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Recursive Method for Computation of Cumulants

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Recursive Method for Computation of Cumulants: Part I

Matrix Formulation for Third-Order Diagonal-Slice Cumulants

Abstract

Recursive third-order diagonal-slice cumulant computations for both stationary and non-stationary systems are derived that are based on a state-space model. The connection between stationary and non-stationary cumulant computations is also considered; it depends upon the assumption of a stable system. An ARMA model is given as an example which shows computational aspects of the methods. Results are also verified by analysis of AR and MA models. Simulations are included.

1. Introduction

Cumulants have made it possible to identify non-minimum phase and non-causal systems. An overview of cumulant applications can be found in [1]. How to compute cumulants effectively is the problem that we address. In this report, we present a method for computing third-order cumulants recursively.

In this report, we concern ourselves with a single-input/single-output system expressed in state-space format, as

$$x(k+1) = \Phi x(k) + \gamma w(k) \tag{1}$$

Measurement $y(k)$ is related to $x(k)$ by means of the measurement equation

$$y(k) = h^T x(k) + v(k) \tag{2}$$

where $x(k)$ is the state vector with n elements. We assume that the model represented in Eqs.(1) and (2) is time-invariant, and additionally, $v(k)$ and $w(k)$ are white random signals that are mutually independent. Without loss of generality we assume that

$$E\{x(k)\} = 0. \tag{3}$$

In practice, if $E\{x\} \neq 0$ then we must first transform $x(k)$ to a zero mean vector, by using the known mean; i.e., we let

$$x(k) \equiv x(k) - E\{x(k)\}$$

In the following sections of this report, we discuss the computation of 3rd-order cumulants. The 3rd-order cumulants of a system output are defined as

$$c(k;m,s) = E\{y(k)y(k+m)y(k+s)\} \tag{4}$$

In many cases, one sets $m = s$ (eg. [2],[3],[4]). We will use $c(k;m)$ to represent this kind of diagonal slice 1-D cumulant, i.e.,

$$c(k;m) = E\{y(k)y^2(k+m)\} \tag{5}$$

2. Cumulant Computations for Stationary Systems

In this section, we assume that our system is stationary, and that

$$E\{v^2(k)\} = \sigma_v^2$$

$$E\{w^2(k)\} = \sigma_w^2$$

and

$$E\{w(k)w(m)w(s)\} = \begin{cases} \gamma_w & k = m = s \\ 0 & \text{otherwise} \end{cases}$$

In the stationary case, $c(k;m)$ does not depend on k, i.e.,

$$c(k;m) \rightarrow c(m) = E\{y(k)y^2(k+m)\} \tag{6}$$

We now proceed to evaluate $c(m)$ for the state-variable model in (1) and (2).

A. Recursive Formulas

From Eqs.(6) and (2), we have

$$\begin{aligned} c(m) &= E\{y(k)y^2(k+m)\} \\ &= E\{[h^T x(k) + v(k)][h^T x(k+m) + v(k+m)]^2\} \\ &= E\{[h^T x(k) + v(k)][h^T x(k+m)h^T x(k+m) + v^2(k+m) + 2h^T x(k+m)v(k+m)]\} \\ &= E\{h^T x(k)[h^T x(k+m)]^2 + h^T x(k)v^2(k+m) + 2h^T x(k)h^T x(k+m)v(k+m) \\ &\quad +v(k)[h^T x(k+m)]^2 + v(k)v^2(k+m) + 2v(k)h^T x(k+m)v(k+m)\} \end{aligned} \tag{7}$$

Applying the stationarity assumptions about $v(k)$ and $w(k)$, as well as the fact that $m_x = 0$, to Eq.(7), we find

$$\begin{aligned} c(m) &= E\{h^T x(k)[h^T x(k+m)]^2\} + \sigma_v^2 h^T m_x + 2h^T E\{x(k)h^T x(k+m)v(k+m)\} \\ &\quad + E\{v(k)[h^T x(k+m)]^2\} + E\{v(k)v^2(k+m)\} + 2\sigma_v^2 \delta(m) h^T m_x \\ &= E\{h^T x(k)[h^T x(k+m)]^2\} + E\{v(k)v^2(k+m)\} \end{aligned} \quad (8)$$

Observe that

$$\begin{aligned} E\{h^T x(k)[h^T x(k+m)]^2\} &= E\{\sum_{i=1}^{n} h_i x_i(k)[h^T x(k+m)]^2\} \\ &= h^T \sum_{i=1}^{n} h_i E\{x_i(k)x(k+m)x^T(k+m)\}h \end{aligned} \quad (9)$$

Since $x(k)$ is stationary, we can also represent Eq.(9) as (useful, later, when $m < 0$)

$$\begin{aligned} E\{h^T x(k)[h^T x(k+m)]^2\} &= E\{[h^T x(k)]^2 h^T x(k-m)\} \\ &= h^T \sum_{i=1}^{n} h_i E\{x_i(k-m)x(k)x^T(k)\}h \end{aligned} \quad (10)$$

For notational convenience, we define the state cumulant matrix $C_i^x(m)$, as

$$\begin{aligned} C_i^x(m) &= E\{x_i(k)x(k+m)x^T(k+m)\} \\ &= E\{x_i(k-m)x(k)x^T(k)\}; \end{aligned} \quad (11)$$

consequently, Eq.(8) can be written as

$$c(m) = h^T \sum_{i=1}^{n} h_i C_i^x(m) h + E\{v(k)v^2(k+m)\} \quad (12)$$

Next, we show how to compute $C_i^x(m)$ recursively with respect to m. For convenience, we divide the computation into two cases: $m \geq 0$ and $m \leq 0$.

Case 1. $m \geq 0$. We begin with the equation

$$C_i^x(m) = E\{x_i(k)x(k+m)x^T(k+m)\}$$

Considering Eq.(1), we obtain:

$$\begin{aligned} C_i^x(m+1) &= E\{x_i(k)x(k+m+1)x^T(k+m+1)\} \\ &= E\{x_i(k)[\Phi x(k+m) + r w(k+m)][\Phi x(k+m) + r w(k+m)]^T\} \\ &= E\{x_i(k)\Phi x(k+m)x^T(k+m)\Phi^T\} \\ &\quad + \Phi E\{x_i(k)x(k+m)w(k+m)\} r^T \\ &\quad + r E\{x_i(k)w(k+m)x^T(k+m)\}\Phi^T \\ &\quad + r E\{x_i(k)w(k+m)w(k+m)\} r^T \\ &= \Phi E\{x_i(k)x(k+m)x^T(k+m)\}\Phi^T + r r^T m_{x_i} \sigma_w^2 \end{aligned}$$

From Eqs.(3) and (11), it follows that $C_i^x(m+1)$ reduces to

$$C_i^x(m+1) = \Phi C_i^x(m)\Phi^T \tag{13}$$

Case 2. $m \leq 0$. Here we begin with

$$C_i^x(m) = E\{x_i(k-m)x(k)x^T(k)\}.$$

Let $s = -m$, so that $s > 0$, in which case

$$C_i^x(m) = C_i^x(-s) = E\{x_i(k+s)x(k)x^T(k)\} \tag{14}$$

From Eq.(1), we have

$$x_i(k+1) = \sum_{j=1}^{n} \phi_{ij} x_j(k) + r_i w(k);$$

hence,

$$\begin{aligned} C_i^x[-(s+1)] &= E\{x_i(k+s+1)x(k)x^T(k)\} \\ &= \textstyle\sum_{j=1}^{n} \phi_{ij} E\{x_j(k+s)x(k)x^T(k)\} \\ &\quad + E\{r_i w(k+s)x(k)x^T(k)\} \end{aligned}$$

so that

$$C_i^x(m-1) = \sum_{j=1}^{n} \phi_{ij} C_j^x(m) \tag{15}$$

From a comparision of Eqs.(13) and (15), it is clear that

$$C_i^x(m) \neq C_i^x(-m)$$

For the special case of $m = 0$, it follows from (13) and (15), that

$$C_i^x(1) = \Phi C_i^x(0)\Phi^T \tag{16}$$

and

$$C_i^x(-1) = \sum_{j=1}^{n} \phi_{ij} C_j^x(0) \tag{17}$$

From these equations, it is clear that we need the initial condition matrices, $C_i^x(0)$.

B. Computation of $C_i^x(0)$

In this section, we explain how to compute $C_i^x(0)$ $(i = 1, 2, \cdots, n)$

For a stationary system,

$$C_i^x(0) = E\{x_i(k+1)x(k+1)x^T(k+1)\} = E\{x_i(k)x(k)x^T(k)\}; \tag{18}$$

hence,

$$\begin{aligned} C_i^x(0) &= E\{[\sum_{j=1}^n \phi_{ij}x_j(k) + r_i w(k)][\Phi x(k) + r w(k)][\Phi x(k) + r w(k)]^T\} \\ &= \sum_{j=1}^n \phi_{ij}\Phi E\{x_j(k)x(k)x^T(k)\}\Phi^T + \sum_{j=1}^n \phi_{ij}\Phi E\{x_j(k)x(k)w(k)\}r^T \\ &\quad + \sum_{j=1}^n \phi_{ij} r E\{x_j(k)w(k)x(k)\}\Phi^T + \sum_{j=1}^n \phi_{ij} r E\{x_j(k)w(k)w(k)\}r^T \\ &\quad + r_i\Phi E\{w(k)x(k)x^T(k)\}\Phi^T + r_i\Phi E\{w(k)x(k)w(k)\}r^T \\ &\quad + r_i r E\{w(k)w(k)x^T(k)\}\Phi^T + r_i\gamma_w r r^T \\ &= \sum_{j=1}^n \phi_{ij}\Phi E\{x_j(k)x(k)x^T(k)\}\Phi^T \\ &\quad + r_i\gamma_w r r^T \end{aligned}$$

Let

$$D_i = r_i\gamma_w r r^T; \tag{19}$$

then,

$$C_i^x(0) = \sum_{j=1}^n \phi_{ij}\Phi C_j^x(0)\Phi^T + D_i \tag{20}$$

i.e.

$$C_i^x(0) = \begin{bmatrix} \phi_{i1}\Phi & \phi_{i2}\Phi & \cdots & \phi_{in}\Phi \end{bmatrix} \begin{bmatrix} C_1^x(0) \\ C_2^x(0) \\ \vdots \\ C_n^x(0) \end{bmatrix} \Phi^T + D_i \tag{21}$$

Setting $i \in \{1, 2, ..., n\}$ in Eq.(21) and collecting the results, we have the $n^2 \times n$ matrix equation

$$\begin{bmatrix} C_1^x(0) \\ C_2^x(0) \\ \vdots \\ C_n^x(0) \end{bmatrix} = \begin{bmatrix} \phi_{1,1}\Phi & \phi_{1,2}\Phi & \cdots & \phi_{1,n}\Phi \\ \phi_{2,1}\Phi & \phi_{2,2}\Phi & \cdots & \phi_{2,n}\Phi \\ \vdots & \vdots & & \vdots \\ \phi_{n,1}\Phi & \phi_{n,2}\Phi & \cdots & \phi_{n,n}\Phi \end{bmatrix} \begin{bmatrix} C_1^x(0) \\ C_2^x(0) \\ \vdots \\ C_n^x(0) \end{bmatrix} \Phi^T + \begin{bmatrix} D_1 \\ D_2 \\ \vdots \\ D_n \end{bmatrix} \tag{22}$$

Eq.(22) is a special form of the linear equation,

$$A_1 X B_1 + A_2 X B_2 = C \tag{23}$$

which can be easily solved [5] for $C_i^x(0)$, $i = 1, 2, \cdots, n$.

For a stable system, Eq.(22) can be solved in a more straight forward way. Actually, when the system is assumed to be stable, the eigenvalues of matrix Φ are all inside the unit circle; also, the eigenvalues of the $n^2 \times n^2$ matrix, $\Phi \otimes \Phi$, the Kronecker product of the matrix Φ, are all inside the unit circle(see [8]). Noticing that

$$\begin{bmatrix} \phi_{1,1}\boldsymbol{\Phi} & \phi_{1,2}\boldsymbol{\Phi} & \cdots & \phi_{1,n}\boldsymbol{\Phi} \\ \phi_{2,1}\boldsymbol{\Phi} & \phi_{2,2}\boldsymbol{\Phi} & \cdots & \phi_{2,n}\boldsymbol{\Phi} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{n,1}\boldsymbol{\Phi} & \phi_{n,2}\boldsymbol{\Phi} & \cdots & \phi_{n,n}\boldsymbol{\Phi} \end{bmatrix} = \boldsymbol{\Phi} \otimes \boldsymbol{\Phi}$$

and letting $C_i^x(0)^j$ denote the jth iterative result, we rewrite Eq.(22) as

$$\begin{bmatrix} C_1^x(0)^{j+1} \\ C_2^x(0)^{j+1} \\ \vdots \\ C_n^x(0)^{j+1} \end{bmatrix} = \boldsymbol{\Phi} \otimes \boldsymbol{\Phi} \begin{bmatrix} C_1^x(0)^j \\ C_2^x(0)^j \\ \vdots \\ C_n^x(0)^j \end{bmatrix} \boldsymbol{\Phi}^T + \begin{bmatrix} D_1 \\ D_2 \\ \vdots \\ D_n \end{bmatrix} \tag{22'}$$

Then, since both $\boldsymbol{\Phi}$ and $\boldsymbol{\Phi} \otimes \boldsymbol{\Phi}$ have their eigenvalues inside the unit circle, with any initial guess $C_i^x(0)^0$, we can solve Eq.(22′) iteratively and be guaranteed that

$$C_i^x(0) = C_i^x(0)^j$$

for $j > N$ where N is a sufficiently large number.

Note: In the stationary case, the assumption $E\{x(0)\} = 0$ is not necessary. Since

$$E\{x(k)\} = E\{x(k+1)\}$$

from Eq.(1), we find

$$E\{x(k)\} = \boldsymbol{\Phi} E\{x(k)\},$$

which leads to the conclusion

$$E\{x(k)\} = 0.$$

3. Cumulant Computations for Non-Stationary Systems

All the analyses in Section 2 are based on the assumptions that the system and input signal are stationary. In this section, we analyze the non-stationary case.

The system model is still in the form of Eqs.(1) and (2). Now, however, $v(k)$ and $w(k)$ are assumed to be non-stationary white, with

$$E\{v^2(k)\} = \sigma_v^2(k)$$

$$E\{w^2(k)\} = \sigma_w^2(k)$$

and

$$E\{w(k)w(m)w(s)\} = \begin{cases} \gamma_w(k) & k = m = s \\ 0 & \text{otherwise} \end{cases}$$

Analogous to Eq.(12), we have

$$c(k;m) = h^T \sum_{i=1}^{n} h_i C_i^x(k;m) h + c_v(k;m) \tag{24}$$

where

$$c_v(k;m) = E\{v(k)v^2(k+m)\} \tag{25}$$

and

$$C_i^x(k;m) = E\{x_i(k)x(k+m)x^T(k+m)\} \tag{26}$$

Note that if measurement noise $v(k)$ is Gaussian, then $C_v(k;m) = 0$. As in the stationary case, the main problem is to find a recursive representation for $C_i^x(k;m)$. Because of the non-stationary nature of $x(k)$, recursions can be with respect to variables k or m.

