Aggregation Using the Linguistic Weighted Average and Interval Type-2 Fuzzy Sets

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***Abstract*—The focus of this paper is the *linguistic weighted average* (LWA), where the weights are always words modeled as *interval type-2 fuzzy sets* (IT2 FSs), and the attributes may also (but do not have to) be words modeled as IT2 FSs; consequently, the output of the LWA is an IT2 FS. The LWA can be viewed as a generalization of the *fuzzy weighted average* (FWA) where the type-1 fuzzy inputs are replaced by IT2 FSs. This paper presents the theory, algorithms, and an application of the LWA. It is shown that finding the LWA can be decomposed into finding two FWAs. Since the LWA can model more uncertainties, it should have wide applications in *distributed and hierarchical decision-making*.**

***Index Terms*—Aggregation, computing with words, distributed and hierarchical decision-making, fuzzy weighted average, interval type-2 fuzzy sets, linguistic weighted average.**

I. Introduction

ZADEH proposed the paradigm of *computing with words* (CWW) [36], [37], i.e., CWW is "a methodology in which the objects of computation are words and propositions drawn from a natural language." Nikravesh [27] further pointed out that CWW "is fundamentally different from the traditional expert systems which are simply tools to 'realize' an intelligent system, but are not able to process natural language which is imprecise, uncertain and partially true."

Words in the CWW paradigm may be modeled by type-1 (T1) fuzzy sets (FSs) [34] or their extension, type-2 (T2) FSs [35]. The main difference between the two kinds of FSs is that the memberships of a T1 FS are crisp numbers whereas the memberships of a T2 FS are T1 FSs; hence, a T2 FS can model more uncertainties. To date the most widely used T2 FSs are interval T2 (IT2) FSs (see Section II-A).

CWW using T1 FSs has been studied by many researchers, e.g., [8], [11], [15], [17]–[19], [28], [30], [31], [33], and [36]; however, the limitations of using T1 FSs in CWW have also been pointed out by several researchers. Herrera and Herrera-Viedma [7] noticed that "formally speaking, it seems difficult to accept that all individuals should agree on the same membership function (T1 FS) associated to linguistic terms." Türkşen [29] also pointed out that "type-1 representation is a 'reductionist' approach for it discards the spread of membership values by averaging or curve fitting techniques and hence, camouflages the 'uncertainty' embedded in the spread of membership values. Therefore, Type-1 representation does not provide a good approximation to meaning representation of words and does not allow computing with words a richer platform." Mendel [19] notes that "words mean different things to different people and so are uncertain. We therefore need a FS model for a word that has the potential to capture its uncertainties, and an IT2 FS should be used as a FS model of a word." The discussions in this paper are therefore limited to IT2 FSs.

Manuscript received April 22, 2006; revised November 20, 2006. This work was supported by the Center of Excellence for Research and Academic Training on Interactive Smart Oilfield Technologies (CiSoft), a joint University of Southern California/Chevron initiative.

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Digital Object Identifier 10.1109/TFUZZ.2007.896325

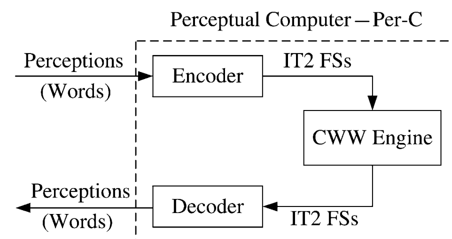

Fig. 1. Specific architecture for CWW, the perceptual computer.

A specific architecture for CWW using IT2 FSs was proposed in [18] (Fig. 1), called a *perceptual computer* (Per-C). The Per-C consists of three components: encoder, decoder, and CWW engine. Perceptions (i.e., granulated terms, words) activate the Per-C and are also output by the Per-C; so, it is possible for a human to interact with the Per-C just using a vocabulary of words. In Fig. 1, the encoder[1] transforms linguistic perceptions into IT2 FSs that activate a CWW engine. The decoder[2] maps the output of the CWW engine into a word. Usually a vocabulary (codebook) is available, in which every word is modeled as an IT2 FS. The output of the CWW engine is mapped into a word (in that vocabulary) most similar to it.

How to transform linguistic perceptions into IT2 FSs, i.e., the encoding problem, has been considered in [23]–[25]. The decoding problem, i.e., how to map an IT2 FS $\tilde{A}$ into a word (linguistic label), has been discussed in [32]. The basic idea of decoding is to first compute the similarities between $\tilde{A}$ and all words in the codebook and then find the largest similarity, to which the corresponding word is assigned to $\tilde{A}$. In [32], a vector similarity measure (VSM) is proposed, whose two elements measure the similarity in shape and proximity, respectively. A crisp similarity measure can be obtained as the product of the two components of the VSM.

[1]Zadeh calls this *constraint explicitation* in [36] and [37]. In some of his recent talks, he calls this *precisiation*.

[2]Zadeh calls this *linguistic approximation* in [36] and [37].

This paper proposes a CWW engine, called the *linguistic weighted average* (LWA),[3] which is an extension of the *fuzzy weighted average* (FWA) when the inputs become IT2 FSs, i.e.,

$$\tilde{Y}_{\text{LWA}} = \frac{\sum_{i=1}^{n} \tilde{X}_i \tilde{W}_i}{\sum_{i=1}^{n} \tilde{W}_i} \tag{1}$$

where $\tilde{X}_i$ and $\tilde{W}_i$ are words modeled by IT2 FSs. The application of the LWA can be illustrated by the following:

Example: Consider the following *distributed and hierarchical decision-making situation*. There are n judges (or experts, managers, commanders, referees, etc.) who have to provide a subjective decision or judgement $\tilde{Y}$ about a situation (e.g., quality of a submitted journal article). They will do this by providing a linguistic evaluation (i.e., a word, term, or phrase) for each of m prespecified and preranked measures, $M_1, M_2, \ldots, M_m$, using a prespecified vocabulary of t_i terms $(i = 1, 2, \ldots, m)$, because it may be too problematic to provide a numerical score for these categories. For a submitted journal article, the categories might be *importance, content, depth, presentation,* etc.; and, e.g., for *presentation,* the terms might be *excellent, good, adequate, marginal,* and *poor*.

Assume that each of the category terms has been modeled *a priori* as an IT2 FS $\tilde{X}$; so, for each M_i, there are t_i associated IT2 FSs $\tilde{X}_{M_i}$. Additionally, assume that the m evaluation categories have also been linguistically rank-ordered *a priori*, so that each M_i is associated with a linguistic weight, modeled as the IT2 FS $\tilde{W}_{M_i}$. The judges do not have to be concerned with any of the *a priori* rankings and modeling; it has all been done before they have been asked to judge.

After the judges have chosen a linguistic term for the m categories, the following LWA is automatically computed:

$$\tilde{Y}_j = \frac{\sum_{i=1}^{m} \tilde{W}_{M_i} \tilde{X}_{j,M_i}}{\sum_{i=1}^{m} \tilde{W}_{M_i}} \qquad j = 1, 2, \ldots, n \tag{2}$$

where $\tilde{X}_{j,M_i}$ is Judge j's choice on M_i. These n IT2 FSs are then sent to a control (command) center (e.g., the associate editor); however, because judges may not be of equal expertise, it is also assumed that each judge's level of expertise has been prespecified using a linguistic term $\tilde{W}_j$ provided by the judge from a small vocabulary of terms (e.g., *low expertise, moderate expertise, high expertise*). The linguistic evaluations from the n judges $\tilde{Y}_j$ are then aggregated using a second LWA as

$$\tilde{Y} = \frac{\sum_{j=1}^{n} \tilde{W}_j \tilde{Y}_j}{\sum_{j=1}^{n} \tilde{W}_j}. \tag{3}$$

This second LWA is also sent to the control (command) center. Using $\tilde{Y}_j$ $(j = 1, 2, \ldots, n)$ and/or $\tilde{Y}$, a final decision or judgement is made at the control (command) center. Exactly how that is done is not the subject of this paper. This example is pursued in greater detail in Section V. □

[3]The phrases *linguistic weighted averaging* and *linguistic weighted aggregation* were first used in [7], where T1 FSs were considered. Although our LWA is very different from the linguistic weighted averaging in [7], it is in the spirit of their linguistic weighted aggregation operators; hence, we also use the LWA acronym in this paper. Note that in this paper, LWA is always connected to IT2 FSs.

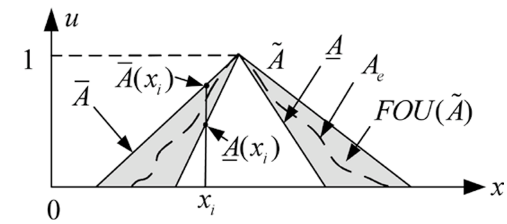

Fig. 2. An IT2 FS. A_e is an embedded T1 FS.

The rest of this paper is organized as follows. Section II reviews the background needed to derive the LWA algorithms, i.e., IT2 FSs, α-cuts, and the FWA. Section III provides theorems for the LWA, which are the bases for the LWA computational algorithms proposed in Section IV. Section V presents an application. Section VI draws conclusions.

II. Background

A. Interval Type-2 Fuzzy Sets (It2 FSs)

An IT2 FS $\tilde{A}$ is to date the most widely used kind of T2 FS and is the only kind of T2 FS that is considered in this paper. It is described as[4]

$$\tilde{A} = \int_{x \in X} \int_{u \in J_x} 1/(x, u) = \int_{x \in X} \left[\int_{u \in J_x} 1/u \right] \Big/ x \tag{4}$$

where x is the *primary variable,* $J_x \subseteq [0, 1]$ is the *primary membership* of x, u is the *secondary variable,* and $\int_{u \in J_x} 1/u$ is the *secondary membership function* (MF) at x. Note that (4) means $\tilde{A} : X \to \{[a, b] : 0 \le a \le b \le 1\}$. Uncertainty about $\tilde{A}$ is conveyed by the union of all of the primary memberships, called the *footprint of uncertainty* of $\tilde{A}$ [FOU($\tilde{A}$)], i.e.,

$$\begin{aligned} \text{FOU}(\tilde{A}) &= \bigcup_{x \in X} J_x \\ &= \{(x, y) : y \in J_x = [\underline{A}(x), \bar{A}(x)] \subseteq [0, 1]\}. \end{aligned} \tag{5}$$

An IT2 FS is shown in Fig. 2. The FOU is shown as the shaded region. It is bounded by[5] an *upper MF* (UMF) $\bar{A}(x) \equiv \bar{A}$ and a *lower MF* (LMF) $\underline{A}(x) \equiv \underline{A}$, both of which are T1 FSs; consequently, the membership grade of each element of an IT2 FS is an interval $[\underline{A}(x), \bar{A}(x)]$.

Note that an IT2 FS can also be represented as

$$\tilde{A} = 1/\text{FOU}(\tilde{A}) \tag{6}$$

with the understanding that this means putting a secondary grade of one at all points of FOU($\tilde{A}$).

For discrete universes of discourse X and U, an *embedded T1 FS* A_e has N elements, one each from $J_{x_1}, J_{x_2}, \ldots, J_{x_N}$, namely $u_1, u_2, \ldots, u_N$, i.e.,

$$A_e = \sum_{i=1}^{N} u_i / x_i \quad u_i \in J_{x_i} \subseteq U = [0, 1]. \tag{7}$$

[4]This background material is taken from [26]. See also [16].

[5]It is also customary to use $\underline{\mu}_{\tilde{A}}(x)$ and $\overline{\mu}_{\tilde{A}}(x)$ for the LMF and UMF of $\tilde{A}$. Here, a simpler notation is used.

Examples of A_e are $\underline{A}$ and $\bar{A}$; see also Fig. 2. Note that if each u_i is discretized into M_i levels, there will be a total of n_A A_e, where

$$n_A = \prod_{i=1}^{N} M_i. \tag{8}$$

Mendel and John [20] have presented a Representation Theorem for a general T2 FS, which when specialized to an IT2 FS can be expressed as follows.

Representation Theorem for an IT2 FS: Assume that primary variable x of an IT2 FS $\tilde{A}$ is sampled at N values, $x_1, x_2, \ldots, x_N$, and at each of these values its primary memberships u_i are sampled at M_i values, $u_{i1}, u_{i2}, \ldots, u_{iM_i}$. Let A_e^j denote the jth embedded T1 FS for $\tilde{A}$. Then $\tilde{A}$ is represented by (6), in which[6]

$$\begin{aligned}\text{FOU}(\tilde{A}) &= \bigcup_{j=1}^{n_A} A_e^j = \{\underline{A}(x), \ldots, \bar{A}(x)\} \\ &\equiv [\underline{A}(x), \bar{A}(x)]. \end{aligned} \tag{9}$$

□

This representation of an IT2 FS, in terms of simple T1 FSs, the embedded T1 FSs, is very useful for deriving theoretical results; however, it is not recommended for computational purposes because it would require the enumeration of the n_A embedded T1 FSs and n_A [given in (8)] can be astronomical. Equation (9) is heavily used in the sequel when the LWA is derived.

B. α-Cuts and Decomposition Theorem

Given a T1 FS A defined on its universe of discourse X, and any number $\alpha \in [0,1]$, the α-cut of A, ${}^{\alpha}A$, is the crisp set [10]

$${}^{\alpha}A = \{x | \mu_A(x) \geq \alpha\} \qquad \forall x \in X. \tag{10}$$

Let $I_{{}^{\alpha}A}(x)$ be an *indicator function* of the crisp set ${}^{\alpha}A$, i.e.

$$I_{{}^{\alpha}A}(x) = \begin{cases} 1 & \forall x \in {}^{\alpha}A \\ 0 & \forall x \notin {}^{\alpha}A \end{cases}. \tag{11}$$

The MF of T1 FS ${}_{\alpha}A$ is then defined as

$$\mu_{\alpha A}(x) \equiv \alpha I_{{}^{\alpha}A}(x). \tag{12}$$

Decomposition Theorem: [10] Let A and ${}_{\alpha}A$ be T1 FSs in X with ${}_{\alpha}A$ defined in (12). Then

$$A = \bigcup_{\alpha \in [0,1]} {}_{\alpha}A \tag{13}$$

where $\bigcup$ denotes the standard fuzzy union (i.e., sup over $\alpha \in [0,1]$). Note that because a T1 FS is described by its MF, (13) is a commonly used shorthand for

$$\mu_A(x) = \mu_{\bigcup_{\alpha \in [0,1]} {}_{\alpha}A}(x) = \sup_{\alpha \in [0,1]} \{\mu_{\alpha A}(x)\} \quad \forall x \in X. \tag{14}$$

□

Observe from this theorem that, if the α-cuts of a T1 FS can be determined, for $\forall \alpha \in [0,1]$, the T1 FS itself can be specified; therefore, determining a T1 FS is equivalent to determining its α-cuts for $\forall \alpha \in [0,1]$.

