Type-2 Fuzzistics for Symmetric Interval Type-2 Fuzzy Sets: Part 1, Forward Problems

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***Abstract*—Interval type-2 fuzzy sets (T2 FS) play a central role in fuzzy sets as models for words and in engineering applications of T2 FSs. These fuzzy sets are characterized by their footprints of uncertainty (FOU), which in turn are characterized by their boundaries—upper and lower membership functions (MF). In this two-part paper, we focus on *symmetric* interval T2 FSs for which the centroid (which is an interval type-1 FS) provides a measure of its uncertainty. Intuitively, we anticipate that geometric properties about the FOU, such as its area and the center of gravities (centroids) of its upper and lower MFs, will be associated with the amount of uncertainty in such a T2 FS. The main purpose of this paper (Part 1) is to demonstrate that our intuition is correct and to quantify the centroid of a symmetric interval T2 FS, and consequently its uncertainty, with respect to such geometric properties. It is then possible, for the first time, to formulate and solve *forward problems*, i.e., to go from parametric interval T2 FS models to data with associated uncertainty bounds. We provide some solutions to such problems. These solutions are used in Part 2 to solve some inverse problems, i.e., to go from uncertain data to parametric interval T2 FS models (*T2 fuzzistics*).**

***Index Terms*—Centroid, fuzzistics, interval type-2 fuzzy sets, type-2 fuzzy sets.**

I. INTRODUCTION

A. Prolog

PROBABILITY is replete with parametric models that let us characterize random uncertainty. These models, e.g., probability density functions (pdf) such as the Normal, Bernoulli, Poisson, Exponential, Gamma, Rayleigh, etc., can be used for both forward and inverse problems. In a *forward problem*, a pdf model is chosen and all of its parameters are numerically specified; then, the model is used to generate random data. In an *inverse problem*, data are measured, a pdf model is chosen, and its parameters are then determined so that the model fits the data in some sense, e.g., using the principle of maximum-likelihood the model's parameters are chosen so that the model is most likely to have generated the measured data. For both the forward and the inverse problems, statistics can be used. In the forward problem, statistics can be used, e.g., to establish the sample mean and standard deviation of the data that have been generated from the model. In the inverse problem, statistics can be used, e.g., to establish properties of the parameter estimates such as their means and standard deviations. Clearly, for random uncertainty, probability and statistics go hand-in-hand whenever random data are present.

Fuzzy sets are replete with models that let us characterize linguistic uncertainty. These are membership function (MF) models, e.g., type-1 (triangle, trapezoidal, Gaussian, etc.) and type-2 (interval, noninterval), and these models can also be used for forward and inverse problems. In a *forward problem*, MF models are chosen (to characterize the words that are associated with a term, e.g., low pressure, medium pressure, and high pressure) and all of their parameters are numerically specified; then, the MF models can be used to generate words or MF values that are associated with numerical values of a primary variable. In an *inverse problem*, data are measured (e.g., MF values or intervals that are associated with MF levels[1]), a MF model is chosen and all of its parameters are then determined so that the model fits the data in some sense, e.g., using the principle of least-squares the MF model's parameters are chosen so that the MF model fits the data in a least-squares sense. Although statistics does not seem to play a role in the forward problem, because the data obtained are not random, statistics does play an important role in the inverse problem, because MF data collected from a group of subjects or even from a single subject at different times are random. In [6], we have coined the word *fuzzistics* to represent the interplay between fuzzy sets and statistics. Earlier works in the fuzzy literatures that focus on collecting type-1 MF data (e.g., [1]) represent *type-1 fuzzistics*. This two-part paper is about *type-2 fuzzistics*.

Recently, Mendel [6] argued that, because words are uncertain, type-2 fuzzy sets (FSs) should be used to model them. He then proposed an FS model for words that is based on collecting data from people—*person membership functions (MFs)*—that reflect *intra-* and *interlevels of uncertainties* about a word, in which a word FS is the union of all such person FSs. The *intrauncertainty* about a word is modeled using interval type-2 (T2) person FSs, and the *interuncertainty* about a word is modeled using an equally weighted union of each person's interval T2 FS. Even if it may not be practical to collect such person MF data from subjects today, it is practical to collect other kinds of data about words (this is explained later in Part 2) for which the uncertainties about the data can also be modeled using an interval T2 FS.

[1]Many methods have been developed for doing this for type-1 fuzzy sets (e.g., [1] and the many references in it), such as polling, direct rating, reverse rating and interval estimation.

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Because an interval T2 FS plays such an important role in the models for words as well as in many engineering applications of T2 FSs (e.g., [5]), we need to understand as much as possible about such sets, how they model uncertainties, and how their parameters can be expressed in terms of data that are collected from subjects. We do the former in Part 1 and the latter in Part 2.

B. Basics of an Interval T2 FS

Recall that an *interval T2 FS* $\tilde{A}$ is characterized as [5], [9]

$$\begin{aligned}\tilde{A} &= \int_{x\in X}\int_{u\in J_x\subseteq[0,1]} 1/(x,u)\\ &= \int_{x\in X}\left[\int_{u\in J_x\subseteq[0,1]} 1/u\right]\Big/ x \end{aligned} \tag{1}$$

where x, the *primary variable*, has domain X; u, the *secondary variable*, has domain J_x at each $x \in X$; J_x is called the *primary membership* of x; and the *secondary grades* of $\tilde{A}$ all equal 1. Uncertainty about $\tilde{A}$ is conveyed by the union of all of the primary memberships, which is called the *footprint of uncertainty* (FOU) of $\tilde{A}$, i.e.,

$$\mathrm{FOU}(\tilde{A}) = \cup_{x\in X} J_x. \tag{2}$$

The *upper membership function* (UMF) and *lower membership function* (LMF) of $\tilde{A}$ are two type-1 MFs that bound the FOU (e.g., see Fig. 1). The UMF is associated with the upper bound of $\mathrm{FOU}(\tilde{A})$ and is denoted $\bar{\mu}_{\tilde{A}}(x), \forall x \in X$, and the LMF is associated with the lower bound of $\mathrm{FOU}(\tilde{A})$ and is denoted $\underline{\mu}_{\tilde{A}}(x), \forall x \in X$, i.e.,

$$\bar{\mu}_{\tilde{A}}(x) \equiv \overline{\mathrm{FOU}(\tilde{A})} \qquad \forall x\in X \tag{3}$$

$$\underline{\mu}_{\tilde{A}}(x) \equiv \underline{\mathrm{FOU}(\tilde{A})} \qquad \forall x\in X. \tag{4}$$

For continuous universes of discourse X and U, an embedded interval T2 FS $\tilde{A}_e$ is

$$\tilde{A}_e = \int_{x\in X} [1/\theta]/x \quad \theta\in J_x \subseteq U=[0,1]. \tag{5}$$

Set $\tilde{A}_e$ is embedded in $\tilde{A}$ such that at each x it only has one secondary variable, and there are an uncountable number of embedded interval T2 FSs. Examples of $\tilde{A}_e$ are[2] $1/\bar{\mu}_{\tilde{A}}(x)$ and $1/\underline{\mu}_{\tilde{A}}(x), \forall x \in X$. Associated with each $\tilde{A}_e$ is an *embedded T1 FS* A_e, where

$$A_e = \int_{x\in X} \theta/x \quad \theta\in J_x \subseteq U=[0,1]. \tag{6}$$

Set A_e, which acts as the domain for $\tilde{A}_e$, is the union of all the primary memberships of the set $\tilde{A}_e$ in (5), and there are an uncountable number of A_e. Examples of A_e are $\bar{\mu}_{\tilde{A}}(x)$ and $\underline{\mu}_{\tilde{A}}(x), \forall x \in X$.

[2]In this notation, it is understood that the secondary grade equals 1 at all elements in $\mathrm{FOU}(\tilde{A})$.

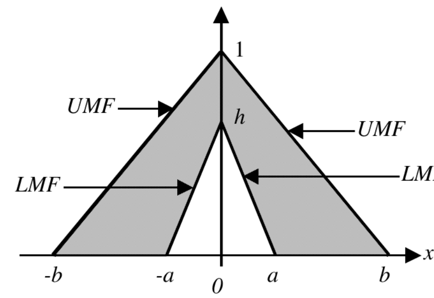


Fig. 1. Symmetrical triangular FOU.

In this paper, we focus on an important sub-class of an interval T2 FS, namely the *symmetric* interval T2 FS, for which the FOU is symmetrical about $x = m$, i.e.,

$$\bar{\mu}_{\tilde{A}}(m+x) = \bar{\mu}_{\tilde{A}}(m-x) \tag{7}$$

$$\underline{\mu}_{\tilde{A}}(m+x) = \underline{\mu}_{\tilde{A}}(m-x). \tag{8}$$

For notational simplicity, in the rest of this paper we let $\bar{\mu}_{\tilde{A}} \equiv \bar{\mu}$ and $\underline{\mu}_{\tilde{A}} \equiv \underline{\mu}$.

C. Goals and Coverage

Recall that probability lets us characterize random uncertainty using measures such as the *mean* (expected value), *standard deviation*, *entropy*, etc., and that statistics lets us estimate these measures from data using the *sample mean*, *sample standard deviation*, *sample entropy*, etc. What are the measures that characterize linguistic uncertainty? One such measure is the centroid of an interval T2 FS [4], which is an interval type-1 (T1) FS (why this is a legitimate measure of linguistic uncertainty is explained in Section II).