A. Computing $C_i^x(k+1;m)$ in Terms of $C_i^x(k;m)$

Considering the equations

$$x_i(k+1) = \sum_{j=1}^{n} \phi_{ij} x_j(k) + r_i w(k)$$

and

$$x(k+m+1) = \Phi x(k+m) + r w(k+m),$$

it follows that

$$\begin{aligned}
x_i(k+1)x(k+m+1)x^T(k+m+1) = {} & [\sum_{j=1}^{n} \phi_{ij} x_j(k) + r_i w(k)] \\
& [\Phi x(k+m)x^T(k+m)\Phi^T + \Phi x(k+m) r^T w(k+m) \\
& + r w(k+m) x^T(k+m)\Phi^T + r r^T w^2(k+m)] \\
= {} & \textstyle\sum_{j=1}^{n} \phi_{ij} x_j(k) \Phi x(k+m) x^T(k+m) \Phi^T \\
& + \textstyle\sum_{j=1}^{n} \phi_{ij} x_j(k) \Phi x(k+m) r^T w(k+m) \\
& + \textstyle\sum_{j=1}^{n} \phi_{ij} x_j(k) r w(k+m) x^T(k+m) \Phi^T \\
& + \textstyle\sum_{j=1}^{n} \phi_{ij} x_j(k) r r^T w^2(k+m) \\
& + r_i w(k) \Phi x(k+m) x^T(k+m) \Phi^T \\
& + r_i w(k) \Phi x(k+m) r^T w(k+m) \\
& + r_i w(k) r w(k+m) x^T(k+m) \Phi^T \\
& + r_i w(k) r r^T w^2(k+m);
\end{aligned}$$

hence,

$$
\begin{aligned}
C_i^x(k+1;m) = & \textstyle\sum_{j=1}^{n} \phi_{ij} \Phi E\{x_j(k) x(k+m) x^T(k+m)\} \Phi^T \\
& + \textstyle\sum_{j=1}^{n} \phi_{ij} \Phi E\{x_j(k) x(k+m) w(k+m)\} r^T \\
& + \textstyle\sum_{j=1}^{n} \phi_{ij} r E\{x_j(k) w(k+m) x^T(k+m)\} \Phi^T \\
& + \textstyle\sum_{j=1}^{n} \Phi_{ij} E\{x_j(k) w^2(k+m)\} r r^T \\
& + r_i \Phi E\{w(k) x(k+m) x^T(k+m)\} \Phi^T \\
& + r_i \Phi E\{w(k) x(k+m) w(k+m)\} r^T \\
& + r_i r E\{w(k) w(k+m) x^T(k+m)\} \Phi^T \\
& + r_i E\{w(k) w^2(k+m)\} r r^T
\end{aligned}
\tag{27}
$$

The first term on the right-hand side of Eq.(27) can be expressed as

$$
\sum_{j=1}^{n} \phi_{ij} \Phi E\{x_j(k) x(k+m) x^T(k+m)\} \Phi^T = \sum_{j=1}^{n} \phi_{ij} \Phi C_j^x(k;m) \Phi^T
$$

We let all the other terms on the right-hand side of Eq.(27) equal $\Theta_i(k;m)$; i.e.,

$$
\begin{aligned}
\Theta_i(k;m) = & \textstyle\sum_{j=1}^{n} \phi_{ij} \Phi E\{x_j(k) x(k+m) w(k+m)\} r^T \\
& + \textstyle\sum_{j=1}^{n} \phi_{ij} r E\{x_j(k) w(k+m) x^T(k+m)\} \Phi^T \\
& + \textstyle\sum_{j=1}^{n} \phi_{ij} E\{x_j(k) w^2(k+m)\} r r^T \\
& + r_i \Phi E\{w(k) x(k+m) x^T(k+m)\} \Phi^T \\
& + r_i \Phi E\{w(k) x(k+m) w(k+m)\} r^T \\
& + r_i r E\{w(k) w(k+m) x^T(k+m)\} \Phi^T \\
& + r_i E\{w(k) w^2(k+m)\} r r^T
\end{aligned}
\tag{28}
$$

Consequently,

$$
C_i^x(k+1;m) = \sum_{j=1}^{n} \phi_{ij} \Phi C_j^x(k;m) \Phi^T + \Theta_i(k;m) \tag{29}
$$

In Eq.(29) $k \geq 0$ when $m \geq 0$ and $k \leq -m$ when $m < 0$, because the argument of $x(k+m+1)$ must be ≥ 0, or else we assume $x(k+m+1) = 0$.

In order to obtain a simplified $\Theta_i(k;m)$, we analyze it by considering three cases.

Case 1. $m > 0$: From the assumptions for the system model, it is obvious that the all terms on the right-hand side of Eq.(28) vanish except the fourth term, so that

$$
\Theta_i(k;m) = r_i \gamma_w(k) \Phi^m r r^T (\Phi^m)^T \tag{30}
$$

In deriving Eq.(30), we have expressed $x(k+m)$ as a function of $x(k)$.

Case 2. $m = 0$: In this case, only the last term in Eq.(28) is non-zero, so that

$$
\Theta_i(k;0) = r_i \gamma_w(k) r r^T \tag{31}
$$

Case 3. $m < 0$: This is the most complex of the three cases. Observe that

$$x(k) = \Phi^{-m} x(k+m) + \sum_{i=-m}^{1} \Phi^{i-1} r w(k-i)$$

so that $x_j(k)$, the jth element of vector $x(k)$ is

$$\begin{aligned} x_j(k) &= [x(k)]_j \\ &= [\Phi^{-m} x(k+m)]_j + [\textstyle\sum_{i=-m}^{1} \Phi^{i-1} r w(k-i)]_j \\ &= [\Phi^{-m}]_j x(k+m) + \textstyle\sum_{i=-m}^{1} [\Phi^{i-1}]_j r w(k-i) \end{aligned}$$

where $[A]_j$ indicates the jth row of matrix A. Note that $x(k+m)$ is independent of $w(k+m)$ and $w(k)$, so all the terms vanish except the 3rd term of Eq.(28) which is

$$\text{3rd term} = \sum_{j=1}^{n} \phi_{ij} [\Phi^{-m-1}]_j r \gamma_w(k+m) r r^T$$

i.e., in this case:

$$\Theta_i(k;m) = \gamma_w(k+m) (\sum_{j=1}^{n} \phi_{ij} \Phi^{-m-1}]_j r) r r^T \tag{32}$$

From Eq.(29), we observe that we must know the initial matrix $C_i^x(0;m)$ before the recursive computations can begin. It is extremely difficult for us to know cumulant values before we obtain a measurement of $y(k)$ or $x(k)$, since there is no observed values for $y(k)$ and $x(k)$, when $k < 0$. Fortunately, we have the following result.

Proposition 1. *If the system represented in Eqs.(1) and (2) is asymptotically stable, then, for large k, $C_i^x(k;m)$ is independent of initial conditions $C_i^x(0;m)$.*

Proof: By the assumption of system stability, we know that

$$\lim_{k \to \infty} \Phi^k = 0$$

In practice, we can always consider

$$\Phi^k = 0$$

when $k > N$, where N is a sufficiently large number. By Eq.(29) (see Appendix A for a derivation of Eq.(33)),

$$\begin{aligned} C_i^x(k;m) &= \textstyle\sum_{j=1}^{n} \phi_{ij} \Phi C_j^x(k-1;m) \Phi^T + \Theta_i(k-1;m) \\ &= \textstyle\sum_{j_1=1}^{n} \sum_{j_2=1}^{n} \cdots \sum_{j_k=1}^{n} \phi_{ij_1} \phi_{j_1 j_2} \cdots \phi_{j_{k-1} j_k} \Phi^k C_{j_k}^x(0;m) (\Phi^k)^T \\ &\quad + \Theta_i(k-1;m) + \textstyle\sum_{j_1=1}^{n} \phi_{ij_1} \Phi \Theta_{j_1}(k-2;m) \Phi^T + \cdots \\ &\quad + \textstyle\sum_{j_1=1}^{n} \sum_{j_2=1}^{n} \cdots \sum_{j_{k-1}=1}^{n} \phi_{ij_1} \phi_{j_1 j_2} \cdots \phi_{j_{k-2} j_{k-1}} \Phi^{k-1} \Theta_{j_{k-1}}(0,m) (\Phi^{k-1})^T \end{aligned} \tag{33}$$

When k is very large, then $\Phi^k = 0$; hence, $(\Phi^k)_{ij} = 0$. Because

$$(\Phi^k)_{ij} = \sum_{j_1=1}^{n} \sum_{j_2=1}^{n} \cdots \sum_{j_{k-1}=1}^{n} \phi_{ij_1}\phi_{j_1j_2}\cdots\phi_{j_{k-1}j};$$

then,

$$\text{1st term in Eq.(33)} = \sum_{j_k=1}^{n} (\Phi^k)_{ij_k} \Phi^k C_{j_k}^x(0;m) = 0$$

All the other terms in Eq.(33) are independent of $C_i^x(0;m)$; therefore, the proposition is proved.

B. Computing $C_i^x(k;m+1)$ in Terms of $C_i^x(k;m)$

As in the stationary case, we divide the computations into two cases:

Case 1. $m \geq 0$. The recursive formula is similar to Eq.(13), i.e.,

$$C_i^x(k;m+1) = \Phi C_i^x(k;m)\Phi^T \tag{34}$$

As mentioned above, $k \geq 0$.

Case 2. $m \leq 0$. Let $k' = k + m$, then

$$C_i^x(k;m) = C_i^x(k'-m;m) = E\{x_i(k'-m)x(k')x^T(k')\} \tag{35}$$

and, similar to the derivation of Eq.(15), the recursive formula is

$$C_i^x(k'-(m-1);m-1) = \sum_{j=1}^{n} \phi_{ij} C_j^x(k'-m;m),$$

i.e.,

$$C_i^x(k+1;m-1) = \sum_{j=1}^{n} \phi_{ij} C_j^x(k;m) \tag{36}$$

If $C_i^x(k;m)$ is given, we can recursively compute the values of $C_i^x(k+1;m-1)$ by Eq.(36). Again, $k \geq -m$ is the constraint of this case.

If Φ is non-singular, then from Eq.(29), Eq.(36) can be reexpressed as

$$C_i^x(k;m-1) = \Phi^{-1}[C_i^x(k;m) - \Theta_i(k;m)][\Phi^{-1}]^T \tag{37}$$

where $\Theta_i(k;m)$ is given by Eq.(32). Whereas Eq.(36) is doubly recursive, Eq.(37) is only recursive in m. The price paid to use Eq.(37) is computation of Φ^{-1}.

Note: When the variable k is very large, $C_i^x(k;m)$ reduces to the results in the stationary case; i.e., Eq.(34) reduces to Eq.(13) and Eq.(36) reduces to Eq.(15), because $C_i^x(k;m)$ no longer depends on k.

4. Examples

In Sections 2 and 3 we developed recursive formulas for cumulant computations. In this section we give some examples for stationary systems. Specifically, we illustrate the computation of $C_i^x(m)$ for the following ARMA model [6]:

$$y(t) + a_1 y(t-1) + \cdots + a_p y(t-p) = b_1 w(t-1) + b_2 w(t-2) + \cdots + b_p w(t-p) + v(t). \tag{38}$$

AR and MA models are also analyzed.

A. ARMA Model Example

The ARMA model, in Eq.(38), can be represented in state-variable canonical form, as in Eqs.(1) and (2)[7], where

$$\Phi = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_p & -a_{p-1} & -a_{p-2} & \cdots & -a_1 \end{bmatrix},$$

$$\gamma = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \text{ and } h = \begin{bmatrix} b_p \\ b_{p-1} \\ \vdots \\ b_2 \\ b_1 \end{bmatrix}$$

From Eqs.(13) and (15), we find that, for $m \geq 0$

$$C_i^x(m+1) = \Phi C_i^x(m) \Phi^T \tag{39}$$

whereas, for $m \leq 0$

$$C_i^x(m-1) = \begin{cases} C_{i+1}^x(m) & \text{when } i < p; \\ -\sum_{j=1}^{p} a_{p-j+1} C_j^x(m) & \text{when } i = p \end{cases} \tag{40}$$

For the ARMA model, Eq.(22) simplifies to

$$\begin{bmatrix} C_1^x(0) \\ C_2^x(0) \\ \vdots \\ C_{p-1}^x(0) \\ C_p^x(0) \end{bmatrix} = \begin{bmatrix} 0 & \Phi & 0 & \cdots & 0 & 0 \\ 0 & 0 & \Phi & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \Phi \\ -a_p\Phi & -a_{p-1}\Phi & -a_{p-2}\Phi & \cdots & -a_2\Phi & -a_1\Phi \end{bmatrix} \begin{bmatrix} C_1^x(0) \\ C_2^x(0) \\ \vdots \\ C_{p-1}^x(0) \\ C_p^x(0) \end{bmatrix} \Phi^T + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ S \end{bmatrix}$$

where

$$S = \gamma_w \begin{bmatrix} 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}$$

Consequently, for $i < p$,

$$C_i^x(0) = \Phi C_{i+1}^x(0)\Phi^T \tag{41}$$

and

$$\begin{aligned} C_p^x(0) &= -\textstyle\sum_{i=1}^p a_{p-i+1}\Phi C_i^x(0)\Phi^T + S \\ &= -\textstyle\sum_{i=1}^p a_i \Phi^i C_p^x(0)(\Phi^i)^T + S \end{aligned} \tag{42}$$

Comparing Eq.(42) with Eq.(22), we now to need solve a lower-order linear equation, since the unknown variables are reduced from p^3 to p^2. The following numerical iterative method can be used to solve Eq.(42), if Φ has all its eigenvalues inside the unit circle. Let $C_p^x(0)^n$ be the nth iterative result; then we propose to solve Eq.(42) by the following recursive formula,

$$\begin{aligned} C_p^x(0)^{n+1} &= -\textstyle\sum_{i=1}^p a_i \Phi^i C_p^x(0)^n (\Phi^i)^T + S \\ &= S - \Phi[a_1 C_p^x(0)^n + \Phi[a_2 C_p^x(0)^n + \Phi[\cdots[a_{p-1}C_p^x(0)^n + \Phi a_n C_p^x(0)^n \Phi^T]\cdots]\Phi^T]\Phi^T]\Phi^T \end{aligned}$$

The right-hand side of this equation can easily be computed as it is another iterative form. Convergence of this procedure is obtained when $n \geq N$, where N is a large enough number, because Φ is a matrix with all its eigenvalues inside the unit circle. In this procedure we only have nth order matrix computations, because $C_p^x(0)$ is an $n \times n$ matrix.

In this example, c(m), given in Eq.(12), can be expressed as

$$c(m) = [\ b_p \quad b_{p-1} \quad \cdots \quad b_1\ ] \sum_{i=1}^{p} b_{p-i+1} C_i^x(m) \begin{bmatrix} b_p \\ b_{p-1} \\ \vdots \\ b_1 \end{bmatrix} + E\{v(k)v^2(k+m)\} \tag{43}$$

B. MA Model Analysis

The MA model is a special case of the ARMA model, when $a_i = 0$, $i = 1, 2, \cdots, p$, and

$$\Phi = \begin{bmatrix} 0 & I_{p-1} \\ 0 & 0^T \end{bmatrix}$$

where I_{p-1} is a $(p-1) \times (p-1)$ identity matrix. The purpose of this example is to demonstrate that our formulas for calculation of $c(m)$ reduce to those given in [4], which were obtained by using the impulse response of the MA model.

For representation convenience, we assume S_i to be a $p \times p$ matrix whose elements are all zero except the iith element, $[S_i]_{ii} = \gamma_w$. Additionally, we define $S_i = 0$ when $i \leq 0$, or $i > p$. When $i \leq p$

$$S_{i-1} = \begin{bmatrix} 0 & I_{p-1} \\ 0 & 0^T \end{bmatrix} S_i \begin{bmatrix} 0 & I_{p-1} \\ 0 & 0^T \end{bmatrix}^T ;$$

then, from Eqs. (41) and (42), the initial matrices become

$$\begin{aligned} C_p^x(0) &= S_p = S \\ C_{p-1}^x(0) &= S_{p-1} \\ &\vdots \\ C_1^x(0) &= S_1 \end{aligned} \tag{44}$$

Using Eq.(44), the computation of $C_i^x(m)$ is easy. For the case $m \geq 0$, from Eq.(39),

$$C_i^x(m) = S_{i-m} \tag{45}$$

For the case of $m \leq 0$, from Eq.(40), we have

$$C_i^x(m) = S_{i+|m|} = S_{i-m} \tag{46}$$

Combining Eqs.(45) and (46), we have a common result suitable for both $m \leq 0$ and $m \geq 0$, namely,

$$C_i^x(m) = S_{i-m} \tag{47}$$

Inserting Eq.(47) into Eq.(43), we obtain

$$c(m) = \sum_{i=1}^{p} b_{p-i+1}[\ b_p \quad b_{p-1} \quad \cdots \quad b_1\ ]S_{i-m} \begin{bmatrix} b_p \\ b_{p-1} \\ \vdots \\ b_1 \end{bmatrix} + E\{v(k)v^2(k+m)\} \tag{48}$$

i.e.,

$$c(m) = \gamma_w \sum_{i=max(1,-m)}^{min(p,p-m)} b_{i-m}^2 b_i + E\{v(k)v^2(k+m)\} \tag{49}$$

which is the same as Eq.(3.26) in [4], when, as assumed in [4], $v(k) = 0$.

C. AR Model Analysis

For an AR model, the coefficients of Eq.(38) have $b_1 = 1$ and $b_i = 0$ for $i = 2, 3, \cdots, p$, in which case

$$c(m) = [\ 0 \ \cdots \ 0 \ \ 1\ ] C_p^x(m) \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} + E\{v(k)v^2(k+m)\} \tag{43'}$$

Considering $m \le 0$, from Eqs.(40) and (43′), we know

$$\begin{aligned} c(-1) - E\{v(k)v^2(k-1)\} &= [C_p^x(-1)]_{pp} = [C_{p-1}^x(-2)]_{pp} = \cdots = [C_1^x(-p)]_{pp} \\ c(-2) - E\{v(k)v^2(k-2)\} &= [C_p^x(-2)]_{pp} = [C_2^x(-p)]_{pp} \\ c(-3) - E\{v(k)v^2(k-3)\} &= [C_p^x(-3)]_{pp} = [C_3^x(-p)]_{pp} \\ &\vdots \\ c(-p) - E\{v(k)v^2(k-p)\} &= [C_p^x(-p)]_{pp} \end{aligned} \tag{50}$$

Consequently,

$$\begin{aligned} c(-p-1) - E\{v(k)v^2(k-p-1)\} &= [C_p^x(-p-1)]_{pp} \\ &= -\textstyle\sum_{j=1}^{p} a_{p-j+1}[C_j^x(p)]_{pp} \\ &= -\textstyle\sum_{j=1}^{p} a_{p-j+1}c(-j) \\ &\quad +\textstyle\sum_{j=1}^{p} a_{p-j+1}E\{v(k)v^2(k-j)\} \\ c(-p-2) - E\{v(k)v^2(k-p-2)\} &= [C_p^x(-p-2)]_{pp} \\ &= -\textstyle\sum_{j=1}^{p} a_{p-j+1}c(-j-1) \\ &\quad +\textstyle\sum_{j=1}^{p} a_{p-j+1}E\{v(k)v^2(k-j-1)\} \\ &\vdots \\ c(-p-i) - E\{v(k)v^2(k-p-i)\} &= -\textstyle\sum_{j=1}^{p} a_{p-j+1}c(-j-i+1) \\ &\quad +\textstyle\sum_{j=1}^{p} a_{p-j+1}E\{v(k)v^2(k-j-i+1)\} \\ &\vdots \end{aligned} \tag{51}$$

which is similar to Eq.(3.38) in [4]. When $v(k) = 0$, then it exactly matches Eq.(3.38) in [4]

5. Algorithms and Simulations

In this section, we review the results in the previous sections. Numerical algorithms are proposed and some simulations are presented to demonstrate the algorithms. For covenience, only stable systems are considered.