One important application of the α-cut Decomposition Theorem is to compute some function of a T1 FS, or between several T1 FSs [10]; it gives exactly the same result as the one obtained by using Zadeh's Extension Principle. For example, when the function f is the FWA of n attributes X_i and the corresponding n weights W_i, it is true that f fuzzified by the Extension Principle satisfies

$$\begin{aligned} &{}^{\alpha}f(X_1, \cdots, X_n, W_1, \ldots, W_n) \\ &\qquad = f({}^{\alpha}X_1, \ldots, {}^{\alpha}X_n, {}^{\alpha}W_1, \ldots, {}^{\alpha}W_n) \end{aligned} \tag{15}$$

i.e., an α-cut on f is computed by finding the corresponding α-cuts on X_i and W_i first and then substituting them into f. When all ${}^{\alpha}f$ $(0 \leq \alpha \leq 1)$ are obtained, f can be constructed by using (13). Equation (15) is heavily used in the sequel.

C. The Fuzzy Weighted Average (FWA)

The FWA is defined as [2]

$$Y_{\text{FWA}} = \frac{\sum_{i=1}^{n} X_i W_i}{\sum_{i=1}^{n} W_i} \equiv f(x_1, \ldots, x_n, w_1, \ldots, w_n). \tag{16}$$

In (16), all W_i and X_i are T1 FSs; consequently, Y_{FWA} is also a T1 FS.

The FWA has been studied in multiple criteria decision making [1]–[6], [12], [13] and computing the generalized centroid of an IT2 FS [9], [16], [22], [21]. Beginning in 1987, various solutions to computing the FWA have been proposed, all of which use (15).

To compute the FWA, first the complete range of the membership [0,1] of the FSs $X_1, X_2, \ldots, X_n$ and $W_1, W_2, \ldots, W_n$ is discretized into m α-cuts, $\alpha_1, \ldots, \alpha_m$. For each α_j, the corresponding intervals for x_i in X_i and w_i in W_i $(i = 1, 2, \ldots, n)$ are found, i.e.,

$$x_i \in [a_i(\alpha_j), b_i(\alpha_j)] \tag{17}$$
$$w_i \in [c_i(\alpha_j), d_i(\alpha_j)]. \tag{18}$$

The output of the FWA algorithm for this particular α-cut, $Y_{\text{FWA}}(\alpha_j)$, is an interval, i.e.,

$$\begin{aligned} Y_{\text{FWA}}(\alpha_j) &= \frac{\sum_{i=1}^{n} X_i(\alpha_j) W_i(\alpha_j)}{\sum_{i=1}^{n} W_i(\alpha_j)} \\ &= [f_L(\alpha_j), f_R(\alpha_j)] \end{aligned} \tag{19}$$

[6]Although there are a finite number of embedded T1 FSs, it is customary to represent FOU($\tilde{A}$) as an interval set $[\underline{A}(x), \bar{A}(x)]$ at each x. Doing this is equivalent to discretizing with infinitesimally many small values and letting the discretizations approach zero.

where

$$f_L(\alpha_j) = \min_{\forall\, w_i \in [c_i(\alpha_j),\, d_i(\alpha_j)]} \frac{\sum_{i=1}^n a_i(\alpha_j) w_i(\alpha_j)}{\sum_{i=1}^n w_i(\alpha_j)} \quad (20)$$

$$f_R(\alpha_j) = \max_{\forall\, w_i \in [c_i(\alpha_j),\, d_i(\alpha_j)]} \frac{\sum_{i=1}^n b_i(\alpha_j) w_i(\alpha_j)}{\sum_{i=1}^n w_i(\alpha_j)}. \quad (21)$$

These results are easy to prove because $X_i(\alpha_j)$ appear only in the numerator of (19), and so the smallest (largest) values of $X_i(\alpha_j)$ are used to find the smallest (largest) value of (19). It is now well known that $f_L(\alpha_j)$ and $f_R(\alpha_j)$ can be represented as

$$f_L(\alpha_j) = \frac{\sum_{i=1}^{k_L} a_i(\alpha_j) d_i(\alpha_j) + \sum_{i=k_L+1}^{n} a_i(\alpha_j) c_i(\alpha_j)}{\sum_{i=1}^{k_L} d_i(\alpha_j) + \sum_{i=k_L+1}^{n} c_i(\alpha_j)} \quad (22)$$

$$f_R(\alpha_j) = \frac{\sum_{i=1}^{k_R} b_i(\alpha_j) c_i(\alpha_j) + \sum_{i=k_R+1}^{n} b_i(\alpha_j) d_i(\alpha_j)}{\sum_{i=1}^{k_R} c_i(\alpha_j) + \sum_{i=k_R+1}^{n} d_i(\alpha_j)} \quad (23)$$

where k_L and k_R are switch points satisfying [14]

$$a_{k_L}(\alpha_j) \le f_L(\alpha_j) \le a_{k_L+1}(\alpha_j) \quad (24)$$

$$b_{k_R}(\alpha_j) \le f_R(\alpha_j) \le b_{k_R+1}(\alpha_j). \quad (25)$$

Note that $\{a_1(\alpha_j), \ldots, a_n(\alpha_j)\}$ and $\{b_1(\alpha_j), \ldots, b_n(\alpha_j)\}$ have been sorted in increasing order, respectively; hence, in the sequel, it is assumed that

$$a_1(\alpha_j) \le a_2(\alpha_j) \le \cdots \le a_n(\alpha_j) \quad (26)$$

$$b_1(\alpha_j) \le b_2(\alpha_j) \le \cdots \le b_n(\alpha_j). \quad (27)$$

Each of the published FWA computational algorithms computes k_L and k_R but in different ways. When all m intervals $[f_L(\alpha_j), f_R(\alpha_j)]$ are found, the MF of $Y_{\text{FWA}}, \mu_{Y_{\text{FWA}}}(y)$, is computed as [see (14) and (12)]

$$\mu_{Y_{\text{FWA}}}(y) = \sup_{\forall \alpha_j (j=1,\ldots,m)} \alpha_j I_{Y_{\text{FWA}}(\alpha_j)}(y) \quad (28)$$

where [see (11)]

$$I_{Y_{\text{FWA}}(\alpha_j)}(y) = \begin{cases} 1 & \forall y \in [f_L(\alpha_j), f_R(\alpha_j)] \\ 0 & \forall y \notin [f_L(\alpha_j), f_R(\alpha_j)] \end{cases} \quad (29)$$

is the *indicator function* of $Y_{\text{FWA}}(\alpha_j)$.

An example of the FWA is shown in Fig. 9. The T1 FS shown by dashed lines in Fig. 9(c) is the FWA of T1 FSs X_i [dashed lines in Fig. 9(a)] and W_i [dashed lines in Fig. 9(b)].

III. LWA Theory

A. LWA Preliminaries

In (1), because all $\tilde{X}_i$ and $\tilde{W}_i$ are IT2 FSs, $\tilde{Y}_{\text{LWA}}$ is also an IT2 FS, and therefore [see (6) and (9)]

$$\tilde{Y}_{\text{LWA}} = 1/\text{FOU}(\tilde{Y}_{\text{LWA}}) \equiv 1/[\underline{Y}_{\text{LWA}}, \bar{Y}_{\text{LWA}}] \quad (30)$$

where $\underline{Y}_{\text{LWA}}$ and $\bar{Y}_{\text{LWA}}$ are the LMF and UMF of $\tilde{Y}_{\text{LWA}}$, respectively. Because the FOU of $\tilde{Y}_{\text{LWA}}$ is completely determined by $\underline{Y}_{\text{LWA}}$ and $\bar{Y}_{\text{LWA}}$, computing $\tilde{Y}_{\text{LWA}}$ is equivalent to computing $\underline{Y}_{\text{LWA}}$ and $\bar{Y}_{\text{LWA}}$.

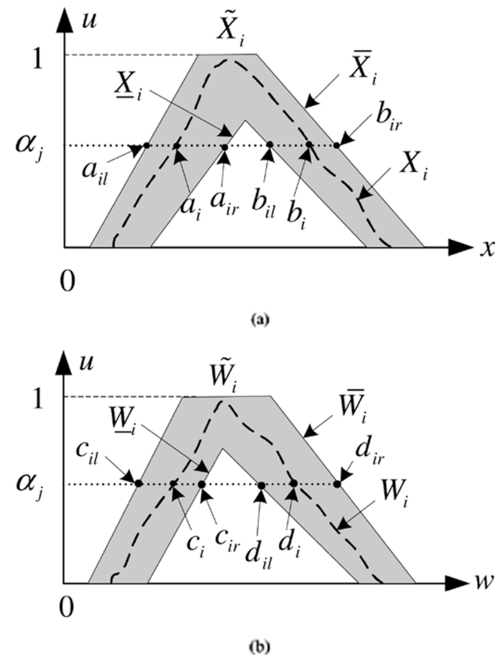

Fig. 3. (a) Variables for $\tilde{X}_i$:$a_{il}(\alpha_j)$, $a_{ir}(\alpha_j)$, $b_{il}(\alpha_j)$, and $b_{ir}(\alpha_j)$. (b) Variables for $\tilde{W}_i$:$c_{il}(\alpha_j)$, $c_{ir}(\alpha_j)$, $d_{il}(\alpha_j)$ and $d_{ir}(\alpha_j)$. The dashed curves are embedded T1 FSs.

Using (6) and (9) applied to each $\tilde{X}_i$ and $\tilde{W}_i$, it follows that

$$\tilde{X}_i = 1/\text{FOU}(\tilde{X}_i) = 1/[\underline{X}_i, \bar{X}_i] = 1/\bigcup_{j_i=1}^{n_{X_i}} X_{e_i}^{j_i} \quad (31)$$

$$\tilde{W}_i = 1/\text{FOU}(\tilde{W}_i) = 1/[\underline{W}_i, \bar{W}_i] = 1/\bigcup_{k_i=1}^{n_{W_i}} W_{e_i}^{k_i} \quad (32)$$

where $\underline{X}_i$ and $\bar{X}_i$ ($\underline{W}_i$ and $\bar{W}_i$) are lower and upper MFs of $\tilde{X}_i(\tilde{W}_i)$, respectively.

In (1), $\tilde{X}_i$ only appears in the numerator of $\tilde{Y}_{\text{LWA}}$; hence

$$\underline{Y}_{\text{LWA}} = \min_{\forall\, W_i \in [\underline{W}_i, \bar{W}_i]} \frac{\sum_{i=1}^n \underline{X}_i W_i}{\sum_{i=1}^n W_i} \quad (33)$$

$$\bar{Y}_{\text{LWA}} = \max_{\forall\, W_i \in [\underline{W}_i, \bar{W}_i]} \frac{\sum_{i=1}^n \bar{X}_i W_i}{\sum_{i=1}^n W_i}. \quad (34)$$

One method to find $\underline{Y}_{\text{LWA}}$ is to compute the totality of all FWAs that can be formed from all of the embedded T1 FSs $W_{e_i}^{k_i}$; however, this is impractical because there can be infinite many $W_{e_i}^{k_i}$. In the following, an α-cut based approach is proposed, which eliminates the need to enumerate and evaluate all embedded T1 FSs.

B. Computing the LWA Using α-Cuts

To compute $\underline{Y}_{\text{LWA}}$ and $\bar{Y}_{\text{LWA}}$ using α-cuts, the complete range of the membership $[0, 1]$ is first discretized into m points, $\alpha_1, \ldots, \alpha_m$; then, for each α_j, the α-cuts on $\tilde{X}_i$ and $\tilde{W}_i$ are used to compute the corresponding α-cut on $\tilde{Y}_{\text{LWA}}$.

The notations in Fig. 3(a) and (b) will be used in the derivations of the LWA. For notational simplicity, dependence of all variables on α_j is omitted in all the figures in this paper. Normal

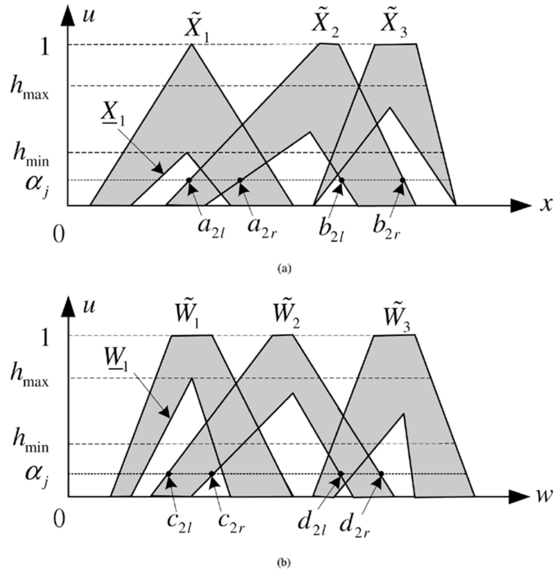

Fig. 4. Case 1: $0 \leq \alpha_j < h_{\min}$. (a) Variables for $\tilde{X}_i$:$a_{il}(\alpha_j), a_{ir}(\alpha_j)$, $b_{il}(\alpha_j)$, and $b_{ir}(\alpha_j)$; and, (b) Variables for $\tilde{W}_i$:$c_{il}(\alpha_j)$, $c_{ir}(\alpha_j), d_{il}(\alpha_j)$, and $d_{ir}(\alpha_j)$.

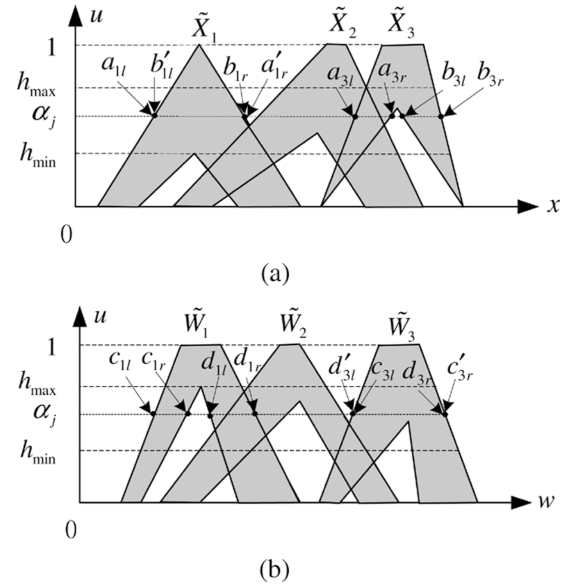

Fig. 5. Case 2: $h_{\min} \leq \alpha_j \leq h_{\max}$. (a) Variables for $\tilde{X}_i$:$a_{il}(\alpha_j)$, $a_{ir}(\alpha_j), b_{il}(\alpha_j)$, $b_{ir}(\alpha_j), a'_{ir}(\alpha_j)$, and $b'_{il}(\alpha_j)$. (b) Variables for $\tilde{W}_i$:$c_{il}(\alpha_j), c_{ir}(\alpha_j), d_{il}(\alpha_j), d_{ir}(\alpha_j), c'_{ir}(\alpha_j)$, and $d'_{il}(\alpha_j)$.