Intuitively, we anticipate that geometric properties about the FOU, such as its area and the center of gravities (centroids) of its upper and lower MFs, will be associated with the amount of uncertainty in an interval T2 FS, e.g., the larger (smaller) the area of the FOU the larger (smaller) the uncertainty of $\tilde{A}$. The main purposes of this two-part paper are to demonstrate that our intuition is correct, to quantify the centroid of an interval T2 FS with respect to the geometric properties of its FOU in Part 1 (this is associated with *forward problems*), and to then formulate and solve *inverse problems*, i.e., to go from uncertain data (that can be elicited from subjects) to parametric interval T2 FS models in Part 2.

A FOU can be symmetric or nonsymmetric, and, as we mentioned above, in this paper we focus exclusively on the case of a symmetric FOU. The case of a nonsymmetric FOU is treated in a separate paper because it requires analyses more general than those used herein for the symmetric FOU. Knowledge that a FOU is symmetric acts, in effect, like a constraint on the more general nonsymmetric FOU, and we have found that such a constraint should be used at the front-end of the analyses in order to obtain the most useful results for such a FOU.

In Section II, we justify the use of the centroid of an interval T2 FS as a legitimate measure of the uncertainty of $\tilde{A}$, after which we provide a mathematical interpretation for the

Karnik-Mendel method for computing the centroid of a general (symmetric or non-symmetric) interval T2 FS and review some new results for such a centroid. In Section III we establish upper and lower bounds for the two end-points of the centroid of a symmetric interval T2 FS and express them in terms of geometric properties of the FOU for such a FS. The difference between the upper and lower bounds for each of the end-points of the centroid is called an *uncertainty interval*. In Section IV, we provide formulas that connect the parameters of specific FOUs to their uncertainty intervals. These examples represent *forward problems* and their solutions. Conclusions and directions for future work are given in Section V.

II. Centroid of an Interval Type-2 Fuzzy Set

A. Introduction

Recall that the centroid $C_{\tilde{A}}$ of the interval T2 FS $\tilde{A}$ is an interval set $[c_l, c_r]$ that is completely specified by its left and right end-points[3], $c_l(\tilde{A}) \equiv c_l$ and $c_r(\tilde{A}) \equiv c_r$, respectively, i.e., [4], [5]

$$\begin{aligned} C_{\tilde{A}} &= [c_l, c_r] \\ &= \int_{\theta_1 \in J_{x_1}} \cdots \int_{\theta_N \in J_{x_N}} 1 \Big/ \frac{\sum_{i=1}^{N} x_i \theta_i}{\sum_{i=1}^{N} \theta_i}. \end{aligned} \tag{9}$$

In this equation, which represents the union of the centroids of all of the embedded type-1 fuzzy sets of $\tilde{A}$, primary variable x has been discretized for computational purposes, such that $x_1 < x_2 < \cdots < x_N$. Unfortunately, no closed-form formulas exist to compute c_l and c_r; however, Karnik and Mendel [4] have developed iterative procedures for computing these end-points exactly, and recently Mendel [8] proved that given a FOU for a symmetric interval T2 FS, then the centroid of such a T2 FS is also symmetrical about $x = m$. For such a T2 FS it is therefore only necessary to compute either c_l or c_r, resulting in a 50% savings in computation.

Before we summarize the Karnik–Mendel (KM) procedures in a form that will be very useful to us, we must first justify the use of the length $c_r - c_l$ as a legitimate measure of the uncertainty of $\tilde{A}$. Wu and Mendel [12] noted that according to information theory uncertainty of a random variable is measured by its entropy [2]. Recall that a one-dimensional random variable that is uniformly distributed over a region has entropy equal to the logarithm of the *length of the region*. Comparing the MF, $\mu_C(x)$, of an interval FS C, where

$$\mu_C(x) = \begin{cases} 1, & x \in [c_l, c_r] \\ 0, & \text{otherwise} \end{cases} \tag{10}$$

with probability density function, $p_Y(y)$, of a random variable Y, which is uniformly distributed over $[c_l, c_r]$, where

$$p_Y(y) = \begin{cases} 1/(c_r - c_l), & y \in [c_l, c_r] \\ 0, & \text{otherwise} \end{cases} \tag{11}$$

we find that they are almost the same except for their amplitudes. Therefore, it is reasonable to consider the *extent of the uncertainty* of the FS C to be the same as (or proportional to) that of the random variable Y. Since the centroid of a T2 FS is an interval set, its length can therefore be used to measure the extent of the T2 FS's uncertainty[4].

B. KM Procedures: Interpretation

The KM iterative procedures for computing c_l and c_r (the details of which are not needed in this paper) can be interpreted for the purposes of this paper as follows [12]. Let $A_e(l)$ denote an embedded T1 FS for which

$$\mu_{A_e(l)}(x) = \begin{cases} \bar{\mu}(x), & \text{if } x \le l \\ \underline{\mu}(x), & \text{if } x > l \end{cases} \tag{12}$$

where l is a *switch point*, i.e., the value of x at which $A_e(l)$ switches from $\bar{\mu}(x)$ to $\underline{\mu}(x)$. Then

$$c_l(\tilde{A}) = \min_{l \in X} \text{ centroid}(A_e(l)) \tag{13}$$

where[5]

$$\text{centroid}(A_e(l)) = \frac{\int_{-\infty}^{l} x\bar{\mu}(x)dx + \int_{l}^{\infty} x\underline{\mu}(x)dx}{\int_{-\infty}^{l} \bar{\mu}(x)dx + \int_{l}^{\infty} \underline{\mu}(x)dx}. \tag{14}$$

Similarly, let $A_e(r)$ denote an embedded T1 FS for which

$$\mu_{A_e(r)}(x) = \begin{cases} \underline{\mu}(x), & \text{if } x \le r \\ \bar{\mu}(x), & \text{if } x > r \end{cases} \tag{15}$$

where r is another *switch point*, i.e., the value of x at which $A_e(r)$ switches from $\underline{\mu}(x)$ to $\bar{\mu}(x)$. Then

$$c_r(\tilde{A}) = \max_{r \in X} \text{ centroid}(A_e(r)) \tag{16}$$

where

$$\text{centroid}(A_e(r)) = \frac{\int_{-\infty}^{r} x\underline{\mu}(x)dx + \int_{r}^{\infty} x\bar{\mu}(x)dx}{\int_{-\infty}^{r} \underline{\mu}(x)dx + \int_{r}^{\infty} \bar{\mu}(x)dx}. \tag{17}$$

The solutions for l and r in (13) and (16) are found by using the KM iterative procedures. That $A_e(l)$ and $A_e(r)$, which lead to c_l and c_r, only involve the lower and upper MFs of $\tilde{A}$, and there is only one switch between them, are theoretical results that are proven by Karnik and Mendel [4].

C. Centroid Facts for General Interval T2 FSs

Because Karnik and Mendel developed their iterative procedures by first discretizing X and J_x, they were apparently un-

[3]We use the notation $c_l(\tilde{A}) \equiv c_l$ and $c_r(\tilde{A}) \equiv c_r$ interchangeably.

[4]The *entropy* of a T2 FS should also provide a measure of the uncertainty of $\tilde{A}$; however, to-date, such entropy has not yet appeared in the FS literature. Because many entropies of a T1 FS have been published (e.g., [3]), each giving a different numerical value of entropy, we expect a similar situation to occur for entropy of a T2 FS. On the other hand, the centroid of $\tilde{A}$ provides a unique measure, since there is only one such centroid.

[5]Although the KM procedures are derived and usually stated in discrete form, for the purposes of this paper it is more convenient to summarize them in continuous form.

aware of the following important results that serve as the bases for the rest of this paper.

Theorem 1: Let $\tilde{A}$ be an interval T2 FS defined on X with lower MF $\underline{\mu}(x)$ and upper MF $\bar{\mu}(x)$. The centroid of $\tilde{A}$ is an interval T1 FS that is characterized by its left and right end-points c_l and c_r, where

$$c_l(\tilde{A}) = \frac{\int_{-\infty}^{c_l} x\bar{\mu}(x)dx + \int_{c_l}^{\infty} x\underline{\mu}(x)dx}{\int_{-\infty}^{c_l} \bar{\mu}(x)dx + \int_{c_l}^{\infty} \underline{\mu}(x)dx} \tag{18}$$

$$c_r(\tilde{A}) = \frac{\int_{-\infty}^{c_r} x\underline{\mu}(x)dx + \int_{c_r}^{\infty} x\bar{\mu}(x)dx}{\int_{-\infty}^{c_r} \underline{\mu}(x)dx + \int_{c_r}^{\infty} \bar{\mu}(x)dx}. \tag{19}$$

Proof: See Appendix C. ■

The results in (18) and (19) are very interesting, because they show that for c_l, when the value of l is found that minimizes centroid$(A_e(l))$ it will be $l = c_l$; and, for c_r, when the value of r is found that maximizes centroid$(A_e(r))$ it will be $r = c_r$. Of course, if X is discretized (for computational purposes) then $l \to L \approx c_l$ but L does not exactly equal c_l, and $r \to R \approx c_r$ but R does not exactly equal c_r, which probably explains why (18) and (19) were not observed by Karnik and Mendel.

