$, in the context of fuzzification of max and min operations. Though their method of proof is very similar to ours, they prove the result for a special case, where f_1 and f_2 have at most three points of intersection and one needs to keep track of the points of intersection of f_1 and f_2 to use their theorem. We reprove this theorem in a general setting in Appendix A. We believe that our statements of $\mu_{F_1 \sqcup F_2}(\theta)$ and $\mu_{F_1 \sqcap F_2}(\theta)$ are more amenable to computer

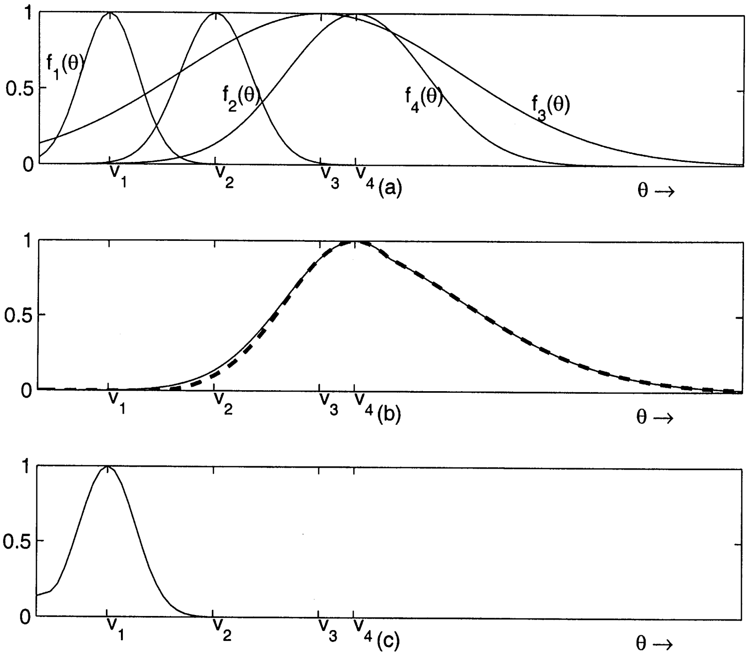

Fig. 2. An illustration of Theorems 1 and 2: *Join* and *meet* operations between Gaussians: (a) Participating Gaussians; (b) *join* under minimum t-norm (solid line) and product t-norm (thick dashed line); and (c) *meet* under minimum t-norm.

implementations than those of Dubois and Prade. Generalization to the case $n>2$ is also difficult using Dubois and Prade's result, but is provided by us in Theorem 1.

As a consequence of Theorem 1, we have the following interesting result:

Corollary 1. *If we have n convex, normal, type-*1 *fuzzy sets* $F_1,\ldots,F_n$ *characterized by membership functions* $f_1,\ldots,f_n$, *respectively, such that* $f_i(\theta)=f_1(\theta-k_i)$, *and* $0=k_1\leqslant k_2\leqslant\cdots\leqslant k_n$; *then, using max t-conorm and min t-norm,* $\bigsqcup_{i=1}^{n} F_i=F_n$ *and* $\bigsqcap_{i=1}^{n} F_i=F_1$.

The proof follows by a direct application of Theorem 1 to the n type-1 sets described in the statement of the corollary.

Fig. 3 shows an example of union and intersection of type-2 sets, using the results in this section.

2.2. Join under product t-norm

For type-1 fuzzy sets satisfying the requirements of Theorem 1, *join* operation under product t-norm gives exactly the same result as (5) with product replacing the minimum.

Theorem 2. *Suppose that we have n convex, normal, type-*1 *real fuzzy sets* $F_1,\ldots,F_n$ *characterized by membership functions* $f_1,\ldots,f_n$, *respectively. Let* $v_1,v_2,\ldots,v_n$ *be real numbers such that* $v_1\leqslant v_2\leqslant\cdots\leqslant v_n$ *and* $f_1(v_1)=f_2(v_2)=\cdots=f_n(v_n)=1$. *Then, the membership function of* $\bigsqcup_{i=1}^{n} F_i$ *using max t-conorm and*

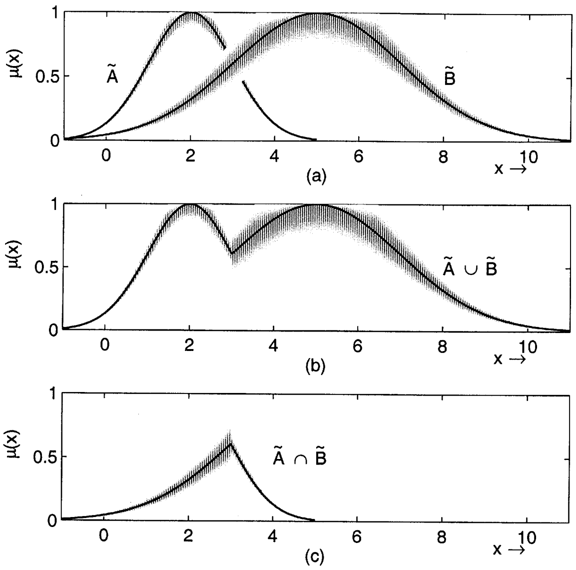

Fig. 3. Union and Intersection of two type-2 fuzzy sets using the pictorial representation introduced in Fig. 1, and, for maximum t-conorm and minimum t-norm: (a) Participating sets; (b) union; and (c) intersection.

product t-norm, can be expressed as

$$\mu_{\sqcup_{i=1}^{n} F_i}(\theta) = \begin{cases} \prod_{i=1}^{n} f_i(\theta), & \theta < v_1, \\ \prod_{i=k+1}^{n} f_i(\theta), & v_k \leqslant \theta < v_{k+1}, \ 1 \leqslant k \leqslant n-1, \\ \bigvee_{i=1}^{n} f_i(\theta), & \theta \geqslant v_n. \end{cases} \tag{7}$$

The proof proceeds in the same way as the proof of part (I) in Theorem 1 (Appendix A), with the minimum operation replaced by product. See [11] for details. Fig. 2 shows an example of an application of Theorem 2 for the case $n = 4$.

2.3. Meet under product t-norm

The *meet* operation under the product t-norm between two convex normal type-1 sets F_1 and F_2 can be represented as

$$F_1 \sqcap F_2 = \int_v \int_w [f_1(v) f_2(w)]/(vw). \tag{8}$$

Ref. [14] gives a similar result while discussing multiplication of fuzzy numbers (see Section 3 for algebraic operations on fuzzy sets).

If θ is an element of $F_1 \sqcap F_2$, then the membership grade of θ can be found by finding all the pairs $\{v, w\}$ such that $v \in F_1$, $w \in F_2$ and $vw = \theta$; multiplying the membership grades of v and w in each pair; and then finding the maximum of these products of membership grades. The possible admissible $\{v, w\}$ pairs whose product is θ are $\{v, \theta/v\}$ ($v \in \Re$, $v \neq 0$) for $\theta \neq 0$ and $\{v, 0\}$ or $\{0, w\}$, where $v, w \in \Re$ for $\theta = 0$, i.e.,

$$
\begin{aligned}
\mu_{F_1 \sqcap F_2}(\theta) &= \sup_{v \in \Re, v \neq 0} f_1(v) f_2\left(\frac{\theta}{v}\right); \quad \theta \in \Re,\ \theta \neq 0, \\
\mu_{F_1 \sqcap F_2}(0) &= \left[\sup_{v \in \Re} f_1(v) f_2(0)\right] \vee \left[\sup_{w \in \Re} f_1(0) f_2(w)\right] \\
&= \left[f_2(0) \sup_{v \in \Re} f_1(v)\right] \vee \left[f_1(0) \sup_{w \in \Re} f_2(w)\right] \\
&= f_1(0) \vee f_2(0).
\end{aligned} \tag{9}
$$

Since the *meet* operation under product t-norm is commutative (see Section 4), we get the same result whether we substitute $\theta/w = v$ or $\theta/v = w$.

As is apparent from (9), the result is very much dependent on functions f_1 and f_2 and does not easily generalize like the *join* and *meet* operations considered earlier, and generally, it is very difficult to obtain a closed-form expression for the result of the *meet* operation. Even if both the fuzzy sets involved have Gaussian membership functions, it is difficult to obtain a nice closed-form expression for the result of the *meet* operation.