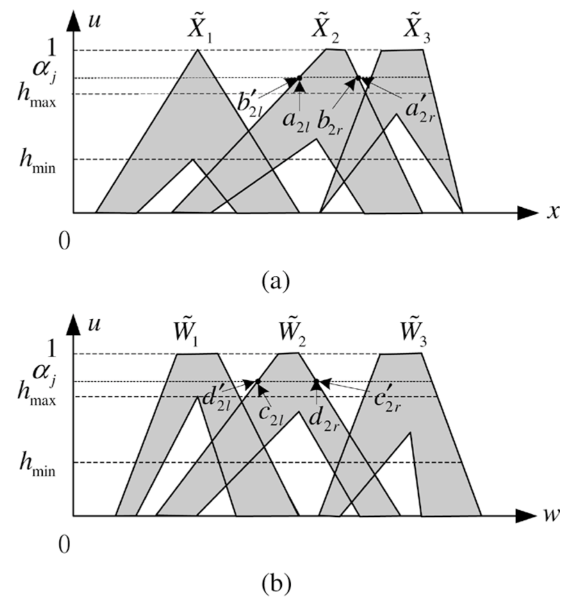

Fig. 6. Case 3: $h_{\max} < \alpha_j \leq 1$. (a) Variables for $\tilde{X}_i$:$a_{il}(\alpha_j), b_{ir}(\alpha_j)$, $a'_{ir}(\alpha_j)$ and $b'_{il}(\alpha_j)$. (b) Variables for $\tilde{W}_i$:$c_{il}(\alpha_j)$, $d_{ir}(\alpha_j), c'_{ir}(\alpha_j)$ and $d'_{il}(\alpha_j)$.

IT2 FSs are always used, i.e., the maximum membership grades of the UMFs of all T2 FSs equal unity. This means that each α-cut on the UMFs will produce an interval for $\alpha \neq 1$, or, a crisp point or an interval for $\alpha = 1$, as shown in Fig. 3(a) and (b).

Generally, the LMFs of $\tilde{X}_i$ and $\tilde{W}_i$ have different heights (maximum membership grades), as shown in Fig. 4(a) and (b). Denote the height of $\underline{X}_i$ as $h_{\underline{X}_i}$ and the height of $\underline{W}_i$ as $h_{\underline{W}_i}$, respectively. Assume the maximum (minimum) height of all $\underline{X}_i$ and all $\underline{W}_i$ is $h_{\max}(h_{\min})$, i.e.,

$$h_{\max} = \max\{\max_{\forall\, i\in[1,\, n]} h_{\underline{X}_i},\ \max_{\forall i\in[1,n]} h_{\underline{W}_i}\} \tag{35}$$

$$h_{\min} = \min\{\min_{\forall i\in[1,n]} h_{\underline{X}_i},\ \min_{\forall i\in[1,n]} h_{\underline{W}_i}\}. \tag{36}$$

In the Fig. 4 example, $h_{\min} = h_{\underline{X}_1}$ and $h_{\max} = h_{\underline{W}_1}$, and for clarity $h_{\min}$ and $h_{\max}$ are shown as dashed lines in both parts of that figure.

Depending on the position of the α-cut, there can be three different cases.

1) α-cuts on all UMFs and LMFs exist, e.g., when $0 \leq \alpha_j = h_{\min}$ in Fig. 4.
2) α-cuts on all UMFs exist while α-cuts on some LMFs do not exist, e.g., when $h_{\min} < \alpha_j \leq h_{\max}$ in Fig. 5, $\underline{X}_1, \underline{X}_2$, and $\underline{W}_3$ have no α-cut when $\alpha = \alpha_j$.
3) α-cuts on all UMFs exist, but none exists on the LMFs, e.g., when $h_{\max} < \alpha_j \leq 1$ in Fig. 6.

Because Cases 1 and 3 can be treated as special cases of Case 2, the approach here is to consider Case 2 first and then specialize its results to Cases 1 and 3.[7]

C. Case 2: α-Cuts on All UMFS Exist While α-Cuts on Some LMFs Do Not Exist

Observe in Fig. 3(a) that if the α-cut on the LMF of $\tilde{X}_i$ exists, the interval $[a_{il}(\alpha_j), b_{ir}(\alpha_j)]$ is divided into three subintervals: $[a_{il}(\alpha_j), a_{ir}(\alpha_j)]$, $(a_{ir}(\alpha_j), b_{il}(\alpha_j))$, and $[b_{il}(\alpha_j), b_{ir}(\alpha_j)]$. In this case, $a_i(\alpha_j) \in [a_{il}(\alpha_j), a_{ir}(\alpha_j)]$ and $a_i(\alpha_j)$ cannot assume a value larger than $a_{ir}(\alpha_j)$. Similarly, $b_i(\alpha_j) \in [b_{il}(\alpha_j), b_{ir}(\alpha_j)]$ and $b_i(\alpha_j)$ cannot assume a value smaller than $b_{il}(\alpha_j)$. However, if the α-cut on the LMF of $\tilde{X}_i$ does not exist [e.g., $\tilde{X}_1$ and $\tilde{X}_2$ in Fig. 5(a) for $\alpha_j > h_{\min}$], then both $a_i(\alpha_j)$ and $b_i(\alpha_j)$ can assume values freely in the entire interval $[a_{il}(\alpha_j), b_{ir}(\alpha_j)]$, i.e.,

$$a_i(\alpha_j) \in [a_{il}(\alpha_j), a'_{ir}(\alpha_j)] \tag{37}$$

$$b_i(\alpha_j) \in [b'_{il}(\alpha_j), b_{ir}(\alpha_j)] \tag{38}$$

[7]The following three sections are very technical. The reader who is not interested in the details can go directly to Section III-F.

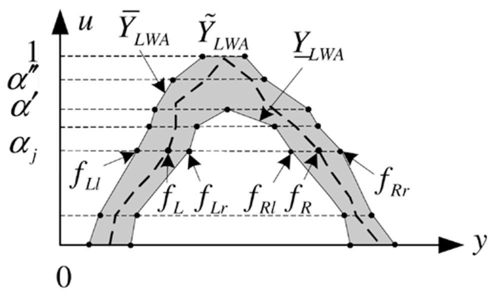

Fig. 7. Variables for $\tilde{Y}_{\mathrm{LWA}}$: $f_{Ll}(\alpha_j), f_{Lr}(\alpha_j), f_{Rl}(\alpha_j)$, and $f_{Rr}(\alpha_j)$. The dashed curve represents an embedded T1 FS.

where

$$a'_{ir}(\alpha_j) = \begin{cases} b_{ir}(\alpha_j), & h_{\underline{X}_i} < \alpha_j \\ a_{ir}(\alpha_j), & h_{\underline{X}_i} \ge \alpha_j \end{cases} \tag{39}$$

$$b'_{il}(\alpha_j) = \begin{cases} a_{il}(\alpha_j), & h_{\underline{X}_i} < \alpha_j \\ b_{il}(\alpha_j), & h_{\underline{X}_i} \ge \alpha_j \end{cases}. \tag{40}$$

Thus, the effect of an α-cut on such an LMF of $\tilde{X}_i$ is to further constrain the ranges of $a_i(\alpha_j)$ and $b_i(\alpha_j)$.

Similarly, observe from Fig. 5(b) that

$$c_i(\alpha_j) \in [c_{il}(\alpha_j), c'_{ir}(\alpha_j)] \tag{41}$$

$$d_i(\alpha_j) \in [d'_{il}(\alpha_j), d_{ir}(\alpha_j)] \tag{42}$$

where

$$c'_{ir}(\alpha_j) = \begin{cases} d_{ir}(\alpha_j), & \forall\, h_{\underline{W}_i} < \alpha_j \\ c_{ir}(\alpha_j), & \forall\, h_{\underline{W}_i} \ge \alpha_j \end{cases} \tag{43}$$

$$d'_{il}(\alpha_j) = \begin{cases} c_{il}(\alpha_j), & \forall\, h_{\underline{W}_i} < \alpha_j \\ d_{il}(\alpha_j), & \forall\, h_{\underline{W}_i} \ge \alpha_j \end{cases}. \tag{44}$$

Thus, the effect of an α-cut on such an LMF of $\tilde{W}_i$ is to further constrain the range of $c_i(\alpha_j)$ and $d_i(\alpha_j)$.

Note that in (22) and (23) for the FWA, $a_i(\alpha_j), b_i(\alpha_j)$, $c_i(\alpha_j)$, and $d_i(\alpha_j)$ are crisp numbers; consequently, $f_L(\alpha_j)$ and $f_R(\alpha_j)$ computed from them are also crisp numbers. However, in Case 2 of the LWA, $a_i(\alpha_j), b_i(\alpha_j)$, $c_i(\alpha_j)$, and $d_i(\alpha_j)$ can assume values continuously in their corresponding α-cut intervals. Numerous different combinations of $a_i(\alpha_j), b_i(\alpha_j)$, $c_i(\alpha_j)$, and $d_i(\alpha_j)$ can be formed. $f_L(\alpha_j)$ and $f_R(\alpha_j)$ need to be computed for all the combinations. By collecting all $f_L(\alpha_j)$, a continuous interval $[f_{Ll}(\alpha_j), f_{Lr}(\alpha_j)]$ is obtained, and by collecting all $f_R(\alpha_j)$ a continuous interval $[f_{Rl}(\alpha_j), f_{Rr}(\alpha_j)]$ is also obtained (see Fig. 7), i.e.,

$$\underline{Y}_{\mathrm{LWA}}(\alpha_j) = [f_{Lr}(\alpha_j), f_{Rl}(\alpha_j)] \tag{45}$$

$$\bar{Y}_{\mathrm{LWA}}(\alpha_j) = [f_{Ll}(\alpha_j), f_{Rr}(\alpha_j)] \tag{46}$$

where $f_{Lr}(\alpha_j), f_{Rl}(\alpha_j), f_{Ll}(\alpha_j)$, and $f_{Rr}(\alpha_j)$ are illustrated in Fig. 7. Clearly, to find $\underline{Y}_{\mathrm{LWA}}(\alpha_j)$ and $\bar{Y}_{\mathrm{LWA}}(\alpha_j)$, $f_{Ll}(\alpha_j), f_{Lr}(\alpha_j), f_{Rl}(\alpha_j)$, and $f_{Rr}(\alpha_j)$ need to be found.

Consider $f_{Ll}(\alpha_j)$ first. Note that it lies on $\bar{Y}_{\mathrm{LWA}}$, and is the minimum of $f_L(\alpha_j)$ but now $a_i \in [a_{il}, a'_{ir}]$, $c_i \in [c_{il}, c'_{ir}]$, and $d_i \in [d'_{il}, d_{ir}]$, i.e.,

$$f_{Ll}(\alpha_j) = \min_{\substack{\forall a_i \in [a_{il}, a'_{ir}] \\ \forall c_i \in [c_{il}, c'_{ir}], \forall d_i \in [d'_{il}, d_{ir}]}} f_L(\alpha_j). \tag{47}$$

Substituting $f_L(\alpha_j)$ from (22) into (47), it follows that we have (48) as shown at the bottom of the page. Observe that $a_i(\alpha_j)$ only appears in the numerator of (48); thus, $a_{il}(\alpha_j)$ should be used to calculate $f_{Ll}(\alpha_j)$, i.e., (49) at the bottom of the page. Following a similar line of reasoning, $f_{Lr}(\alpha_j), f_{Rl}(\alpha_j)$, and $f_{Rr}(\alpha_j)$ can also be expressed as shown in (50)–(52) at the bottom of the page.

So far, only $a_i(\alpha_j)$ are fixed for $f_{Ll}(\alpha_j)$ and $f_{Lr}(\alpha_j)$, and $b_i(\alpha_j)$ are fixed for $f_{Rl}(\alpha_j)$ and $f_{Rr}(\alpha_j)$. As will be shown, it is also possible to fix $c_i(\alpha_j)$ and $d_i(\alpha_j)$ for $f_{Ll}(\alpha_j)$,

$$f_{Ll}(\alpha_j) \equiv \min_{\substack{\forall a_i \in [a_{il}, a'_{ir}] \\ \forall c_i \in [c_{il}, c'_{ir}], \forall d_i \in [d'_{il}, d_{ir}]}} \frac{\sum_{i=1}^{k_{L1}} a_i(\alpha_j) d_i(\alpha_j) + \sum_{i=k_{L1}+1}^{n} a_i(\alpha_j) c_i(\alpha_j)}{\sum_{i=1}^{k_{L1}} d_i(\alpha_j) + \sum_{i=k_{L1}+1}^{n} c_i(\alpha_j)} \tag{48}$$

$$f_{Ll}(\alpha_j) = \min_{\substack{\forall c_i \in [c_{il}, c'_{ir}] \\ \forall d_i \in [d'_{il}, d_{ir}]}} \frac{\sum_{i=1}^{k_{L1}} a_{il}(\alpha_j) d_i(\alpha_j) + \sum_{i=k_{L1}+1}^{n} a_{il}(\alpha_j) c_i(\alpha_j)}{\sum_{i=1}^{k_{L1}} d_i(\alpha_j) + \sum_{i=k_{L1}+1}^{n} c_i(\alpha_j)} \tag{49}$$

$$f_{Lr}(\alpha_j) = \max_{\substack{\forall c_i \in [c_{il}, c'_{ir}] \\ \forall d_i \in [d'_{il}, d_{ir}]}} \frac{\sum_{i=1}^{k_{L2}} a'_{ir}(\alpha_j) d_i(\alpha_j) + \sum_{i=k_{L2}+1}^{n} a'_{ir}(\alpha_j) c_i(\alpha_j)}{\sum_{i=1}^{k_{L2}} d_i(\alpha_j) + \sum_{i=k_{L2}+1}^{n} c_i(\alpha_j)} \tag{50}$$

$$f_{Rl}(\alpha_j) = \min_{\substack{\forall c_i \in [c_{il}, c'_{ir}] \\ \forall d_i \in [d'_{il}, d_{ir}]}} \frac{\sum_{i=1}^{k_{R1}} b'_{il}(\alpha_j) c_i(\alpha_j) + \sum_{i=k_{R1}+1}^{n} b'_{il}(\alpha_j) d_i(\alpha_j)}{\sum_{i=1}^{k_{R1}} c_i(\alpha_j) + \sum_{i=k_{R1}+1}^{n} d_i(\alpha_j)} \tag{51}$$

$$f_{Rr}(\alpha_j) = \max_{\substack{\forall c_i \in [c_{il}, c'_{ir}] \\ \forall d_i \in [d'_{il}, d_{ir}]}} \frac{\sum_{i=1}^{k_{R2}} b_{ir}(\alpha_j) c_i(\alpha_j) + \sum_{i=k_{R2}+1}^{n} b_{ir}(\alpha_j) d_i(\alpha_j)}{\sum_{i=1}^{k_{R2}} c_i(\alpha_j) + \sum_{i=k_{R2}+1}^{n} d_i(\alpha_j)} \tag{52}$$

$f_{Lr}(\alpha_j), f_{Rl}(\alpha_j)$, and $f_{Rr}(\alpha_j)$; thus, there will be no need to enumerate and evaluate all of $\tilde{W}_i$'s embedded T1 FSs to find $\underline{Y}_{\text{LWA}}$ and $\bar{Y}_{\text{LWA}}$.