A. Stationary Systems

Algorithm 1. (Stationary System)

Step 1. Guess initial matrices $C_i^x(0)^0$ and compute Eq.(22′). Then $C_i^x(0) = C_i^x(0)^j$ for $j \geq N$.

Step 2. For $m > 0$, $C_i^x(m)$ can be computed from Eq.(13).

Step 3. For $m < 0$, $C_i^x(m)$ can be computed from Eq.(15), i.e.,

$$\begin{bmatrix} C_1^x(m-1) \\ C_2^x(m-1) \\ \vdots \\ C_n^x(m-1) \end{bmatrix} = \Phi \otimes I_{n\times n} \begin{bmatrix} C_1^x(m) \\ C_2^x(m) \\ \vdots \\ C_n^x(m) \end{bmatrix} \tag{52}$$

Step 4. The output cumulant can be computed from Eq.(12), in which all terms are either known or have been computed in Steps 1 to 3.

As an example, we chose a system for which

$$\Phi = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0.1279 & -0.88666 & 1.4358 \end{bmatrix}$$

$$h = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \text{ and } r = \begin{bmatrix} 1 \\ 1.3 \\ 1.4 \end{bmatrix};$$

Input $w(k)$ is exponentially distributed white noise, with $\sigma_w^2 = 1$ and $\gamma_w = 2$. Measurement noise $v(k)$ is Gaussian and white, with $\sigma_v^2 = 1$ and $E\{v(k)v^2(k+m)\} = 0$. The eigenvalues of Φ are 0.2 and $0.6176 \pm i0.5076$, which are all inside the unit circle. The output cumulant $c(m), m = -4, -3, \cdots, 3, 4$ was computed by Algorithm 1. Results are summarized in Table 1. $c(m)$ values obtained via Algorithm 1 compare very well with sampled values, which were obtained for data of 3 lengths(512, 1024 and 4096 samples), averaged over 100 realizations.

B. Non-Stationary Systems

The non-stationary case is more complex than the stationary case.

Algorithm 2. (Non-Stationary, Iteration with Time for a Fixed Lag)

Step 1. Guess the value of $C_i^x(0, m)$, $i = 1, 2, \cdots, p$.

Step 2. Compute $C_i^x(k, m)$ using Eq.(29)

$i = 1, 2, \cdots, p$.

Step 3. Compute $c(k; m)$ using Eq.(24)

Table 1 c(m) Comparisons. Algorithm 1 versus Sampled Values

c(m) m	using Algorithm 1.	sample 512	sample 1024	sample 4096
-4	-2.6466	-2.5536 ± 0.9902	-2.7198 ± 0.7174	-2.6040 ± 0.3598
-3	-1.7910	-1.7131 ± 0.8385	-1.8370 ± 0.7016	-7.7850 ± 0.3279
-2	0.6820	0.6423 ± 0.7365	0.6547 ± 0.5943	0.6542 ± 0.2616
-1	4.1421	3.8984 ± 1.1910	4.1186 ± 1.0333	4.0876 ± 0.3838
0	7.0553	6.5715 ± 1.7841	6.9342 ± 1.4808	7.0041 ± 0.5807
1	2.6461	2.3734 ± 1.2299	2.5941 ± 1.1032	2.5892 ± 0.3304
2	0.4084	0.2157 ± 0.8704	0.3507 ± 0.6287	0.3733 ± 0.3303
3	0.6393	0.5709 ± 0.8611	0.6044 ± 0.5713	0.6090 ± 0.3106
4	1.0175	0.9267 ± 0.7666	1.0570 ± 0.6402	1.0123 ± 0.2919

Algorithm 3. (Non-Stationary, Iteration with Lag for a Fixed Time)

Step 1. Take $C_i^x(k,m)$ from Step 2 of Algorithm 2.

Step 2. For $m \geq 0$, compute $C_i^x(k,m+1)$ using Eq.(34). For $m \leq 0$ compute $C_i^x(k+1,m-1)$ or $C_i^x(k,m-1)$ using Eqs.(36) or (37).

Step 3. Compute $c(k;m)$ using Eq.(24).

Combining Algorithms 2 and 3, we obtain the following computational strategy. First, using Algorithm 2, we obtain $C_i^x(k,0)$. Then we can compute $C_i^x(k,m)$ using Algorithm 3, for any value of m. In Step 2 of Algorithm 3 when $C_i^x(k,-m)$ is needed we recommend use of Eq.(36); i.e., compute $C_i^x(k+1,m-1)$ by using $C_i^x(k,m)$. This avoids having to compute the inverse of matrix Φ. Unfortunately, Eq.(36) cannot be used for $k \geq -m$. Of course, if matrix Φ is non-singular, we can use Eq.(37) to obtain the results. The same constraint $k \geq -m$, also affects Eq.(37).

We illustrate these algorithms for the following first-order system,

$$x(k+1) = 0.75x(k) + w(k)$$

$$y(k) = x(k) + v(k)$$

The statistics of $w(k)$ are assumed variable with time, i.e.,

$$\sigma_w^2(k) = (1 + 2e^{-0.005k} sin(0.1k))^2$$

and

$$\gamma_w(k) = (1 + 2e^{-0.005k} sin(0.1k))^3$$

Additionally, $v(k)$ is assumed to be stationary white noise, with $\sigma_v^2(k) = 1$ and $E\{v(k)v^2(k+m)\} = 0$.

Using Algorithm 2 we obtained the results of $c(k,0)$ shown in Figure 1, which illustrates that initial matrix $C_i^x(0,0)$ is not important when k is large. Regardless of whether the initial value of $C_i^x(0,0)$ is large or small, convergence occurs after several steps of computation. The solid curve corresponds to $C(0,0) = -100$ and the dashed curve corresponds to $C(0,0) = 100$. Observe that, even though the initial condition are quite different, the two curves converge within 10 steps. Figure 1 demonstrates the truth of Proposition 1.

Using Algorithm 3, we easily obtain $c(k,m)$ for $m \in \{\cdots -1, 0, 1, \cdots\}$. Some results are shown in Figures 2 to 4, in which, the initial conditions are taken as $C(0,0) = 100$ (the same initial condition as the dashed curve in Fig. 1). Figure 2 shows $c(k;m)$ where $m = 0,1,2,3,4$. Figure 3 depicts $c(k,m)$ where $m = 0,-1,-2,-3,-4$. The two results are combined as a 3-D graph in Figure 4.

C. A Practical Example

We take a ninth-order air gun system from Section 7.5 of [9] as an example for the comparison of the cumulant computation method in this report with the traditional impulse response cumulant computation method. Figure 5 shows the impulse response of the system. Cumulant computation using the impulse response method requires infinite (or a very large number of) samples, as the formula for this computation is

$$c(m) = \gamma_v \sum_{i=0}^{\infty} h(i)h^2(i+m),$$

where $h(i)$ is the impule response of the system. In the following simulation, a zero mean exponential white noise $v(t)$ with $\gamma_v = 0.2$ is used as the system input. Figure 6 is the third-order cumulant waveform computed from 1000 impulse response samples. Using the method provided in this report, we obtain the cumulant easily; it is depicted in Figure 7. The difference between these two computation results is shown in Figure 8. It is extremely small.

From this example we see that with the method provided in this report we can accurately compute the cumulant without having to use a large number number of impulse response samples.

6. Conclusions

A recursive approach for computation of third-order diagonal-slice cumulant has been developed. All the analyses are based on a state-space model. The special case of a stationary system is discussed. Recursive formulas for the non-stationary system are also derived. An ARMA model is considered as a special example. The analyses of AR and MA models have verified the recursive formulas. Computational algorithms have been given and simulation examples for both stationary and non-stationary systems have been presented.

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Appendix A. Derivation of Eq.(33)

From Eq.(29), we know,

$$C_i^x(k;m) = \sum_{j=1}^{n} \phi_{ij} \Phi C_j^x(k-1;m)\Phi^T + \Theta_i(k-1;m) \qquad (A.1)$$

and

$$\begin{aligned}
C_i^x(k;m) &= \textstyle\sum_{j_1=1}^n \phi_{ij_1}\Phi[\sum_{j_2=1}^n \phi_{j_1j_2}\Phi C_{j_2}^x(k-2;m)\Phi^T]\Phi^T \\
&\quad + \textstyle\sum_{j_1=1}^n \phi_{ij_1}\Phi\Theta_{j_1}(k-2;m)\Phi^T + \Theta_i(k-1;m) \\
&= \textstyle\sum_{j_1=1}^n \sum_{j_2=1}^n \phi_{ij_1}\phi_{j_1j_2}\Phi^2 C_{j_2}^x(k-2;m)(\Phi^2)^T \\
&\quad + \textstyle\sum_{j_1=1}^n \phi_{ij_1}\Phi\Theta_{j_1}(k-2;m)\Phi^T + \Theta_i(k-1;m)
\end{aligned}$$

In the same way, we can express $C_i^x(k;m)$ in terms of $C_i^x(k-3;m)$, as

$$\begin{aligned}
C_i^x(k;m) &= \textstyle\sum_{j_1=1}^n \sum_{j_2=1}^n \sum_{j_3=1}^n \phi_{ij_1}\phi_{j_1j_2}\phi_{j_2j_3}\Phi^{\,3} C_{j_3}^x(k-3;m)(\Phi^3)^T \\
&\quad + \textstyle\sum_{j_1=1}^n \sum_{j_2=1}^n \phi_{ij_1}\phi_{j_1j_2}\Phi^2\Theta_{j_2}(k-3;m)(\Phi^2)^T \\
&\quad + \textstyle\sum_{j_1=1}^n \phi_{ij_1}\Phi\Theta_{j_1}(k-2;m)\Phi^T + \Theta_i(k-1;m)
\end{aligned}$$

Eq.(33) is obtained by continuing this derivation.