To determine the membership of a particular point θ in $F \sqcap G$, we find all the pairs $\{v, w\}$ such that $v \in \Re$, $w \in \Re$ and $vw = \theta$; and multiply the memberships of each pair. The membership grade of θ is given by the supremum of the set of all these products. For example, if $\theta = 20$, all the pairs $\{v, w\}$ that give 20 as their product are v and $20/v$ ($v \in \Re, v \neq 0$). So, the membership grade of 20 is given by the supremum of the set of all the products $f(v)g(20/v)$ ($v \in \Re, v \neq 0$). Fig. 4(c) shows how $f(v)g(20/v)$ looks for F and G in Fig. 4(a). Clearly, it is no easy matter to represent (9) visually.

Because of the complexity of (9) for general membership functions (MFs), we concentrate on Gaussian MFs. For a derivation of $\mu_{F_1 \sqcap F_2}$ when both F_1 and F_2 are Gaussian type-1 sets, see [11]. It is very complicated, does not lead to a closed-form expression, and does not generalize to more than two MFs. Since the *meet* operation under product t-norm is heavily used in the development of a type-2 FLS (because it is widely used in type-1 FLSs), here we develop a Gaussian approximation to this operation that will make it practical.

If there are n Gaussian fuzzy sets $F_1, F_2, \ldots, F_n$ with means $m_1, m_2, \ldots, m_n$ and standard deviations $\sigma_1, \sigma_2, \ldots, \sigma_n$, respectively, then, under the product t-norm,

$$
\mu_{F_1 \sqcap F_2 \sqcap \cdots \sqcap F_n}(\theta) \approx \mathrm{e}^{-(1/2)((\theta - m_1 m_2 \cdots m_n)/\bar{\sigma})^2}. \tag{10}
$$

where $(i = 1, \ldots, n)$

$$
\bar{\sigma} = \sqrt{\sigma_1^2 \prod_{i; i \neq 1} m_i^2 + \cdots + \sigma_j^2 \prod_{i; i \neq j} m_i^2 + \cdots + \sigma_n^2 \prod_{i;\, i \neq n} m_i^2}. \tag{11}
$$

For the derivation of this approximation, see Appendix B. One important feature of this approximation is that it is Gaussian; hence, it is reproducible, i.e., Gaussians remain Gaussians.

Fig. 5 shows some examples of the *meet* operation under product t-norm, and the approximation in (10). See [9] for a triangular approximation for "triangular type-2 sets", i.e., type-2 sets in which all the secondary membership functions are triangular.

A popular type-2 fuzzy set is the interval type-2 set, i.e., the type-2 set all of whose secondary MFs are interval sets. Note that an interval $[l, r]$ can also be represented in terms of its mean and spread as $[m - s, m + s]$,

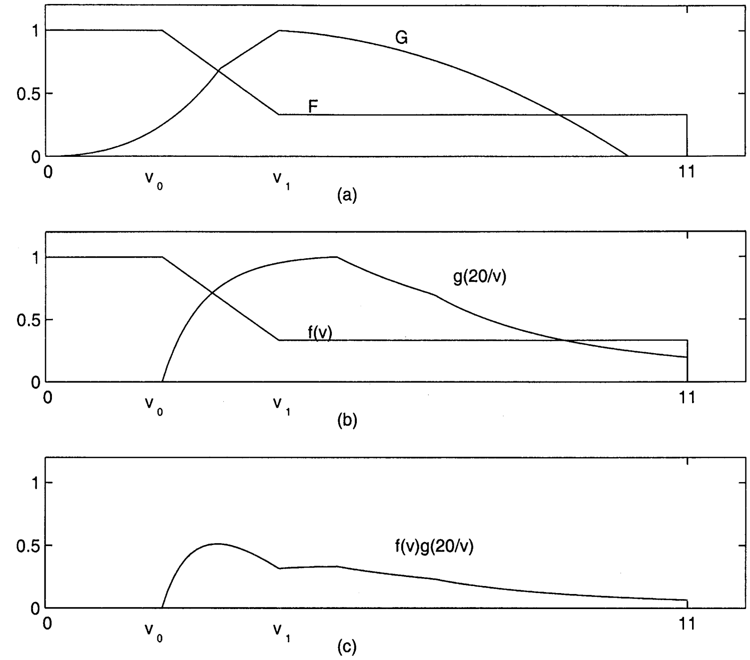

Fig. 4. An example showing how $f(v)g(20/v)$ looks for the membership functions f and g of type-1 sets F and G, respectively, in (a); (b) $f(v)$, which is the same as that in (a) and $g(20/v)$; and, (c) the product $f(v)g(20/v)$.

where $m=(l+r)/2$ and $s=(r-l)/2$. The meet of interval sets under product/minimum t-norm has been given by Schwartz [26] and Kaufmann and Gupta [14], and is discussed in detail in Liang and Mendel [17].

2.4. Negation

From the definition of the *negation* operation, it follows that

Theorem 3. *If a type-1 real fuzzy set F has a membership function* $f(\theta)$, $\neg F$ *has a membership function* $f(1-\theta)$.

The proof follows by substituting $\theta=1-u$ in (3).

3. Algebraic operations

Just as the t-conorm and t-norm operations have been extended to the membership grades of type-2 sets using the Extension Principle, algebraic operations between type-1 sets have been defined using the Extension Principle [4,14,11]. A binary operation "$*$" defined for crisp numbers, can be extended to two type-1 sets, $F_1=\int_v f_1(v)/v$ and $F_2=\int_w f_2(w)/w$, as follows:

$$F_1 * F_2 = \int_v \int_w [f_1(v) \bigstar f_2(w)]/(v * w) \tag{12}$$

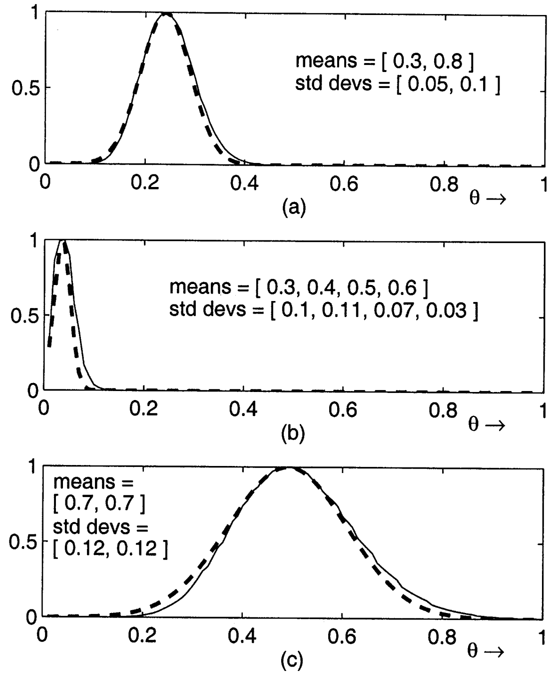

Fig. 5. Some examples of the *meet* between Gaussians under product t-norm (solid line) and the approximation in (10) (thick dashed line). Means and standard deviations of participating Gaussians in each case are also shown on the figure. In each case, the solid curve was obtained numerically, using (9).

where $\star$ indicates the t-norm used. We will mainly be interested in multiplication and addition of type-1 sets. Observe, from (8) and (12), that multiplication of two type-1 sets under product t-norm is the same as the *meet* of the two type-1 sets under product t-norm; therefore, all our earlier discussion about the *meet* under product t-norm applies to the multiplication of type-1 sets under product t-norm.

Theorem 4. *Given n interval type-1 numbers $F_1,\ldots,F_n$, with means $m_1,m_2,\ldots,m_n$ and spreads $s_1,s_2,\ldots,s_n$, their affine combination $\sum_{i=1}^{n}\alpha_i F_i+\beta$, where α_i $(i=1,\ldots,n)$ and β are crisp constants, is also an interval type-1 number with mean $\sum_{i=1}^{n}\alpha_i m_i+\beta$, and spread $\sum_{i=1}^{n}|\alpha_i|s_i$.*

See Appendix C for the proof of Theorem 4.