Theorem 1: The following are true.

a) $f_{Ll}(\alpha_j)$ in (49) can be specified as

$$f_{Ll}(\alpha_j) = \frac{\sum_{i=1}^{k_{Ll}} a_{il}(\alpha_j)d_{ir}(\alpha_j) + \sum_{i=k_{Ll}+1}^{n} a_{il}(\alpha_j)c_{il}(\alpha_j)}{\sum_{i=1}^{k_{Ll}} d_{ir}(\alpha_j) + \sum_{i=k_{Ll}+1}^{n} c_{il}(\alpha_j)} \quad (53)$$

where k_{Ll} is the switch point satisfying $a_{k_{Ll},l}(\alpha_j) \le f_{Ll}(\alpha_j) \le a_{k_{Ll}+1,l}(\alpha_j)$.

b) $f_{Lr}(\alpha_j)$ in (50) can be specified as

$$f_{Lr}(\alpha_j) = \frac{\sum_{i=1}^{k_{Lr}} a'_{ir}(\alpha_j)d'_{il}(\alpha_j) + \sum_{i=k_{Lr}+1}^{n} a'_{ir}(\alpha_j)c'_{ir}(\alpha_j)}{\sum_{i=1}^{k_{Lr}} d'_{il}(\alpha_j) + \sum_{i=k_{Lr}+1}^{n} c'_{ir}(\alpha_j)} \quad (54)$$

where k_{Lr} is the switch point satisfying $a'_{k_{Lr},r}(\alpha_j) \le f_{Lr}(\alpha_j) \le a'_{k_{Lr}+1,r}(\alpha_j)$.

c) $f_{Rl}(\alpha_j)$ in (51) can be specified as

$$f_{Rl}(\alpha_j) = \frac{\sum_{i=1}^{k_{Rl}} b'_{il}(\alpha_j)c'_{ir}(\alpha_j) + \sum_{i=k_{Rl}+1}^{n} b'_{il}(\alpha_j)d'_{il}(\alpha_j)}{\sum_{i=1}^{k_{Rl}} c'_{ir}(\alpha_j) + \sum_{i=k_{Rl}+1}^{n} d'_{il}(\alpha_j)} \quad (55)$$

where k_{Rl} is the switch point satisfying $b'_{k_{Rl},l}(\alpha_j) \le f_{Rl}(\alpha_j) \le b'_{k_{Rl}+1,l}(\alpha_j)$.

d) $f_{Rr}(\alpha_j)$ in (52) can be specified as

$$f_{Rr}(\alpha_j) = \frac{\sum_{i=1}^{k_{Rr}} b_{ir}(\alpha_j)c_{il}(\alpha_j) + \sum_{i=k_{Rr}+1}^{n} b_{ir}(\alpha_j)d_{ir}(\alpha_j)}{\sum_{i=1}^{k_{Rr}} c_{il}(\alpha_j) + \sum_{i=k_{Rr}+1}^{n} d_{ir}(\alpha_j)} \quad (56)$$

where k_{Rr} is the switch point satisfying $b_{k_{Rr},r}(\alpha_j) \le f_{Rr}(\alpha_j) \le b_{k_{Rr}+1,r}(\alpha_j)$. □

Proof: See Appendix A. □

Comment 1: Theorem 1(a) can be understood in the following way. When an FWA algorithm is used to compute $f_L(\alpha_j)$ in (22), if $d_i(\alpha_j) \in [d'_{il}(\alpha_j), d_{ir}(\alpha_j)]$ and $c_i(\alpha_j) \in [c_{il}(\alpha_j), c'_{ir}(\alpha_j)]$, then $\max d_i(\alpha_j) = d_{ir}(\alpha_j)$, and $\min c_i(\alpha_j) = c_{il}(\alpha_j)$; so, $d_i(\alpha_j)$ should be replaced by $d_{ir}(\alpha_j)$ and $c_i(\alpha_j)$ should be replaced by $c_{il}(\alpha_j)$ to get $f_{Ll}(\alpha_j)$. Theorem 1(b)–(d) can be understood in a similar way. □

Comment 2: Note that k_{Ll} in (53), k_{Lr} in (54), k_{Rl} in (55), and k_{Rr} in (56) have to be determined by an FWA algorithm, such as the KM algorithm [9]. □

Comment 3: Observe from Theorem 1(a) and (d) that $f_{Ll}(\alpha_j)$ and $f_{Rr}(\alpha_j)$ only depend on $\bar{W}_i$ [see Fig. 3(b)]. □

Since Cases 1 and 3 can be viewed as special cases of Case 2, Theorem 1 can also be used in Cases 1 and 3 by properly setting the parameters of (37), (38), (41), and (42), as will be shown next.

D. Case 1: α-Cuts on All UMFs and LMFs Exist

When $0 \le \alpha_j \le h_{\min}$, the α-cuts on all UMFs and LMFs exist. Consequently, in Theorem 1, set

$$\left.\begin{array}{ll} a'_{ir}(\alpha_j) = a_{ir}(\alpha_j) & \forall\, i \in [1,n] \\ b'_{il}(\alpha_j) = b_{il}(\alpha_j) & \forall\, i \in [1,n] \\ c'_{ir}(\alpha_j) = c_{ir}(\alpha_j) & \forall\, i \in [1,n] \\ d'_{il}(\alpha_j) = d_{il}(\alpha_j) & \forall\, i \in [1,n] \end{array}\right\} \quad (57)$$

and keep all other quantities unchanged.

Corollary 1: It is true that when $0 \le \alpha_j \le h_{\min}$, Theorem 1(a) and (d) remain unchanged, and Theorem 1(b) and (c) can be simplified to the following.

(b') $f_{Lr}(\alpha_j)$ can be specified as

$$f_{Lr}(\alpha_j) = \frac{\sum_{i=1}^{k_{Lr}} a_{ir}(\alpha_j)d_{il}(\alpha_j) + \sum_{i=k_{Lr}+1}^{n} a_{ir}(\alpha_j)c_{ir}(\alpha_j)}{\sum_{i=1}^{k_{Lr}} d_{il}(\alpha_j) + \sum_{i=k_{Lr}+1}^{n} c_{ir}(\alpha_j)} \quad (58)$$

where k_{Lr} is the switch point satisfying $a_{k_{Lr},r}(\alpha_j) \le f_{Lr}(\alpha_j) \le a_{k_{Lr}+1,r}(\alpha_j)$.

(c') $f_{Rl}(\alpha_j)$ can be specified as

$$f_{Rl}(\alpha_j) = \frac{\sum_{i=1}^{k_{Rl}} b_{il}(\alpha_j)c_{ir}(\alpha_j) + \sum_{i=k_{Rl}+1}^{n} b_{il}(\alpha_j)d_{il}(\alpha_j)}{\sum_{i=1}^{k_{Rl}} c_{ir}(\alpha_j) + \sum_{i=k_{Rl}+1}^{n} d_{il}(\alpha_j)} \quad (59)$$

where k_{Rl} is the switch point satisfying $b_{k_{Rl},l}(\alpha_j) \le f_{Rl}(\alpha_j) \le b_{k_{Rl}+1,l}(\alpha_j)$. □

Comment 4: Corollary 1 shows that in Case 1, $f_{Lr}(\alpha_j)$ and $f_{Rl}(\alpha_j)$, that define $\underline{Y}_{\text{LWA}}(\alpha_j)$, only depend on the LMFs of $\tilde{X}_i$ and $\tilde{W}_i$. □

E. Case 3: α-Cuts on All UMFs Exist But None Exists on the LMFs

When $h_{\max} < \alpha_j \le 1$, the α-cuts on all UMFs exist but none of the α-cuts on the LMFs of $\tilde{X}_i$ and $\tilde{W}_i$ exists. Consequently, in Theorem 1, set

$$\left.\begin{array}{ll} a'_{ir}(\alpha_j) = b_{ir}(\alpha_j) & \forall\, i \in [1,n] \\ b'_{il}(\alpha_j) = a_{il}(\alpha_j) & \forall\, i \in [1,n] \\ c'_{ir}(\alpha_j) = d_{ir}(\alpha_j) & \forall\, i \in [1,n] \\ d'_{il}(\alpha_j) = c_{il}(\alpha_j) & \forall\, i \in [1,n] \end{array}\right\} \quad (60)$$

and keep all other quantities unchanged.

Corollary 2: It is true that when $h_{\max} < \alpha_j \le 1$, Theorem 1(a) and (d) remain unchanged and Theorem 1(b) and (c) can be simplified to (b') $f_{Ll}(\alpha_j) = f_{Rl}(\alpha_j)$ and (c') $f_{Lr}(\alpha_j) = f_{Rr}(\alpha_j)$. □

Proof: See Appendix B. □

Comment 5: Corollary 2 shows that in Case 3, $f_{Lr}(\alpha_j)$ and $f_{Rl}(\alpha_j)$, that define $\underline{Y}_{\text{LWA}}(\alpha_j)$, only depend on the UMFs of $\tilde{X}_i$ and $\tilde{W}_i$. □

F. Observations

Observe, from Fig. 7, that when α_j is small, there may be a gap $(f_{Lr}(\alpha_j), f_{Rl}(\alpha_j))$ between the left-hand interval $f_L(\alpha_j) = [f_{Ll}(\alpha_j), f_{Lr}(\alpha_j)]$ and the right-hand interval

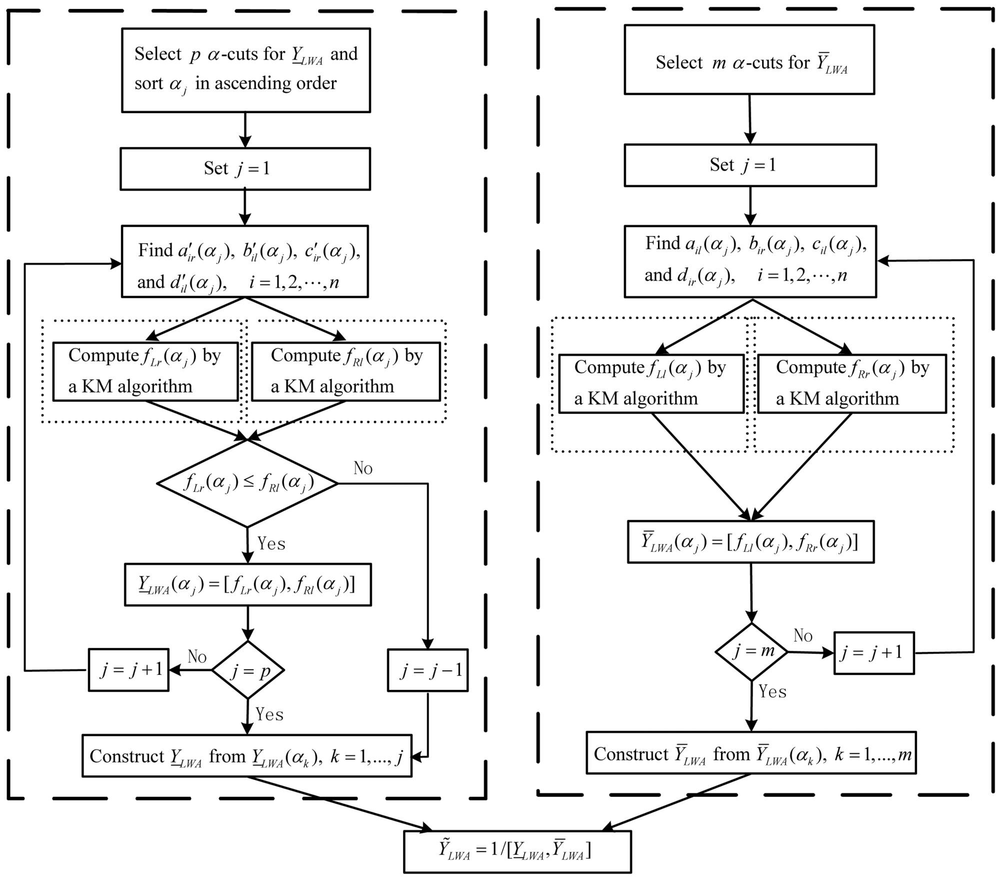

Fig. 8. Flowchart for computing the LWA.

$f_R(\alpha_j)=[f_{Rl}(\alpha_j), f_{Rr}(\alpha_j)]$. However, for large values of α_j, there may not be such a gap.

Theorem 2:

a) When $0\le\alpha_j\le h_{\min}$, $f_{Lr}(\alpha_j)\le f_{Rl}(\alpha_j)$, i.e., there is always a gap $(f_{Lr}(\alpha_j), f_{Rl}(\alpha_j))$ within $f_{Ll}(\alpha_j)$ and $f_{Rr}(\alpha_j)$.

b) When $h_{\min}<\alpha_j\le h_{\max}$, $f_{Lr}(\alpha_j)$ may be smaller than, equal to, or larger than $f_{Rl}(\alpha_j)$ depending on specific values of α_j, e.g., if for an α_j, $f_{Lr}(\alpha_j)>f_{Rl}(\alpha_j)$, then there is no gap within $f_{Ll}(\alpha_j)$ and $f_{Rr}(\alpha_j)$.

c) When $h_{\max}<\alpha_j\le 1$, there is never a gap between $f_{Ll}(\alpha_j)$ and $f_{Rr}(\alpha_j)$, and hence there is no need to compute $f_{Lr}(\alpha_j)$ and $f_{Rl}(\alpha_j)$. □

Proof: See Appendix C. □

Comment 6: Theorem 2 demonstrates that the height of $\underline{Y}_{\text{LWA}}$ cannot be larger than $h_{\max}$ because when $h_{\max}<\alpha_j\le 1$, there is never a gap between $f_{Ll}(\alpha_j)$ and $f_{Rr}(\alpha_j)$. □

Comment 7: Because it is impossible to know in Case 2 whether or not $f_{Lr}(\alpha_j)$ is larger than $f_{Rl}(\alpha_j)$ without computing their values, the following two steps are still needed in this case to determine the corresponding α-cut on $\tilde{Y}_{\text{LWA}}$.