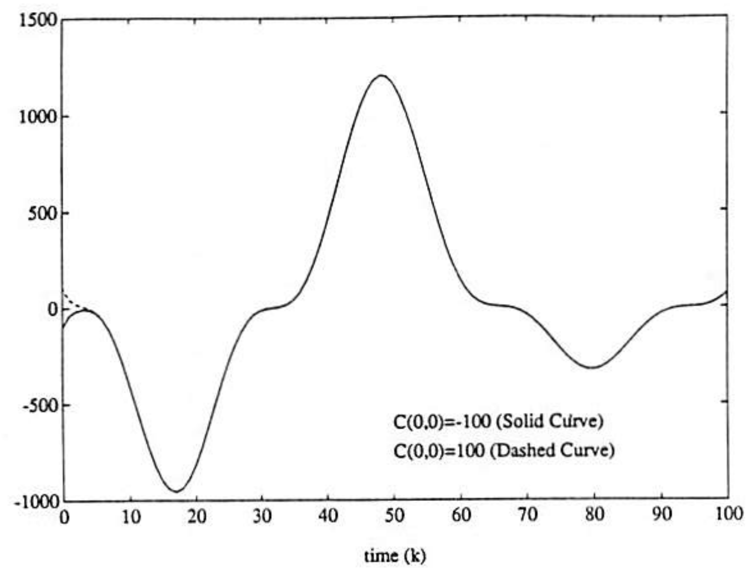

Figure 1. $c_y(k;0)$ for nonstationary system with different $C(0,0)$

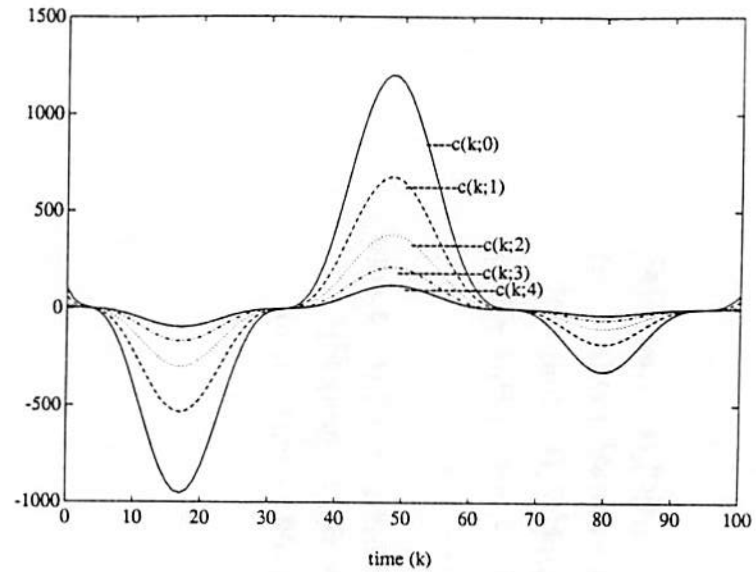

Figure 2. $c_y(k;0) - -c_y(k;4)$, $C(0,0) = 100$

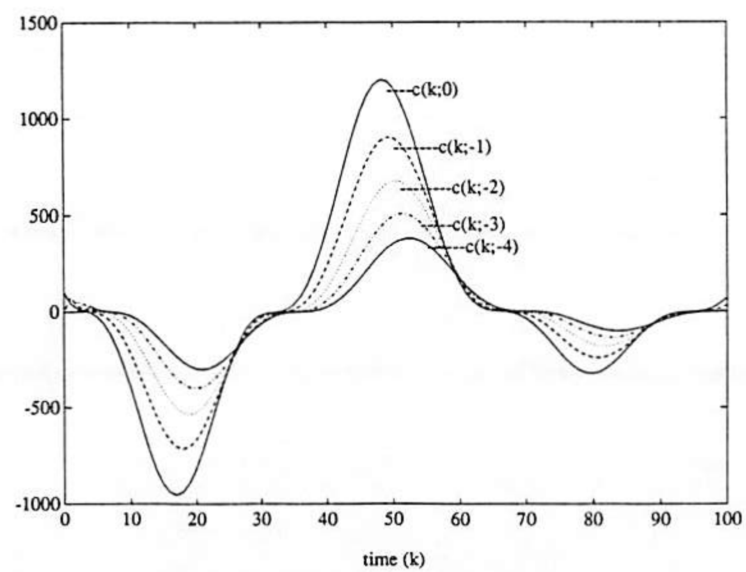

Figure 3. $c_y(k;0) - -c_y(k;-4)$, $C(0,0) = 100$

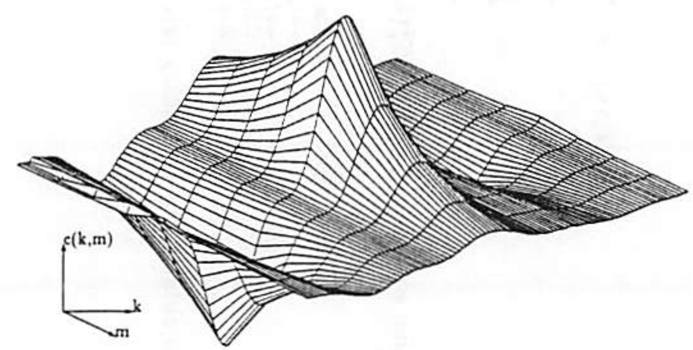

Figure 4. Output cumulants $c_y(k;-4) - -c_y(k;4)$

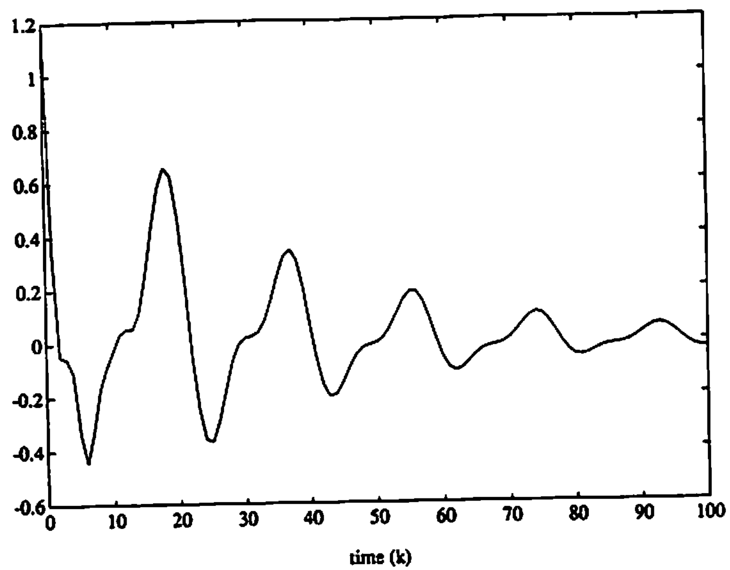

Figure 5. Impulse response of an air gun system

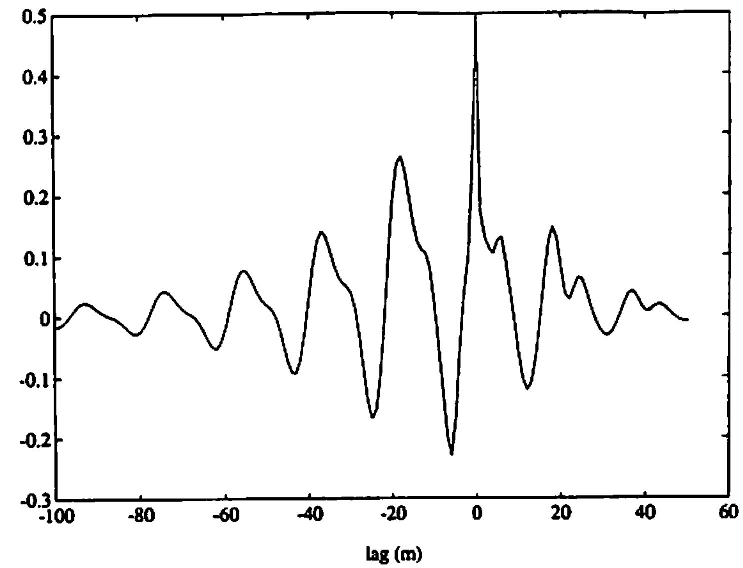

Figure 6. Cumulants computed by impulse response response method

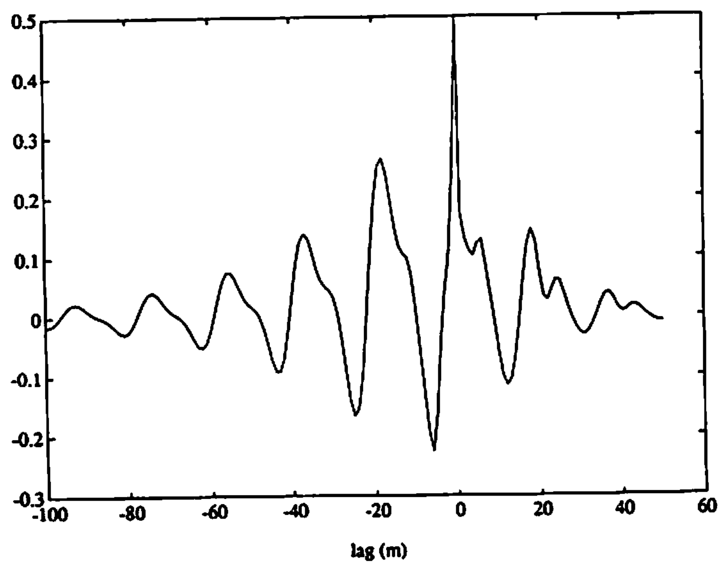

Figure 7. Cumulants computed by the method in this report

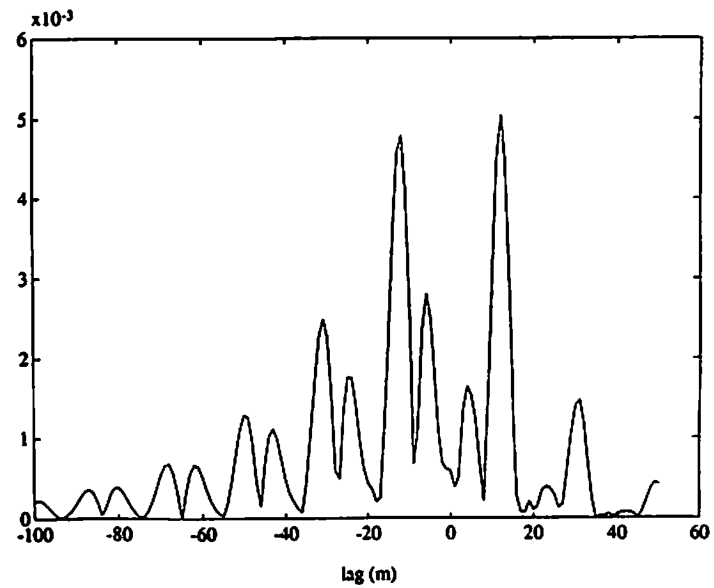

Figure 8. The difference between two methods

Recursive Method for Computation of Cumulants: Part II

Kronecker Product Formulation for Third- and Fourth-Order Diagonal-Slice Cumulants

Abstract

Recursive formulas for third- and fourth-order diagonal-slice cumulant computations of a state-variable model are developed. The computation formulas can be used for both stationary and non-stationary systems. By using Kronecker products, elegant formulas are obtained. Some examples and simulations are given.

1. Introduction

In part I of this report we have derived a number of formulas for recursive cumulant computations; however, the representation of the results are complex. In this part, the Kronecker product is used and more compact formulas are obtained.

Our system is assumed to be a SISO (Single Input Single Output) state-variable model

$$x(k+1) = \Phi x(k) + r w(k) \tag{1}$$

$$y(k) = h^T x(k) + v(k) \tag{2}$$

where $w(k)$ and $v(k)$ are zero mean white noises and $w(k)$ is uncorrelated with $v(k)$. For simplicity $x(0) = 0$ is assumed. When $x(0) \neq 0$, a simple transformation of variables can be used to remove the mean.

In this report, we obtain formulas to generate third- and fourth-order diagonal-slice cumulants from a given state-variable model. The third- and fourth-order cumulants for a zero mean random variable are defined as (we take the definition from [3]),

$$c(x_1, x_2, x_3) = E\{x_1 x_2 x_3\} \tag{3}$$

$$\begin{aligned} c(x_1, x_2, x_3, x_4) = \; & E\{x_1 x_2 x_3 x_4\} - E\{x_1 x_2\}E\{x_3 x_4\} \\ & - E\{x_1 x_3\}E\{x_2 x_4\} - E\{x_1 x_4\}E\{x_2 x_3\} \end{aligned} \tag{4}$$

For the state-variable model, since $w(k)$ and $v(k)$ are zero mean and $x(0) = 0$, $y(k)$ is a zero mean random process (r.p.). The third- and fourth-order cumulants can be defined as

$$c_{3y}(k; m_1, m_2) = E\{y(k)y(k+m_1)y(k+m_2)\} \tag{5}$$

and

$$\begin{aligned} c_{4y}(k; m_1, m_2, m_3) &= E\{y(k)y(k+m_1)y(k+m_2)y(k+m_3)\} \\ &\quad -E\{y(k)y(k+m_1)\}E\{y(k+m_2)y(k+m_3)\} \\ &\quad -E\{y(k)y(k+m_2)\}E\{y(k+m_1)y(k+m_3)\} \\ &\quad -E\{y(k)y(k+m_3)\}E\{y(k+m_1)y(k+m_2)\} \end{aligned} \tag{6}$$

In many estimation problems only 1-D diagonal-slice cumulants are used [1]. For this reason, only 1-D diagonal-slice cumulants are discussed in this report. To obtain such cumulants, we let $m = m_1 = m_2$ and $m = m_1 = m_2 = m_3$ for the third- and fourth-order cumulants, respectively.

We derive recursive formulas for the third-order cumulant in Section 2. In Section 3, we develop similar formulas for fourth-order cumulants. The definitions of the cumulants in Eqs.(3), (4), (5) and (6) only involve scalar functions; however, vector functions are involved in this report; hence, third- and fourth-order cumulants are defined for such functions in Sections 2 and 3. Based on these definitions, we show how to compute cumulants for multi-input multi-output (MIMO) state-variable models in Section 4. Some examples and simulations are presented in Section 5. Most of the detailed derivations appear in appendices.

Before deriving our results, we list some useful properties of Kronecker products [2].

$$AB \otimes CD = (A \otimes C)(B \otimes D). \tag{7}$$

$$(A \otimes B) \otimes C = A \otimes (B \otimes C). \tag{8}$$

$$(A + B) \otimes (C + D) = A \otimes C + A \otimes D + B \otimes C + B \otimes D. \tag{9}$$

$$(A \otimes B)^T = A^T \otimes B^T. \tag{10}$$

Additionally, if the eigenvalues of A and B are inside the unit circle, the eigenvalues of $A \otimes B$ are also inside the unit circle.

2. Third-Order Cumulant Computations

We use the state-variable model to determine the cumulant of output $y(k)$. From Eq.(2) we have

$$\begin{aligned} c_{3y}(k;m) &= E\{y(k)y^2(k+m)\} \\ &= E\{[h^T x(k)+v(k)][h^T x(k+m)+v(k+m)]^2\} \\ &= E\{h^T x(k)h^T x(k+m)h^T x(k+m)\} \\ &\quad +2E\{h^T x(k)h^T x(k+m)v(k+m)\} \\ &\quad +E\{h^T x(k)v^2(k+m)\} \\ &\quad +E\{v(k)h^T x(k+m)h^T x(k+m)\} \\ &\quad +2E\{v(k)h^T x(k+m)v(k+m)\} \\ &\quad +E\{v(k)v^2(k+m)\} \end{aligned} \tag{11}$$

Let

$$C_{3x}(k;m) = E\{x(k)\otimes x(k+m)x^T(k+m)\} \tag{12}$$

which is an $n^2 \times n$ matrix; and

$$\sigma_v^2(k) = E\{v^2(k)\} \tag{13}$$

Lemma 1. *The third-order cumulant of output $y(k)$ can be represented as*

$$c_{3y}(k;m) = [h\otimes h]^T C_{3x}(k;m)h + E\{v(k)v^2(k+m)\} \tag{14}$$

The proof of this lemma is given in Appendix A. Since the statistics of $v(k)$ are assumed to be known, it is obvious that the key problem for computing $c_{3y}(k;m)$ is computing $C_{3x}(k;m)$.

2.1. Recursive Computation Formulas

Proposition 1. (Recursive with Lag m). *$C_{3x}(k;m)$ can be computed recursively with respect to lag m as follows:*

$$C_{3x}(k;m+1) = [I\otimes \Phi]C_{3x}(k;m)\Phi^T, \quad m \geq 0 \tag{15}$$

$$C_{3x}(k;m-1) = [\Phi\otimes I]C_{3x}(k-1;m), \quad m \leq 0 \tag{16}$$

Proof : (a) When $m \geq 0$,

$$\begin{aligned} C_{3x}(k;m+1) &= E\{x(k)\otimes x(k+m+1)x^T(k+m+1)\} \\ &= E\{x(k)\otimes \Phi x(k+m)x^T(k+m)\Phi^T\} \\ &\quad +E\{x(k)\otimes \Phi x(k+m)\Gamma^T w(k+m)\} \\ &\quad +E\{x(k)\otimes \Gamma w(k+m)x^T(k+m)\Phi^T\} \\ &\quad +E\{x(k)\otimes \Gamma w(k+m)\Gamma^T w(k+m)\} \end{aligned}$$

For $m \geq 0$, $w(k+m)$ is uncorrelated with $x(k)$ and $x(k+m)$. Additionally, $w(k)$ and $x(k)$ are zero mean; hence,

$$C_{3x}(k; m+1) = [I \otimes \Phi] C_{3x}(k; m) \Phi^T$$

Note that to obtain this result we have used the following:

$$\begin{aligned} & E\{x(k) \otimes \Phi x(k+m) x^T(k+m) \Phi^T\} \\ = \; & E\{I x(k) \otimes \Phi x(k+m) x^T(k+m) \Phi^T\} \\ = \; & [I \otimes \Phi] E\{x(k) \otimes x(k+m) x^T(k+m)\} \Phi^T \end{aligned}$$

The last line follows from Eq.(7). This type of expansion is used repeatedly in all the derivations below and in the appendices.

(b) When $m \leq 0$,

$$\begin{aligned} C_{3x}(k; m-1) &= E\{x(k-1+1) \otimes x(k-1+m) x^T(k-1+m)\} \\ &= E\{\Phi x(k-1) \otimes x(k-1+m) x^T(k-1+m)\} \\ &\quad + E\{r w(k-1) \otimes x(k-1+m) x^T(k-1+m)\} \\ &= [\Phi \otimes I] C_{3x}(k-1; m) \end{aligned}$$

This completes the proof of Proposition 1. Q.E.D.

Proposition 2. (**Recursive with time k**) . *$C_{3x}(k+1; m)$ can be computed recursively with respect to time k, as*

$$C_{3x}(k+1; m) = [\Phi \otimes \Phi] C_{3x}(k; m) \Phi^T + \Gamma(k; m) \tag{17}$$

where

$$\Gamma(k; m) = \begin{cases} \gamma_w(k) [I \otimes \Phi]^m [r \otimes r r^T] (\Phi^T)^m & \text{when } m \geq 0 \\ \gamma_w(k+m) [\Phi^{-m} \otimes I] [r \otimes r r^T] & \text{when } m \leq 0 \end{cases} \tag{18}$$

Proof : See Appendix B.

In the above two propositions, the formulas are recursive. Observe that these formulas require the initial condition matrix $C_{3x}(0; m)$. Fortunately, we have the following:

Proposition 3. *For an asymptoticaly stable system, the value of $C_{3x}(k; m)$ is independent of initial condition matix $C_{3x}(0; m)$ for large k.*

Proof : The proof follows directly from Eq.(17) in Proposition 2. Since the system is asymptotically stable, for a large enough k,

$$\Phi^k = 0$$

By Eq.(7) we have

$$[\Phi \otimes \Phi]^k = [\Phi^k \otimes \Phi^k]$$

so,

$$[\Phi \otimes \Phi]^k = 0$$

for a large enough k. Then, from Eq.(17) we can represent $C_{3x}(k;m)$ as

$$\begin{aligned} C_{3x}(k;m) &= [\Phi \otimes \Phi]^k C_{3x}(0;m)[\Phi^k]^T \\ &+ \sum_{i=0}^{k} [\Phi \otimes \Phi]^{k-i} \Gamma(i;m)[\Phi^{k-i}]^T \end{aligned}$$

When k is large enough the first term in the equation vanishes and the second term does not depend upon $C_{3x}(0;m)$; hence, $C_{3x}(k;m)$ is independent of $C_{3x}(0;m)$. Q.E.D.

2.2. Some Comments

In the stationary case (i.e. stationary input and stationary system), the recursive formulas simplify. In this case all the variables in Propositions 1 and 2 are independent of temporal variable k. The recursive formulas in Proposition 1 remain the same, except that variable k has no meaning. We let $C_{3x}(m)$ denote the stationary cumulant of state-vector x and $\Gamma(m)$ be the stationary version of $\Gamma(k;m)$. Then, initial matrix $C_{3x}(0)$ can be obtained by solving the equation,

$$C_{3x}(0) = [\Phi \otimes \Phi] C_{3x}(0) \Phi^T + \Gamma(0) \tag{19}$$

.

The computation of $C_{3x}(k;m-1)$ when $m \leq 0$ by Eq.(16) requires the existence of $C_{3x}(k-1;m)$. We can only compute $c_{3y}(k;m)$, when $m \leq 0$, for all $m \geq -k$, if the starting initial condition is $C_{3x}(0;0)$. We assume that $y(k)$ exists in the domain $[0, \infty)$ and is zero for $k < 0$. If we need to compute $c_{3y}(k;m)$ for $m < -k$ then, by the definition of the cumulant, we have to compute

$$c_{3y}(k;m) = E\{y(k)y^2(k+m)\}.$$

In this equation if $k+m < 0$, $y(k+m)$ is set equal to zero. The computation of the negative lags of $C_{3y}(k;m)$ will be along sloped lines. Equation (16) provides a way to compute $C_{3x}(1,-1), C_{3x}(2,-2), \cdots, C_{3x}(n,-n), \cdots$ from initial condition $C_{3x}(0,0)$. Combining the formulas in Propositions 1 and 2, we can compute all the $C_{3x}(k;m)$ when $k \geq 0, m \geq 0$, and, $k \geq 0, m \geq -k$.

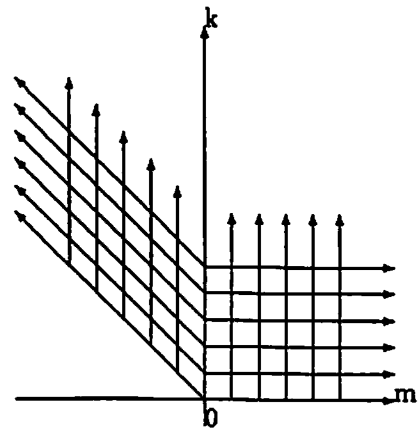

Figure 1. Illustration of Cumulant Computations

Figure 1 gives an illustration of the cumulant-computation directions. All the values of $C_{3x}(k;m+1)$ along a horizontal direction can be computed from Eq.(15). Equation (16) makes it possible to compute along the negative diagonal direction. Values along the vertical direction (i.e., k) can be computed from Eq.(17). The initial value is assumed to be $C_{3x}(0;0)$.