Theorem 5. *Given n type-1 Gaussian fuzzy numbers $F_1,\ldots,F_n$, with means $m_1,m_2,\ldots,m_n$ and standard deviations $\sigma_1,\sigma_2,\ldots,\sigma_n$, their affine combination $\sum_{i=1}^{n}\alpha_i F_i+\beta$, where α_i $(i=1,\ldots,n)$ and β are crisp constants, is also a Gaussian fuzzy number with mean $\sum_{i=1}^{n}\alpha_i m_i+\beta$, and standard deviation Σ', where*

$$\Sigma' = \begin{cases} \sqrt{\sum_{i=1}^{n}\alpha_i^2\sigma_i^2} & \text{if product t-norm is used,} \\ \sum_{i=1}^{n}|\alpha_i|\sigma_i & \text{if minimum t-norm is used.} \end{cases} \tag{13}$$

See Appendix D for the proof of Theorem 5.

Table 1
Properties of membership grades. In the type-2 case, we assume *convex* and *normal* membership grades; and for type-2 laws, $\sqcup$, $\sqcap$ and $\subseteq$ replace $\vee$, $\bigstar$ and $\leqslant$, respectively. For product t-norm, the laws that are satisfied in the type-1 case, but not in the type-2 case, are highlighted. Results for the minimum t-norm are taken from [23]

Set-theoretic laws		Minimum t-norm		Product t-norm	
		Type-1	Type-2	Type-1	Type-2
Reflexive	$\mu_A \leqslant \mu_A$	Yes	Yes	Yes	Yes
Anti-symmetric	$\mu_A \leqslant \mu_B,\ \mu_B \leqslant \mu_A \Rightarrow \mu_A = \mu_B$	Yes	Yes	Yes	Yes
Transitive	$\mu_A \leqslant \mu_B,\ \mu_B \leqslant \mu_C \Rightarrow \mu_A \leqslant \mu_C$	Yes	Yes	Yes	Yes
Idempotent	$\mu_A \vee \mu_A = \mu_A$	Yes	Yes	Yes	**No**
	$\mu_A \bigstar \mu_A = \mu_A$	Yes	Yes	No	No
Commutative	$\mu_A \vee \mu_B = \mu_B \vee \mu_A$	Yes	Yes	Yes	Yes
	$\mu_A \bigstar \mu_B = \mu_B \bigstar \mu_A$	Yes	Yes	Yes	Yes
Associative	$(\mu_A \vee \mu_B) \vee \mu_C = \mu_A \vee (\mu_B \vee \mu_C)$	Yes	Yes	Yes	Yes
	$(\mu_A \bigstar \mu_B) \bigstar \mu_C = \mu_A \bigstar (\mu_B \bigstar \mu_C)$	Yes	Yes	Yes	Yes
Absorption	$\mu_A \bigstar (\mu_A \vee \mu_B) = \mu_A$	Yes	Yes	No	No
	$\mu_A \vee (\mu_A \bigstar \mu_B) = \mu_A$	Yes	Yes	Yes	**No**
Distributive	$\mu_A \bigstar (\mu_B \vee \mu_C) = (\mu_A \bigstar \mu_B) \vee (\mu_A \bigstar \mu_C)$	Yes	Yes	Yes	**No**
	$\mu_A \vee (\mu_B \bigstar \mu_C) = (\mu_A \vee \mu_B) \bigstar (\mu_A \vee \mu_C)$	Yes	Yes	No	No
Involution	$\mu_{\bar{\bar{A}}} = \mu_A$	Yes	Yes	Yes	Yes
De Morgan's Laws	$\overline{\mu_A \vee \mu_B} = \mu_{\bar{A}} \bigstar \mu_{\bar{B}}$	Yes	Yes	No	No
	$\overline{\mu_A \bigstar \mu_B} = \mu_{\bar{A}} \vee \mu_{\bar{B}}$	Yes	Yes	No	No
Identity	$\mu_A \vee 0 = \mu_A$	Yes	Yes	Yes	Yes
	$\mu_A \bigstar 1 = \mu_A$	Yes	Yes	Yes	Yes
	$\mu_A \vee 1 = 1$	Yes	Yes	Yes	Yes
	$\mu_A \bigstar 0 = 0$	Yes	Yes	Yes	Yes
Complement	$\mu_A \vee \mu_{\bar{A}} \neq 1$	Yes	Yes	Yes	Yes
(Failure)	$\mu_A \bigstar \mu_{\bar{A}} \neq 0$	Yes	Yes	Yes	Yes

4. Properties of membership grades

Mizumoto and Tanaka [23] discuss properties of membership grades of type-2 sets in detail, under the minimum t-norm and maximum t-conorm. We examined these properties under product t-norm and maximum t-conorm for convex, normal membership grades. Our findings are summarized in Table 1. See Chapter 3 of [11] for details, which we have omitted here because they are rather straight-forward.

In proving the *identity law*, we use the concepts of 0 and 1 membership grades for type-2 sets [23]; they are represented as $1/0$ and $1/1$, respectively, which is in accordance with our earlier discussion about type-1 sets being a special case of type-2 sets. An element is said to have a zero membership in a type-2 set if it has a secondary membership equal to 1 corresponding to the primary membership of 0, and if it has all other secondary memberships equal to 0. Similarly, an element is said to have a membership grade equal to 1 in a type-2 set, if it has a secondary membership equal to 1 corresponding to the primary membership of 1 and if all other secondary memberships are zero.

All our operations on type-2 sets collapse to their type-1 counterparts, i.e., if all the type-2 sets are replaced by their principal membership functions (assuming principal membership function can be defined for all of them), all our results remain valid. We, therefore, conclude that *if there are any set-theoretic laws that are not satisfied by type-1 fuzzy sets, we can safely say that type-2 sets will not satisfy those laws either*; however, the converse of this statement may not be true. *If any condition is satisfied by type-1 sets, it may or may not be satisfied by type-2 sets.*

Observe, from Table 1, that the {max, product} t-conorm/t-norm pair, for type-2 sets, does not satisfy *idempotent*, *absorption*, *distributive*, *De Morgan's* and *complement* laws; hence, if the design of a type-2 fuzzy logic system (controller) involves the use of these laws, it will be in error. Observe also that some laws that are satisfied in the type-1 case under {max, product} t-conorm/t-norm pair are not satisfied in the type-2 case.

5. Type-2 fuzzy relations, their compositions, and cartesian product

5.1. Composition of type-2 fuzzy relations

Let $X_1, X_2, \ldots, X_n$ be n universes. A crisp relation in $X_1 \times X_2 \times \cdots \times X_n$ is a crisp subset of the product space. Similarly, a type-1 fuzzy relation in $X_1 \times X_2 \times \cdots \times X_n$ is a type-1 fuzzy subset of the product space and a type-2 fuzzy relation in $X_1 \times X_2 \times \cdots \times X_n$ is a type-2 fuzzy subset of the product space. We concentrate on binary type-2 fuzzy relations.

Unions and intersections of type-2 relations on the same product space can be obtained exactly in the same manner as unions and intersections of ordinary type-2 sets are obtained. If $\tilde{\mathrm{R}}(U,V)$ and $\tilde{\mathrm{S}}(U,V)$ are two type-2 relations on the same product space $U \times V$, their union and intersection are determined as $\mu_{\tilde{\mathrm{R}}\cup\tilde{\mathrm{S}}}(u,v) = \mu_{\tilde{\mathrm{R}}}(u,v) \sqcup \mu_{\tilde{\mathrm{S}}}(u,v)$ and $\mu_{\tilde{\mathrm{R}}\cap\tilde{\mathrm{S}}}(u,v) = \mu_{\tilde{\mathrm{R}}}(u,v) \sqcap \mu_{\tilde{\mathrm{S}}}(u,v)$.

Consider two different product spaces, $U \times V$ and $V \times W$, that share a common set and let $R(U,V)$ and $S(V,W)$ be two *crisp* relations on these spaces. The composition of these relations is defined [15] as "a subset $T(U,W)$ of $U \times W$ such that $(u,w) \in T$ if and only if $(u,v) \in R$ and $(v,w) \in S$". This can be expressed as a *max–min*, *max-product* or, in general, as the *sup-star* composition $\mu_{\mathrm{R}\circ\mathrm{S}}(u,w) = \sup_{v\in V} [\mu_{\mathrm{R}}(u,v) \bigstar \mu_{\mathrm{S}}(v,w)]$ where $\bigstar$ indicates any suitable t-norm operation. The validity of the *sup-star* composition for crisp sets is shown in [28]. If R and S are two crisp relations on $U \times V$ and $V \times W$, respectively, then the membership for any pair (u,w), $u \in U$ and $w \in W$, is 1 if and only if there exists at least one $v \in V$ such that $\mu_{\mathrm{R}}(u,v) = 1$ and $\mu_{\mathrm{S}}(v,w) = 1$. In [28], it is shown that this condition is equivalent to having the *sup-star* composition equal to 1.