1) Use the α-cuts on the UMFs of $\tilde{X}_i$ and $\tilde{W}_i$ to calculate the interval $[f_{Ll}(\alpha_j), f_{Rr}(\alpha_j)]$ without considering whether or not there is a gap.
2) Calculate $f_{Lr}(\alpha_j)$ and $f_{Rl}(\alpha_j)$ to determine whether there is a gap. If $f_{Lr}(\alpha_j)<f_{Rl}(\alpha_j)$, then there is a gap $(f_{Lr}(\alpha_j), f_{Rl}(\alpha_j))$ which should be removed from the interval $[f_{Ll}(\alpha_j), f_{Rr}(\alpha_j)]$; otherwise, there is no gap and the FOU of $\tilde{Y}_{\text{LWA}}$ fills in the entire interval $[f_{Ll}(\alpha_j), f_{Rr}(\alpha_j)]$. □

IV. LWA ALGORITHMS

A flowchart for computing the LWA is shown in Fig. 8. Observe that $\underline{Y}_{\text{LWA}}$ and $\overline{Y}_{\text{LWA}}$ can be computed in parallel, and that the two FWA algorithms that are used to compute $\underline{Y}_{\text{LWA}}$ and $\overline{Y}_{\text{LWA}}$ can also be computed in parallel. Some blocks of Fig. 8 are explained in detail next.

A. Computation of $\underline{Y}_{\text{LWA}}$

To compute $\underline{Y}_{\text{LWA}}$.

1) Calculate $f_{Lr}(\alpha_j)$ and $f_{Rl}(\alpha_j)$, $j=1,\dots,p$. To do this:
   a) Determine $h_{\underline{X}_i}$ and $h_{\underline{W}_i}$, $i=1,\dots,n$ and $h_{\max}$, which is the maximum of all $h_{\underline{X}_i}$ and $h_{\underline{W}_i}$.
   b) Select appropriate p α-cuts for $\underline{Y}_{\text{LWA}}$ (e.g., divide $[0,h_{\max}]$ into $p-1$ intervals and set $\alpha_j=h_{\max}(j-1)/(p-1)$, $j=1,2,...,p$).
   c) Find the corresponding α-cuts $[a'_{ir}(\alpha_j), b'_{il}(\alpha_j)]$ and $[c'_{ir}(\alpha_j), d'_{il}(\alpha_j)]$ on $\underline{X}_i$ and $\underline{W}_i$ [see Fig. 5(a) and (b); $a'_{ir}(\alpha_j)$, $b'_{il}(\alpha_j)$, $c'_{ir}(\alpha_j)$ and $d'_{il}(\alpha_j)$ are defined in (39), (40), (43), and (44)], respectively].
   d) Use a KM algorithm to find $f_{Lr}(\alpha_j)$ in (54) and $f_{Rl}(\alpha_j)$ in (55).

e) IF $f_{Lr}(\alpha_j) \leq f_{Rl}(\alpha_j)$, THEN keep $\underline{Y}_{LWA}(\alpha_j) = [f_{Lr}(\alpha_j), f_{Rl}(\alpha_j)]$; otherwise, discard it. The last value of α_j for which this test is passed is called $\alpha_{\max}$; denote the number of α_j smaller than or equal to $\alpha_{\max}$ by p', and go to Step 2). Otherwise, go to Step f).

f) Repeat Steps c)—e) until the test in Step e) is failed, or until $j = p$.

2) Construct $\underline{Y}_{\text{LWA}}$ from the p' α-cuts. To do this:

a) Store the left-coordinates $(f_{Lr}(\alpha_j), \alpha_j)$, $j = 1, ..., p'$.

b) Store the right-coordinates $(f_{Rl}(\alpha_j), \alpha_j)$, $j = 1, ..., p'$.

c) (Optional) Fit a spline curve through the $2p'$ coordinates just stored.

B. Computation of $\bar{Y}_{\text{LWA}}$

Computation of $\bar{Y}_{\text{LWA}}$ is simpler than that of $\underline{Y}_{\text{LWA}}$. To compute $\bar{Y}_{\text{LWA}}$:

1) Calculate $f_{Ll}(\alpha_j)$ and $f_{Rr}(\alpha_j)$, $j = 1, \ldots, m$. To do this:

a) Select appropriate m α-cuts for $\bar{Y}_{\text{LWA}}$ (e.g., divide [0, 1] into $m-1$ intervals and set $\alpha_j = (j-1)/(m-1)$, $j = 1, 2, ..., m$).

b) Find the corresponding α-cuts $[a_{il}(\alpha_j), b_{ir}(\alpha_j)]$ and $[c_{il}(\alpha_j), d_{ir}(\alpha_j)]$ on $\bar{X}_i$ and $\bar{W}_i$ [see Fig. 3(a) and (b)].

c) Use a KM algorithm to find $f_{Ll}(\alpha_j)$ in (53) and $f_{Rr}(\alpha_j)$ in (56).

d) Repeat Steps b) and c) for every α_j $(j = 1, \ldots, m)$, and then go to Step 2).

2) Construct $\bar{Y}_{\text{LWA}}$ from the m α-cuts. To do this:

a) Store the left-coordinates $(f_{Ll}(\alpha_j), \alpha_j), j = 1, ..., m$.

b) Store the right-coordinates $(f_{Rr}(\alpha_j), \alpha_j)$, $j = 1, ..., m$.

c) (Optional) Fit a spline curve through the $2m$ coordinates just stored.

Once $\underline{Y}_{\text{LWA}}$ and $\bar{Y}_{\text{LWA}}$ are obtained, $\tilde{Y}_{\text{LWA}}$ is determined, i.e. its FOU is the area between $\underline{Y}_{\text{LWA}}$ and $\bar{Y}_{\text{LWA}}$.

C. Example

An example of the LWA is shown in Fig. 9. $\tilde{X}_i$ and $\tilde{W}_i$ are obtained by blurring the corresponding T1 FSs X_i and W_i, which are shown as the dashed lines in the FOUs. Note that the LMFs of $\tilde{X}_i$ and $\tilde{W}_i$ have different heights. In each figure 201 equally spaced α-cuts were used. Observe from Fig. 9(c) that the dashed curve Y_{FWA} is not located symmetrically in the FOU of $\tilde{Y}_{\text{LWA}}$. The nonsymmetrical $\tilde{X}_i$ and $\tilde{W}_i$ FOUs provide a nonsymmetrical $\tilde{Y}_{\text{LWA}}$.

V. Application

As pointed out in the Introduction, a promising application for the LWA is *distributed and hierarchical decision-making*. In this section, the paper evaluation process (that was described in the Introduction) for a generic journal is used as an application to illustrate how the LWA can be employed. This application is representative of other distributed and hierarchical decision-making applications; so, its results should be extendable to them.

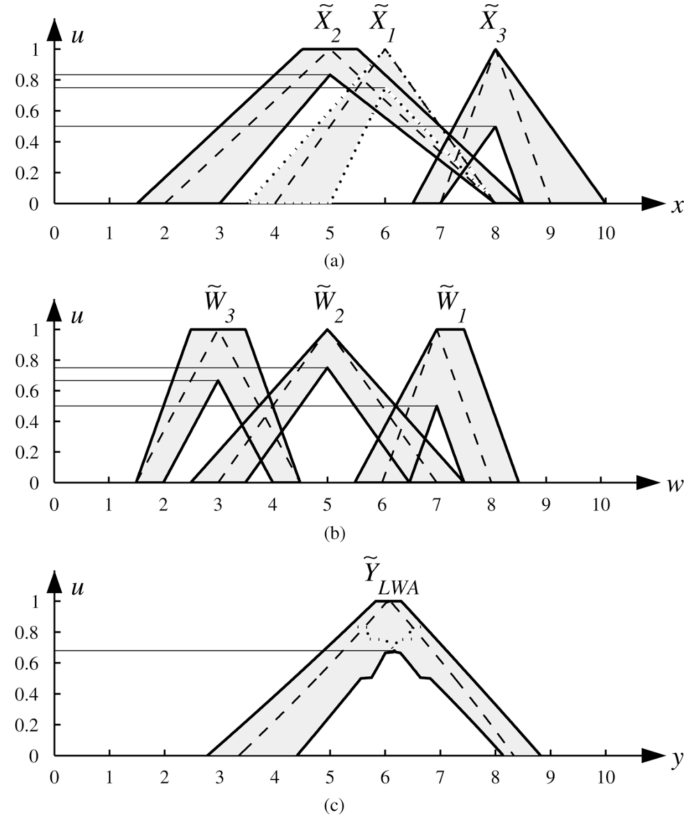

Fig. 9. (a) $\tilde{X}_i$. To distinguish between $\tilde{X}_1$ and $\tilde{X}_2$, the UMF and LMF of $\tilde{X}_1$ are plotted in dotted lines; (b) $\tilde{W}_i$; (c) $\tilde{Y}_{\text{LWA}}$. The dotted curve in (c) indicates the overlapped area where $f_{Lr}(\alpha_j) > f_{Rl}(\alpha_j)$. The dashed lines in the FOUs are T1 FSs (a) X_i, (b) W_i, and (c) the corresponding Y_{FWA}.

ASSESSMENT:	Poor	Marginal	Adequate	Good	Excellent
Importance	( )	( )	( )	( )	( )
Content	( )	( )	( )	( )	( )
Depth	( )	( )	( )	( )	( )
Presentation	( )	( )	( )	( )	( )
EXPERTISE:	Low		Moderate		High
Your Expertise	( )		( )		( )

Fig. 10. The review form for a generic journal.

A. Introduction

When an author submits a paper to a journal, the Editor usually assigns its review to an Associate Editor (AE), who then sends it to at least three reviewers. These reviewers send their reviews back to the AE, who then makes a final decision based on their opinions. In addition to the "comments for the author(s)," each reviewer usually has to complete a form similar to the one shown in Fig. 10, in which the reviewer has to evaluate the paper based on four[8] measures: *importance, content, depth,* and *presentation*. For each of the four measures, there are five assessment levels that range from the best to the worst, namely: *excellent, good, adequate, marginal,* and *poor*. A reviewer must check off an appropriate level for every measure. Usually, the reviewer is asked to give an overall evaluation of the paper and

[8]Four measures are chosen for illustration purposes and to save space; there could be arbitrary many measures. It is also possible to have hierarchical measures, i.e., measures with some submeasures associated with each of them.

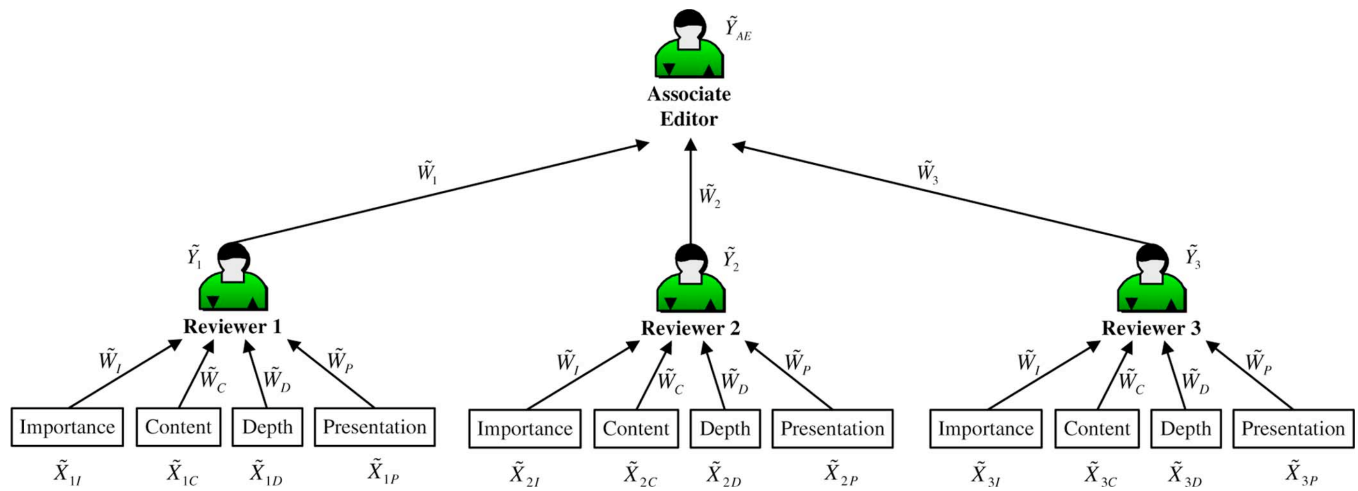

Fig. 11. The paper evaluation process for a generic journal.

make a recommendation to the AE. The AE then makes a final decision based on the opinions of the three reviewers. The entire hierarchical process is shown in Fig. 11.

Sometimes a reviewer may feel it is difficult to give an overall evaluation of a paper because it gets high scores on some of the measures but does poorly on the others. In that case, the reviewer may give an evaluation based on the reputation of the author(s) or randomly choose an evaluation from several comparable evaluations. A similar situation may also occur at the AE level, e.g., if one reviewer suggests rejection of the paper, another suggests a revision of the paper, and the third reviewer suggests acceptance of the paper, what should the final decision be?

B. The Automatic Paper Evaluation Process

Because the above evaluation process is often difficult and subjective, it may be better to leave it to a computer, i.e., each reviewer should only be asked to provide a subjective evaluation of a paper for each of the four measures, after which the LWA would automatically compute the reviewer's overall opinion (judgement) of the paper. Once the opinions of all the reviewers are obtained, another LWA would compute a final aggregated (fused) opinion for the AE. This automatic process has the potential to relieve much of the burden of the reviewers and the AE, and, moreover, it may be more accurate and less subjective.

Because the paper evaluation process is distributed and hierarchical (see Fig. 11), and linguistic evaluations are used to reach a conclusion, this seems to be an excellent application for the LWA; however, before the LWA can be implemented, the following need to be established.

1) A five-word[9] *vocabulary for each of the four measures*, e.g., *excellent, good, adequate, marginal,* and *poor,* as in the review form shown in Fig. 10.
2) *IT2 FSs corresponding to the five-word vocabulary for each of the four measures,* so that once a reviewer selects an appropriate word for a measure, the corresponding IT2 FS can be activated.

In this application it is assumed that all IT2 FSs are established on a 0–10 scale. Their FOUs can be found by surveying the AEs (and reviewers). Because "*words mean different things to different people*" [16], it is almost certain that the AEs (and reviewers) will have different opinions about the ranges of the five words for each measure, i.e., there will be uncertainties for every word. An IT2 FS can then be used to model the uncertainties for each word. Mendel and Wu [23]–[25] have proposed a fuzzistic methodology[10] to model words by IT2 FSs. Here it is assumed that FOUs have already been established for the five words.

3) *Four IT2 FSs corresponding to the weights for the four measures,* shown in Fig. 11 as $\tilde{W}_I, \tilde{W}_C, \tilde{W}_D$, and $\tilde{W}_P$. These weights are necessary because usually the four measures are not equally important. The IT2 FSs corresponding to the weights can also be established by the fuzzistics methodology described in [23]–[25]. Here it is assumed that FOUs for these four weights have already been established.
4) *Three IT2 FSs corresponding to the weights for the three reviewers*, shown in Fig. 11 as $\tilde{W}_1, \tilde{W}_2$, and $\tilde{W}_3$. These weights are necessary because the opinion of a reviewer with high expertise should be considered more seriously than the opinion of a reviewer with low expertise. In the review form shown in Fig. 10, a reviewer's expertise is divided into three levels: *high, moderate,* and *low*. The IT2 FSs corresponding to the three words can also be established by the fuzzistics methodology described in [23] and [24]. Every reviewer would be asked to indicate his/her expertise by checking off one of the three words. The word's IT2 FS would be used as the reviewer's weight. Once again, it is assumed that the FOUs for these three levels of expertise have been established.