Theorem 2: Let $\tilde{A}$ be an interval T2 FS defined on X, and $\tilde{A}'$ be $\tilde{A}$ shifted by[6] Δm along X, i.e.,

$$\underline{\mu}_{\tilde{A}'}(x) = \underline{\mu}_{\tilde{A}}(x - \Delta m) \tag{20}$$

$$\bar{\mu}_{\tilde{A}'}(x) = \bar{\mu}_{\tilde{A}'}(x - \Delta m). \tag{21}$$

Then, the centroid of $\tilde{A}'$, $[c_l(\tilde{A}'), c_r(\tilde{A}')]$, is the same as the centroid of $\tilde{A}$, $[c_l(\tilde{A}), c_r(\tilde{A})]$, shifted by Δm, i.e.,

$$c_l(\tilde{A}') = c_l(\tilde{A}) + \Delta m \tag{22}$$

$$c_r(\tilde{A}') = c_r(\tilde{A}) + \Delta m. \tag{23}$$

Proof: See [11]. ■

Theorem 2 demonstrates that the span of the centroid set of an interval T2 FS is *shift-invariant*. This means that regardless of where along X the FOU of $\tilde{A}$ occurs, as long as the FOU for $\tilde{A}$ is unchanged, then

$$\begin{aligned}\text{span[centroid}(\tilde{A}')] &= c_r(\tilde{A}') - c_l(\tilde{A}') \\ &= \text{span[centroid}(\tilde{A})] = c_r(\tilde{A}) - c_l(\tilde{A}).\end{aligned} \tag{24}$$

Theorem 2 also justifies our shifting the FOU of $\tilde{A}$ to a possibly more convenient point along X for the actual computation of the centroid. When we do that, we are computing $c_l(\tilde{A}')$ and $c_r(\tilde{A}')$, after which we can compute the centroid of $\tilde{A}$, using (22) and (23), as $c_l(\tilde{A}) = c_l(\tilde{A}') - \Delta m$ and $c_r(\tilde{A}) = c_r(\tilde{A}') - \Delta m$.

D. Centroid Fact for a Symmetrical Interval T2 FS

Theorems 1 and 2 are valid for all interval T2 FSs, symmetrical or non-symmetrical. For the centroid of a symmetrical interval T2 FS, we also have the following.

[6] Δm may be positive or negative.

Theorem 3: If the interval T2 FS $\tilde{A}$ defined on X is symmetrical about $m \in X$, then

$$c_l \le m \tag{25}$$

$$c_r \ge m. \tag{26}$$

More specifically

$$c_r = 2m - c_l. \tag{27}$$

Proof: See [11] for the proofs of (25) and (26). Equation (27) follows from Mendel [7], [8] in which, as mentioned above, it is proved that given a FOU for a symmetric interval T2 FS, then the centroid of such a T2 FS is also symmetrical about $x = m$; hence, $(c_r + c_l)/2 = m$ from which (27) follows. ■

III. Bounds on c_l and c_r for a Symmetric FOU

Because closed-form formulas do not exist for c_l and c_r we have not been able to study how these end-points explicitly depend upon geometric properties of the FOU, namely on the area of the FOU and the centroids of the upper and lower MFs of the FOU. The approach taken in the rest of this paper is to obtain bounds for both c_l and c_r, and to then examine the explicit dependencies of these bounds on the geometric properties of the FOU.

The geometric properties that we shall make use of, for a FOU that is symmetric about m, are

- A_{UMF}: Area under the upper MF;
- A_{LMF}: Area under the lower MF;
- A_{FOU}: Area of the FOU, where

$$\begin{aligned}A_{\mathrm{FOU}} &= A_{\mathrm{UMF}} - A_{\mathrm{LMF}} \\ &= 2\int_m^{\infty} [\bar{\mu}(x) - \underline{\mu}(x)]dx;\end{aligned} \tag{28}$$

- $c_{\mathrm{HFOU}}(\tilde{A})$: Centroid of half of $FOU(\tilde{A})$, where

$$\begin{aligned}c_{\mathrm{HFOU}}(\tilde{A}) &= \frac{\int_m^{\infty} x[\bar{\mu}(x) - \underline{\mu}(x)]dx}{\int_m^{\infty} [\bar{\mu}(x) - \underline{\mu}(x)]dx} \\ &= \frac{\int_m^{\infty} x[\bar{\mu}(x) - \underline{\mu}(x)]dx}{A_{\mathrm{FOU}}/2}.\end{aligned} \tag{29}$$

c_r is computed based upon (16) and (17), and this involves one switch between $\underline{\mu}(x)$ and $\bar{\mu}(x)$. When the switch occurs at the optimal value of $x = c_r$ then, of course, c_r is at its maximum value. When the switch occurs at any other value of x then centroid$(A_e(r)) \le c_r$; hence, any such switch point can provide a valid lower bound for c_r.

Definition: The centroid of an[7] *arbitrary* embedded T1 FS that switches once between $\underline{\mu}(x)$ and $\bar{\mu}(x)$ provides us with a *valid lower bound*[8] for c_r. A valid bound is not necessarily the best bound, but it may be one that can be expressed in terms of the geometric properties of the FOU. ■

[7] Although "arbitrary," it should be a *carefully chosen* embedded T1 FS, or else the bounds will be too loose to be of much use.

[8] Or a *valid upper bound* for c_l because centroid$(A_e(l)) \ge c_l$.

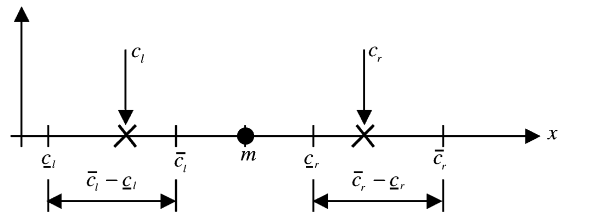


Fig. 2. End-points (X) of the centroid $[c_l, c_r]$ of $\tilde{A}$ for a FOU that is symmetrical about m and the lower and upper bounds (|) for the two end-points.

Imposing symmetry constraints on $\mathrm{FOU}(\tilde{A})$ we obtain the following *valid lower* and *upper bounds* for both c_l and c_r:

Theorem 4: Let $[x_1, x_N]$ be the primary domain of $\tilde{A}$. Then the end-points, c_l and c_r for the centroid of a symmetric interval T2 FS, $\tilde{A}$, are bounded from below and above by (Fig. 2)[9]

$$\max(x_1, \underline{c}_l(\tilde{A})) \le c_l(\tilde{A}) \le \bar{c}_l(\tilde{A}) \tag{30}$$

$$\underline{c}_r(\tilde{A}) \le c_r(\tilde{A}) \le \min(\bar{c}_r(\tilde{A}), x_N) \tag{31}$$

where

$$\underline{c}_r(\tilde{A}) = m + [c_{\mathrm{HFOU}}(\tilde{A}) - m]\frac{A_{\mathrm{FOU}}}{A_{\mathrm{UMF}} + A_{\mathrm{LMF}}} \ge m \tag{32}$$

$$\bar{c}_r(\tilde{A}) = m + [c_{\mathrm{HFOU}}(\tilde{A}) - m]\frac{A_{\mathrm{FOU}}}{2A_{\mathrm{LMF}}} \tag{33}$$

$$\bar{c}_l(\tilde{A}) = 2m - \underline{c}_r(\tilde{A}) \tag{34}$$

$$\underline{c}_l(\tilde{A}) = 2m - \bar{c}_r(\tilde{A}). \tag{35}$$

Proof: See Appendix A. ■

Comment 1: Theorem 4 demonstrates that the bounds and their associated bounding intervals (*uncertainty intervals*)—$\bar{c}_r - \underline{c}_r(\bar{c}_l - \underline{c}_l)$—for the end-points of the centroid of $\tilde{A}$ are indeed expressible in terms of geometric properties of the FOU. It has made use of the *a priori* geometric knowledge about the symmetry of the FOU. ■

Comment 2: The proof of Theorem 4 demonstrates that a valid way to compute $\underline{c}_r(\tilde{A})$ and $\bar{c}_r(\tilde{A})$ is to do it first for the symmetric FOU shifted to the origin, i.e., for *FOU*$(\tilde{A}')$, in which case we set $m = 0$ in (32) and (33) to obtain $\underline{c}_r(\tilde{A}')$ and $\bar{c}_r(\tilde{A}')$. Then, based on Theorem 2 [(22) and (23)], we add m to those results to obtain $\underline{c}_r(\tilde{A})$ and $\bar{c}_r(\tilde{A})$ for the original unshifted FOU. ■

IV. Solutions to Some Forward Problems

In this section, we provide four examples that illustrate Theorem 4. Each example illustrates the solution to a *forward problem*, namely, given $\mathrm{FOU}(\tilde{A}')$ we compute the centroid uncertainty bounds given by (32) and (33) with $m = 0$.

Example 1: Symmetric FOU—Lower MF is Triangular and Upper MF is Trapezoidal or Triangular: In this case (see the figure in the first row of Table I)

$$\underline{\mu}_{\tilde{A}'}(x) = \begin{cases} h(x+a)/a, & \text{if } -a \le x \le 0 \\ h(a-x)/a, & \text{if } 0 \le x \le a \\ 0, & \text{otherwise} \end{cases} \tag{36}$$

$$\bar{\mu}_{\tilde{A}'}(x) = \begin{cases} (x+b)/(b-c), & \text{if } -b \le x \le -c \\ 1, & \text{if } -c \le x \le c \\ (b-x)/(b-c), & \text{if } c \le x \le b \\ 0, & \text{otherwise} \end{cases} \tag{37}$$

where $0 \le a \le b, 0 \le c \le b$ and $0 \le h \le 1$. The quantities that are used in $\underline{c}_r(\tilde{A}')$ and $\bar{c}_r(\tilde{A}')$ are computed as

$$A_{\mathrm{LMF}} = ah \tag{38}$$

$$A_{\mathrm{UMF}} = b + c \tag{39}$$

$$\begin{aligned} c_{\mathrm{HFOU}}(\tilde{A}') &= \frac{2\int_0^\infty x[\bar{\mu}(x) - \underline{\mu}(x)]dx}{A_{\mathrm{FOU}}} \\ &= \frac{2\int_0^\infty x[\bar{\mu}(x) - \underline{\mu}(x)]dx}{A_{\mathrm{UMF}} - A_{\mathrm{LMF}}} \end{aligned} \tag{40}$$

where

$$\begin{aligned} \int_0^\infty x\bar{\mu}(x)dx &= \int_0^c x dx + \int_c^b x\frac{b-x}{b-c}dx \\ &= \frac{1}{2}c^2 + \int_c^b \left(\frac{b}{b-c}x - \frac{x^2}{b-c}\right)dx \\ &= \frac{1}{2}c^2 + \frac{b}{2(b-c)}(b^2 - c^2) \\ &\quad - \frac{1}{3(b-c)}(b^3 - c^3) \\ &= \frac{1}{6}(c^2 + b^2 + bc) \end{aligned} \tag{41}$$

and

$$\begin{aligned} \int_0^\infty x\underline{\mu}(x)dx &= \int_0^a x\frac{h(a-x)}{a}dx \\ &= \frac{h}{a}\int_0^a (ax - x^2)dx \\ &= \frac{1}{6}ha^2. \end{aligned} \tag{42}$$

Consequently, $c_{\mathrm{HFOU}}(\tilde{A}'), \bar{c}_r(\tilde{A}')$ and $\underline{c}_r(\tilde{A}')$ are given by the entries in the first row of Table I.