When we enter the fuzzy domain, set memberships (which are crisp numbers for type-1 sets, and type-1 fuzzy sets for type-2 sets) belong to the interval $[0,1]$ rather than just being 0 or 1. So, now we can think of an element as belonging to a set if it has a non-zero membership in that set. In this respect, the aforementioned condition on the composition of relations can be rephrased as follows (recall the concept of 0 and 1 membership grades in the case of type-2 sets discussed in Section 4):

> If R ($\tilde{\mathrm{R}}$) and S ($\tilde{\mathrm{S}}$) are two type-1 (type-2) fuzzy relations on $U \times V$ and $V \times W$, respectively, then the membership for any pair (u,w), $u \in U$ and $w \in W$, is non-zero if and only if there exists at least one $v \in V$ such that $\mu_{\mathrm{R}}(u,v) \neq 0$ ($\mu_{\tilde{\mathrm{R}}}(u,v) \neq 0$) and $\mu_{\mathrm{S}}(v,w) \neq 1/0$ ($\mu_{\tilde{\mathrm{S}}}(v,w) \neq 1/0$).

It can be easily shown that in the type-1 case, this condition is again equivalent to the *sup-star* composition,

$$\mu_{\mathrm{R}\circ\mathrm{S}}(u,w) = \sup_{v\in V} [\mu_{\mathrm{R}}(u,v) \bigstar \mu_{\mathrm{S}}(v,w)] \tag{14}$$

and, in the type-2 case, it is equivalent to the following "extended" version of the *sup-star* composition.

$$\mu_{\tilde{\mathrm{R}}\circ\tilde{\mathrm{S}}}(u,w) = \bigsqcup_{v\in V} [\mu_{\tilde{\mathrm{R}}}(u,v) \sqcap \mu_{\tilde{\mathrm{S}}}(v,w)]. \tag{15}$$

Proofs of (14) and (15) are very similar. We just give the proof of (15) next. In the proof, we use the following method. Let $\boldsymbol{A}$ be the statement "$\mu_{\tilde{R}\circ\tilde{S}}(u,w) \neq 1/0$", and $\boldsymbol{B}$ be the statement "there exists at least one $v \in V$ such that $\mu_{\tilde{R}}(u,v) \neq 1/0$ and $\mu_{\tilde{S}}(v,w) \neq 1/0$". We prove that "$\boldsymbol{A}$ iff $\boldsymbol{B}$" by first proving that $\bar{\boldsymbol{B}} \Rightarrow \bar{\boldsymbol{A}}$ (which is equivalent to proving $\boldsymbol{A} \Rightarrow \boldsymbol{B}$, i.e., necessity of $\boldsymbol{B}$) and then proving that $\bar{\boldsymbol{A}} \Rightarrow \bar{\boldsymbol{B}}$ (which is equivalent to proving $\boldsymbol{B} \Rightarrow \boldsymbol{A}$, i.e., sufficiency of $\boldsymbol{B}$).

Proof. (*Necessity*) If there exists no $v \in V$ such that $\mu_{\tilde{R}}(u,v) \neq 1/0$ and $\mu_{\tilde{S}}(v,w) \neq 1/0$, then this means that either $\mu_{\tilde{R}}(u,v) = 1/0$ or $\mu_{\tilde{S}}(v,w) = 1/0$ (or both are zero) for every $v \in V$. So, assuming that $\mu_{\tilde{R}}(u,v)$ and $\mu_{\tilde{S}}(v,w)$ are normal, then from the identity law ($\mu_A \sqcap 1/0 = 1/0$), $\mu_{\tilde{R}}(u,v) \sqcap 1/0 = 1/0$ or $1/0 \sqcap \mu_{\tilde{S}}(v,w) = 1/0$, so that we have that $\mu_{\tilde{R}}(u,v) \sqcap \mu_{\tilde{S}}(v,w) = 1/0$ for every $v \in V$, and therefore, in (15), $\bigsqcup_{v\in V} 1/0 = 1/0$.

(*Sufficiency*) If the extended *sup-star* composition is zero, then, from (15),

$$\bigsqcup_{v\in V} [\mu_{\tilde{R}}(u,v) \sqcap \mu_{\tilde{S}}(v,w)] = 1/0. \tag{16}$$

For each value of v, the term in the bracket is a type-1 fuzzy set. Suppose, for example, that there are two such terms, say P and Q, so that $\mu_P \sqcup \mu_Q = 1/0$. Observe, from (1), that for two normal membership grades, $\mu_P = \int_u f(u)/u$ and $\mu_Q = \int_v g(v)/v$,

$$\mu_P \sqcup \mu_Q = 1/0 \Leftrightarrow \int_u \int_v [f(u) \bigstar g(v)]/(u \vee v) = 1/0. \tag{17}$$

For $u \vee v = 0$, $u = v = 0$, in which case, $\mu_P = \mu_Q = 1/0$. This means that each of the two terms in the bracket of (15) equals 1/0.

The extension of this analysis to more than two terms is easy. We conclude, therefore, that $\mu_{\tilde{R}}(u,v) \sqcap \mu_{\tilde{S}}(v,w) = 1/0$ for every $v \in V$, which means that for every $v \in V$, either $\mu_{\tilde{R}}(u,v)$ or $\mu_{\tilde{S}}(v,w)$ (or both) is 1/0. Consequently, there is no $v \in V$ such that $\mu_{\tilde{R}}(u,v) \neq 1/0$ and $\mu_{\tilde{S}}(v,w) \neq 1/0$. $\square$

Dubois and Prade [3,4] give a formula for the composition of type-2 relations under minimum t-norm as an extension of the type-1 sup-min composition. Their formula is the same as (15); however, we have demonstrated the validity of (15) for product as well as minimum t-norms.

Example 2. Consider the type-1 relation "u is close to v" on $U \times V$, where $U = \{u_1, u_2\}$ and $V = \{v_1, v_2, v_3\}$ are given as: $U = \{2, 12\}$, $V = \{1, 7, 13\}$ and

$$\mu_{\text{close}}(u,v) = \begin{array}{c} \\ u_1 \\ u_2 \end{array} \!\!\begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{pmatrix} 0.9 & 0.4 & 0.1 \\ 0.1 & 0.4 & 0.9 \end{pmatrix} \end{array}. \tag{18}$$

Now consider another type-1 fuzzy relation "v is bigger than w" on $V \times W$ where $W = \{w_1, w_2\} = \{4, 8\}$, with the following membership function:

$$\mu_{\text{bigger}}(v,w) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \end{array} \!\!\begin{array}{c} \begin{array}{cc} w_1 & w_2 \end{array} \\ \begin{pmatrix} 0 & 0 \\ 0.6 & 0 \\ 1 & 0.7 \end{pmatrix} \end{array}. \tag{19}$$

As is well known, the statement "u is close to v *and* v is bigger than w" indicates the composition of these two type-1 relations, which can be found by using (14) and the min t-norm, as follows:

$$\mu_{\text{close}\circ\text{bigger}}(u_i,w_j)=[\mu_{\text{close}}(u_i,v_1)\wedge\mu_{\text{bigger}}(v_1,w_j)]\vee[\mu_{\text{close}}(u_i,v_2)\wedge\mu_{\text{bigger}}(v_2,w_j)]\vee[\mu_{\text{close}}(u_i,v_3)\wedge\mu_{\text{bigger}}(v_3,w_j)] \tag{20}$$

where $i=1,2$ and $j=1,2,3$. Using (20), we get

$$\mu_{\text{close}\circ\text{bigger}}(u,w)=\begin{array}{c c} & \begin{array}{cc} w_1 & w_2 \end{array} \\ \begin{array}{c} u_1 \\ u_2 \end{array} & \begin{pmatrix} 0.4 & 0.1 \\ 0.9 & 0.7 \end{pmatrix} \end{array}. \tag{21}$$