Note that a reviewer is not asked to select the weights for the four measures; this is done behind the scenes, ahead of time. Note, also, that the positions of the measures on the review form will indicate their relative order of importance to a reviewer.

[9]Of course, fewer or more than five words could be used, and different words could be used for each measure.

[10]Research on fuzzistic methodology is ongoing, and we expect even better methodologies to be available in the future.

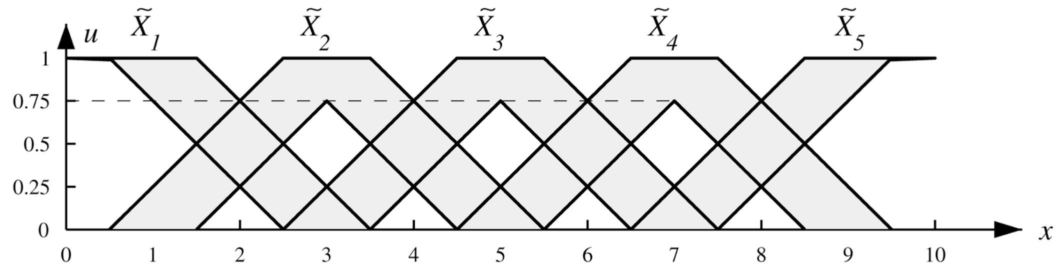

Fig. 12. FOUs for the five words for each measure. $\tilde{X}_1, \tilde{X}_2, \tilde{X}_3, \tilde{X}_4$, and $\tilde{X}_5$ correspond to *poor, marginal, adequate, good,* and *excellent,* respectively.

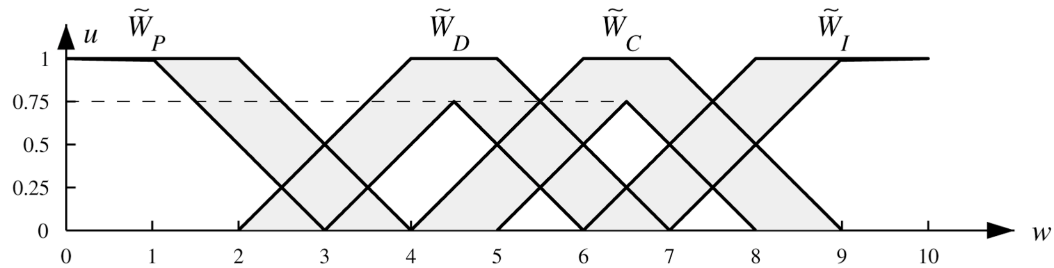

Fig. 13. The weights associated with the four measures. $\tilde{W}_I, \tilde{W}_C, \tilde{W}_D$, and $\tilde{W}_P$ correspond to the weight for *importance, content, depth,* and *presentation*, respectively.

Once the words for the four measures and all of the weights are modeled by IT2 FSs, the entire paper evaluation process can proceed automatically as follows.

1) For each of the four measures *importance, content, depth,* and *presentation,* a reviewer chooses an appropriate linguistic word from the five precalibrated words. Doing this determines $\tilde{X}_{jI}, \tilde{X}_{jC}, \tilde{X}_{jD}$, and $\tilde{X}_{jP} (j = 1, 2, 3)$. Once a reviewer has finished the online evaluation of the four measures, an LWA is computed automatically to provide the IT2 FS FOU representing the reviewer's overall opinion about the paper $\tilde{Y}_j (j = 1, 2, 3)$, where

$$\tilde{Y}_j = \frac{\tilde{X}_{jI}\tilde{W}_I + \tilde{X}_{jC}\tilde{W}_C + \tilde{X}_{jD}\tilde{W}_D + \tilde{X}_{jP}\tilde{W}_P}{\tilde{W}_I + \tilde{W}_C + \tilde{W}_D + \tilde{W}_P}. \quad (61)$$

FOU$(\tilde{Y}_j)$ is obtained for each reviewer using the method described in Section IV. Whether or not $\tilde{Y}_j$ should be revealed to the reviewer is an open question.

2) Not only would the AE receive FOU$(\tilde{Y}_i)$ but an LWA $\tilde{Y}_{\mathrm{AE}}$ would also be computed for the AE that summarizes the aggregated opinions of the three reviewers, where

$$\tilde{Y}_{\mathrm{AE}} = \frac{\tilde{Y}_1\tilde{W}_1 + \tilde{Y}_2\tilde{W}_2 + \tilde{Y}_3\tilde{W}_3}{\tilde{W}_1 + \tilde{W}_2 + \tilde{W}_3}. \quad (62)$$

Again, FOU$(\tilde{Y}_{\mathrm{AE}})$ is obtained by the method described in Section IV.

3) FOU$(\tilde{Y}_{\mathrm{AE}})$ can be mapped to a word in a codebook by using the similarity measure proposed in [32]. Using FOU$(\tilde{Y}_{\mathrm{AE}})$ and/or FOU$(\tilde{Y}_j)(j = 1, 2, 3)$ and a set of rules (which also need to be established *a priori*), the AE makes a final recommendation, e.g., one rule might be "If $\tilde{Y}_i$ is *Poor* and $\tilde{W}_i$ is *High*, then reject the paper." Another rule might be, "If $\tilde{Y}_{\mathrm{AE}}$ is *Good*, then accept the paper." Exactly how to establish such rules is an open research issue, but it is well within state-of-the-art knowledge to do this.

Note that the above process has four LWAs: one for each reviewer [(61)] and one that aggregates the three reviews for the AE [(62)].

C. Examples

Simulation results for the simplified paper evaluation process are presented in this section. The universes of discourse for all the measures and weights used in this example are [0, 10]. The four measures *(importance, content, depth,* and *presentation)* use the same five IT2 FSs $(\tilde{X}_1, \ldots, \tilde{X}_5)$ shown in Fig. 12 to represent the five words, *poor, marginal, adequate, good,* and *excellent*. A reviewer chooses one word for each measure according to the reviewer's subjective decision about the paper's score in that measure. The reviewer's choices for different measures can even be the same word, e.g., the choice for both *importance* and *depth* may be *good* $(\tilde{X}_4)$.

The predetermined weights for the four measures *importance, content, depth,* and *presentation* are $\tilde{W}_I, \tilde{W}_C, \tilde{W}_D$, and $\tilde{W}_P$, respectively, as shown in Fig. 13. Note that there is no need to assign words to these weights (since the weights are not revealed to a reviewer).

The weights for the reviewer's expertise *low, moderate,* and *high* are represented by IT2 FSs $\tilde{W}_L, \tilde{W}_M$, and $\tilde{W}_H$, respectively, as shown in Fig. 14. Every reviewer needs to indicate the reviewer's expertise in the review form. The corresponding predetermined IT2 FS is automatically set as the reviewer's weight.

Example 1: Three completed review forms are shown in Fig. 15(a)–(c). The IT2 FSs corresponding to the reviewer's choices about the four measures are shown in Fig. 15(d)–(f). The weights for the four measures have already been given in Fig. 13. The LWAs computed for the three reviewers are shown in Fig. 15(g). The weights for the three reviewers are shown in Fig. 15(h). The aggregated reviewer's LWA computed for the AE is shown in Fig. 15(i).

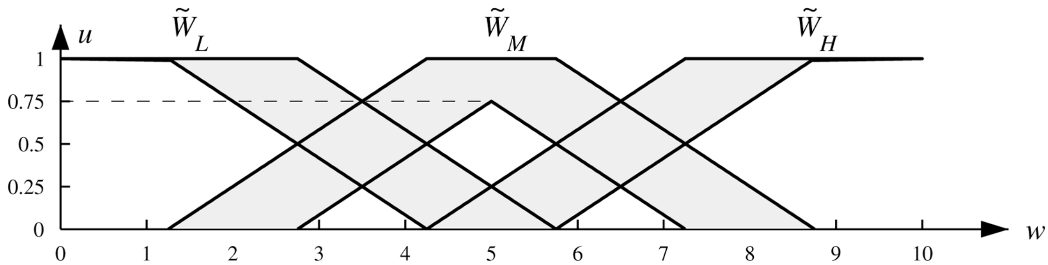

Fig. 14. The weights for the reviewer's expertise: *low* ($\tilde{W}_L$), *moderate* ($\tilde{W}_M$), and *high* ($\tilde{W}_H$)).

ASSESSMENT:	Poor	Marginal	Adequate	Good	Excellent
Importance	(X)	( )	( )	( )	( )
Content	( )	(X)	( )	( )	( )
Depth	( )	( )	(X)	( )	( )
Presentation	( )	( )	( )	(X)	( )
EXPERTISE:	Low		Moderate		High
Your Expertise	( )		( )		(X)

(a)

ASSESSMENT:	Poor	Marginal	Adequate	Good	Excellent
Importance	( )	(X)	( )	( )	( )
Content	(X)	( )	( )	( )	( )
Depth	( )	( )	( )	(X)	( )
Presentation	( )	( )	(X)	( )	( )
EXPERTISE:	Low		Moderate		High
Your Expertise	( )		(X)		( )

(b)

ASSESSMENT:	Poor	Marginal	Adequate	Good	Excellent
Importance	( )	( )	( )	(X)	( )
Content	( )	( )	( )	(X)	( )
Depth	( )	( )	( )	(X)	( )
Presentation	( )	( )	(X)	( )	( )
EXPERTISE:	Low		Moderate		High
Your Expertise	(X)		( )		( )

(c)

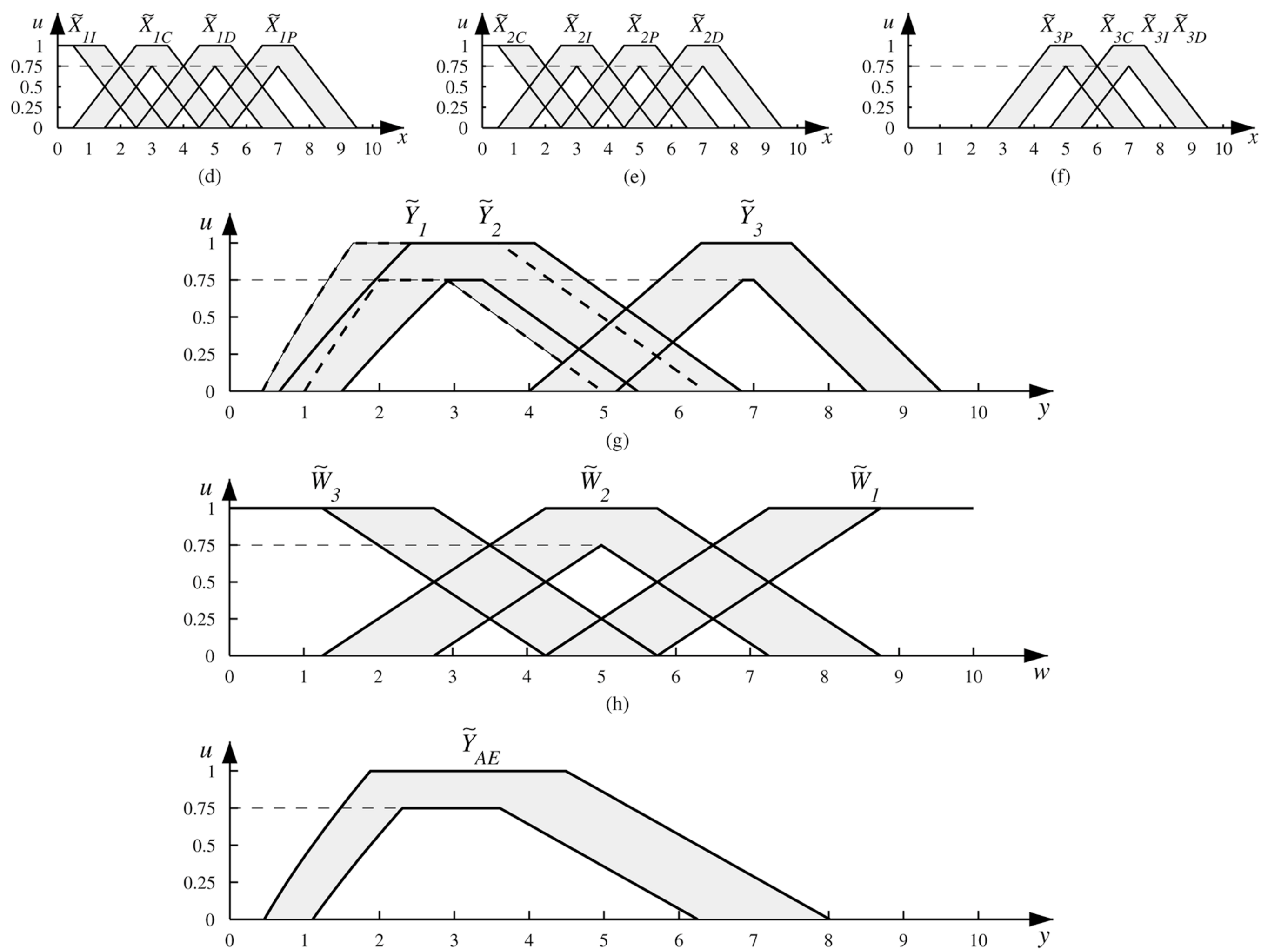

Fig. 15. Example 1: (a)–(c) Reviewer i's ($i = 1, 2, 3$) completed review form; (d)–(f) FOUs for Reviewer i's ($i = 1, 2, 3$) choices for the four measures; (g) FOUs for all three reviewers' opinions (The LMF and UMF of $\tilde{Y}_1$ are dashed); (h) FOUs for the weights corresponding to the three reviewers' expertise; and (i) FOU of the aggregated reviewers' opinions computed by the LWA. The weights for the four measures used to compute $\tilde{Y}_i$ ($i = 1, 2, 3$) are shown in Fig. 13.

Observe from Fig. 15(a) that Reviewer 1's opinions on the first two measures are below *adequate,* on the third measure is exactly *adequate,*, and on the fourth measure is somewhat above *adequate.* Observe also, from Fig. 13, that the first two measures have the largest weights ($\tilde{W}_I$ and $\tilde{W}_C$) and the fourth measure has the smallest weight ($\tilde{W}_P$); consequently, it is expected that

Reviewer 1's overall opinion on the paper will be below *adequate.* This is confirmed by $\tilde{Y}_1$ shown in Fig. 15(g), which is to the left of $\tilde{X}_3$ in Fig. 12. Similarly, it is also expected that Reviewer 2's overall opinion on the paper will be below *adequate,* as confirmed by $\tilde{Y}_2$ shown in Fig. 15(g).