Table I also shows six special cases of these most general results. We include them because their FOUs are geometrically quite different looking than the FOU of the general case. $\bar{c}_r(\tilde{A}')$ and $\underline{c}_r(\tilde{A}')$ are easily obtained for these special cases by making the appropriate substitutions into the results given for the general case. ■

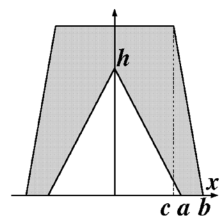
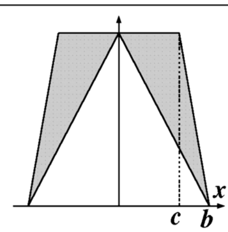
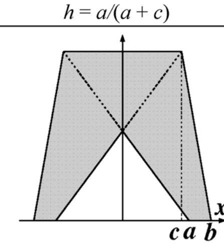
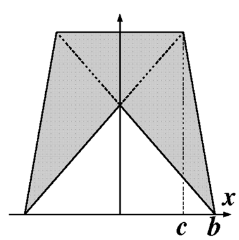
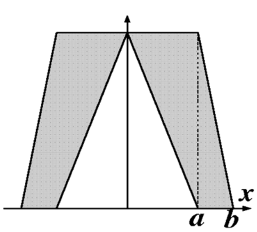
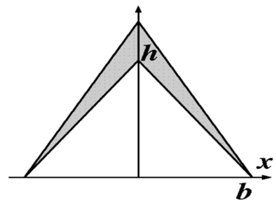
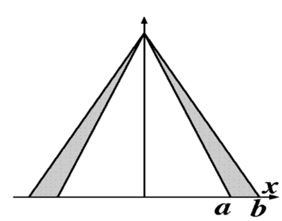
Example 2: Symmetric FOU—Lower MF and Upper MF are Trapezoidal: In this case (see the figure in the first row of Table II)

$$\underline{\mu}_{\tilde{A}'}(x) = \begin{cases} (x+a)/(a-c), & \text{if } -a \le x \le -c \\ 1, & \text{if } -c \le x \le c \\ (a-x)/(a-c), & \text{if } c \le x \le a \\ 0, & \text{otherwise} \end{cases} \tag{43}$$

$$\bar{\mu}_{\tilde{A}'}(x) = \begin{cases} (x+b)/(b-d), & \text{if } -b \le x \le -d \\ 1, & \text{if } -d \le x \le d \\ (b-x)/(b-d), & \text{if } d \le x \le b \\ 0, & \text{otherwise} \end{cases} \tag{44}$$

[9]In general, $-\infty > x_1 > x_N > \infty$, e.g., if the primary MF is Gaussian, then its associated FOU extends to $\pm\infty$. In that case, $\max(x_1, \underline{c}_l(\tilde{A})) = \underline{c}_l(\tilde{A})$ and $\min(\bar{c}_r(\tilde{A}), x_N) = \bar{c}_r(\tilde{A})$. For most other FOUs, x_1 and x_N are finite numbers, and we need to use the more complete bounds in (30) and (31).

TABLE I
THE SYMMETRICAL INTERVAL TYPE-2 FUZZY SET ($\tilde{A}'$) WHOSE LMF IS TRIANGULAR AND UMF IS TRAPEZOIDAL (OR TRIANGULAR)

<table>
<tr><th colspan="2">General case that is specified by four parameters (a, b, c, h)</th></tr>
<tr><td>
</td><td>A_{UMF} $b+c$
A_{LMF} ah
c_{HFOU} $\dfrac{c^2+b^2+bc-ha^2}{3(b+c-ha)}$
$\bar{c}_r$ $\dfrac{c^2+b^2+bc-ha^2}{6ha}$
$\underline{c}_r$ $\dfrac{c^2+b^2+bc-ha^2}{3(b+c+ha)}$</td></tr>
<tr><td>Special case 1 for which
$h = 1$ and $a = b$</td><td>Special case 2 for which
$h = a/(a+c)$</td></tr>
<tr><td>
</td><td>
</td></tr>
<tr><td>Special case 3 for which
$a = b$ and $h = b/(b+c)$</td><td>Special case 4 for which
$h = 1$ and $a = c$</td></tr>
<tr><td>
</td><td>
</td></tr>
<tr><td>Special case 5 for which
$c = 0$ and $a = b$</td><td>Special case 6 for which
$c = 0$ and $h = 1$</td></tr>
<tr><td>
</td><td>
</td></tr>
</table>

where $0 \le c \le a$ and $0 \le d \le b$. The quantities that are used in $\underline{c}_r(\tilde{A}')$ and $\bar{c}_r(\tilde{A}')$ are computed as

$$A_{\mathrm{LMF}} = a + c \tag{45}$$

$$A_{\mathrm{UMF}} = b + d \tag{46}$$

and $c_{\mathrm{HFOU}}(\tilde{A}')$ is given by (40) in which

$$\begin{aligned}\int_0^\infty x\bar{\mu}(x)dx &= \int_0^d xdx + \int_d^b x\frac{b-x}{b-d}dx \\ &= \frac{1}{2}d^2 + \int_d^b \left(\frac{b}{b-d}x - \frac{x^2}{b-d}\right)dx \\ &= \frac{1}{6}(b^2 + d^2 + bd)\end{aligned} \tag{47}$$

and

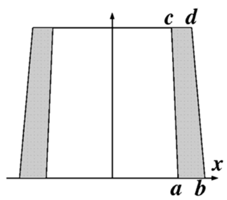
$$\begin{aligned}\int_0^\infty x\underline{\mu}(x)dx &= \int_0^c xdx + \int_c^a x\frac{a-x}{a-c}dx \\ &= \frac{1}{2}c^2 + \int_c^a \left(\frac{a}{a-c}x - \frac{x^2}{a-c}\right)dx \\ &= \frac{1}{6}(a^2 + c^2 + ac).\end{aligned} \tag{48}$$

Consequently, $c_{\mathrm{HFOU}}(\tilde{A}'), \bar{c}_r(\tilde{A}')$ and $\underline{c}_r(\tilde{A}')$ are given by the entries in the first row of Table II.

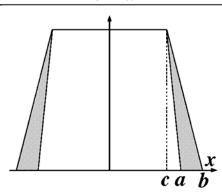
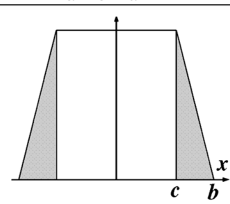
Table II also shows two special cases of the general results. We again include these special cases because their FOUs are geometrically quite different looking than the FOU of the general case. ■

TABLE II
THE SYMMETRICAL INTERVAL TYPE-2 FUZZY SET ($\tilde{A}'$) WHOSE LMF AND UMF ARE TRAPEZOIDAL

General case that is specified by four parameters (a, b, c, d)



A_{UMF}	$b+d$
A_{LMF}	$a+c$
c_{HFOU}	$\dfrac{(b^2+bd+d^2)-(a^2+ac+c^2)}{3(b+d-a-c)}$
$\bar{c}_r$	$\dfrac{(b^2+bd+d^2)-(a^2+ac+c^2)}{6(a+c)}$
$\underline{c}_r$	$\dfrac{(b^2+bd+d^2)-(a^2+ac+c^2)}{3(a+b+c+d)}$

Special case 1 where $c = d$	**Special case 2** where $a = c = d$
	

Example 3: Symmetric FOU—Gaussian With Uncertain Mean and Standard Deviation: In this case (see the figure in the first row of Table III)

$$\underline{\mu}_{\tilde{A}'}(x) = \begin{cases} \exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma_1}\right)^2\right\}, & \text{if } x \le 0 \\ \exp\left\{-\frac{1}{2}\left(\frac{x+\mu}{\sigma_1}\right)^2\right\}, & \text{if } x \ge 0 \end{cases} \tag{49}$$

$$\bar{\mu}_{\tilde{A}'}(x) = \begin{cases} \exp\left\{-\frac{1}{2}\left(\frac{x+\mu}{\sigma_2}\right)^2\right\}, & \text{if } x \le -\mu \\ 1, & \text{if } -\mu \le x \le \mu \\ \exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma_2}\right)^2\right\}, & \text{if } x \ge \mu \end{cases}. \tag{50}$$

The quantities that are used in $\underline{c}_r(\tilde{A}')$ and $\bar{c}_r(\tilde{A}')$ are computed as

$$\begin{aligned} A_{\mathrm{LMF}} &= \int_{-\infty}^{\infty} \underline{\mu}(x)dx = 2\int_0^{\infty} \underline{\mu}(x)dx \\ &= 2\int_0^{\infty} \exp\left\{-\frac{1}{2}\left(\frac{x+\mu}{\sigma_1}\right)^2\right\}dx \\ &= 2\sigma_1 \int_{\mu/\sigma_1}^{\infty} \exp\left(-\frac{1}{2}y^2\right)dy \triangleq 2\sqrt{2\pi}\sigma_1 \Phi\left(\frac{\mu}{\sigma_1}\right) \end{aligned} \tag{51}$$

where

$$\Phi(x) \equiv \frac{1}{\sqrt{2\pi}} \int_x^{\infty} \exp\left(-\frac{1}{2}t^2\right)dt \tag{52}$$

$$\begin{aligned} A_{\mathrm{UMF}} &= \int_{-\infty}^{\infty} \bar{\mu}(x)dx = 2\int_0^{\infty} \bar{\mu}(x)dx \\ &= 2\left[\int_0^{\mu} dx + \int_{\mu}^{\infty} \exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma_2}\right)^2\right\}dx\right] \\ &= 2\left[\mu + \sigma_2 \int_0^{\infty} \exp\left(-\frac{1}{2}y^2\right)dy\right] = 2\mu + \sqrt{2\pi}\sigma_2 \end{aligned} \tag{53}$$

and $c_{\mathrm{HFOU}}(\tilde{A}')$, which is given by (40), is computed in Appendix B.