3. Fourth-Order Cumulant Computations

In this section, we develop fourth-order recursive diagonal-slice cumulant computation formulas. From Eq.(6), we see that

$$c_{4y}(k;m) = E\{y(k)y^3(k+m)\} - 3E\{y(k)y(k+m)\}E\{y^2(k+m)\} \tag{20}$$

By using Eq.(2), we have

$$\begin{aligned} y(k)y^3(k+m) &= [h^T x(k) + v(k)][h^T x(k+m) + v(k+m)]^3 \\ &= h^T x(k)[h^T x(k+m)]^3 + 3h^T x(k)[hx(k+m)]^2 v(k+m) \\ &\quad +3h^T x(k)h^T x(k+m)v^2(k+m) + h^T x(k)v^3(k+m) \\ &\quad +[h^T x(k+m)]^3 v(k) + 3[h^T x(k+m)]^2 v(k+m)v(k) \\ &\quad +3[h^T x(k+m)]v^2(k+m)v(k) + v^3(k+m)v(k) \end{aligned}$$

We recall that $v(k)$ and $w(k)$ are assumed to be zero mean white noises and are uncorrelated. In addition, we assume $w(k)$ is fourth-order uncorrelated; i.e. ,

$$E\{w(k)w(k+t_1)w(k+t_2)w(k+t_3)\} = \gamma_{4w}(k)\delta(t_1,t_2,t_3),$$

where $\delta(t_1, t_2, t_3)$ is the Kronecker delta function; it equals zero except when $t_1 = t_2 = t_3$, in which case it equals unity. Continuing, we have

$$\begin{aligned} E\{y(k)y^3(k+m)\} &= E\{\mathbf{h}^T\mathbf{x}(k)[\mathbf{h}^T\mathbf{x}(k+m)]^3\} \\ &+3E\{\mathbf{h}^T\mathbf{x}(k)\mathbf{h}^T\mathbf{x}(k+m)\}E\{v^2(k+m)\} \\ &+3E\{v(k+m)v(k)\}E\{[\mathbf{h}^T\mathbf{x}(k+m)]^2\} \\ &+E\{v^3(k+m)v(k)\} \end{aligned} \tag{21}$$

Additionally,

$$\begin{aligned} E\{y(k)y(k+m)\}E\{y^2(k+m)\} &= E\{\mathbf{h}^T\mathbf{x}(k)\mathbf{h}^T\mathbf{x}(k+m)\}E\{[\mathbf{h}^T\mathbf{x}(k+m)]^2\} \\ &+E\{\mathbf{h}^T\mathbf{x}(k)\mathbf{h}^T\mathbf{x}(k+m)\}E\{v^2(k+m)\} \\ &+E\{v(k)v(k+m)\}E\{[\mathbf{h}^T\mathbf{x}(k+m)]^2\} \\ &+E\{v(k)v(k+m)\}E\{v^2(k+m)\} \end{aligned} \tag{22}$$

Combining Eqs.(20), (21) and (22) together, we obtain

$$\begin{aligned} c_{4y}(k;m) &= E\{\mathbf{h}^T\mathbf{x}(k)[\mathbf{h}^T\mathbf{x}(k+m)]^3\} \\ &+E\{v^3(k+m)v(k)\} \\ &-3E\{\mathbf{h}^T\mathbf{x}(k)\mathbf{h}^T\mathbf{x}(k+m)\}E\{[\mathbf{h}^T\mathbf{x}(k+m)]^2\} \\ &-3E\{v(k)v(k+m)\}E\{v^2(k+m)\} \end{aligned} \tag{23}$$

The combination of the second and the fourth terms in Eq.(23) equals the fourth-order cumulant of the observation noise, i.e.,

$$c_{4v}(k;m) = E\{v^3(k+m)v(k)\} - 3E\{v(k)v(k+m)\}E\{v^2(k+m)\}, \tag{24}$$

which is assumed to be known. The problem before us is to compute the first and third terms in Eq.(23).

3.1. Representation of the Cumulant of the System's State Vector

The first term in Eq.(23) can be expressed as (see Appendix C)

$$\begin{aligned} & E\{\mathbf{h}^T\mathbf{x}(k)[\mathbf{h}^T\mathbf{x}(k+m)]^3\} \\ = \; & [\mathbf{h}\otimes\mathbf{h}]^T E\{\mathbf{x}(k)\mathbf{x}^T(k+m)\otimes\mathbf{x}(k+m)\mathbf{x}^T(k+m)\}[\mathbf{h}\otimes\mathbf{h}] \end{aligned} \tag{25}$$

The third-term in Eq.(23) can be expressed in three different ways, as (see Appendix D)

$$\begin{aligned} & E\{\mathbf{h}^T\mathbf{x}(k)\mathbf{h}^T\mathbf{x}(k+m)\}E\{[\mathbf{h}^T\mathbf{x}(k+m)]^2\} \\ = \; & [\mathbf{h}\otimes\mathbf{h}]^T E\{\mathbf{x}(k)\mathbf{x}^T(k+m)\}\otimes E\{\mathbf{x}(k+m)\mathbf{x}^T(k+m)\}[\mathbf{h}\otimes\mathbf{h}] && (26.a) \\ = \; & [\mathbf{h}\otimes\mathbf{h}]^T [E\{\mathbf{x}(k)\otimes\mathbf{x}(k+m)\}][E\{\mathbf{x}(k+m)\otimes\mathbf{x}(k+m)\}]^T[\mathbf{h}\otimes\mathbf{h}] && (26.b) \\ = \; & [\mathbf{h}\otimes\mathbf{h}]^T E\{[\mathbf{x}(k)\otimes\mathbf{I}][E\{\mathbf{x}^T(k+m)\otimes\mathbf{x}(k+m)\}][\mathbf{I}\otimes\mathbf{x}^T(k+m)]\}[\mathbf{h}\otimes\mathbf{h}] && (26.c) \end{aligned}$$

Using these results, we now define $C_{4x}(k;m)$ as

$$\begin{aligned} C_{4x}(k;m) &= E\{x(k)x^T(k+m)\otimes x(k+m)x^T(k+m)\} \\ &-E\{x(k)x^T(k+m)\}\otimes E\{x(k+m)x^T(k+m)\} \\ &-[E\{x(k)\otimes x(k+m)\}][E\{x(k+m)\otimes x(k+m)\}]^T \\ &-E\{[x(k)\otimes I][E\{x^T(k+m)\otimes x(k+m)\}][I\otimes x^T(k+m)]\} \end{aligned} \tag{27}$$

which is an $n^2 \times n^2$ matrix.

Lemma 2. *Each element of matrix $C_{4x}(k;m)$ is a cumulant representation.*

The proof of Lemma 2 is given in Appendix E. It apparently requires all three different ways of representing the third term in Eq.(23) to obtain a legitimate fourth-order cumulant matrix for state vector, $x(t)$.

Lemma 3. *The fourth-order cumulant of the system output can be represented as*

$$c_{4y}(k;m) = [h\otimes h]^T C_{4x}(k;m)[h\otimes h] + c_{4v}(k;m) \tag{28}$$

Proof: Combining Eqs.(23) and (24), we have

$$\begin{aligned} c_{4y}(k;m) &= E\{h^T x(k)[h^T x(k+m)]^3\} \\ &-3E\{h^T x(k)h^T x(k+m)\}E\{[h^T x(k+m)]^2\} \\ &+c_{4v}(k;m) \end{aligned} \tag{29}$$

Using Eqs.(26) and (27), we have

$$c_{4y}(k;m) = [h\otimes h]^T C_{4x}(k;m)[h\otimes h] + c_{4v}(k;m)$$

which is Eq.(28). Q.E.D.

The key computation is $C_{4x}(k;m)$.

3.2. Recursive Computation Formulas

Proposition 4. (Recursive with Lag m). *$C_{4x}(k;m+1)$ can be computed recursively with respect to lag m as follows:*

$$C_{4x}(k;m+1) = [I\otimes \Phi]C_{4x}(k;m)[\Phi\otimes\Phi]^T, \quad m\geq 0 \tag{30}$$

$$C_{4x}(k;m-1) = [\Phi\otimes I]C_{4x}(k-1;m), \quad m\leq 0 \tag{31}$$

The proof of Proposition 4, which is given in Appendix F, expands $C_{4x}(k;m+1)$ and $C_{4x}(k;m-1)$ and uses Kronecker product identities to simplify the expanded results.

Proposition 5. (Recursive with Time k). *$C_{4x}(k+1;m)$ can be computed recursively with respect to time k, as*

$$C_{4x}(k+1;m) = [\Phi \otimes \Phi] C_{4x}(k;m) [\Phi \otimes \Phi]^T + \Theta(k;m) \tag{32}$$

where $\Theta(k;m)$ is an $n^2 \times n^2$ matrix, defined as

$$\Theta(k;m) = \begin{cases} c_{4w}(k)[I \otimes \Phi]^m [rr^T \otimes rr^T][\Phi^T \otimes \Phi^T]^m & m \geq 0 \\ c_{4w}(k+m)[\Phi \otimes I]^{-m}[rr^T \otimes rr^T] & m \leq 0 \end{cases} \tag{33}$$

and

$$c_{4w}(k) = E\{w^4(k)\} - 3[E\{w^2(k)\}]^2$$

The proof of Proposition 5 is along the lines of the proof of Proposition 4; it is given in Appendix G.

Our recursive formulas require an initial condition. As in the case of our third-order cumulant, we have:

Proposition 6. *For an asymptoticaly stable system, the value of $C_{4x}(k;m)$ is independent of initial condition matrix $C_{4x}(0;m)$ for large k.*

The proof of Proposition 6 is so similar to the proof of Proposition 3, its details are omitted.

3.3. Stationary Case

In the stationary case our recursive formulas are greatly simplified. In this situation we have

$$C_{4x}(k+1;m) = C_{4x}(k;m)$$

and

$$\Theta(k+1;m) = \Theta(k;m)$$

which means that $C_{4x}(k;m)$ and $\Theta(k;m)$ do not depend on k; hence, we will use $C_{4x}(m)$ and Θ(m) to represent $C_{4x}(k;m)$ and $\Theta(k;m)$ in the stationary case. Analogous to Proposition 4, we have:

Proposition 7. *In the stationary case, $C_{4x}(m)$ can be computed recursively by the following formulas:*

$$C_{4x}(m+1) = [I \otimes \Phi] C_{4x}(m) [\Phi \otimes \Phi]^T, \quad m \geq 0 \tag{34}$$

$$C_{4x}(m-1) = [\Phi \otimes I] C_{4x}(m), \quad m \leq 0 \tag{35}$$

The initial matrix $C_{4x}(0)$ can be computed from the discrete Lyapunov equation

$$C_{4x}(0) = [\Phi \otimes \Phi] C_{4x}(0) [\Phi \otimes \Phi]^T + \Theta(0) \tag{36}$$

where

$$\Theta(0) = c_{4w}(0)[rr^T \otimes rr^T] \tag{37}$$

4. Multi-Input Multi-Output Case

In previous sections we assumed the system was a single-input single-output (SISO) model. Using the definitions of the matrix third- and fourth-order cumulants in Eqs.(12) and (27) respectively, it is easy to extend the recursive formulas to the multi-input multi-output (MIMO) case.

Now our system model is

$$x(k+1) = \Phi x(k) + Rw(k) \tag{38}$$

$$y(k) = Hx(k) + Dv(k) \tag{39}$$

in which $w(k)$, $v(k)$ and $y(k)$ are vectors all with different dimensions. Analogous to the definitions of $C_{3x}(k;m)$ in Eq.(12) and of $C_{4x}(k;m)$ in Eq.(27), we can define the cumulant matrices of vectors $w(k)$, $v(k)$ and $y(k)$ by $C_{3w}(k;m)$, $C_{4w}(k;m)$, $C_{3v}(k;m)$, $C_{4v}(k;m)$, $C_{3y}(k;m)$ and $C_{4y}(k;m)$, respectively. We assume $w(k)$ and $v(k)$ are independent (with time) white noises, so the cumulants of $w(k)$ and $v(k)$ can be expressed, as $C_{3w}(k)$, $C_{4w}(k)$, $C_{3v}(k)$ and $C_{4v}(k)$.

The third-order output cumulant is

$$C_{3y}(k;m) = E\{y(k) \otimes y(k+m)y^T(k+m)\}$$

Using a derivation similar to the one for Lemma 1, we find:

Lemma 4. *Third order output cumulant matrix, $C_{3y}(k;m)$, can be represented as:*

$$C_{3y}(k;m) = [H \otimes H]C_{3x}(k;m)H^T + [D \otimes D]C_{3v}(k)D^T \tag{40}$$

For a MIMO system, the formulas that are recursive with lag are no different than those for a SISO system; i.e., Proposition 1 still holds in the MIMO case. Proposition 2 must be modified slightly, to:

Proposition 8. *For a MIMO system, $C_{3x}(k+1;m)$ can be computed recursively with respect to time k, from*

$$C_{3x}(k+1;m) = [\Phi \otimes \Phi]C_{3x}(k;m)\Phi^T + \Gamma(k;m) \tag{41}$$

where

$$\Gamma(k;m) = \begin{cases} [I \otimes \Phi]^m [R \otimes R] C_{3w}(k) R^T [\Phi^T]^m & \text{when } m \geq 0 \\ [\Phi \otimes I]^{-m} [R \otimes R] C_{3w}(k+m) R^T & \text{when } m \leq 0 \end{cases} \tag{42}$$

The fourth-order output cumulant matrix of a MIMO system is

$$\begin{aligned} C_{4y}(k;m) \;=\; & E\{y(k)y^T(k+m) \otimes y(k+m)y^T(k+m)\} \\ & -E\{y(k)y^T(k+m)\} \otimes E\{y(k+m)y^T(k+m)\} \\ & -[E\{y(k) \otimes y(k+m)][E\{y(k+m) \otimes y(k+m)\}]^T \\ & -E\{[y(k) \otimes I][E\{y^T(k+m) \otimes y(k+m)\}][I \otimes y^T(k+m)]\} \end{aligned} \tag{43}$$

Lemma 5. *The fourth-order output cumulant matrix $C_{4y}(k;m)$ of a MIMO system can be represented as*

$$C_{4y}(k;m) = [H \otimes H]C_{4x}(k;m)[H \otimes H]^T + [D \otimes D]C_{4v}(k)[D \otimes D]^T \tag{44}$$

In the MIMO case Proposition 4 still holds. Proposition 5 changes slightly, to:

Proposition 9. *$C_{4x}(k+1;m)$ can be computed recursively with respect to time, as*

$$C_{4x}(k+1;m) = [\Phi \otimes \Phi]C_{4x}(k;m)[\Phi \otimes \Phi]^T + \Theta(k;m) \tag{45}$$

where

$$\Theta(k;m) = \begin{cases} [I \otimes \Phi]^m [R \otimes R]C_{4w}(k)[R \otimes R]^T[\Phi^m \otimes \Phi^m]^T & \textit{when } m \geq 0 \\ [\Phi \otimes I]^{-m}[R \otimes R]C_{4w}(k+m)[R \otimes R]^T & \textit{when } m \leq 0 \end{cases} \tag{46}$$

Propositions 3 and 6 still hold in the MIMO case.

The lemmas and propositions in this section can be proved by following the proofs of their counterparts in the SISO case, with minor changes of notations and symbols.

5. Examples and Simulations

An air gun is a source of seismic energy for a marine environment. The IR (i.e., "signature") of a real air gun is depicted in Figure 2. In the follwing simulations, a zero mean exponential white noise $v(t)$ with variance $r_v = 0.01$, third-order cumulant $\gamma_{3v} = 0.0002$ and fourth-order cumulant $\gamma_{4v} = 0.0002$ is used as the system input. Its correlation is depicted in Figure 3. In order to compute the third- and fourth-order 1-D cumulants of the system output, we first obtained a state space realization for it using Kung's approximate realization technique [4]. A 10th-order model fit the IR almost perfectly. The 10th-order model was used in Eqs.(15), (16) and (14), with $v(k) = 0$, to obtain the third-order 1-D cumulant depicted in Figure 4. Note that in the stationary case, Eqs.(15) and (16) simplify to

$$C_{3x}(m+1) = [I \otimes \Phi]C_{3x}(m)\Phi^T, \text{ when } m \geq 0 \tag{47}$$

$$C_{3x}(m-1) = [\Phi \otimes I]C_{3x}(m), \text{ when } m \leq 0 \tag{48}$$

In comparison, the third-order cumulant of the system was computed by the impulse response method; results are plotted in Figure 5. The formula

$$c_{3y}(m) = \gamma_{3w} \sum_{i=0}^{\infty} h(i)h^2(i+m) \tag{49}$$

was used in which $h(i)$ is the impulse response of the wavelet. In Eq.(49) infinite samples of the impulse response are needed. In the computation of Figure 5, 1000 sample points were used. The difference of the results of the two methods is shown in Figure 5. Observe that the difference is very small.

Computation of the fourth-order cumulants of this 10th-order system are presented in Figures 7 to 9. Using Eqs.(34), (35) and (28), with $v(k) = 0$, the fourth-order cumulant is shown in Figure 7. Figure 8 depicts the fourth order cumulants computed by

$$c_{4y}(m) = \gamma_{4w} \sum_{i=0}^{\infty} h(i)h^3(i+m) \tag{50}$$

In the computation of Figure 8, 1000 sample points were used. The difference between the two methods in the computation of the fourth-order cumulants is shown in Figure 9.