Now, let us consider the type-2 relations "u is close to v" and "v is bigger than w" which are obtained by adding some uncertainty to the type-1 relations in (18) and (19). Let the membership grades be as follows:

$$\mu_{\widetilde{\text{close}}}(u,v)=\begin{array}{c c} & \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{array}{c} u_1 \\ \\ u_2 \\ \end{array} & \begin{pmatrix} 0.3/0.8+1/0.9 & 0.7/0.3+1/0.4 & 0.5/0+1/0.1 \\ +0.7/1 & +0.1/0.5 & \\ 0.5/0+1/0.1 & 0.7/0.3+1/0.4 & 0.3/0.8+1/0.9 \\ & +0.1/0.5 & +0.7/1 \end{pmatrix} \end{array} \tag{22}$$

and

$$\mu_{\widetilde{\text{bigger}}}(v,w)=\begin{array}{c c} & \begin{array}{cc} w_1 & w_2 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ \\ v_3 \\ \\ \end{array} & \begin{pmatrix} 1/0+0.6/0.1 & 1/0+0.1/0.1 \\ 0.4/0.5+1/0.6 & 1/0+0.8/0.1 \\ +0.9/0.7 & +0.2/0.2 \\ 0.7/0.9+1/1 & 0.5/0.6+1/0.7 \\ & +0.7/0.8 \end{pmatrix} \end{array}. \tag{23}$$

The composition of the type-2 relations "u is close to v and v is bigger than w" can be found by using (15), as follows:

$$\mu_{\widetilde{\text{close}}\circ\widetilde{\text{bigger}}}(u_i,w_j)=[\mu_{\widetilde{\text{close}}}(u_i,v_1)\sqcap\mu_{\widetilde{\text{bigger}}}(v_1,w_j)]\sqcup[\mu_{\widetilde{\text{close}}}(u_i,v_2)\sqcap\mu_{\widetilde{\text{bigger}}}(v_2,w_j)]\sqcup[\mu_{\widetilde{\text{close}}}(u_i,v_3)\sqcap\mu_{\widetilde{\text{bigger}}}(v_3,w_j)] \tag{24}$$

where $i=1,2$ and $j=1,2$. Using (24), (22), (23), (1) and (2), we have (using min t-norm and max t-conorm)

$$\mu_{\widetilde{\text{close}}\circ\widetilde{\text{bigger}}}(u,w)=\begin{array}{c c} & \begin{array}{cc} w_1 & w_2 \end{array} \\ \begin{array}{c} u_1 \\ \\ u_2 \\ \\ \end{array} & \begin{pmatrix} 0.7/0.3+1/0.4 & 0.5/0+1/0.1 \\ & +0.2/0.2 \\ 0.3/0.8+1/0.9 & 0.5/0.6+1/0.7 \\ +0.7/1 & +0.7/0.8 \end{pmatrix} \end{array}. \tag{25}$$

Comparing (25) and (21), we observe that the results in (25) are indeed quite similar to the results of the type-1 *sup-star* composition in (21), i.e., in the type-2 results, primary memberships corresponding to the memberships in the type-1 results, have unity secondary memberships. Eq. (25) provides additional uncertainty information to account for the secondary MFs. □

Next, consider the case where one of the relations involved in the composition is just a fuzzy set. The composition of the type-1 set $\mathrm{R} \in U$ and type-1 fuzzy relation $\mathrm{S}(U, V)$ is given as [16]

$$\mu_{\mathrm{R}\circ\mathrm{S}}(v) = \sup_{u \in U} [\mu_{\mathrm{R}}(u) \star \mu_{\mathrm{S}}(u, v)], \tag{26}$$

which is a type-1 fuzzy set on V. Similarly, in the type-2 case, the composition of a type-2 fuzzy set in $\tilde{\mathrm{R}} \in U$ and a type-2 relation $\tilde{\mathrm{S}}(U, V)$ is given by

$$\mu_{\tilde{\mathrm{R}}\circ\tilde{\mathrm{S}}}(v) = \bigsqcup_{u \in U} [\mu_{\tilde{\mathrm{R}}}(u) \sqcap \mu_{\tilde{\mathrm{S}}}(u, v)]. \tag{27}$$

Eq. (27) plays an important role as the inference mechanism of a rule where antecedents or consequents are type-2 fuzzy sets, and is the fundamental inference mechanism for rules in type-2 FLSs (e.g., [10,11,17]).

5.2. Cartesian product

If U and V are domains of type-1 fuzzy sets F and G, respectively, characterized by certain membership functions, then the Cartesian product of these membership functions will be supported over $U \times V$, and each point $\{(u, v); u \in U, v \in V\}$ in this plane will have a membership grade (a crisp number in $[0, 1]$) which is the result of a t-norm operation between the membership grade of u in F and the membership grade of v in G.

Now consider the case of type-2 sets $\tilde{F}$ and $\tilde{G}$. The membership grades of every element in $\tilde{F}$ and $\tilde{G}$ are now type-1 fuzzy sets. So, now the t-norm operation in the type-1 case has to be replaced by the *meet* operation; therefore, when we find the Cartesian product, it is still supported over $U \times V$ as in the type-1 case; but now each point $(u, v) \in U \times V$ will have a membership grade which is again a type-1 fuzzy set (result of a *meet* operation between membership grades of u and v).

6. Conclusions

We have discussed set-theoretic operations for type-2 sets, properties of membership grades of type-2 sets, and type-2 relations and their compositions, and cartesian product under minimum and product t-norms. We have also developed easily implementable algorithms to perform set-theoretic and algebraic operations of type-2 sets. When actual results are difficult to generalize or implement, we have developed practical approximations. All our results reduce to the correct type-1 results when all the type-2 uncertainties collapse to type-1 uncertainties.

All of the results in this paper are needed to implement type-2 FLSs [11,17], and should be of use to other researchers who explore applications of type-2 fuzzy sets. Software to implement some of the algorithms developed herein are on-line at the URL: http://sipi.usc.edu/~mendel/software.

Appendix A. Proof of Theorem 1

In the proof of Theorem 1, given next, we represent fuzzy sets F_i as $F_i = \int_v f_i(v)/v$ $(i = 1, \ldots, n)$, where v, in general, varies over the real line. If a real number w_0 is not in F_i, $f_i(w_0)$ is zero.

Proof. First we prove (5), and then we prove (6).

(I) We first prove (5) for the case $n = 2$. The *join* operation between F_1 and F_2 can be expressed, as

$$F_1 \sqcup F_2 = \int_v \int_w [f_1(v) \wedge f_2(w)]/(v \vee w). \tag{A.1}$$

Let us see what operations are involved here. For every pair of points $\{v,w\}$, such that $v \in F_1$ and $w \in F_2$, we find the maximum of v and w and the minimum of their memberships, so that $v \vee w$ is an element of $F_1 \sqcup F_2$ and $f_1(v) \wedge f_2(w)$ is the corresponding membership grade. If more than one $\{v,w\}$ pair gives the same maximum (i.e., the same element in $F_1 \sqcup F_2$), we use the maximum of all the corresponding membership grades as the membership of this element. So, every element of the resulting set is obtained as a result of the *max* operation on one or more $\{v,w\}$ pairs, and its membership is the maximum of all the results of the *min* operation on memberships of v and w.

If $\theta \in F_1 \sqcup F_2$, the possible $\{v,w\}$ pairs that can give θ as the result of the maximum operation are $\{v,\theta\}$ where $v \in (-\infty,\theta]$ and $\{\theta,w\}$ where $w \in (-\infty,\theta]$. The process of finding the membership of θ in $F_1 \sqcup F_2$ can be broken into three steps: (1) find the minima between the memberships of all the pairs $\{v,\theta\}$ such that $v \in (-\infty,\theta]$ and then find their supremum; (2) do the same with all the pairs $\{\theta,w\}$ such that $w \in (-\infty,\theta]$; and, (3) find the maximum of the two suprema, i.e.,

$$\mu_{(F_1 \sqcup F_2)}(\theta) = \phi_1(\theta) \vee \phi_2(\theta) \tag{A.2}$$

where

$$\phi_1(\theta) = \sup_{v \in (-\infty,\theta]} \{f_1(v) \wedge f_2(\theta)\} = f_2(\theta) \wedge \sup_{v \in (-\infty,\theta]} f_1(v) \tag{A.3}$$

and

$$\phi_2(\theta) = \sup_{w \in (-\infty,\theta]} \{f_1(\theta) \wedge f_2(w)\} = f_1(\theta) \wedge \sup_{w \in (-\infty,\theta]} f_2(w). \tag{A.4}$$

We break θ into the following three ranges: $\theta < v_1$, $v_1 \leqslant \theta < v_2$ and $\theta \geqslant v_2$ (see Fig. 6). Recall that $f_1(v_1) = 1$ and $f_2(v_2) = 1$. Also, observe that *convexity of* F_i $(i = 1,2)$ *is equivalent to the condition that* f_i *is monotonic non-decreasing in* $(-\infty, v_i]$ *and monotonic non-increasing in* $[v_i, \infty)$.