Quantities $c_{\mathrm{HFOU}}(\tilde{A}'), \bar{c}_r(\tilde{A}')$ and $\underline{c}_r(\tilde{A}')$ are given by the entries in the first part of Table III.

Table III also shows two special cases of the general results. We again include these special cases because their FOUs are geometrically quite different looking than the FOU of the general case. ■

Example 4: Symmetric FOU—Gaussian UMF and Scaled Gaussian LMF: In this case (see the figure in the last row of Table III)

$$\bar{\mu}(x) = \exp\left\{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2\right\} \tag{54}$$

$$\underline{\mu}(x) = s\exp\left\{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2\right\}. \tag{55}$$

Because the calculations of the quantities that are used in $\bar{c}_r(\tilde{A}')$ and $\underline{c}_r(\tilde{A}')$ are very similar to the ones just given in Example 3, we leave them to the reader. Quantities $c_{\mathrm{HFOU}}(\tilde{A}'), \bar{c}_r(\tilde{A}')$ and $\underline{c}_r(\tilde{A}')$ are given by the entries in the last row of Table III. Compared to the results in Example 3, the results for this example are quite simple. ■

Using the results in Tables I–III it is possible to choose a FOU, specify numerical values for its parameters, and then compute $\underline{c}_r$ and $\bar{c}_r$. By varying the parameters of each FOU, it is then possible to observe how $[\underline{c}_r, \bar{c}_r]$ varies, and it is also possible to study when, or if, $x_1 > \underline{c}_l(\tilde{A})$ and $x_N < \bar{c}_r(\tilde{A})$ [see (30) and (31)]. Because such studies are not central to this paper and its companion paper (Part 2), we leave them to the interested reader.

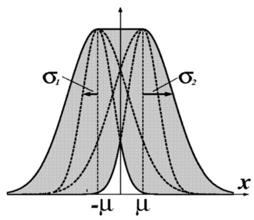
V. CONCLUSION

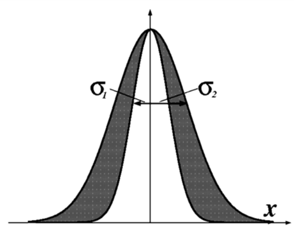
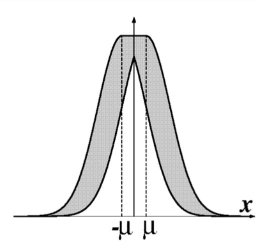
We have demonstrated that the centroid of an interval T2 FS provides a measure of the uncertainty in such a FS. The centroid is a type-1 FS that is completely described by its two end-points. Although it is not possible to obtain closed-form formulas for these end-points, we have established closed-form formulas for their upper and lower bounds. Most importantly, these bounds have been expressed in terms of geometric properties of the FOU, namely its area, the areas under the UMF and the LMF, and the centers of gravity of half of the UMF and half of the LMF (the latter two describe the center of gravity of half of the FOU). As a result, for the first time it is possible to quantify the uncertainty of an interval T2 FS with respect to these geometric properties of its FOU.

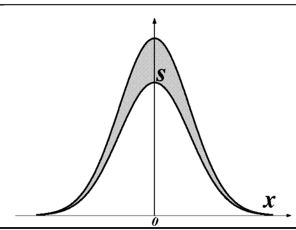
Using the results in this paper, it is possible to examine many "forward" problems, i.e., given a class of footprints of uncertainty (e.g., triangular, trapezoidal, Gaussian) we have established the bounds on the centroid as a function of the parameters that define the FOU, and have summarized many such results in Tables I–III.

In Part 2, we examine "inverse" problems, i.e., given data collected from people about a phrase, and the inherent uncertainties associated with that data (which can be described statistically), we establish parametric FOUs such that their uncertainty bounds are directly connected to statistical uncertainty bounds.

TABLE III
THE SYMMETRICAL INTERVAL TYPE-2 FUZZY SET ($\tilde{A}'$) WHOSE LMF AND UMF ARE BASED ON GAUSSIAN FUNCTIONS

General case where the membership function (MF) is Gaussian with uncertain mean $\in [-\mu, \mu]$ and uncertain standard deviation $\in [\sigma_1, \sigma_2]$		
	A_{UMF}	$2\mu+\sqrt{2\pi}\sigma_2$
	A_{LMF}	$2\sqrt{2\pi}\sigma_1\Phi(\mu/\sigma_1)$
	c_{HFOU}	$\dfrac{\mu^2+2\sigma_2^2+\sqrt{2\pi}\mu\sigma_2-2\sigma_1^2\exp\left\{-\frac{1}{2}(\mu/\sigma_1)^2\right\}+2\sqrt{2\pi}\mu\sigma_1\Phi(\mu/\sigma_1)}{2\mu+\sqrt{2\pi}\sigma_2-2\sqrt{2\pi}\sigma_1\Phi(\mu/\sigma_1)}$
$\overline{c}_r$	$\dfrac{\sigma_1\left[(\mu/\sigma_1)^2+2(\sigma_2/\sigma_1)^2+\sqrt{2\pi}(\mu/\sigma_1)(\sigma_2/\sigma_1)-2\exp\left\{-\frac{1}{2}(\mu/\sigma_1)^2\right\}+2\sqrt{2\pi}(\mu/\sigma_1)\Phi(\mu/\sigma_1)\right]}{4\sqrt{2\pi}\Phi(\mu/\sigma_1)}$	
$\underline{c}_r$	$\dfrac{\sigma_1\left[(\mu/\sigma_1)^2+2(\sigma_2/\sigma_1)^2+\sqrt{2\pi}(\mu/\sigma_1)(\sigma_2/\sigma_1)-2\exp\left\{-\frac{1}{2}(\mu/\sigma_1)^2\right\}+2\sqrt{2\pi}(\mu/\sigma_1)\Phi(\mu/\sigma_1)\right]}{2(\mu/\sigma_1)+\sqrt{2\pi}(\sigma_2/\sigma_1)+2\sqrt{2\pi}\Phi(\mu/\sigma_1)}$	

Special case 1 where $\mu = 0$ so that the MF is Gaussian with uncertain standard deviation $\in [\sigma_1, \sigma_2]$	**Special case 2** where $\sigma_1 = \sigma_2 \equiv \sigma$ so that the MF is Gaussian with uncertain mean $\in [-\mu, \mu]$ and standard deviation σ
	

Symmetrical interval type-2 fuzzy set $\tilde{A}$ whose UMF is Gaussian and whose LMF is the scaled UMF (as in Example 4)		
	A_{UMF}	$\sqrt{2\pi}\sigma$
	A_{LMF}	$s\sqrt{2\pi}\sigma$
	c_{HFOU}	$\sigma\sqrt{2/\pi}$
	$\overline{c}_r$	$\sigma(1-s)/\sqrt{2\pi}s$
	$\underline{c}_r$	$\sigma\sqrt{2/\pi}(1-s)/(1+s)$

How to generalize the results in this paper to a nonsymmetrical FOU is presently under study and will be reported on shortly. Some results for this case have already appeared in [10] (they are based on minimax techniques that are described in [12]). This case is very important because interval data that have already been collected for words demonstrate that for most words *uncertainties about the two end-points are not equal*.

Another interesting avenue of research is to establish tighter bounds on the centroid than we have done in this paper. One way to do this for the upper bound is described in Section A-D, but its details remain to be explored.

APPENDIX A
PROOF OF THEOREM 4

A. *Derivation of* $\underline{c}_r$

Our derivation of $\underline{c}_r(\tilde{A})$ in (32) proceeds in three steps.

Step 1) Let $\tilde{A}'$ be $\tilde{A}$ shifted by $-m$ so that it is symmetrical about the origin. For $\tilde{A}'$ we show that $\underline{c}_r(\tilde{A}')$, given by (32) with $m = 0$, is a valid lower bound for $c_r(\tilde{A}')$ and that $\underline{c}_r(\tilde{A}') \geq 0$.

Step 2) We obtain an expression for $\underline{c}_r(\tilde{A})$.