Fig. 15(c) shows that Reviewer 3's opinions on the first three measures are above *adequate* and Reviewer 3's opinion on the fourth measure is *adequate.* It is hence expected that Reviewer 3's overall opinion about the paper should be above *adequate,* as confirmed by $\tilde{Y}_3$ in Fig. 15(g). Observe, also, that $\tilde{Y}_3$ is visually more similar to $\tilde{X}_{3D}$ shown in Fig. 15(f) (which is the same as $\tilde{X}_{3I}$ and $\tilde{X}_{3C}$). This is reasonable because $\tilde{X}_{3P}$ has the smallest weight, and consequently it has the least influence on $\tilde{Y}_3$.

As shown in Fig. 15(g), $\tilde{Y}_1$ and $\tilde{Y}_2$ are quite similar. Fig. 15(h) shows that the weights associated with $\tilde{Y}_1$ and $\tilde{Y}_2$ are larger than $\tilde{W}_3$, the weight associated with $\tilde{Y}_3$; consequently, it is expected that $\tilde{Y}_{\mathrm{AE}}$ should be close to $\tilde{Y}_1$ or $\tilde{Y}_2$, as confirmed by Fig. 15(i).

Once $\tilde{Y}_{\mathrm{AE}}$ is obtained, it can be mapped to a word in a codebook. Assume the codebook consists of the five words,[11] *poor, marginal, adequate, good,* and *excellent*, whose corresponding IT2 FSs are shown in Fig. 12 as $\tilde{X}_1 - \tilde{X}_5$. Using the VSM proposed in [32], the similarities between $\tilde{Y}_{\mathrm{AE}}$ and $\tilde{X}_i$ are

$$\begin{aligned} s(\tilde{Y}_{\mathrm{AE}}, \tilde{X}_1) &= 0.1161 \\ s(\tilde{Y}_{\mathrm{AE}}, \tilde{X}_2) &= 0.3805 \\ s(\tilde{Y}_{\mathrm{AE}}, \tilde{X}_3) &= 0.3161 \\ s(\tilde{Y}_{\mathrm{AE}}, \tilde{X}_4) &= 0.1096 \\ s(\tilde{Y}_{\mathrm{AE}}, \tilde{X}_5) &= 0.0297. \end{aligned} \tag{63}$$

Observe that all $s(\tilde{Y}_{\mathrm{AE}}, \tilde{X}_i)$ are relatively small and there is no $s(\tilde{Y}_{\mathrm{AE}}, \tilde{X}_i)$ that dominates the others. This is because the codebook consists of only five words. If we consider more diverse FOUs in the codebook, we should be able to map $\tilde{Y}_{\mathrm{AE}}$ to a word with high similarity degree. In this example, we may map $\tilde{Y}_{\mathrm{AE}}$ to *marginal.* $\tilde{Y}_{\mathrm{AE}}$ is most similar to *marginal* but there are still two possibilities: 1) $\tilde{Y}_{\mathrm{AE}}$ is better than *marginal,* i.e., $\tilde{Y}_{\mathrm{AE}}$ is between *marginal* and *good;* and 2) $\tilde{Y}_{\mathrm{AE}}$ is worse than *marginal,* i.e., $\tilde{Y}_{\mathrm{AE}}$ is between *poor* and *marginal.* Since the second largest similarity is $s(\tilde{Y}_{\mathrm{AE}}, \tilde{X}_3)$, we conclude that the quality of the paper is between *marginal* and *good.* This suggests that the paper should be rewritten.

Example 2: Three different completed review forms are shown in Fig. 16(a)–(c). The IT2 FSs corresponding to the reviewer's choices about the four measures are shown in Fig. 16(d)–(f). The weights for the four measures have already been given in Fig. 13. The three LWAs computed for the three reviewers are shown in Fig. 16(g). The weights for the three reviewers are shown in Fig. 16(h). The LWA computed for the AE is shown in Fig. 16(i).

Observe, from Fig. 16(a), (d), and (g), that when a reviewer's choices on all measures are *marginal,* the aggregated opinion is also *marginal* ($\tilde{Y}_1$ is exactly the same as $\tilde{X}_2$, the IT2 FS corresponding to *marginal* in Fig. 12). Observe also, from Fig. 16(b), (e), and (g) [or Fig. 16(c), (f), and (g)] that when a reviewer's choices on the measures are either below or exactly *marginal,* the aggregated FOU corresponding to the reviewer's overall opinion is below *marginal*. This again agrees with intuition.

Fig. 16(g) shows that when all three reviewers' opinions are below *marginal,* the output of the LWA ($\tilde{Y}_{\mathrm{AE}}$), shown in Fig. 16(i), is also below *marginal* (compare $\tilde{Y}_{\mathrm{AE}}$ with $\tilde{X}_2$ in Fig. 12, which corresponds to *marginal*).

Next we map $\tilde{Y}_{\mathrm{AE}}$ to a word. Again, assume the codebook consists of five words shown in Fig. 12. Using the VSM proposed in [32], the similarities between $\tilde{Y}_{\mathrm{AE}}$ and $\tilde{X}_i$ are

$$\begin{aligned} s(\tilde{Y}_{\mathrm{AE}}, \tilde{X}_1) &= 0.2837 \\ s(\tilde{Y}_{\mathrm{AE}}, \tilde{X}_2) &= 0.4878 \\ s(\tilde{Y}_{\mathrm{AE}}, \tilde{X}_3) &= 0.1094 \\ s(\tilde{Y}_{\mathrm{AE}}, \tilde{X}_4) &= 0.0245 \\ s(\tilde{Y}_{\mathrm{AE}}, \tilde{X}_5) &= 0.0044. \end{aligned} \tag{64}$$

Observe that $\tilde{Y}_{\mathrm{AE}}$ is most similar to $\tilde{X}_2$. Since the second largest similarity is $s(\tilde{Y}_{\mathrm{AE}}, \tilde{X}_1)$, the quality of the paper is between *poor* and *marginal.* This suggests that the AE should reject the paper.

VI. Conclusion

The concept of the LWA was introduced in this paper. It was shown that for IT2 FSs, the LWA is also an IT2 FS, and theorems were provided to compute it. α-cuts and FWA algorithms were employed.

Because the LWA is a generalization of the FWA from T1 FSs to IT2 FSs, there is a close relation between them. It is shown that finding the LWA $\tilde{Y}_{\mathrm{LWA}}$ is equivalent to finding its UMF $\overline{Y}_{\mathrm{LWA}}$ and LMF $\underline{Y}_{\mathrm{LWA}}$, each of which may be viewed as an FWA.

The LWA offers a unique property that a weighted average or a FWA does not have, namely, it is able to incorporate the linguistic opinions of a group of people and then reach a decision linguistically. This can be accomplished by mapping $\tilde{Y}_{\mathrm{LWA}}$ into the word whose FOU is most similar to $FOU(\tilde{Y}_{\mathrm{LWA}})$, e.g., [32]. A promising application of the LWA is *distributed and hierarchical decision-making*. As shown in the Introduction, the LWA can also be used as a CWW engine in the Per-C.

Appendix A
Proof of Theorem 1

Because the proofs of Theorem 1(b)–(d) are quite similar to the proof of (a), only the proof of (a) is given here.

Let

$$g_{Ll}(\mathbf{c}(\alpha_j), \mathbf{d}(\alpha_j)) \equiv \frac{\sum_{i=1}^{k_{L1}} a_{il}(\alpha_j) d_i(\alpha_j) + \sum_{i=k_{L1}+1}^{n} a_{il}(\alpha_j) c_i(\alpha_j)}{\sum_{i=1}^{k_{L1}} d_i(\alpha_j) + \sum_{i=k_{L1}+1}^{n} c_i(\alpha_j)} \tag{A-1}$$

where $\mathbf{c}(\alpha_j) \equiv [c_{k_{L1}+1}(\alpha_j), c_{k_{L1}+2}(\alpha_j), \ldots, c_n(\alpha_j)]^T$, $\mathbf{d}(\alpha_j) \equiv [d_1(\alpha_j), d_2(\alpha_j), \ldots, d_{k_{L1}}(\alpha_j)]^T$, $c_i(\alpha_j) \in$

[11]In practice, there can be more words in the codebook, and there are not necessarily the same as those used by the four measures.

ASSESSMENT:	Poor	Marginal	Adequate	Good	Excellent
Importance	( )	(X)	( )	( )	( )
Content	( )	(X)	( )	( )	( )
Depth	( )	(X)	( )	( )	( )
Presentation	( )	(X)	( )	( )	( )
EXPERTISE:	Low		Moderate		High
Your Expertise	(X)		( )		( )

(a)

ASSESSMENT:	Poor	Marginal	Adequate	Good	Excellent
Importance	( )	(X)	( )	( )	( )
Content	(X)	( )	( )	( )	( )
Depth	(X)	( )	( )	( )	( )
Presentation	(X)	( )	( )	( )	( )
EXPERTISE:	Low		Moderate		High
Your Expertise	( )		(X)		( )

(b)

ASSESSMENT:	Poor	Marginal	Adequate	Good	Excellent
Importance	(X)	( )	( )	( )	( )
Content	( )	(X)	( )	( )	( )
Depth	(X)	( )	( )	( )	( )
Presentation	( )	(X)	( )	( )	( )
EXPERTISE:	Low		Moderate		High
Your Expertise	( )		( )		(X)

(c)

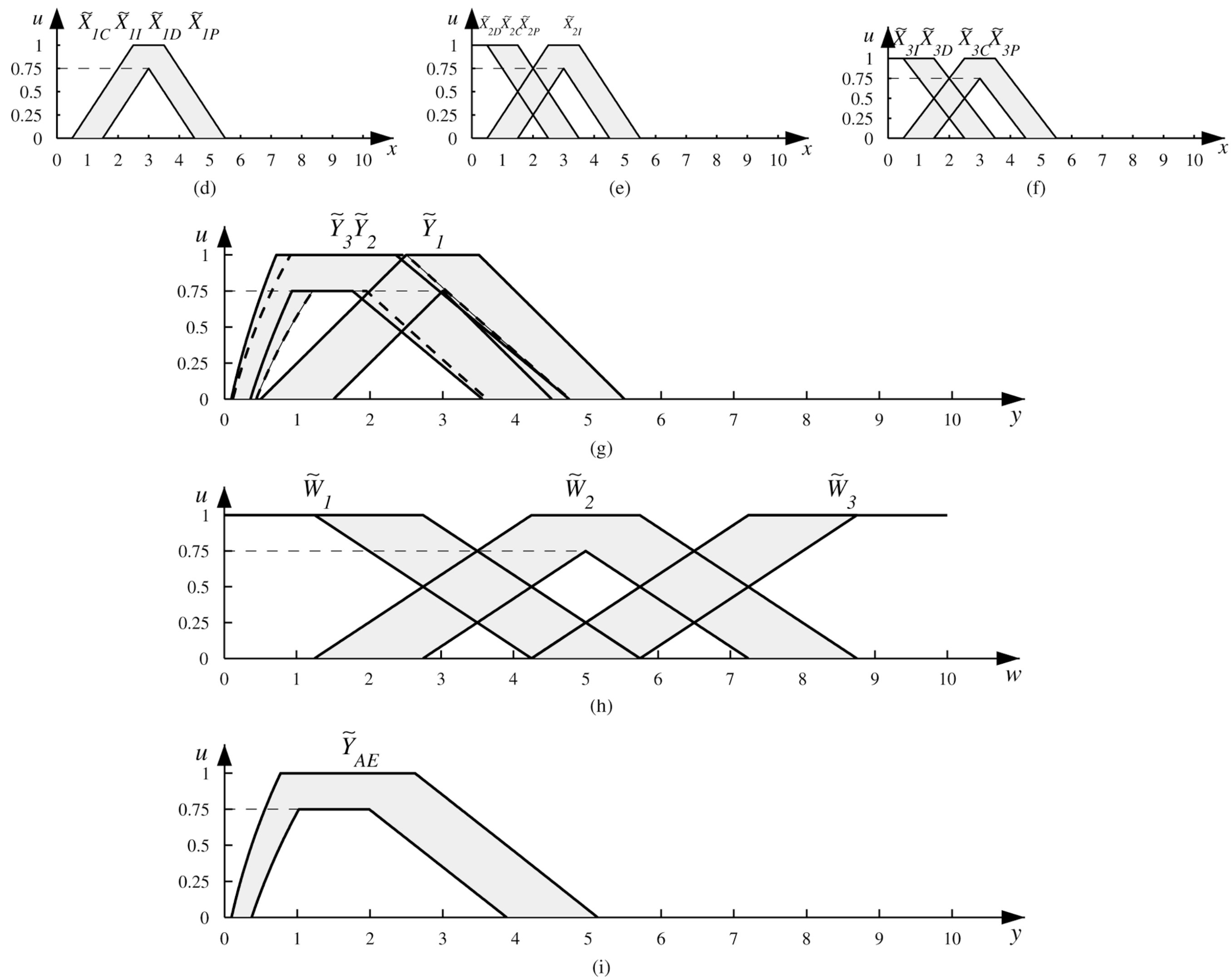

Fig. 16. Example 2: (a)–(c) Reviewer i's $(i = 1, 2, 3)$ completed review form; (d)–(f) FOUs for Reviewer i's $(i = 1, 2, 3)$ choices for the four measures; (g) FOUs for all three reviewers' opinions (the LMF and UMF of $\tilde{Y}_2$ are dashed); (h) FOUs for the weights corresponding to the three reviewers' expertise; and (i) FOU of the aggregated reviewers' opinions computed by the LWA. The weights for the four measures used to compute $\tilde{Y}_i$ $(i = 1, 2, 3)$ are shown in Fig. 13.

$[c_{il}(\alpha_j), c'_{ir}(\alpha_j)]$, and $d_i(\alpha_j) \in [d'_{il}(\alpha_j), d_{ir}(\alpha_j)]$. Then $f_{Ll}(\alpha_j)$ in (49) can be found by the following.

1) enumerating all possible combinations of $(c_{k_{L1}+1}(\alpha_j), \ldots, c_n(\alpha_j), d_1(\alpha_j), \ldots, d_{k_{L1}}(\alpha_j))$ for $c_i(\alpha_j) \in [c_{il}(\alpha_j), c'_{ir}(\alpha_j)]$ and $d_i(\alpha_j) \in [d'_{il}(\alpha_j), d_{ir}(\alpha_j)]$;
2) computing $g_{Ll}(\mathbf{c}(\alpha_j), \mathbf{d}(\alpha_j))$ in (A-1) for each combination; and,
3) setting $f_{Ll}(\alpha_j)$ to the smallest $g_{Ll}(\mathbf{c}(\alpha_j), \mathbf{d}(\alpha_j))$.