$\theta = \theta_1 < v_1$: See Fig. 6(a). Since f_1 and f_2 both are monotonic non-decreasing in $(-\infty, v_1]$, $\sup_{v \in (-\infty,\theta]} f_1(v) = f_1(\theta)$ and $\sup_{w \in (-\infty,\theta]} f_2(w) = f_2(\theta)$; therefore, from (A.2)–(A.4), we have

$$\mu_{(F_1 \sqcup F_2)}(\theta) = f_1(\theta) \wedge f_2(\theta), \quad \theta < v_1. \tag{A.5}$$

$v_1 \leqslant \theta = \theta_2 < v_2$: See Fig. 6(a). Recall that $f_1(v_1) = 1$ and that f_2 is monotonic non-decreasing in $(-\infty, v_2]$. Therefore, $\sup_{v \in (-\infty,\theta]} f_1(v) = 1$ and $\sup_{w \in (-\infty,\theta]} f_2(w) = f_2(\theta)$. Using these facts in (A.2)–(A.4), we have

$$\mu_{(F_1 \sqcup F_2)}(\theta) = f_2(\theta) \vee [f_1(\theta) \wedge f_2(\theta)] = f_2(\theta), \quad v_1 \leqslant \theta < v_2 \tag{A.6}$$

where we have made use of the fact that $a \vee [a \wedge b] = a$, for real a and b.

$\theta = \theta_3 \geqslant v_2$: For θ in this range [see Fig. 6(a)], both f_1 and f_2 have already attained their maximum values; therefore, $\sup_{v \in (-\infty,\theta]} f_1(v) = \sup_{w \in (-\infty,\theta]} f_2(w) = 1$. Consequently, from (A.2)–(A.4), we have

$$\mu_{(F_1 \sqcup F_2)}(\theta) = f_1(\theta) \vee f_2(\theta), \quad \theta \geqslant v_2. \tag{A.7}$$

From (A.5)–(A.7), we get

$$\mu_{(F_1 \sqcup F_2)}(\theta) = \begin{cases} f_1(\theta) \wedge f_2(\theta), & \theta < v_1, \\ f_2(\theta), & v_1 \leqslant \theta < v_2, \\ f_1(\theta) \vee f_2(\theta), & \theta \geqslant v_2. \end{cases} \tag{A.8}$$

Observe that, since F_1 and F_2 both are convex and normal with $f_1(v_1) = f_2(v_2) = 1$: (1) $f_1 \wedge f_2$ is monotonic non-decreasing in $(-\infty, v_1]$; (2) $f_1(v_1) \wedge f_2(v_1) = f_2(v_1)$; (3) f_2 is monotonic non-decreasing in $[v_1, v_2]$; and, (4) $f_1 \vee f_2$ is monotonic non-increasing in $[v_2, \infty)$. From these three observations, we see that $\mu_{(F_1 \sqcup F_2)}$ is

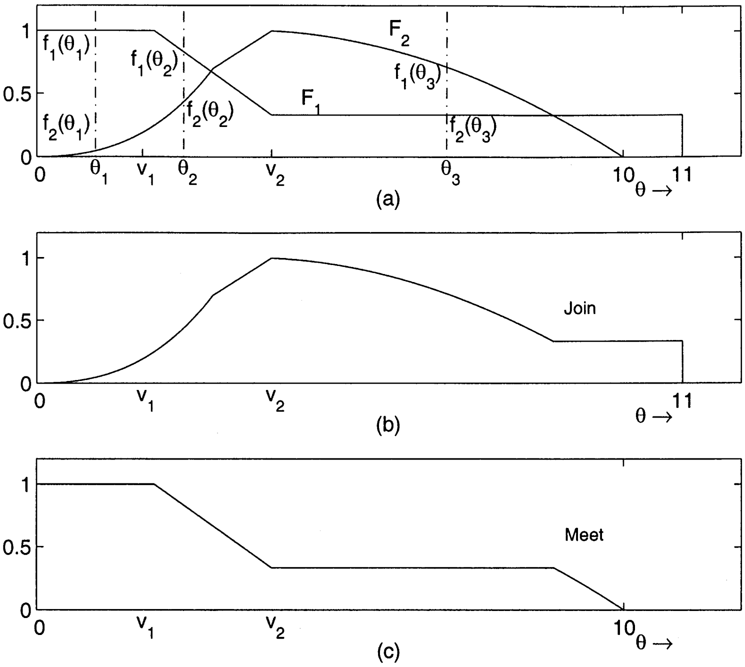

Fig. 6. An example of two general membership functions, f_1 and f_2, that satisfy the requirements of Theorem 1. Observe that for set F_1, any of the points at which f_1 attains its maximum value of unity may be chosen as v_1. We arbitrarily chose $v_1 = 1.8$: (a) The three possibilities: $\theta_1 < v_1$, $v_1 \leqslant \theta_2 < v_2$, $\theta_3 \geqslant v_2$. (b) Result of the *join* operation. (c) Result of the *meet* operation. The t-norm used is *min*.

monotonic non-decreasing in $(-\infty, v_2]$ and monotonic non-increasing in $[v_2, \infty)$, which implies that $F_1 \sqcup F_2$ is convex [see Fig. 6(b)]. Also, $F_1 \sqcup F_2$ is normal with $\mu_{(F_1 \sqcup F_2)}(v_2) = 1$.

Next, we prove (5) for $n > 2$. Using the associative law [23], we have $F_1 \sqcup F_2 \sqcup F_3 = (F_1 \sqcup F_2) \sqcup F_3$. Since, as explained in the previous paragraph, $F_1 \sqcup F_2$ is also convex and normal, from (A.8) we have [recall that $f_3(v_3) = 1$ and that $v_3 \geqslant v_2$]

$$\mu_{(F_1 \sqcup F_2 \sqcup F_3)}(\theta) = \begin{cases} \mu_{(F_1 \sqcup F_2)}(\theta) \wedge f_3(\theta), & \theta < v_2, \\ f_3(\theta), & v_2 \leqslant \theta < v_3, \\ \mu_{(F_1 \sqcup F_2)}(\theta) \vee f_3(\theta), & \theta \geqslant v_3. \end{cases} \tag{A.9}$$

Since $v_2 \leqslant v_3$, (A.8) and (A.9) can be rewritten as

$$\mu_{(F_1 \sqcup F_2)}(\theta) = \begin{cases} f_1(\theta) \wedge f_2(\theta), & \theta < v_1, \\ f_2(\theta), & v_1 \leqslant \theta < v_2, \\ f_1(\theta) \vee f_2(\theta), & v_2 \leqslant \theta < v_3, \\ f_1(\theta) \vee f_2(\theta), & \theta \geqslant v_3, \end{cases} \tag{A.10}$$

$$\mu_{(F_1 \sqcup F_2 \sqcup F_3)}(\theta) = \begin{cases} \mu_{(F_1 \sqcup F_2)}(\theta) \wedge f_3(\theta), & \theta < v_1, \\ \mu_{(F_1 \sqcup F_2)}(\theta) \wedge f_3(\theta), & v_1 \leqslant \theta < v_2, \\ f_3(\theta), & v_2 \leqslant \theta < v_3, \\ \mu_{(F_1 \sqcup F_2)}(\theta) \vee f_3(\theta), & \theta \geqslant v_3. \end{cases} \tag{A.11}$$

Substituting for $\mu_{(F_1 \sqcup F_2)}(\theta)$ from (A.10) into (A.11), we have

$$\mu_{(F_1 \sqcup F_2 \sqcup F_3)}(\theta) = \begin{cases} f_1(\theta) \wedge f_2(\theta) \wedge f_3(\theta), & \theta < v_1, \\ f_2(\theta) \wedge f_3(\theta), & v_1 \leqslant \theta < v_2, \\ f_3(\theta), & v_2 \leqslant \theta < v_3, \\ f_1(\theta) \vee f_2(\theta) \vee f_3(\theta), & \theta \geqslant v_3, \end{cases} \tag{A.12}$$

$$= \begin{cases} \bigwedge_{i=1}^{3} f_i(\theta), & \theta < v_1, \\ \bigwedge_{i=k+1}^{3} f_i(\theta), & v_k \leqslant \theta < v_{k+1}, \ 1 \leqslant k \leqslant 2, \\ \bigvee_{i=1}^{3} f_i(\theta), & \theta \geqslant v_3. \end{cases} \tag{A.13}$$

It is straightforward to show that $F_1 \sqcup F_2 \sqcup F_3$ is also a convex and normal set. Therefore, (A.8) can be applied again to obtain $\mu_{(F_1 \sqcup F_2 \sqcup F_3 \sqcup F_4)}$. Continuing in this fashion, we get (5).