Step 3) We show that $\underline{c}_r(\tilde{A})$ is a valid lower bound for $c_r(\tilde{A})$ when $\tilde{A}$ is symmetrical about an arbitrary m, and that $\underline{c}_r(\tilde{A}) \geq m$ ■

Step 1) We focus first on the interval T2 FS $\tilde{A}'$ that is symmetrical about the origin (i.e., $m = 0$). Consider a special embedded T1 FS A'_e [from the class of embedded T1 FSs whose MFs are described by (15)], defined as

$$\mu_{A'_e}(x) = \begin{cases} \underline{\mu}(x), & \text{if } x \leq 0 \\ \overline{\mu}(x), & \text{if } x > 0. \end{cases} \tag{A-1}$$

Because A'_e is an embedded T1 FS of $\tilde{A}'$, centroid(A'_e) is a valid lower bound for $c_r(\tilde{A}')$, where the concept of a "valid lower bound" has been defined in Section III. It follows that

$$\begin{aligned}\text{centroid}(A'_e) &= \frac{\int_{-\infty}^{0} x\underline{\mu}(x)dx + \int_{0}^{\infty} x\bar{\mu}(x)dx}{\int_{-\infty}^{0} \underline{\mu}(x)dx + \int_{0}^{\infty} \bar{\mu}(x)dx}\\ &= \frac{\int_{0}^{\infty} -x\underline{\mu}(-x)dx + \int_{0}^{\infty} x\bar{\mu}(x)dx}{\int_{0}^{\infty} \underline{\mu}(-x)dx + \int_{0}^{\infty} \bar{\mu}(x)dx}\\ &= \frac{\int_{0}^{\infty} x[\bar{\mu}(x) - \underline{\mu}(x)]dx}{\int_{0}^{\infty} \underline{\mu}(x)dx + \int_{0}^{\infty} \bar{\mu}(x)dx}. \qquad \text{(A-2)}\end{aligned}$$

Note that in going from line 2 to line 3 of this derivation we have made use of the symmetry of $\underline{\mu}(x)$ about the origin, namely that $\underline{\mu}(x) = \underline{\mu}(-x)$. We can re-express the last line of (A-2) as

$$\begin{aligned}\text{centroid}(A'_e) &= \frac{\int_{0}^{\infty} x[\bar{\mu}(x) - \underline{\mu}(x)]dx}{\int_{0}^{\infty} [\bar{\mu}(x) - \underline{\mu}(x)]dx}\\ &\quad\times \frac{\int_{0}^{\infty} [\bar{\mu}(x) - \underline{\mu}(x)]dx}{\int_{0}^{\infty} [\bar{\mu}(x) + \underline{\mu}(x)]dx}\\ \text{centroid}(A'_e) &= c_{\text{HFOU}}(\tilde{A}')\frac{A_{\text{FOU}}/2}{(A_{\text{UMF}} + A_{\text{LMF}})/2}\\ &= c_{\text{HFOU}}(\tilde{A}')\frac{A_{\text{FOU}}}{A_{\text{UMF}} + A_{\text{LMF}}} \equiv \underline{c}_r(\tilde{A}')\end{aligned}$$

(A-3)

where $c_{\text{HFOU}}(\tilde{A}')$ is computed using (29) in which $m = 0$. Note that centroid(A'_e) is exactly the same as $\underline{c}_r(\tilde{A}')$ given in (32) with $m = 0$. Due to the definition of a *valid bound*, we conclude that $\underline{c}_r(\tilde{A}')$ is a valid lower bound for $c_r(\tilde{A}')$.

That $\underline{c}_r(\tilde{A}') \geq 0$ follows from the last line of (A-2) because $\bar{\mu}(x) \geq \underline{\mu}(x)$ for $\forall x \in X$.

Step 2) Recall that $\tilde{A}'$ is $\tilde{A}$ shifted by $-m$; therefore, the lower and upper MFs of $\tilde{A}$ and $\tilde{A}'$ are related according to (20) and (21) (in which $\Delta m = -m$) as:

$$\underline{\mu}_{\tilde{A}'}(x) = \underline{\mu}_{\tilde{A}}(x+m) \qquad \text{(A-4)}$$
$$\bar{\mu}_{\tilde{A}'}(x) = \bar{\mu}_{\tilde{A}}(x+m) \qquad \text{(A-5)}$$

Here we show that $\underline{c}_r(\tilde{A}) = \underline{c}_r(\tilde{A}') + m$.

The areas A_{LMF}, A_{UMF} and A_{FOU}, which appear in (A-3) as well as in (32) (for arbitrary values of m) are the same for both $\tilde{A}$ and $\tilde{A}'$. We now prove that $c_{\text{HFOU}}(\tilde{A}') = c_{\text{HFOU}}(\tilde{A}) - m$, as follows:

$$\begin{aligned}c_{\text{HFOU}}(\tilde{A}') &= \frac{\int_{0}^{\infty} x[\bar{\mu}_{\tilde{A}'}(x) - \underline{\mu}_{\tilde{A}'}(x)]dx}{\int_{0}^{\infty} [\bar{\mu}_{\tilde{A}'}(x) - \underline{\mu}_{\tilde{A}'}(x)]dx}\\ &= \frac{\int_{0}^{\infty} x[\bar{\mu}_{\tilde{A}}(x+m) - \underline{\mu}_{\tilde{A}}(x+m)]dx}{\int_{0}^{\infty} [\bar{\mu}_{\tilde{A}}(x+m) - \underline{\mu}_{\tilde{A}}(x+m)]dx}\\ &= \frac{\int_{0}^{\infty} (x+m)[\bar{\mu}_{\tilde{A}}(x+m) - \underline{\mu}_{\tilde{A}}(x+m)]dx}{\int_{0}^{\infty} [\bar{\mu}_{\tilde{A}}(x+m) - \underline{\mu}_{\tilde{A}}(x+m)]dx} - m\\ &= \frac{\int_{m}^{\infty} y[\bar{\mu}_{\tilde{A}}(y) - \underline{\mu}_{\tilde{A}}(y)]dy}{\int_{m}^{\infty} [\bar{\mu}_{\tilde{A}}(y) - \underline{\mu}_{\tilde{A}}(y)]dy} - m\\ &= c_{\text{HFOU}}(\tilde{A}) - m. \qquad \text{(A-6)}\end{aligned}$$

Consequently, beginning with (A-3) and using (A-6), we find

$$\begin{aligned}\underline{c}_r(\tilde{A}') &= c_{\text{HFOU}}(\tilde{A}')\frac{A_{\text{FOU}}}{A_{\text{UMF}} + A_{\text{LMF}}}\\ &= [c_{\text{HFOU}}(\tilde{A}) - m]\frac{A_{\text{FOU}}}{A_{\text{UMF}} + A_{\text{LMF}}}. \qquad \text{(A-7)}\end{aligned}$$

In order to complete this part of the proof, we pause to state and prove the following:

Fact: If $\tilde{A}'$ is $\tilde{A}$ shifted by $-m$ along X and we have computed $\underline{c}_r(\tilde{A}')$ and $\bar{c}_r(\tilde{A}')$, then

$$\underline{c}_r(\tilde{A}) = \underline{c}_r(\tilde{A}') + m \qquad \text{(A-8)}$$
$$\bar{c}_r(\tilde{A}) = \bar{c}_r(\tilde{A}') + m. \qquad \text{(A-9)}$$

Similar results hold for $\underline{c}_l(\tilde{A})$ and $\bar{c}_l(\tilde{A})$.

Proof: Using (23) with $\Delta m = -m$, it is clear that

$$c_r(\tilde{A}) = c_r(\tilde{A}') + m. \qquad \text{(A-10)}$$

Consequently, upper and lower bounds for $c_r(\tilde{A})$ can be expressed in terms of comparable quantities for $c_r(\tilde{A}')$, as

$$\text{u.b.}[c_r(\tilde{A})] = \text{u.b.}[c_r(\tilde{A}')] + m \qquad \text{(A-11)}$$
$$\text{l.b.}[c_r(\tilde{A})] = l.\text{b.}[c_r(\tilde{A}')] + m. \qquad \text{(A-12)}$$

Note that these bounds are not necessarily greatest-lower or least-upper bounds; they are just any bounds. That said, we know, e.g., that $\underline{c}_r(\tilde{A}')$ is a lower bound on $c_r(\tilde{A}')$; hence, by (A-12), $\underline{c}_r(\tilde{A}') + m$ is the comparable lower bound on $c_r(\tilde{A})$. We call this lower bound on $c_r(\tilde{A}), \underline{c}_r(\tilde{A})$. A similar argument leads us to $\bar{c}_r(\tilde{A})$. ■

Substituting (A-12) into (A-7), we find that

$$\underline{c}_r(\tilde{A}) = m + [c_{\text{HFOU}}(\tilde{A}) - m]\frac{A_{\text{FOU}}}{A_{\text{UMF}} + A_{\text{LMF}}} \qquad \text{(A-13)}$$

which is (32).

Step 3) In Step 1), we have proven that $\underline{c}_r(\tilde{A}') \geq 0$ is a valid lower bound for $c_r(\tilde{A}')$ when $\tilde{A}'$ is symmetrical about the origin. From (A-8) and $\underline{c}_r(\tilde{A}') \geq 0$, we see that

$$\underline{c}_r(\tilde{A}) \geq m. \qquad \text{(A-14)}$$

We conclude, therefore, that $\underline{c}_r(\tilde{A}) \geq m$ is a valid lower bound for $c_r(\tilde{A})$ when $\tilde{A}$ is symmetrical about an arbitrary m.

B. Derivation of $\bar{c}_r$

From (9), it is clear that $\sum_{i=1}^{N} x_i\theta_i / \sum_{i=1}^{N} \theta_i \leq x_N$, which is why in (31) $c_r(\tilde{A}) \leq \min(\bar{c}_r(\tilde{A}), x_N)$.

Our derivation of $\bar{c}_r(\tilde{A})$ in (33) also proceeds in three steps.