Similar computations were performed for second- through tenth- order air gun models. Two observation appear to be common for all of the results: (1) positive-lag values of the cumulant have a higher frequency content than do negative-lag values, and (2) the cumulant is much more sensitive to model order than is the correlation. The correlation plot reached a "steady state" as a function of model order for a much lower order than did the cumulant.

Conclusions

Recursive diagonal-slice cumulant computations for a state-variable model have been discussed in this report. 1-D cumulant computation formulas for both third- and fourth-order formulas have been developed. Our formulas were developed for a SISO system; however simple modifications make them suitable for MIMO systems. Kronecker product make these results very compact, even though derivations of the results are quite tedious. The tediousness of the derivation is to be expected, because we are computing third- and fourth-order statistics.

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Appendix A. Proof of Lemma 1

Proof: We begin with Eq.(11). Let

$$\begin{aligned}\rho(k;m) = \; & 2E\{h^T x(k) h^T x(k+m) v(k+m)\} \\ & + E\{h^T x(k) v^2(k+m)\} \\ & + E\{v(k) h^T x(k+m) h^T x(k+m)\} \\ & + 2E\{v(k) h^T x(k+m) v(k+m)\} \\ & + E\{v(k) v^2(k+m)\}\end{aligned} \qquad (A.1)$$

so that

$$C_{3y}(k;m) = E\{h^T x(k) h^T x(k+m) h^T x(k+m)\} + \rho(k;m)$$

Since $v(k)$ is uncorrelated with $x(k)$, and, $E\{v(k)\} = 0$ and $E\{x(k)\} = 0$, the first four terms in Eq.(A.1) vanish; hence,

$$\rho(k;m) = E\{v(k) v^2(k+m)\} \qquad (A.2)$$

By the definition of a Kronecker product for scalars a and b, it is true that

$$a \otimes b = ab;$$

hence,

$$\begin{aligned} & E\{h^T x(k) h^T x(k+m) h^T x(k+m)\} \\ = \; & E\{[h^T x(k)] \otimes [h^T x(k+m) x^T(k+m) h]\} \\ = \; & E\{[h^T \otimes h^T][x(k) \otimes x(k+m) x^T(k+m)] h\} \\ = \; & [h \otimes h]^T E\{x(k) \otimes x(k+m) x^T(k+m)\} h \end{aligned} \qquad (A.3)$$

In going from the second to the third line of Eq.(A.3), we used Eq.(7). Using the definition of $C_{3x}(k;m)$ in Eq.(12), we have

$$c_{3y}(k;m) = [h \otimes h]^T C_{3x}(k;m) h + \rho(k;m) \qquad (A.4)$$

where $\rho(k;m)$ is given in Eq.(A.2). Eq.(A.4) matches Eq.(14); hence Lemma 1 is proved.

Appendix B. Proof of Proposition 2

Proof : We expand $C_{3x}(k+1;m)$, as follows:

$$
\begin{aligned}
C_{3x}(k+1;m) &= E\{[\Phi x(k) + r w(k)] \otimes [\Phi x(k+m) + r w(k+m)][\Phi x(k+m) + r w(k+m)]^T\} \\
&= E\{\Phi x(k) \otimes \Phi x(k+m) x^T(k+m) \Phi^T\} \\
&\quad + E\{\Phi x(k) \otimes \Phi x(k+m) r^T w(k+m)\} \\
&\quad + E\{\Phi x(k) \otimes r w(k+m) x^T(k+m) \Phi^T\} \\
&\quad + E\{\Phi x(k) \otimes r w(k+m) r^T w(k+m)\} \\
&\quad + E\{r w(k) \otimes \Phi x(k+m) x^T(k+m) \Phi^T\} \\
&\quad + E\{r w(k) \otimes \Phi x(k+m) r^T w(k+m)\} \\
&\quad + E\{r w(k) \otimes r w(k+m) x^T(k+m) \Phi^T\} \\
&\quad + E\{r w(k) \otimes r w(k+m) r^T w(k+m)\}
\end{aligned} \qquad (B.1)
$$

To analyze this equation, we considered the cases $m > 0$,$m = 0$ and $m < 0$ seperately. The first term in Eq.(B.1) remains in all three cases, so that

$$
C_{3x}(k+1;m) = [\Phi \otimes \Phi] C_{3x}(k;m) \Phi^T + \Gamma(k;m) \qquad (B.2)
$$

In the following analyses of $\Gamma(k;m)$, we use the facts that $E\{x(k)\} = 0, E\{w(k)\} = 0, w(k)$ is white, and, $x(k)$ is uncorrelated with $w(s)$ if $k \le s$.

When $m > 0$:

$$
\begin{aligned}
\Gamma(k;m) &= E\{r w(k) \otimes \Phi x(k+m) x^T(k+m) \Phi^T\} \\
&= E\{r w(k) \otimes \Phi^m r w(k) [\Phi^m r]^T w(k)\} \\
&= \gamma_w(k) [r \otimes \Phi^m r r^T (\Phi^m)^T] \\
&= \gamma_w(k) [I \otimes \Phi]^m [r \otimes r r^T] (\Phi^T)^m
\end{aligned} \qquad (B.3)
$$

When $m = 0$:

$$
\begin{aligned}
\Gamma(k;m) &= E\{r w(k) \otimes r w(k+m) r^T w(k+m)\} \\
&= \gamma_w(k) [r \otimes r r^T]
\end{aligned} \qquad (B.4)
$$

When $m < 0$:

$$
\begin{aligned}
\Gamma(k;m) &= E\{\Phi x(k) \otimes r w(k+m) r^T w(k+m)\} \\
&= E\{\Phi^{-m} r w(k+m) \otimes r w(k+m) r^T w(k+m)\} \\
&= \gamma_w(k+m) [\Phi \otimes I]^{-m} [r \otimes r r^T]
\end{aligned} \qquad (B.5)
$$

Combining Eqs.(B.2) - (B.5), we obtain Eqs.(17) and (18); hence, Proposition 2 is proved.

Appendix C. Derivation of Eq.(25)

Because $a \otimes b = ab$ for scalar a and b,

$$\begin{aligned}
& [h^T x(k) h^T x(k+m)][h^T x(k+m) h^T x(k+m)] \\
= \; & [h^T x(k) x^T(k+m) h] \otimes [h^T x(k+m) x^T(k+m) h] \\
= \; & [h^T \otimes h^T][x(k) x^T(k+m) \otimes x(k+m) x^T(k+m)][h \otimes h] \\
= \; & [h \otimes h]^T [x(k) x^T(k+m) \otimes x(k+m) x^T(k+m)][h \otimes h]
\end{aligned}$$

i.e.

$$E\{h^T x(k)[h^T x(k+m)]^3\} = [h \otimes h]^T E\{x(k) x^T(k+m) \otimes x(k+m) x^T(k+m)\}[h \otimes h]$$

which is Eq.(25).

Appendix D. Derivation of Eq.(26)

$$\begin{aligned}
& E\{h^T x(k) h^T x(k+m)\} E\{h^T x(k+m) h^T x(k+m)\} \\
= \; & E\{h^T x(k) x^T(k+m) h\} \otimes E\{h^T x(k+m) x^T(k+m) h\} \\
= \; & h^T E\{x(k) x^T(k+m)\} h \otimes h^T E\{x(k+m) x^T(k+m)\} h \\
= \; & [h \otimes h]^T [E\{x(k) x^T(k+m)\} \otimes E\{x(k+m) x^T(k+m)\}][h \otimes h]
\end{aligned}$$

which is Eq.(26.a).

$$\begin{aligned}
& E\{h^T x(k) h^T x(k+m)\} E\{h^T x(k+m) h^T x(k+m)\} \\
= \; & E\{h^T x(k) \otimes h^T x(k+m)\} E\{h^T x(k+m) \otimes h^T x(k+m)\} \\
= \; & [h^T \otimes h^T] E\{x(k) \otimes x(k+m)\} E\{x^T(k+m) \otimes x(k+m)\}[h \otimes h]
\end{aligned}$$

which is Eq.(26.b). Finally,

$$\begin{aligned}
& E\{h^T x(k) h^T x(k+m)\} E\{h^T x(k+m) h^T x(k+m)\} \\
= \; & E\{h^T x(k) \otimes E\{h^T x(k+m) x^T(k+m) h\} \otimes x^T(k+m) h\} \\
= \; & E\{h^T x(k) 1 \otimes [h^T I E\{x(k+m) x^T(k+m)\} I h] \otimes 1 x^T(k+m) h\} \\
= \; & E\{[h^T x(k) \otimes h^T I][1 \otimes E\{x(k+m) x^T(k+m)\} \otimes 1][I h \otimes x^T(k+m) h]\} \\
= \; & E\{[h^T \otimes h^T][x(k) \otimes I] E\{x(k+m) x^T(k+m)\}[I \otimes x^T(k+m)][h \otimes h]\} \\
= \; & [h^T \otimes h^T] E\{[x(k) \otimes I] E\{x(k+m) x^T(k+m)\}[I \otimes x^T(k+m)]\}[h \otimes h]
\end{aligned}$$

which is Eq.(26.c)

Appendix E. Proof of Lemma 2

In this appendix, we denote an $n \times n$ matrix A with element a_{ij} as $A = [a_{ij}]_{n \times n}$

From the definition of a Kronecker product, we know that the Kronecker product of $A = [a_{ij}]_{n\times n}$ and $B = [b_{ls}]_{n\times n}$ is

$$C = A \otimes B = [c_{pq}]_{n^2\times n^2} \tag{E.1}$$

where $c_{pq} = a_{ij}b_{ls}$ and $p = (i-1)n + l$ and $q = (j-1)n + s$.

Exactly like the construction of Eq.(E.1), we have

$$\begin{aligned} A = [a_{pq}]_{n^2\times n^2} &\equiv E\{x(k)x^T(k+m)\} \otimes E\{x(k+m)x^T(k+m)\} \\ &= [E\{x_i(k)x_j(k+m)\}E\{x_l(k+m)x_s(k+m)\}]_{n^2\times n^2} \end{aligned} \tag{E.2}$$

and

$$\begin{aligned} E = [e_{pq}]_{n^2\times n^2} &\equiv E\{x(k)x^T(k+m) \otimes x(k+m)x^T(k+m)\} \\ &= [E\{x_i(k)x_j(k+m)x_l(k+m)x_s(k+m)\}]_{n^2\times n^2} \end{aligned} \tag{E.3}$$

in which $p = (i-1)n + l$ and $q = (j-1)n + s$. The symbol "$\equiv$" is used as "define to equal".

Making use of Eq.(7) we also have

$$\begin{aligned} D = [d_{pq}]_{n^2\times n^2} &\equiv E\{[x(k) \otimes I]E\{x^T(k+m) \otimes x(k+m)\}[I \otimes x^T(k+m)]\} \\ &= [E\{x_i(k)E\{x_j(k+m)x_l(k+m)\}x_s(k+m)\}]_{n^2\times n^2} \\ &= [E\{x_i(k)x_s(k+m)\}E\{x_j(k+m)x_l(k+m)\}]_{n^2\times n^2} \end{aligned} \tag{E.4}$$

where $p = (i-1)n + j$ and $q = (l-1)n + s$.

Next, let

$$\begin{aligned} B' = [b'_p]_{n^2\times 1} &\equiv E\{x(k) \otimes x(k+m)\} \\ &= [E\{x_i(k)x_l(k+m)\}]_{n^2\times 1} \end{aligned} \tag{E.5}$$

in which, $p = (i-1)n + l$ and

$$\begin{aligned} B'' = [b'_q]_{1\times n^2} &\equiv E\{x^T(k+m) \otimes x^T(k+m)\} \\ &= [E\{x_j(k)x_s(k+m)\}]_{1\times n^2} \end{aligned} \tag{E.6}$$

where $q = (j-1)n + s$. Then

$$\begin{aligned} B = [b_{pq}]_{n^2\times n^2} &\equiv B'B'' \\ &= [E\{x_i(k)x_l(k+m)\}E\{x_j(k+m)x_s(k+m)\}]_{n^2\times n^2} \end{aligned} \tag{E.7}$$

From Eq.(27) and Eqs.(E.2), (E.3), (E.4), and (E.7), we have the pqth element of $C_{4x}(k;m)$, as

$$\begin{aligned} [C_{4x}(k;m)]_{pq} &= e_{pq} - a_{pq} - b_{pq} - d_{pq} \\ &= E\{x_i(k)x_j(k+m)x_l(k+m)x_s(k+m)\} \\ &\quad -E\{x_i(k)x_j(k+m)\}E\{x_l(k+m)x_s(k+m)\} \\ &\quad -E\{x_i(k)x_s(k+m)\}E\{x_j(k+m)x_l(k+m)\} \\ &\quad -E\{x_i(k)x_l(k+m)\}E\{x_j(k+m)x_s(k+m)\} \end{aligned} \tag{E.8}$$

The right-hand side of Eq.(E.8) is in the proper form of a fourth-order cumulant representation, as defined in Eq.(4); hence, Lemma 2 is proved.

Appendix F. Proof of Proposition 4

Proof: First, we derive Eq.(30). To begin, we expand some of the elements in the representation of $C_{4x}(k; m+1)$, using Eq.(27):

$$x(k)x^T(k+m+1) = x(k)x^T(k+m)\Phi^T + x(k)r^T w(k+m) \qquad (F.1)$$

$$\begin{aligned} x(k+m+1)x^T(k+m+1) &= \Phi x(k+m)x^T(k+m)\Phi^T \\ &+\Phi x(k+m)r^T w(k+m) \\ &+rw(k+m)x^T(k+m)\Phi^T \\ &+w^2(k+m)rr^T \end{aligned} \qquad (F.2)$$

$$\begin{aligned} x(k)\otimes x(k+m+1) &= x(k)\otimes \Phi x(k+m) \\ &+x(k)\otimes rw(k+m) \\ &= [I\otimes\Phi][x(k)\otimes x(k+m)] \\ &+x(k)\otimes rw(k+m) \end{aligned} \qquad (F.3)$$

$$\begin{aligned} x(k+m+1)\otimes x(k+m+1) &= \Phi x(k+m)\otimes \Phi x(k+m) \\ &+\Phi x(k+m)\otimes rw(k+m) \\ &+rw(k+m)\otimes \Phi x(k+m) \\ &+w^2(k+m)r\otimes r \end{aligned} \qquad (F.4)$$

$$\begin{aligned} x^T(k+m+1)\otimes x(k+m+1) &= x^T(k+m)\Phi^T \otimes \Phi x(k+m) \\ &+x^T(k+m)\Phi^T\otimes rw(k+m) \\ &+r^T w(k+m)\otimes \Phi x(k+m) \\ &+w^2(k+m)r^T\otimes r \end{aligned} \qquad (F.5)$$

Combining Eqs.(F.1) and (F.2) together, we obtain

$$\begin{aligned} & x(k)x^T(k+m+1)\otimes x(k+m+1)x^T(k+m+1) \\ = \; & x(k)x^T(k+m)\Phi^T \otimes \Phi x(k+m)x^T(k+m)\Phi^T \\ & +x(k)r^T w(k+m)\otimes \Phi x(k+m)x^T(k+m)\Phi^T \\ & +x(k)x^T(k+m)\Phi^T\otimes \Phi x(k+m)r^T w(k+m) \\ & +x(k)r^T w(k+m)\otimes \Phi x(k+m)r^T w(k+m) \\ & +x(k)x^T(k+m)\Phi^T\otimes rw(k+m)x^T(k+m)\Phi^T \\ & +x(k)r^T w(k+m)\otimes rw(k+m)x^T(k+m)\Phi^T \\ & +x(k)x^T(k+m)\Phi^T\otimes w^2(k+m)rr^T \\ & +x(k)r^T w(k+m)\otimes w^2(k+m)rr^T \end{aligned} \qquad (F.6)$$

Because $E\{w(k)\}=0$, $E\{x(k)\}=0$ and, when $m \geq 0$ $w(k+m)$ is uncorrelated with $x(k+m)$ and $x(k)$, we have:

$$\begin{aligned}
& E\{x(k)x^T(k+m+1)\otimes x(k+m+1)x^T(k+m+1)\} \\
= \; & E\{x(k)x^T(k+m)\Phi^T \otimes \Phi x(k+m)x^T(k+m)\Phi^T\} \\
& +E\{x(k)r^T w(k+m)\otimes \Phi x(k+m)r^T w(k+m)\} \\
& +E\{x(k)r^T w(k+m)\otimes r w(k+m)x^T(k+m)\Phi^T\} \\
& +E\{x(k)x^T(k+m)\Phi^T \otimes w^2(k+m)rr^T\}
\end{aligned} \quad (F.7)$$

Using Kronecker product properties, we have

$$\begin{aligned}
& E\{x(k)x^T(k+m+1)\otimes x(k+m+1)x^T(k+m+1) \\
= \; & E\{[I\otimes\Phi][x(k)x^T(k+m)\otimes x(k+m)x^T(k+m)][\Phi\otimes\Phi]^T\} \\
& +E\{[x(k)\otimes \Phi x(k+m)][r^T w(k+m)\otimes r^T w(k+m)]\} \\
& +E\{[x(k)\otimes I][r^T w(k+m)\otimes r w(k+m)][I\otimes x^T(k+m)\Phi^T]\} \\
& +E\{x(k)x^T(k+m)\Phi^T\}\otimes E\{w^2(k+m)rr^T\} \\
= \; & [I\otimes\Phi]E\{[x(k)x^T(k+m)\otimes x(k+m)x^T(k+m)]\}[\Phi\otimes\Phi]^T \\
& +E\{[x(k)\otimes \Phi x(k+m)][r^T w(k+m)\otimes r^T w(k+m)]\} \\
& +E\{[x(k)\otimes I][r^T w(k+m)\otimes r w(k+m)][I\otimes x^T(k+m)\Phi^T]\} \\
& +E\{x(k)x^T(k+m)\Phi^T\}\otimes E\{w^2(k+m)rr^T\}
\end{aligned} \quad (F.8)$$

By the definition of Eq.(27), we have

$$\begin{aligned}
C_{4x}(k;m+1) \; = \; & E\{x(k)x^T(k+m+1)\otimes x(k+m+1)x^T(k+m+1)\} \\
& -E\{x(k)x^T(k+m+1)\}\otimes E\{x(k+m+1)x^T(k+m+1)\} \\
& -[E\{x(k)\otimes x(k+m+1)\}][E\{x(k+m+1)\otimes x(k+m+1)\}]^T \\
& -E\{[x(k)\otimes I][E\{x^T(k+m+1)\otimes x(k+m+1)\}][I\otimes x^T(k+m+1)]\}
\end{aligned} \quad (F.9)$$

and Eq.(F.8) is the first term of Eq.(F.9). Proceeding in a similar manner we can determine the other three terms of $C_{4x}(k;m+1)$.