(II) For the case, $n = 2$, the *meet* operation between F_1 and F_2 can be represented as

$$F_1 \sqcap F_2 = \int_v \int_w [f_1(v) \wedge f_2(w)]/(v \wedge w). \tag{A.14}$$

The only difference between this operation and the *join* operation in (A.1) is that every $\theta \in F_1 \sqcup F_2$ is the result of the minimum operation on some pair $\{v, w\}$, such that $v \in F_1$ and $w \in F_2$; the possible such pairs being $\{v, \theta\}$ where $v \in [\theta, \infty)$ and $\{\theta, w\}$ where $w \in [\theta, \infty)$. The process of finding the membership of θ in $F_1 \sqcup F_2$ can be broken into three steps: (1) find the minima between the memberships of all the pairs $\{v, \theta\}$ such that $v \in [\theta, \infty)$ and then find their supremum; (2) do the same with all the pairs $\{\theta, w\}$ such that $w \in [\theta, \infty)$; and, (3) find the maximum of the two suprema.

By proceeding in a manner very similar to that in part (I) of the proof, we get the result in (6). For details, see [11].

Appendix B. Solving for the Gaussian meet approximation

Consider two Gaussian type-1 sets F_1 and F_2 characterized by membership functions f_1 and f_2, having means m_1 and m_2 and standard deviations σ_1 and σ_2. Since we will perform the *join* and *meet* operations on the membership grades of type-2 sets when computing unions and intersections of type-2 sets, the type-1 sets involved in the operations will have $[0, 1]$ as their domains. The membership functions of these type-1 sets may, therefore, be clipped (i.e., any portion of the membership function lying outside $[0, 1]$ is cut off); and, this clipping may affect the result of the *meet* operation. This constraint needs to be considered in the derivation of the actual result of the *meet* between two Gaussian type-1 sets under the product t-norm; however, we will ignore it for simplicity, while deriving the Gaussian approximation. For a detailed discussion about the effects of this constraint on the actual result, as well as on the Gaussian approximation and its upper and lower bounds, see [11].

Applying (9) for Gaussian membership functions, we have

$$\mu_{F_1 \sqcap F_2}(\theta) = \sup_{v \in \Re, v \neq 0} \exp\left\{ -\frac{1}{2}\left[\left(\frac{v - m_1}{\sigma_1} \right)^2 + \left(\frac{\theta/v - m_2}{\sigma_2} \right)^2 \right] \right\}, \quad \theta \in \Re, \ \theta \neq 0, \tag{B.1}$$

$$\mu_{F_1 \sqcap F_2}(0) = \exp\left\{-\frac{1}{2}\left(\frac{m_1}{\sigma_1}\right)^2\right\} \vee \exp\left\{-\frac{1}{2}\left(\frac{m_2}{\sigma_2}\right)^2\right\}. \tag{B.2}$$

Finding $\mu_{F_1 \sqcap F_2}(\theta)$ for $\theta \neq 0$, requires minimization of

$$J(v) = \left(\frac{v - m_1}{\sigma_1}\right)^2 + \left(\frac{\theta/v - m_2}{\sigma_2}\right)^2 = \left(\frac{v - m_1}{\sigma_1}\right)^2 + \left(\frac{\theta - m_2 v}{\sigma_2 v}\right)^2. \tag{B.3}$$

Observe that J is non-convex and difficult to minimize. Therefore, we simplify the problem by replacing the v in the denominator of the second term of $J(v)$ by a constant k. This gives us the following objective function to minimize:

$$H(v) = \left(\frac{v - m_1}{\sigma_1}\right)^2 + \left(\frac{\theta - m_2 v}{k\sigma_2}\right)^2. \tag{B.4}$$

Observe that H is convex ($H'' = 2/\sigma_1^2 + 2m_2^2/\sigma_2^2 > 0$). Therefore, equating H' to zero and assuming that the infimum is obtained at $v = v^*$, we get

$$2\left(\frac{v^* - m_1}{\sigma_1}\right)\left(\frac{1}{\sigma_1}\right) + 2\left(\frac{\theta - m_2 v^*}{k\sigma_2}\right)\left(\frac{-m_2}{k\sigma_2}\right) = 0 \Rightarrow \ v^* = \frac{\theta m_2 \sigma_1^2 + m_1 k^2 \sigma_2^2}{m_2^2 \sigma_1^2 + k^2 \sigma_2^2}. \tag{B.5}$$

Substituting (B.5) into (B.4), we get

$$\inf_v H(v) = \left(\frac{\theta - m_1 m_2}{\sqrt{m_2^2 \sigma_1^2 + k^2 \sigma_2^2}}\right)^2. \tag{B.6}$$

Recall that the type-1 sets involved in the *meet* operation have $[0,1]$ as their domains. Since the constant k in (B.4) replaces v that varies between 0 and 1, it seems apparent that some value of k between 0 and 1 will give a good approximation to the actual result of the *meet* operation. Observe, from (B.5), that replacing k by m_1, makes the Gaussian approximation commutative [i.e., if we interchange $\{m_1, \sigma_1\}$ and $\{m_2, \sigma_2\}$, we still get the same result] just as the actual *meet* operation is (see Section 4). Therefore, by replacing k with m_1, we obtain the Gaussian approximation.

$$\mu_{F_1 \sqcap F_2}(\theta) \approx \exp\left\{-\frac{1}{2}\left(\frac{\theta - m_1 m_2}{\sqrt{m_2^2 \sigma_1^2 + m_1^2 \sigma_2^2}}\right)^2\right\}. \tag{B.7}$$

Generalization of this result to the *meet* of n Gaussian type-1 sets is straightforward and gives the result in (10). For a detailed derivation of bounds on the approximation error between (10) and the actual value of $\mu_{F_1 \sqcap F_2 \sqcap \cdots \sqcap F_n}(\theta)$, see [11].

Appendix C. Proof of Theorem 4

Consider $F_i = [m_i - s_i, m_i + s_i]$. Multiplying F_i by a crisp constant α_i ($= 1/\alpha_i$) yields (see (12))

$$\alpha_i F_i = \int_v 1/(\alpha_i v), \quad v \in [m_i - s_i, m_i + s_i]. \tag{C.1}$$

Adding a crisp constant β $(=1/\beta)$ to $\alpha_i F_i$ yields

$$\alpha_i F_i + \beta = \int_v 1/(\alpha_i v + \beta), \quad v \in [m_i - s_i, m_i + s_i]. \tag{C.2}$$

Substituting $w = \alpha_i v + \beta$, (C.2) gives us

$$\alpha_i F_i + \beta = \int_w 1/w, \quad w \in [\alpha_i m_i + \beta - |\alpha_i| s_i,\ \alpha_i m_i + \beta + |\alpha_i| s_i]. \tag{C.3}$$

Since F_i can be represented as $[l_i, r_i]$, where $l_i = m_i - s_i$ and $r_i = m_i + s_i$, then

$$\sum_{i=1}^{n} F_i = \left[\sum_{i=1}^{n} m_i - \sum_{i=1}^{n} s_i,\ \sum_{i=1}^{n} m_i + \sum_{i=1}^{n} s_i \right]. \tag{C.4}$$

Using (C.3) and (C.4), we get the result in Theorem 4.