Step 1) As before, let $\tilde{A}'$ be $\tilde{A}$ shifted by $-m$ so that it is symmetrical about the origin. For $\tilde{A}'$ we show that

$\bar{c}_r(\tilde{A}')$ given in (33) with $m=0$ is a valid upper bound for $c_r(\tilde{A}\prime)$.

Step 2) We obtain an expression for $\bar{c}_r(\tilde{A})$.

Step 3) We show that $\bar{c}_r(\tilde{A})$ is a valid upper bound for $c_r(\tilde{A})$ when $\tilde{A}$ is symmetrical about an arbitrary m. ■

Step 1) When $\tilde{A}'$ is symmetrical about the origin, because $c_r(\tilde{A}\prime) \geq 0$ (Theorem 3, with $m=0$) we can rewrite $c_r(\tilde{A}')$ in (19) as shown in (A-15) at the bottom of the page. Because $\bar{\mu}_{\tilde{A}'}(x) \geq \underline{\mu}_{\tilde{A}'}(x)$ for $\forall x \in X$, it follows that

$$\int_0^{c_r} x\left[\bar{\mu}_{\tilde{A}'}(x) - \underline{\mu}_{\tilde{A}'}(x)\right]dx \geq 0 \tag{A-16}$$

$$\int_{c_r}^{\infty} \left[\bar{\mu}_{\tilde{A}'}(x) - \underline{\mu}_{\tilde{A}'}(x)\right]dx \geq 0. \tag{A-17}$$

Using (A-16) and (A-17) in (A-15), the latter becomes (for discussions about a tighter upper bound, see Section D)

$$c_r(\tilde{A}') \leq \frac{\int_0^\infty x[\bar{\mu}_{\tilde{A}'}(x) - \underline{\mu}_{\tilde{A}'}(x)]dx}{2\int_0^\infty \underline{\mu}_{\tilde{A}'}(x)dx} \tag{A-18}$$

which can be re-expressed, using the formula for $c_{\mathrm{HFOU}}(\tilde{A}')$ given in (29) when $m=0$, as

$$\begin{aligned} c_r(\tilde{A}') &\leq \frac{\int_0^\infty x[\bar{\mu}_{\tilde{A}'}(x) - \underline{\mu}_{\tilde{A}'}(x)]dx}{\int_0^\infty [\bar{\mu}_{\tilde{A}'}(x) - \underline{\mu}_{\tilde{A}'}(x)]dx} \\ &\quad\times \frac{\int_0^\infty [\bar{\mu}_{\tilde{A}'}(x) - \underline{\mu}_{\tilde{A}'}(x)]dx}{2\int_0^\infty \underline{\mu}_{\tilde{A}'}(x)dx} \\ &= c_{\mathrm{HFOU}}(\tilde{A}')\frac{A_{\mathrm{FOU}}}{2A_{\mathrm{LMF}}} \equiv \bar{c}_r(\tilde{A}'). \end{aligned} \tag{A-19}$$

Observe that the right-hand side of (A-19) is exactly the same as $\bar{c}_r(\tilde{A}')$ in (33) when $m=0$.

Step 2) Because the details for this step are so similar to those given in Step 2) of Subsection 1-A, we leave them for the reader. Of course, the starting point now is (33), and we also use (A-9).

Step 3) In Step 1), we have proven that $\bar{c}_r(\tilde{A}')$ is a valid upper bound for $c_r(\tilde{A}')$ when $\tilde{A}'$ is symmetrical about the origin. In Theorem 2 and Step 2, we have proven that when $\tilde{A}$ is shifted by $-m$ both $c_r(\tilde{A})$ and $\bar{c}_r(\tilde{A})$ are also shifted by $-m$. We conclude, therefore, that $\bar{c}_r(\tilde{A})$ is a valid upper bound for $c_r(\tilde{A})$ when $\tilde{A}$ is symmetrical about an arbitrary m.

C. Derivations of $\bar{c}_l(\tilde{A})$ and $\underline{c}_l(\tilde{A})$

From (9), it is also true that $\sum_{i=1}^N x_i\theta_i / \sum_{i=1}^N \theta_i \geq x_1$, which is why in (30) $c_l(\tilde{A}) \geq \max(x_1, \underline{c}_l(\tilde{A}))$.

Because $\tilde{A}$ is symmetrical about m, it follows [see Fig. 2 and (27)] that

$$\frac{\bar{c}_l(\tilde{A}) + \underline{c}_r(\tilde{A})}{2} = m \tag{A-20}$$

$$\frac{\underline{c}_l(\tilde{A}) + \bar{c}_r(\tilde{A})}{2} = m \tag{A-21}$$

from which it is easy to obtain the results in (34) and (35) for $\bar{c}_l(\tilde{A})$ and $\underline{c}_l(\tilde{A})$.

D. A Tighter Upper Bound

It is theoretically possible to obtain an even tighter upper bound on $c_r(\tilde{A}')$ by using the already computed value of $\underline{c}_r(\tilde{A}')$, as follows. Using the fact that $\underline{c}_r(\tilde{A}') \leq c_r(\tilde{A}')$, re-express the last line of (A-15) as shown in (A-22) at the bottom of the next page. It then follows that

$$\begin{aligned} c_r(\tilde{A}') \leq & \frac{\int_0^\infty x[\bar{\mu}_{\tilde{A}'}(x) - \underline{\mu}_{\tilde{A}'}(x)]dx}{2\int_0^\infty \underline{\mu}_{\tilde{A}'}(x)dx} \\ & - \frac{\int_0^{\underline{c}_r(\tilde{A}')} x[\bar{\mu}_{\tilde{A}'}(x) - \underline{\mu}_{\tilde{A}'}(x)]dx}{2\int_0^\infty \underline{\mu}_{\tilde{A}'}(x)dx} \end{aligned} \tag{A-23}$$

which can be expressed as

$$\begin{aligned} c_r(\tilde{A}') \leq & c_{\mathrm{HFOU}}(\tilde{A}')\frac{A_{\mathrm{FOU}}}{2A_{\mathrm{LMF}}} \\ & - \frac{\int_0^{\underline{c}_r(\tilde{A}')} x[\bar{\mu}_{\tilde{A}'}(x) - \underline{\mu}_{\tilde{A}'}(x)]dx}{A_{\mathrm{LMF}}}. \end{aligned} \tag{A-24}$$

The first term in (A-24) is the same as our upper bound in (A-19), whereas the second term, which must always be positive, can be interpreted as a correction term to (A-19). Note that

$$\begin{aligned} c_r(\tilde{A}') &= \frac{\int_{-\infty}^{c_r} x\underline{\mu}_{\tilde{A}'}(x)dx + \int_{c_r}^\infty x\bar{\mu}_{\tilde{A}'}(x)dx}{\int_{-\infty}^{c_r} \underline{\mu}_{\tilde{A}'}(x)dx + \int_{c_r}^\infty \bar{\mu}_{\tilde{A}'}(x)dx} \\ &= \frac{\int_{-\infty}^{0} x\underline{\mu}_{\tilde{A}'}(x)dx + \int_0^{c_r} x\underline{\mu}_{\tilde{A}'}(x)dx + \int_0^\infty x\bar{\mu}_{\tilde{A}'}(x)dx - \int_0^{c_r} x\bar{\mu}_{\tilde{A}'}(x)dx}{\int_{-\infty}^{0} \underline{\mu}_{\tilde{A}'}(x)dx + \int_0^{c_r} \underline{\mu}_{\tilde{A}'}(x)dx + \int_0^\infty \bar{\mu}_{\tilde{A}'}(x)dx - \int_0^{c_r} \bar{\mu}_{\tilde{A}'}(x)dx} \\ &= \frac{\int_0^\infty x[\bar{\mu}_{\tilde{A}'}(x) - \underline{\mu}_{\tilde{A}'}(x)]dx - \int_0^{c_r} x[\bar{\mu}_{\tilde{A}'}(x) - \underline{\mu}_{\tilde{A}'}(x)]dx}{\int_0^\infty [\bar{\mu}_{\tilde{A}'}(x) + \underline{\mu}_{\tilde{A}'}(x)]dx - \int_0^{c_r} [\bar{\mu}_{\tilde{A}'}(x) - \underline{\mu}_{\tilde{A}'}(x)]dx} \\ &= \frac{\int_0^\infty x[\bar{\mu}_{\tilde{A}'}(x) - \underline{\mu}_{\tilde{A}'}(x)]dx - \int_0^{c_r} x[\bar{\mu}_{\tilde{A}'}(x) - \underline{\mu}_{\tilde{A}'}(x)]dx}{2\int_0^\infty \underline{\mu}_{\tilde{A}'}(x)dx + \int_0^\infty [\bar{\mu}_{\tilde{A}'}(x) - \underline{\mu}_{\tilde{A}'}(x)]dx - \int_0^{c_r} [\bar{\mu}_{\tilde{A}'}(x) - \underline{\mu}_{\tilde{A}'}(x)]dx} \\ &= \frac{\int_0^\infty x[\bar{\mu}_{\tilde{A}'}(x) - \underline{\mu}_{\tilde{A}'}(x)]dx - \int_0^{c_r} x[\bar{\mu}_{\tilde{A}'}(x) - \underline{\mu}_{\tilde{A}'}(x)]dx}{2\int_0^\infty \underline{\mu}_{\tilde{A}'}(x)dx + \int_{c_r}^\infty [\bar{\mu}_{\tilde{A}'}(x) - \underline{\mu}_{\tilde{A}'}(x)]dx} \end{aligned} \tag{A-15}$$

to compute the tighter upper bound for $c_r(\tilde{A})$ from (A-24), we must: (1) calculate $\underline{c}_r(\tilde{A}')$; (2) calculate the right-hand side of (A-24), which now becomes our new $\bar{c}_r(\tilde{A}')$; and (3) add m to this value, giving us the tighter $\bar{c}_r(\tilde{A})$.