Note that k_{L1} corresponding to the smallest $g_{Ll}(\mathbf{c}(\alpha_j), \mathbf{d}(\alpha_j))$ in step 3) is k_{Ll} in Theorem 1. In the following proof, the fact that there always exists such a k_{Ll} is used; however, there is no need to know the value of it (its value can be computed by a FWA algorithm). Equation (49) can be expressed as

$$f_{Ll}(\alpha_j) = \min_{\substack{\forall c_i \in [c_{il}, c'_{ir}] \\ \forall d_i \in [d'_{il}, d_{ir}]}} g_{Ll}(\mathbf{c}(\alpha_j), \mathbf{d}(\alpha_j)). \tag{A-2}$$

In [14], it is proved that $f_{Ll}(\alpha_j)$ has a value in the interval $[a_{k_{Ll},l}(\alpha_j), a_{k_{Ll}+1,l}(\alpha_j)]$; hence, at least one $g_{Ll}(\mathbf{c}(\alpha_j), \mathbf{d}(\alpha_j))$ must assume a value in the same interval. In general there can be numerous $g_{Ll}(\mathbf{c}(\alpha_j), \mathbf{d}(\alpha_j))$ satisfying

$$a_{k_{Ll},l}(\alpha_j) \le g_{Ll}(\mathbf{c}(\alpha_j), \mathbf{d}(\alpha_j)) \le a_{k_{Ll}+1,l}(\alpha_j). \tag{A-3}$$

$$f_{Lr}(\alpha_j) = \frac{\sum_{i=1}^{k_{Lr}} b_{ir}(\alpha_j)c_{il}(\alpha_j) + \sum_{i=k_{Lr}+1}^{n} b_{ir}(\alpha_j)d_{ir}(\alpha_j)}{\sum_{i=1}^{k_{Lr}} c_{il}(\alpha_j) + \sum_{i=k_{Lr}+1}^{n} d_{ir}(\alpha_j)} \tag{B-1}$$

$$f_{Rl}(\alpha_j) = \frac{\sum_{i=1}^{k_{Rl}} a_{il}(\alpha_j)d_{ir}(\alpha_j) + \sum_{i=k_{Rl}+1}^{n} a_{il}(\alpha_j)c_{il}(\alpha_j)}{\sum_{i=1}^{k_{Rl}} d_{ir}(\alpha_j) + \sum_{i=k_{Rl}+1}^{n} c_{il}(\alpha_j)}. \tag{B-2}$$

The remaining $g_{Ll}(\mathbf{c}(\alpha_j), \mathbf{d}(\alpha_j))$ must be larger than $a_{k_{Ll}+1,l}(\alpha_j)$, i.e., they must assume values in one of the intervals $(a_{k_{Ll}+1,l}(\alpha_j), a_{k_{Ll}+2,l}(\alpha_j)]$, $(a_{k_{Ll}+2,l}(\alpha_j), a_{k_{Ll}+3,l}(\alpha_j)]$, etc. Because the minimum of $g_{Ll}(\mathbf{c}(\alpha_j), \mathbf{d}(\alpha_j))$ is of interest, only those $g_{Ll}(\mathbf{c}(\alpha_j), \mathbf{d}(\alpha_j))$ satisfying (A-3) will be considered in this proof.

Next it is shown that when $g_{Ll}(\mathbf{c}(\alpha_j), \mathbf{d}(\alpha_j))$ achieves its minimum, i) $d_i(\alpha_j) = d_{ir}(\alpha_j)$ for $i \le k_{Ll}$ and ii) $c_i(\alpha_j) = c_{il}(\alpha_j)$ for $i \ge k_{Ll}+1$.

i) When $i \le k_{Ll}$, it is straightforward to show that the derivative of $g_{Ll}(\mathbf{c}(\alpha_j), \mathbf{d}(\alpha_j))$ with respect to $d_i(\alpha_j)$, computed from (A-1), is

$$\frac{\partial g_{Ll}(\mathbf{c}(\alpha_j), \mathbf{d}(\alpha_j))}{\partial d_i(\alpha_j)} = \frac{a_{il}(\alpha_j) - g_{Ll}(\mathbf{c}(\alpha_j), \mathbf{d}(\alpha_j))}{\sum_{i=1}^{k_{Ll}} d_i(\alpha_j) + \sum_{i=k_{Ll}+1}^{n} c_i(\alpha_j)}. \tag{A-4}$$

Using the left-hand side of (A-3), it follows that

$$-g_{Ll}(\mathbf{c}(\alpha_j), \mathbf{d}(\alpha_j)) \le -a_{k_{Ll},l}(\alpha_j). \tag{A-5}$$

Hence, in the numerator of (A-4)

$$a_{il}(\alpha_j) - g_{Ll}(\mathbf{c}(\alpha_j), \mathbf{d}(\alpha_j)) \le a_{il}(\alpha_j) - a_{k_{Ll},l}(\alpha_j) \le 0. \tag{A-6}$$

In obtaining (A-6), the fact that $a_{il}(\alpha_j) \le a_{k_{Ll},l}(\alpha_j)$ when $i \le k_{Ll}$ [due to the *a priori* increased ordering of the $a_{il}(\alpha_j)$; see (26)] was used. Consequently, using (A-6) in (A-4), it follows that

$$\frac{\partial g_{Ll}(\mathbf{c}(\alpha_j), \mathbf{d}(\alpha_j))}{\partial d_i(\alpha_j)} \le \frac{a_{il}(\alpha_j) - a_{k_{Ll},l}(\alpha_j)}{\sum_{i=1}^{k_{Ll}} d_i(\alpha_j) + \sum_{i=k_{Ll}+1}^{n} c_i(\alpha_j)} \le 0. \tag{A-7}$$

Equation (A-3) indicates that the first derivative of $g_{Ll}(\mathbf{c}(\alpha_j), \mathbf{d}(\alpha_j))$ with respect to $d_i(\alpha_j)$ $(i \le k_{Ll})$ is negative; thus, $g_{Ll}(\mathbf{c}(\alpha_j), \mathbf{d}(\alpha_j))$ decreases when $d_i(\alpha_j)(i \le k_{Ll})$ increases. Consequently, the minimum of $g_{Ll}(\mathbf{c}(\alpha_j), \mathbf{d}(\alpha_j))$ must use maximum possible $d_i(\alpha_j)$ for $i \le k_{Ll}$, i.e., $d_i(\alpha_j) = d_{ir}(\alpha_j)$ for $i \le k_{Ll}$, as stated in (53).

ii) When $i \ge k_{Ll}+1$, it is straightforward to show that the derivative of $g_{Ll}(\mathbf{c}(\alpha_j), \mathbf{d}(\alpha_j))$ with respect to $c_i(\alpha_j)$, computed from (A-1), is

$$\frac{\partial g_{Ll}(\mathbf{c}(\alpha_j), \mathbf{d}(\alpha_j))}{\partial c_i(\alpha_j)} = \frac{a_{il}(\alpha_j) - g_{Ll}(\mathbf{c}(\alpha_j), \mathbf{d}(\alpha_j))}{\sum_{i=1}^{k_{Ll}} d_i(\alpha_j) + \sum_{i=k_{Ll}+1}^{n} c_i(\alpha_j)}. \tag{A-8}$$

Using the right-hand side of (A-3), it follows that

$$-g_{Ll}(\mathbf{c}(\alpha_j), \mathbf{d}(\alpha_j)) \ge -a_{k_{Ll}+1,l}(\alpha_j). \tag{A-9}$$

Hence, in the numerator of (A-8)

$$a_{il}(\alpha_j) - g_{Ll}(\mathbf{c}(\alpha_j), \mathbf{d}(\alpha_j)) \ge a_{il}(\alpha_j) - a_{k_{Ll}+1,l}(\alpha_j) \ge 0. \tag{A-10}$$

In obtaining (A-10), the fact that $a_{il}(\alpha_j) \ge a_{k_{Ll}+1,l}(\alpha_j)$ when $i \ge k_{Ll}+1$ [due to the *a priori* increased ordering of the $a_{il}(\alpha_j)$; see (26)] was used. Consequently, using (A-10) in (A-8), it follows that

$$\frac{\partial g_{Ll}(\mathbf{c}(\alpha_j), \mathbf{d}(\alpha_j))}{\partial c_i(\alpha_j)} \ge \frac{a_{il}(\alpha_j) - a_{k_{Ll}+1,l}(\alpha_j)}{\sum_{i=1}^{k_{Ll}} d_i(\alpha_j) + \sum_{i=k_{Ll}+1}^{n} c_i(\alpha_j)} \ge 0. \tag{A-11}$$

Equation (A-11) indicates that the first derivative of $g_{Ll}(\mathbf{c}(\alpha_j), \mathbf{d}(\alpha_j))$ with respect to $c_i(\alpha_j)$ $(i \ge k_{Ll}+1)$ is positive; thus, $g_{Ll}(\mathbf{c}(\alpha_j), \mathbf{d}(\alpha_j))$ decreases when $c_i(\alpha_j)(i \ge k_{Ll}+1)$ decreases. Consequently, the minimum of $g_{Ll}(\mathbf{c}(\alpha_j), \mathbf{d}(\alpha_j))$ must use minimum possible $c_i(\alpha_j)$ for $i \ge k_{Ll}+1$, i.e., $c_i(\alpha_j) = c_{il}(\alpha_j)$ for $i \ge k_{Ll}+1$, as stated in (53).

Appendix B
Proof of Corollary 2

Substitute (60) into (53)–(56) in Theorem 1, and observe that (53) and (56) remain unchanged and (54) and (55) change to (B-1) and (B-2) as shown at the top of the page. Note that $f_{Ll}(\alpha_j)$ and $f_{Rr}(\alpha_j)$, which determine the α-cut on $\tilde{Y}_{\text{LWA}}$, are calculated by (53) and (56), respectively. Comparing (B-1) with (56), it is observed that $f_{Lr}(\alpha_j)$ in (B-1) is the same as $f_{Rr}(\alpha_j)$ in (56).[12] Additionally, $f_{Rl}(\alpha_j)$ in (B-2) is also the same as $f_{Lr}(\alpha_j)$ in (53).

Appendix C
Proof of Theorem 2

A. *Proof of Theorem 2(a)*

This proof shows that $f_{Lr}(\alpha_j)$ and $f_{Rl}(\alpha_j)$ in Case 1, computed from (54) and (55), respectively, equal a generalized centroid [9], [16], [22], for which $f_{Lr}(\alpha_j)$ is its left bound and $f_{Rl}(\alpha_j)$ is its right bound. Consequently, $f_{Lr}(\alpha_j) \le f_{Rl}(\alpha_j)$.

[12]The switch point in (B-1) is denoted as k_{Lr} and that in (56) is denoted as k_{Rr}; however, because all $b_{ir}(\alpha_j)$, $c_{il}(\alpha_j)$, and $d_{ir}(\alpha_j)$ are the same in (B-1) and (56), when an FWA algorithm is used to compute (B-1) and (56), the resulting switch points will be the same. Consequently, (B-1) and (56) are the same.

Let

$$x_i = [a_{ir}(\alpha_j), b_{il}(\alpha_j)] \tag{C-1}$$

and

$$w_i = [c_{ir}(\alpha_j), d_{il}(\alpha_j)]. \tag{C-2}$$

It is known that in Case 1

$$a_{ir}(\alpha_j) \le b_{il}(\alpha_j) \tag{C-3}$$
$$c_{ir}(\alpha_j) \le d_{il}(\alpha_j). \tag{C-4}$$

The generalized centroid of x_i and w_i is [9], [16], [22]

$$y = \frac{\sum_{i=1}^n x_i w_i}{\sum_{i=1}^n w_i} = [y_l, y_r] \tag{C-5}$$

where

$$y_l = \frac{\sum_{i=1}^{k_l} a_{ir}(\alpha_j) d_{il}(\alpha_j) + \sum_{i=k_l+1}^{n} a_{ir}(\alpha_j) c_{ir}(\alpha_j)}{\sum_{i=1}^{k_l} d_{il}(\alpha_j) + \sum_{i=k_l+1}^{n} c_{ir}(\alpha_j)} \tag{C-6}$$

$$y_r = \frac{\sum_{i=1}^{k_r} b_{il}(\alpha_j) c_{ir}(\alpha_j) + \sum_{i=k_r+1}^{n} b_{il}(\alpha_j) d_{il}(\alpha_j)}{\sum_{i=1}^{k_r} c_{ir}(\alpha_j) + \sum_{i=k_r+1}^{n} d_{il}(\alpha_j)}. \tag{C-7}$$

The switch points are determined by [14]

$$a_{k_l,r}(\alpha_j) \le y_l \le a_{k_l+1,r}(\alpha_j) \tag{C-8}$$
$$b_{k_r,l}(\alpha_j) \le y_r \le b_{k_r+1,l}(\alpha_j). \tag{C-9}$$

Observe that (54) is the same as (C-6) and (55) is the same as (C-7), i.e.,

$$f_{Lr}(\alpha_j) = y_l \tag{C-10}$$
$$f_{Rl}(\alpha_j) = y_r. \tag{C-11}$$

Because $a_{ir}(\alpha_j) \le b_{il}(\alpha_j)$ and $c_{ir}(\alpha_j) \le d_{il}(\alpha_j)$, when FWA algorithms are used to calculate y_l and y_r, they will give $y_l \le y_r$. Consequently, $f_{Lr}(\alpha_j) \le f_{Rl}(\alpha_j)$ is always true for Case 1.

Here $a_{ir}(\alpha_j) \le b_{il}(\alpha_j)$ and $c_{ir}(\alpha_j) \le d_{il}(\alpha_j)$ are emphasized since they guarantee $y_l \le y_r$; otherwise, this may not be true, as happens in Case 2.

B. Proof of Theorem 2(b)

The correctness of Theorem 2(b) can be demonstrated by the example shown in Fig. 9. Observe from Fig. 9(a) and (b) that Case 2 corresponds to $0.5 \le \alpha_j \le 0.85$. Observe from Fig. 9(c) that when $0.5 \le \alpha_j \le 0.68$, there is a gap in the corresponding α-cut of $\tilde{Y}_{\text{LWA}}$, and when $0.68 < \alpha_j \le 0.85$, there is no gap in the corresponding α-cut of $\tilde{Y}_{\text{LWA}}$.

C. Proof of Theorem 2(c)

Corollary 2 indicates that, in Case 3

$$f_{Ll}(\alpha_j) = f_{Rl}(\alpha_j) \tag{C-12}$$
$$f_{Lr}(\alpha_j) = f_{Rr}(\alpha_j). \tag{C-13}$$

Consequently

$$\begin{aligned}[f_{Ll}(\alpha_j), f_{Lr}(\alpha_j)] &= [f_{Rl}(\alpha_j), f_{Rr}(\alpha_j)] \\ &= [f_{Ll}(\alpha_j), f_{Rr}(\alpha_j)].\end{aligned} \tag{C-14}$$

Equation (C-14) means that the FOU of $\tilde{Y}_{\text{LWA}}$ fills in the entire interval $[f_{Ll}(\alpha_j), f_{Rr}(\alpha_j)]$ (see α'' in Fig. 7), which is completely determined by the α-cuts on the UMFs. Consequently, there is no need to compute $f_{Lr}(\alpha_j)$ and $f_{Rl}(\alpha_j)$ in this case.

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