From Eq.(F.1) we have

$$E\{x(k)x^T(k+m+1)\} = E\{x(k)x^T(k+m)\Phi^T\} \quad (F.10)$$

and, from Eq.(F.2) we obtain

$$E\{x(k+m+1)x^T(k+m+1)\} = \Phi E\{x(k+m)x^T(k+m)\}\Phi^T + E\{w^2(k+m)rr^T\} \quad (F.11)$$

Consequently, the second term of Eq.(F.9) is

$$\begin{aligned}
& E\{x(k)x^T(k+m+1)\} \otimes E\{x(k+m+1)x^T(k+m+1)\} \\
= \;& E\{x(k)x^T(k+m)\}\Phi^T \otimes \Phi E\{x(k+m)x^T(k+m)\}\Phi^T \\
& +E\{x(k)x^T(k+m)\Phi^T\} \otimes E\{w^2(k+m)rr^T\} \\
= \;& [I \otimes \Phi][E\{x(k)x^T(k+m)\} \otimes E\{x(k+m)x^T(k+m)\}][\Phi \otimes \Phi]^T \\
& +E\{x(k)x^T(k+m)\Phi^T\} \otimes E\{w^2(k+m)rr^T\}
\end{aligned} \qquad (F.12)$$

From Eq.(F.3) we get

$$E\{x(k) \otimes x(k+m+1)\} = [I \otimes \Phi][E\{x(k) \otimes x(k+m)\}] \qquad (F.13)$$

From Eq.(F.4) we find that

$$E\{x(k+m+1) \otimes x(k+m+1)\} = [\Phi \otimes \Phi][E\{x(k+m) \otimes x(k+m)\}] + E\{w^2(k+m)r \otimes r\} \qquad (F.14)$$

Consequently, the third term of Eq.(F.1) is

$$\begin{aligned}
& [E\{x(k) \otimes x(k+m+1)\}][E\{x(k+m+1) \otimes x(k+m+1)\}]^T \\
= \;& [I \otimes \Phi][E\{x(k) \otimes x(k+m)\}][E\{x(k+m) \otimes x(k+m)\}]^T[\Phi \otimes \Phi]^T \\
& +[I \otimes \Phi][E\{x(k) \otimes x(k+m)\}]E\{w^2(k+m)r^T \otimes r^T\}
\end{aligned} \qquad (F.15)$$

From Eq.(F.5), we see that

$$\begin{aligned}
E\{x^T(k+m+1) \otimes x(k+m+1)\} \;=\; & \Phi[E\{x^T(k+m) \otimes x(k+m)\}]\Phi^T \\
& +E\{w^2(k+m)r^T \otimes r\}
\end{aligned} \qquad (F.16)$$

Because $E\{x^T(k+m+1) \otimes x(k+m+1)\}$ is a deterministic matrix function, we have

$$\begin{aligned}
& E\{[x(k) \otimes I][E\{x^T(k+m+1) \otimes x(k+m+1)\}][I \otimes x^T(k+m+1)]\} \\
= \;& E\{[x(k) \otimes I][E\{x^T(k+m+1) \otimes x(k+m+1)\}][I \otimes x^T(k+m)\Phi^T]\} \\
= \;& E\{[x(k) \otimes I]\Phi[E\{x^T(k+m) \otimes x(k+m)\}]\Phi^T[I \otimes x^T(k+m)\Phi^T]\} \\
& +E\{[x(k) \otimes I]E\{w^2(k+m)r^T \otimes r\}[I \otimes x^T(k+m)\Phi^T]\}
\end{aligned} \qquad (F.17)$$

By the properties of the Kronecker product, we have

$$\begin{aligned}
[x(k) \otimes I]\Phi \;&=\; [x(k) \otimes I][1 \otimes \Phi] \\
&=\; [x(k) \otimes \Phi] \\
&=\; [Ix(k) \otimes \Phi I] \\
&=\; [I \otimes \Phi][x(k) \otimes I]
\end{aligned} \qquad (F.18)$$

and

$$\begin{aligned}
\Phi^T[I \otimes x^T(k+m)\Phi^T] \;&=\; [\Phi^T \otimes 1][I \otimes x^T(k+m)\Phi^T] \\
&=\; [\Phi^T \otimes x^T(k+m)\Phi^T] \\
&=\; [I\Phi^T \otimes x^T(k+m)\Phi^T] \\
&=\; [I \otimes x^T(k+m)][\Phi \otimes \Phi]^T
\end{aligned} \qquad (F.19)$$

Inserting Eqs.(F.18) and (F.19) into Eq.(17), the fourth term of Eq.(F.9) becomes

$$\begin{aligned} & E\{[x(k)\otimes I][E\{x^T(k+m+1)\otimes x(k+m+1)\}][I\otimes x^T(k+m+1)]\} \\ = \; & [I\otimes \Phi]E\{[x(k)\otimes I][E\{x^T(k+m)\otimes x(k+m)\}][I\otimes x^T(k+m)]\}[\Phi\otimes\Phi]^T \\ & +E\{[x(k)\otimes I]E\{w^2(k+m)r^T\otimes r\}[I\otimes x^T(k+m)\Phi^T]\} \end{aligned} \tag{F.20}$$

Finally, substituting Eqs.(F.8), (F.12), (F.15) and (F.20) into Eq.(F.9), we have

$$\begin{aligned} C_{4x}(k;m+1) \; = \; & [I\otimes\Phi]E\{[x(k)x^T(k+m)\otimes x(k+m)x^T(k+m)]\}[\Phi\otimes\Phi]^T \\ & -[I\otimes\Phi][E\{x(k)x^T(k+m)\}\otimes E\{x(k+m)x^T(k+m)\}][\Phi\otimes\Phi]^T \\ & -[I\otimes\Phi][E\{x(k)\otimes x(k+m)\}][E\{x(k+m)\otimes x(k+m)\}]^T[\Phi\otimes\Phi]^T \\ & -[I\otimes\Phi]E\{[x(k)\otimes I][E\{x^T(k+m)\otimes x(k+m)\}][I\otimes x^T(k+m)]\}[\Phi\otimes\Phi]^T \\ = \; & [I\otimes\Phi]C_{4x}(k;m)[\Phi\otimes\Phi]^T \end{aligned} \tag{F.21}$$

Next, we derive Eq.(31), which is easier to do than deriving Eq.(30). Noticing that

$$\begin{aligned} C_{4x}(k;m-1) \; = \; & E\{x(k-1+1)x^T(k-1+m)\otimes x(k-1+m)x^T(k-1+m)\} \\ & -E\{x(k-1+1)x^T(k-1+m)\}\otimes E\{x(k-1+m)x^T(k-1+m)\} \\ & -[E\{x(k-1+1)\otimes x(k-1+m)\}][E\{x(k-1+m)\otimes x(k-1+m)\}]^T \\ & -E\{[x(k-1+1)\otimes I][E\{x^T(k-1+m)\otimes x(k-1+m)\}][I\otimes x^T(k-1+m)]\} \end{aligned} \tag{F.22}$$

and by making use of

$$x(k-1+1) = \Phi x(k-1) + rw(k-1)$$

we have

$$\begin{aligned} C_{4x}(k;m-1) \; = \; & E\{\Phi x(k-1)x^T(k-1+m)\otimes x(k-1+m)x^T(k-1+m)\} \\ & +E\{rw(k-1)x^T(k-1+m)\otimes x(k-1+m)x^T(k-1+m)\} \\ & -E\{\Phi x(k-1)x^T(k-1+m)\}\otimes E\{x(k-1+m)x^T(k-1+m)\} \\ & -E\{rw(k-1)x^T(k-1+m)\}\otimes E\{x(k-1+m)x^T(k-1+m)\} \\ & -[E\{\Phi x(k-1)\otimes x(k-1+m)\}][E\{x(k-1+m)\otimes x(k-1+m)\}]^T \\ & -[E\{rw(k-1)\otimes x(k-1+m)\}][E\{x(k-1+m)\otimes x(k-1+m)\}]^T \\ & -E\{[\Phi x(k-1)\otimes I][E\{x^T(k-1+m)\otimes x(k-1+m)\}][I\otimes x^T(k-1+m)]\} \\ & -E\{[rw(k-1)\otimes I][E\{x^T(k-1+m)\otimes x(k-1+m)\}][I\otimes x^T(k-1+m)]\} \end{aligned} \tag{F.23}$$

For $m \le 0$, $w(k-1)$ is uncorrelated with $x(k-1+m)$; hence, using Kronecker product properties, it is straightforward to show, that

$$
\begin{aligned}
C_{4x}(k;m-1) &= [I\otimes\Phi]E\{x(k-1)x^T(k-1+m)\otimes x(k-1+m)x^T(k-1+m)\} \\
&\quad -[I\otimes\Phi]E\{x(k-1)x^T(k-1+m)\}\otimes E\{x(k-1+m)x^T(k-1+m)\} \\
&\quad -[I\otimes\Phi][E\{x(k-1)\otimes x(k-1+m)\}][E\{x(k-1+m)\otimes x(k-1+m)\}]^T \\
&\quad -[I\otimes\Phi]E\{[x(k-1)\otimes I][E\{x^T(k-1+m)\otimes x(k-1+m)\}][I\otimes x^T(k-1+m)]\} \\
&= [I\otimes\Phi]C_{4x}(k-1;m)
\end{aligned}
\tag{F.24}
$$

which completes the proof of Proposition 4.

Appendix G. Proof of Proposition 5

Proof: As we did in Appendix F, we begin by listing some of the equations that will be used later:

$$
\begin{aligned}
x(k+1)x^T(k+m+1) &= \Phi x(k)x^T(k+m)\Phi^T \\
&\quad +\Phi x(k)r^T w(k+m) \\
&\quad +rw(k)x^T(k+m)\Phi^T \\
&\quad +rr^T w(k)w(k+m)
\end{aligned}
\tag{G.1}
$$

$$
\begin{aligned}
x(k+1)\otimes x(k+m+1) &= [\Phi\otimes\Phi][x(k)\otimes x(k+m)] \\
&\quad +[\Phi\otimes I][x(k)\otimes rw(k+m)] \\
&\quad +[I\otimes\Phi][rw(k)\otimes x(k+m)] \\
&\quad +w(k)w(k+m)r\otimes r
\end{aligned}
\tag{G.2}
$$

Since

$$
\begin{aligned}
C_{4x}(k+1;m) &= E\{x(k+1)x^T(k+m+1)\otimes x(k+m+1)x^T(k+m+1)\} \\
&\quad -E\{x(k+1)x^T(k+m+1)\}\otimes E\{x(k+m+1)x^T(k+m+1)\} \\
&\quad -[E\{x(k+1)\otimes x(k+m+1)\}][E\{x(k+m+1)\otimes x(k+m+1)\}]^T \\
&\quad -E\{[x(k+1)\otimes I][E\{x^T(k+m+1)\otimes x(k+m+1)\}][I\otimes x^T(k+m+1)]\}
\end{aligned}
\tag{G.3}
$$

our approach is to expand the four terms of Eq.(G.3) and make the result match Eq.(32).

Making use of Eqs.(G.1) and (F.2), the first term in Eq.(G.3) becomes

$$\begin{aligned}
& E\{x(k+1)x^T(k+m+1) \otimes x(k+m+1)x^T(k+m+1)\} \\
= \; & [\Phi \otimes \Phi][E\{x(k)x^T(k+m) \otimes x(k+m)x^T(k+m)\}][\Phi \otimes \Phi]^T \\
& +[\Phi \otimes \Phi][E\{x(k)x^T(k+m) \otimes x(k+m)r^T w(k+m)\}][\Phi \otimes I]^T \\
& +[\Phi \otimes I][E\{x(k)x^T(k+m) \otimes rw(k+m)x^T(k+m)\}][\Phi \otimes \Phi]^T \\
& +[\Phi \otimes I][E\{x(k)x^T(k+m) \otimes w^2(k+m)rr^T\}][\Phi \otimes I]^T \\
& +[\Phi \otimes \Phi][E\{x(k)r^T w(k+m) \otimes x(k+m)x^T(k+m)\}][I \otimes \Phi]^T \\
& +[\Phi \otimes \Phi][E\{x(k)r^T w(k+m) \otimes x(k+m)r^T w(k+m)\}] \\
& +[\Phi \otimes I][E\{x(k)r^T w(k+m) \otimes rw(k+m)x^T(k+m)\}][I \otimes \Phi]^T \\
& +[\Phi \otimes I][E\{x(k)r^T w(k+m) \otimes w^2(k+m)rr^T\}] \qquad (G.4) \\
& +[I \otimes \Phi][E\{rw(k)x^T(k+m) \otimes x(k+m)x^T(k+m)\}][\Phi \otimes \Phi]^T \\
& +[I \otimes \Phi][E\{rw(k)x^T(k+m) \otimes x(k+m)r^T w(k+m)\}][\Phi \otimes I]^T \\
& +[E\{rw(k)x^T(k+m) \otimes rw(k+m)x^T(k+m)\}][\Phi \otimes \Phi]^T \\
& +[E\{rw(k)x^T(k+m) \otimes w^2(k+m)rr^T\}][\Phi \otimes I]^T \\
& +[I \otimes \Phi][E\{w(k)w(k+m)rr^T \otimes x(k+m)x^T(k+m)\}][I \otimes \Phi]^T \\
& +[I \otimes \Phi][E\{w(k)w(k+m)rr^T \otimes x(k+m)r^T w(k+m)\}] \\
& +[E\{w(k)w(k+m)rr^T \otimes rw(k+m)x^T(k+m)\}][I \otimes \Phi]^T \\
& +[E\{w(k)w(k+m)rr^T \otimes w^2(k+m)rr^T\}]
\end{aligned}$$

We shall analyze Eq.(G.4) separately for $m > 0$, $m = 0$ and $m < 0$. Also, we keep in mind that $E\{w(k)\} = 0$ and $E\{x(k)\} = 0$. When $m > 0$, $w(k+m)$ is uncorrelated with $x(k)$, $x(k+m)$ and $w(k)$; hence, in Eq.(G.4) the 2nd, 3rd, 5th, 8th, 10th, 11th, 13th and 16th terms vanish. When $m = 0$, $w(k) = w(k+m)$ is uncorrelated with $x(k) = x(k+m)$, so that the 2nd, 3rd, 5th, 8th, 9th, 12th, 14th and 15th terms vanish. When $m < 0$, $w(k)$ is uncorrelated with $w(k+m)$, $x(k)$ and $x(k+m)$. In this case, the 9th through 15th terms vanish.