Inequality (A-24) represents a novel use of the already-computed lower bound $\underline{c}_r(\tilde{A}')$; however, computing the correction term is in most cases much more demanding than computing (A-19); hence, in this paper, we use (A-19) and leave further explorations of the use of (A-24) as an interesting open research problem.

Appendix B
Computation of $c_{\mathrm{HFOU}}(\tilde{A}')$ for Example 3

We compute the two integrals that are in (40) for Example 3

$$\begin{aligned}
a &\triangleq \int_0^\infty x\bar{\mu}(x)dx \\
&= \int_0^\mu x dx + \int_\mu^\infty x \exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma_2}\right)^2\right\} dx \\
&= \frac{1}{2}\mu^2 + \int_\mu^\infty (x-\mu) \exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma_2}\right)^2\right\} dx \\
&\quad + \mu \int_\mu^\infty \exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma_2}\right)^2\right\} dx \qquad \text{(B-1)} \\
&= \frac{1}{2}\mu^2 + \sigma_2^2 \int_0^\infty \exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma_2}\right)^2\right\} d\frac{1}{2}\left(\frac{x-\mu}{\sigma_2}\right)^2 \\
&\quad + \mu\sigma_2 \int_0^\infty \exp\left(-\frac{1}{2}y^2\right)dy \\
&= \frac{1}{2}\mu^2 + \sigma_2^2 + \frac{1}{2}\sqrt{2\pi}\mu\sigma_2 \\
b &\triangleq \int_0^\infty x\underline{\mu}(x)dx \\
&= \int_0^\infty x \exp\left\{-\frac{1}{2}\left(\frac{x+\mu}{\sigma_1}\right)^2\right\}dx \\
&= \int_0^\infty (x+\mu) \exp\left\{-\frac{1}{2}\left(\frac{x+\mu}{\sigma_1}\right)^2\right\}dx \\
&\quad - \mu \int_0^\infty \exp\left\{-\frac{1}{2}\left(\frac{x+\mu}{\sigma_1}\right)^2\right\}dx \\
b & \\
&= \sigma_1^2 \int_{(\mu/\sigma_1)^2/2}^\infty \exp\left\{-\frac{1}{2}\left(\frac{x+\mu}{\sigma_1}\right)^2\right\} d\frac{1}{2}\left(\frac{x+\mu}{\sigma_1}\right)^2 \\
&\quad - \mu\sigma_1 \int_{\mu/\sigma_1}^\infty \exp\left(-\frac{1}{2}y^2\right)dy \\
&= \sigma_1^2 \exp\left\{-\frac{1}{2}\left(\frac{\mu}{\sigma_1}\right)^2\right\} - \sqrt{2\pi}\mu\sigma_1\Phi\left(\frac{\mu}{\sigma_1}\right). \qquad \text{(B-2))}
\end{aligned}$$

Appendix C
Proof of Theorem 1

Because the proof of Theorem 1 only appears in [11], which is not yet published, we provide a condensed version of it here. The proof of[10] (18), due to Mendel and Wu, proceeds in two steps.

Step 1) Let $c_l(\alpha)$ be given by (18) in which c_l in the limits of integration are replaced by α. A necessary condition for finding $\min_\alpha c_l(\alpha)$ is that the derivative of $c_l(\alpha)$ with respect to α must be zero when evaluated at c_l, i.e.,

$$\frac{d}{d\alpha}\left\{\left[\int_{-\infty}^{\alpha} x\bar{\mu}(x)dx + \int_{\alpha}^{\infty} x\underline{\mu}(x)dx\right] \Big/ \left[\int_{-\infty}^{\alpha} \bar{\mu}(x)dx + \int_{\alpha}^{\infty} \underline{\mu}(x)dx\right]\right\}\Bigg|_{\alpha=c_l}. \qquad \text{(C-1)}$$

This equation expands to

$$\begin{aligned}
&\frac{[c_l\bar{\mu}(c_l) - c_l\underline{\mu}(c_l)]\left[\int_{-\infty}^{c_l}\bar{\mu}(x)dx + \int_{c_l}^{\infty}\underline{\mu}(x)dx\right]}{\left[\int_{-\infty}^{c_l}\bar{\mu}(x)dx + \int_{c_l}^{\infty}\underline{\mu}(x)dx\right]^2} \\
&\quad - \frac{[\bar{\mu}(c_l) - \underline{\mu}(c_l)]\left[\int_{-\infty}^{c_l}x\bar{\mu}(x)dx + \int_{c_l}^{\infty}x\underline{\mu}(x)dx\right]}{\left[\int_{-\infty}^{c_l}\bar{\mu}(x)dx + \int_{c_l}^{\infty}\underline{\mu}(x)dx\right]^2} \\
&\quad = 0 \qquad \text{(C-2)}
\end{aligned}$$

from which it follows that

$$\begin{aligned}
[\bar{\mu}(c_l) - \underline{\mu}(c_l)]&\left\{c_l\left[\int_{-\infty}^{c_l}\bar{\mu}(x)dx + \int_{c_l}^{\infty}\underline{\mu}(x)dx\right]\right. \\
&\left.- \left[\int_{-\infty}^{c_l}x\bar{\mu}(x)dx + \int_{c_l}^{\infty}x\underline{\mu}(x)dx\right]\right\} = 0. \qquad \text{(C-3)}
\end{aligned}$$

Step 2) We now show that c_l can only be as in (18). Let $\alpha_1 \in X$ for which

$$\bar{\mu}(\alpha_1) - \underline{\mu}(\alpha_1) = 0 \qquad \text{(C-4)}$$

[10]We do not need the proof of (19) because of the assumed symmetry of $FOU(\tilde{A})$, i.e., knowing c_l we can compute c_r using (27).

$$\begin{aligned}
c_r(\tilde{A}') &= \frac{\int_0^\infty x[\bar{\mu}_{\tilde{A}'}(x) - \underline{\mu}_{\tilde{A}'}(x)]dx - \int_0^{c_r} x[\bar{\mu}_{\tilde{A}'}(x) - \underline{\mu}_{\tilde{A}'}(x)]dx}{2\int_0^\infty \underline{\mu}_{\tilde{A}'}(x)dx + \int_{c_r}^\infty [\bar{\mu}_{\tilde{A}'}(x) - \underline{\mu}_{\tilde{A}'}(x)]dx} \\
&= \frac{\int_0^\infty x[\bar{\mu}_{\tilde{A}'}(x) - \underline{\mu}_{\tilde{A}'}(x)]dx - \int_0^{\underline{c}_r(\tilde{A}')} x[\bar{\mu}_{\tilde{A}'}(x) - \underline{\mu}_{\tilde{A}'}(x)]dx}{2\int_0^\infty \underline{\mu}_{\tilde{A}'}(x)dx + \int_{c_r}^\infty [\bar{\mu}_{\tilde{A}'}(x) - \underline{\mu}_{\tilde{A}'}(x)]dx} \\
&\quad - \frac{\int_{\underline{c}_r(\tilde{A}')}^{c_r} x[\bar{\mu}_{\tilde{A}'}(x) - \underline{\mu}_{\tilde{A}'}(x)]dx}{2\int_0^\infty \underline{\mu}_{\tilde{A}'}(x)dx + \int_{c_r}^\infty [\bar{\mu}_{\tilde{A}'}(x) - \underline{\mu}_{\tilde{A}'}(x)]dx} \qquad \text{(A-22)}
\end{aligned}$$

and $\alpha_2 \in X$ for which

$$\alpha_2 \left[\int_{-\infty}^{\alpha_2} \bar{\mu}(x)dx + \int_{\alpha_2}^{\infty} \underline{\mu}(x)dx\right] = \left[\int_{-\infty}^{\alpha_2} x\bar{\mu}(x)dx + \int_{\alpha_2}^{\infty} x\underline{\mu}(x)dx\right]. \quad \text{(C-5)}$$

Observe that (C-5) can be solved for α_2, as

$$\alpha_2 = \frac{\int_{-\infty}^{\alpha_2} x\bar{\mu}(x)dx + \int_{\alpha_2}^{\infty} x\underline{\mu}(x)dx}{\int_{-\infty}^{\alpha_2} \bar{\mu}(x)dx + \int_{\alpha_2}^{\infty} \underline{\mu}(x)dx}. \quad \text{(C-6)}$$

Since both α_1 and α_2 satisfy (C-3), either or both of them may be c_l. One must now show that

$$c_l(\alpha_i) \geq c_l(\alpha_2) \quad \text{for } \forall \alpha_i \neq \alpha_2. \quad \text{(C-7)}$$

Because c_l is the minimum of $c_l(\alpha)$, it therefore cannot be $c_l(\alpha_1)$, but must be $c_l(\alpha_2)$.

If it happens that $\alpha_1 = \alpha_2 = c_l$, then it may happen that $\bar{\mu}(c_l) = \underline{\mu}(c_l)$, in which case (C-3) is simultaneously satisfied by *both* of its terms equaling zero. By these arguments we see that (C-3) can never be satisfied by $\bar{\mu}(c_l) = \underline{\mu}(c_l)$ alone. Note that the condition $\bar{\mu}(c_l) = \underline{\mu}(c_l)$ means that at $x = c_l$ the upper and lower MFs touch each other, something that is perfectly permissible in the $\mathrm{FOU}(\tilde{A})$.

The proof of (C-7) must be done for both for $\alpha_1 < \alpha_2$ and $\alpha_1 > \alpha_2$. Both parts use some relatively simple inequality analysis applied to $c_l(\alpha)$. Because of space limitations, these details are not included here but can be found in [11].

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