Appendix D. Proof of Theorem 5

We follow a method similar to the one used in the proof of Theorem 4 in Appendix C. We prove the theorem in two parts: (a) we prove that $\alpha_i F_i + \beta$ is a Gaussian fuzzy number with mean $\alpha_i m_i + \beta$ and standard deviation $|\alpha_i|\sigma_i$; and (b) we prove that $\sum_{i=1}^{n} F_i$ is a Gaussian fuzzy number with mean $\sum_{i=1}^{n} m_i$ and standard deviation Σ'', where

$$\Sigma'' = \begin{cases} \sqrt{\sum_{i=1}^{n} \sigma_i^2} & \text{if product t-norm is used,} \\ \sum_{i=1}^{n} \sigma_i & \text{if minimum t-norm is used.} \end{cases} \tag{D.1}$$

(a) Consider

$$F_i = \int_v \exp\left\{ -\frac{1}{2} \left(\frac{v - m_i}{\sigma_i} \right)^2 \right\} \Big/ v. \tag{D.2}$$

Multiplying F_i by a constant α_i $(=1/\alpha_i)$ and adding a constant β $(=1/\beta)$ to $\alpha_i F_i$, yields [see (12)]

$$\begin{aligned} \alpha_i F_i + \beta &= \int_v \left[\exp\left\{ -\frac{1}{2} \left(\frac{v - m_i}{\sigma_i} \right)^2 \right\} \star 1 \right] \Big/ (\alpha_i v + \beta) \\ &= \int_v \exp\left\{ -\frac{1}{2} \left(\frac{v - m_i}{\sigma_i} \right)^2 \right\} \Big/ (\alpha_i v + \beta). \end{aligned} \tag{D.3}$$

Let $\alpha_i v + \beta = v'$; this gives $v = (v' - \beta)/\alpha_i$, which when substituted into (D.3), leads to

$$\begin{aligned} \alpha_i F_i + \beta &= \int_{v'} \exp\left\{ -\frac{1}{2} \left[\frac{(v' - \beta)/\alpha_i - m_i}{\sigma_i} \right]^2 \right\} \Big/ v' \\ &= \int_{v'} \exp\left\{ -\frac{1}{2} \left[\frac{v' - (\alpha_i m_i + \beta)}{\alpha_i \sigma_i} \right]^2 \right\} \Big/ v' \end{aligned} \tag{D.4}$$

which shows that $\alpha_i F_i + \beta$ is a Gaussian fuzzy number with mean $\alpha_i m_i + \beta$ and standard deviation $|\alpha_i|\sigma_i$. Note that this result does not depend on the kind of t-norm used, since α_i and β are crisp numbers.

(b) Consider F_1 and F_2, with means m_1 and m_2 and standard deviations σ_1 and σ_2, respectively. The sum of these two fuzzy numbers can be expressed as [see (12)]

$$F_1 + F_2 = \int_{v\in F_1}\int_{w\in F_2} \exp\left\{-\frac{1}{2}\left(\frac{v-m_1}{\sigma_1}\right)^2\right\} \star \exp\left\{-\frac{1}{2}\left(\frac{w-m_2}{\sigma_2}\right)^2\right\} \bigg/ [v+w] \tag{D.5}$$

where $\star$ indicates the chosen t-norm.

(i) *Product t-norm*: In this case, (D.5) reduces to

$$F_1 + F_2 = \int_{v\in F_1}\int_{w\in F_2} \exp\left\{-\frac{1}{2}\left(\frac{v-m_1}{\sigma_1}\right)^2\right\} \exp\left\{-\frac{1}{2}\left(\frac{w-m_2}{\sigma_2}\right)^2\right\} \bigg/ [v+w]. \tag{D.6}$$

If θ is an element of $F_1 + F_2$, the membership grade of θ in $F_1 + F_2$ can be obtained by considering all the $\{v, w\}$ pairs such that $v \in F_1$ and $w \in F_2$ and $v + w = \theta$, multiplying the memberships of v and w in every pair, and, choosing the maximum of all these membership products. In other words,

$$\mu_{(F_1+F_2)}(\theta) = \sup_v \exp\left\{-\frac{1}{2}\left[\left(\frac{v-m_1}{\sigma_1}\right)^2 + \left(\frac{(\theta-v)-m_2}{\sigma_2}\right)^2\right]\right\}. \tag{D.7}$$

Let v^* maximize the expression on the RHS of (D.7). v^* is obtained by minimizing

$$J(v) = \left(\frac{v-m_1}{\sigma_1}\right)^2 + \left[\frac{(\theta-v)-m_2}{\sigma_2}\right]^2. \tag{D.8}$$

Since J is convex ($J'' = 1/\sigma_1^2 + 1/\sigma_2^2 > 0$), equating the first derivative of J to zero, we get

$$\begin{aligned} &2\left(\frac{v^*-m_1}{\sigma_1}\right)\left(\frac{1}{\sigma_1}\right) + 2\left[\frac{(\theta-v^*)-m_2}{\sigma_2}\right]\left(-\frac{1}{\sigma_2}\right) = 0,\\ &v^*\left(\frac{1}{\sigma_1^2}+\frac{1}{\sigma_2^2}\right) = \frac{m_1}{\sigma_1^2} + \frac{\theta-m_2}{\sigma_2^2},\\ &v^* = \frac{m_1\sigma_2^2 + (\theta-m_2)\sigma_1^2}{\sigma_1^2+\sigma_2^2}. \end{aligned} \tag{D.9}$$

Substituting (D.9) into (D.7), we get

$$\mu_{(F_1+F_2)}(\theta) = \exp\left\{-\frac{1}{2}\left[\frac{\theta-(m_1+m_2)}{\sqrt{\sigma_1^2+\sigma_2^2}}\right]^2\right\}. \tag{D.10}$$

This result generalizes easily to the sum of n Gaussian type-1 sets, $F_1, \ldots, F_n$; hence, $\sum_{i=1}^n F_i$, under product t-norm, is a Gaussian fuzzy number with mean $\sum_{i=1}^n m_i$ and standard deviation $\sqrt{\sum_{i=1}^n \sigma_i^2}$.

(ii) *Minimum t-norm*: In this case, (D.5) reduces to

$$F_1 + F_2 = \int_{v\in F_1}\int_{w\in F_2} \exp\left\{-\frac{1}{2}\left(\frac{v-m_1}{\sigma_1}\right)^2\right\} \wedge \exp\left\{-\frac{1}{2}\left(\frac{w-m_2}{\sigma_2}\right)^2\right\} \bigg/ [v+w]. \tag{D.11}$$

If θ is an element of $F_1 + F_2$, the membership grade of θ in $F_1 + F_2$ can be obtained by considering all the $\{v, w\}$ pairs such that $v \in F_1$ and $w \in F_2$ and $v + w = \theta$, finding the minimum of the memberships of v and w in every pair, and, choosing the maximum of all these minimums. In other words,

$$\mu_{(F_1+F_2)}(\theta) = \sup_v \left[\exp\left\{ -\frac{1}{2}\left(\frac{v - m_1}{\sigma_1}\right)^2 \right\} \wedge \exp\left\{ -\frac{1}{2}\left[\frac{(\theta - v) - m_2}{\sigma_2}\right]^2 \right\} \right]. \tag{D.12}$$

We make use of the fact that the supremum of the minimum of two Gaussians is reached at their point of intersection lying between their means. This point, v^*, is obtained by equating the equations of the two Gaussians, and is

$$v_* = \frac{m_1\sigma_2 + (\theta - m_2)\sigma_1}{\sigma_1 + \sigma_2}. \tag{D.13}$$

Since v_* is the point of intersection of the two Gaussians, it has the same membership in F_1 or F_2. Therefore, the membership grade of θ in $F_1 + F_2$ is

$$\mu_{(F_1+F_2)}(\theta) = \exp\left\{ -\frac{1}{2}\left(\frac{v_* - m_1}{\sigma_1}\right)^2 \right\} = \exp\left\{ -\frac{1}{2}\left(\frac{\theta - (m_1 + m_2)}{\sigma_1 + \sigma_2}\right)^2 \right\}. \tag{D.14}$$

This result generalizes easily to the case of n Gaussian type-1 sets, $F_1, \ldots, F_n$; hence, $\sum_{i=1}^n F_i$, under minimum t-norm, is a Gaussian fuzzy number with mean $\sum_{i=1}^n m_i$ and standard deviation $\sum_{i=1}^n \sigma_i$.

Combining parts (a) and (b), we get the desired result.

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