In the following analyses, we let $G(k;m)$ denote the sum of the 1st, 4th, 6th and 7th terms, since they are common for all three cases; i.e.,

$$\begin{aligned}
G(k;m) \;=\; & [\Phi \otimes \Phi][E\{x(k)x^T(k+m) \otimes x(k+m)x^T(k+m)\}][\Phi \otimes \Phi]^T \\
& +[\Phi \otimes I][E\{x(k)x^T(k+m) \otimes w^2(k+m)rr^T][\Phi \otimes I]^T \\
& +[\Phi \otimes \Phi][E\{x(k)r^T w(k+m) \otimes x(k+m)r^T w(k+m)] \\
& +[\Phi \otimes I][E\{x(k)r^T w(k+m) \otimes rw(k+m)x^T(k+m)\}][I \otimes \Phi]^T
\end{aligned}$$

Using the properties of the Kronecker product, we have

$$\begin{aligned} G(k;m) &= [\Phi \otimes \Phi][E\{x(k)x^T(k+m) \otimes x(k+m)x^T(k+m)\}][\Phi \otimes \Phi]^T \\ &\quad +[\Phi \otimes I][E\{x(k)x^T(k+m)\} \otimes \sigma_w^2(k+m)rr^T][\Phi \otimes I]^T \\ &\quad +[\Phi \otimes \Phi][E\{x(k) \otimes x(k+m)\}][\sigma_w^2(k+m)r^T \otimes r^T] \\ &\quad +[\Phi \otimes I][E\{[x(k) \otimes I][\sigma_w^2(k+m)r^T \otimes r][I \otimes x^T(k+m)]\}][I \otimes \Phi]^T \end{aligned} \tag{G.5}$$

For $m > 0$, we have

$$\begin{aligned} & E\{x(k+1)x^T(k+m+1) \otimes x(k+m+1)x^T(k+m+1)\} \\ = \; & G(k;m) \\ & +[I \otimes \Phi][E\{rw(k)x^T(k+m) \otimes x(k+m)x^T(k+m)\}][\Phi \otimes \Phi]^T \\ & +[E\{rw(k)x^T(k+m) \otimes w^2(k+m)rr^T\}][\Phi \otimes I] \\ & +[I \otimes \Phi][E\{w(k)w(k+m)rr^T \otimes x(k+m)r^T w(k+m)\}] \\ & +[E\{w(k)w(k+m)rr^T \otimes rw(k+m)x^T(k+m)\}][I \otimes \Phi]^T \end{aligned} \tag{G.6}$$

Let $Z(k;m)$ be

$$\begin{aligned} Z(k;m) &\equiv E\{rw(k)x^T(k+m) \otimes x(k+m)x^T(k+m)\} \\ &= E\{rw(k)[\Phi^m x(k) + \textstyle\sum_{i=0}^{m-1} \Phi^{m-i-1} rw(k+i)]^T \\ &\quad \otimes[\Phi^m x(k) + \textstyle\sum_{i=0}^{m-1} \Phi^{m-i-1} rw(k+i)][\Phi^m x(k) + \textstyle\sum_{i=0}^{m-1} \Phi^{m-i-1} rw(k+i)]^T\} \end{aligned}$$

From the assumption of $w(k)$, we have

$$\begin{aligned} Z(k;m) &= E\{rw(k)[\Phi^m x(k) + \Phi^{m-1} rw(k)]^T \otimes [\Phi^m x(k) + \Phi^{m-1} rw(k)][\Phi^m x(k) + \Phi^{m-1} rw(k)]^T\} \\ &= E\{rw(k)[\Phi^m x(k)]^T \otimes \Phi^m x(k)[\Phi^m x(k)]^T\} \\ &\quad +E\{rw(k)[\Phi^m x(k)]^T \otimes \Phi^m x(k)[\Phi^{m-1} rw(k)]^T\} \\ &\quad +E\{rw(k)[\Phi^m x(k)]^T \otimes \Phi^{m-1} rw(k)[\Phi^m x(k)]^T\} \\ &\quad +E\{rw(k)[\Phi^{m-1} rw(k)]^T \otimes \Phi^m x(k)[\Phi^m x(k)]^T\} \\ &\quad +E\{rw(k)[\Phi^m x(k)]^T \otimes \Phi^{m-1} rw(k)[\Phi^{m-1} rw(k)]^T\} \\ &\quad +E\{rw(k)[\Phi^{m-1} rw(k)]^T \otimes \Phi^m x(k)[\Phi^{m-1} rw(k)]^T\} \\ &\quad +E\{rw(k)[\Phi^{m-1} rw(k)]^T \otimes \Phi^{m-1} rw(k)[\Phi^m x(k)]^T\} \\ &\quad +E\{rw(k)[\Phi^{m-1} rw(k)]^T \otimes \Phi^{m-1} rw(k)[\Phi^{m-1} rw(k)]^T\} \end{aligned} \tag{G.7}$$

In Eq.(G.7), the 1st, 5th, 6th and 7th terms vanish because $x(k)$ is independent of $w(k)$; so, Eq.(G.7) simplifies to

$$
\begin{aligned}
Z(k;m) &= E\{rw(k)[\Phi^m x(k)]^T \otimes \Phi^m x(k)[\Phi^{m-1} r w(k)]^T\} \\
&\quad + E\{rw(k)[\Phi^m x(k)]^T \otimes \Phi^{m-1} r w(k)[\Phi^m x(k)]^T\} \\
&\quad + E\{rw(k)[\Phi^{m-1} r w(k)]^T \otimes \Phi^m x(k)[\Phi^m x(k)]^T\} \\
&\quad + E\{rw(k)[\Phi^{m-1} r w(k)]^T \otimes \Phi^{m-1} r w(k)[\Phi^{m-1} r w(k)]^T\} \\
&= \sigma_w^2 [r \otimes I] E\{[\Phi^m x(k)]^T \otimes [\Phi^m x(k)]\}[I \otimes \Phi^m r]^T \\
&\quad + \sigma_w^2 [r \otimes \Phi^{m-1} r][E\{\Phi^m x(k) \otimes \Phi^m x(k)\}]^T \\
&\quad + \sigma_w^2 [r(\Phi^{m-1} r)^T] \otimes E\{\Phi^m x(k)(\Phi^m x(k))^T\} \\
&\quad + \gamma_{4w} [r(\Phi^{m-1} r)^T \otimes \Phi^{m-1} r(\Phi^{m-1} r)^T]
\end{aligned} \tag{G.8}
$$

Observe that

$$E\{x^T(k+m) \otimes x(k+m)\} = E\{[\Phi^m x(k)]^T \otimes \Phi^m x(k)\} + \sigma_w^2 [(\Phi^{m-1} r)^T \otimes \Phi^{m-1} r] \tag{G.9}$$

$$E\{x(k+m) \otimes x(k+m)\} = E\{\Phi^m x(k) \otimes \Phi^m x(k)\} + \sigma_w^2 [\Phi^{m-1} r \otimes \Phi^{m-1} r] \tag{G.10}$$

$$E\{x(k+m) x^T(k+m)\} = E\{\Phi^m x(k)[\Phi^m x(k)]^T\} + \sigma_w^2 [\Phi^{m-1} r[\Phi^{m-1} r]^T] \tag{G.11}$$

Substituting Eqs.(G.9), (G.10) and (G.11) into Eq.(G.8), we obtain

$$
\begin{aligned}
Z(k;m) &= \sigma_w^2 [r \otimes I] E\{x^T(k+m) \otimes x(k+m)\}[I \otimes \Phi^m r]^T \\
&\quad + \sigma_w^2 [r \otimes \Phi^{m-1} r][E\{x(k+m) \otimes x(k+m)\}]^T \\
&\quad + \sigma_w^2 [r(\Phi^{m-1} r)^T] \otimes E\{x(k+m)(x^T(k)\} \\
&\quad - 3[\sigma_w^2]^2 [r(\Phi^{m-1} r)^T \otimes \Phi^{m-1} r(\Phi^{m-1} r)^T] \\
&\quad + \gamma_{4w} [r(\Phi^{m-1} r)^T \otimes \Phi^{m-1} r(\Phi^{m-1} r)^T]
\end{aligned}
$$

Using the properties of Kronecker product, we obtain

$$
\begin{aligned}
Z(k;m) &= \sigma_w^2(k)[rr^T \otimes E\{x(k+m)x^T(k+m)\}][\Phi^{m-1} \otimes I]^T \\
&\quad + \sigma_w^2(k)[I \otimes \Phi^{m-1}] E\{[r \otimes r][x(k+m) \otimes x(k+m)]^T\} \\
&\quad + \sigma_w^2(k) E\{[r \otimes I][x^T(k+m) \otimes x(k+m)][I \otimes r^T][I \otimes \Phi^m]^T\} \\
&\quad - 3[\sigma_w^2(k)]^2 [I \otimes \Phi^{m-1}][rr^T \otimes rr^T][\Phi^{m-1} \otimes \Phi^{m-1}]^T \\
&\quad + \gamma_w(k)[I \otimes \Phi^{m-1}][rr^T \otimes rr^T][\Phi^{m-1} \otimes \Phi^{m-1}]^T
\end{aligned} \tag{G.12}
$$

Then,

$$
\begin{aligned}
&\text{the second term of Eq.(G.6)} \\
&= [I \otimes \Phi] Z(k;m)[\Phi \otimes \Phi]^T \\
&= \sigma_w^2(k)[I \otimes \Phi][rr^T \otimes E\{x(k+m)x^T(k+m)\}][\Phi^m \otimes \Phi]^T \\
&\quad + \sigma_w^2(k)[I \otimes \Phi^m] E\{[r \otimes r][x(k+m) \otimes x(k+m)]^T\}[\Phi \otimes \Phi]^T \\
&\quad + \sigma_w^2(k)[I \otimes \Phi] E\{[r \otimes I][x^T(k+m) \otimes x(k+m)][I \otimes r^T][\Phi \otimes \Phi^m]^T\} \\
&\quad - 3[\sigma_w^2(k)]^2 [I \otimes \Phi^m][rr^T \otimes rr^T][\Phi^m \otimes \Phi^m]^T \\
&\quad + \gamma_w(k)[I \otimes \Phi^m][rr^T \otimes rr^T][\Phi^m \otimes \Phi^m]^T
\end{aligned} \tag{G.13}
$$

The third term of Eq.(G.6) is

$$\begin{aligned} & [E\{rw(k)x^T(k+m)\otimes w^2(k+m)rr^T\}][\Phi\otimes I] \\ = \ & [E\{rw(k)(\Phi^m x(k)+\textstyle\sum_{i-1}^{m-1}\Phi^{m-i-1}rw(m-i))\otimes w^2(k+m)rr^T\}[\Phi\otimes I] \\ = \ & \sigma_w^2(k)\sigma_w^2(k+m)[r(\Phi^{m-1}r)^T\otimes rr^T][\Phi\otimes I] \\ = \ & \sigma_w^2(k)\sigma_w^2(k+m)[rr^T\otimes rr^T][\Phi^m\otimes I] \end{aligned} \tag{G.14}$$

The fourth term of Eq.(G.6)

$$\begin{aligned} & [I\otimes\Phi][E\{w(k)w(k+m)rr^T\otimes x(k+m)r^T w(k+m)\}] \\ = \ & [I\otimes\Phi][E\{w(k)w(k+m)rr^T\otimes(\Phi^m x(k)+\textstyle\sum_{i-1}^{m-1}\Phi^{m-i-1}rw(m-i))r^T w(k+m)\}] \\ = \ & \sigma_w^2(k)\sigma_w^2(k+m)[I\otimes\Phi][rr^T\otimes\Phi^{m-1}r^T] \\ = \ & \sigma_w^2(k)\sigma_w^2(k+m)[I\otimes\Phi^m][rr^T\otimes rr^T] \end{aligned} \tag{G.15}$$

The fifth term of Eq.(G.6)

$$\begin{aligned} & [E\{w(k)w(k+m)rr^T\otimes rw(k+m)x^T(k+m)\}][I\otimes\Phi]^T \\ = \ & [E\{w(k)w(k+m)rr^T\otimes rw(k+m)(\Phi^m x(k)+\textstyle\sum_{i-1}^{m-1}\Phi^{m-i-1}rw(m-i))^T\}][I\otimes\Phi]^T \\ = \ & \sigma_w^2(k)\sigma_w^2(k+m)[rr^T\otimes r(\Phi^{m-1}r)^T][I\otimes\Phi]^T \\ = \ & \sigma_w^2(k)\sigma_w^2(k+m)[rr^T\otimes rr^T][I\otimes\Phi^m]^T \end{aligned} \tag{G.16}$$

Combining Eqs.(G.13) to (G.16) we obtain

$$\begin{aligned} & E\{x(k+1)x^T(k+m+1)\otimes x(k+m+1)x^T(k+m+1)\} \\ = \ & G(k;m) \\ & +\sigma_w^2(k)[I\otimes\Phi][rr^T\otimes E\{x(k+m)x^T(k+m)\}][\Phi^m\otimes\Phi]^T \\ & +\sigma_w^2(k)[I\otimes\Phi^m]E\{[r\otimes r][x(k+m)\otimes x(k+m)]^T\}[\Phi\otimes\Phi]^T \\ & +\sigma_w^2(k)[I\otimes\Phi]E\{[r\otimes I][x^T(k+m)\otimes x(k+m)][I\otimes r^T][\Phi\otimes\Phi^m]^T\} \\ & -3[\sigma_w^2(k)]^2[I\otimes\Phi^m][rr^T\otimes rr^T][\Phi^m\otimes\Phi^m]^T \\ & +\gamma_w(k)[I\otimes\Phi^m][rr^T\otimes rr^T][\Phi^m\otimes\Phi^m]^T \\ & +\sigma_w^2(k)\sigma_w^2(k+m)[rr^T\otimes rr^T][\Phi^m\otimes I] \\ & +\sigma_w^2(k)\sigma_w^2(k+m)[I\otimes\Phi^m][rr^T\otimes rr^T] \\ & +\sigma_w^2(k)\sigma_w^2(k+m)[rr^T\otimes rr^T][I\otimes\Phi^m]^T \end{aligned} \tag{G.17}$$

When $m = 0$.

$$\begin{aligned}
& E\{x(k+1)x^T(k+m+1) \otimes x(k+m+1)x^T(k+m+1)\} \\
= \ & G(k;0) \\
& +[I \otimes \Phi][E\{rw(k)x^T(k+m) \otimes x(k+m)r^T w(k+m)\}][\Phi \otimes I]^T \\
& +[E\{rw(k)x^T(k+m) \otimes rw(k+m)x^T(k+m)\}][\Phi \otimes \Phi]^T \\
& +[I \otimes \Phi][E\{w(k)w(k+m)rr^T \otimes x(k+m)x^T(k+m)\}][I \otimes \Phi]^T \\
& +[E\{w(k)w(k+m)rr^T \otimes w^2(k+m)rr^T\}] \qquad (G.18) \\
= \ & G(k;0) \\
& +\sigma_w^2(k)[I \otimes \Phi][r \otimes I][E\{x^T(k) \otimes x(k)\}][I \otimes r^T][\Phi \otimes I]^T \\
& +\sigma_w^2(k)[r \otimes r][E\{x(k) \otimes x(k)\}]^T[\Phi \otimes \Phi]^T \\
& +\sigma_w^2(k)[I \otimes \Phi][rr^T \otimes E\{x(k)x^T(k)\}][I \otimes \Phi]^T \\
& +\gamma_{4w}(k)[rr^T \otimes rr^T]
\end{aligned}$$

When $m < 0$

$$\begin{aligned}
& E\{x(k+1)x^T(k+m+1) \otimes x(k+m+1)x^T(k+m+1)\} \\
= \ & G(k;m) \\
& +[\Phi \otimes \Phi][E\{x(k)x^T(k+m) \otimes x(k+m)r^T w(k+m)\}][\Phi \otimes I]^T \\
& +[\Phi \otimes I][E\{x(k)x^T(k+m) \otimes rw(k+m)x^T(k+m)\}][\Phi \otimes \Phi]^T \\
& +[\Phi \otimes \Phi][E\{x(k)r^T w(k+m) \otimes x(k+m)x^T(k+m)\}][I \otimes \Phi]^T \\
& +[\Phi \otimes I][E\{x(k)r^T w(k+m) \otimes w^2(k+m)rr^T\}] \qquad (G.19) \\
= \ & G(k;m) \\
& +\sigma_w^2(k+m)[\Phi^{-m} \otimes \Phi][r \otimes I]E\{x^T(k+m) \otimes x(k+m)\}[I \otimes r^T][\Phi \otimes I]^T \\
& +[\Phi^{-m} \otimes I][r \otimes r][E\{x(k+m) \otimes x(k+m)\}]^T[\Phi \otimes \Phi]^T \\
& +[\Phi^{-m} \otimes \Phi][\sigma_w^2(k+m)rr^T \otimes E\{x(k+m)x^T(k+m)\}][I \otimes \Phi]^T \\
& +\gamma_{4w}(k+m)[\Phi^{-m} \otimes I][rr^T \otimes rr^T]
\end{aligned}$$

Next we analyze the remaining three terms of Eq.(G.3). From Eq.(G.1) we obtain

$$E\{x(k+1)x^T(k+m+1)\} = \Phi E\{x(k)x^T(k+m)\}\Phi^T + H_1(k;m) \qquad (G.20)$$

where

$$H_1(k;m) = \begin{cases} \sigma_w^2(k)rr^T(\Phi^T)^m & \text{when } m > 0 \\ \sigma_w^2(k)rr^T & \text{when } m = 0 \\ \sigma_w^2(k+m)\Phi^{-m}rr^T & \text{when } m < 0 \end{cases}$$

From Eq.(F.4), we have

$$E\{x(k+m+1)x^T(k+m+1)\} = \Phi E\{x(k+m)x^T(k+m)\}\Phi^T + \sigma_w^2(k+m)rr^T \qquad (G.21)$$

Combining Eqs.(G.20) and (G.21) together, we have the following formulas:

When $m > 0$:

$$\begin{aligned}
& E\{x(k+1)x^T(k+m+1)\} \otimes E\{x(k+m+1)x^T(k+m+1)\} \\
= \; & [\Phi \otimes \Phi][E\{x(k)x^T(k+m)\} \otimes E\{x(k+m)x^T(k+m)\}][\Phi \otimes \Phi]^T \\
& + \sigma_w^2(k+m)[\Phi \otimes I][E\{x(k)x^T(k+m)\} \otimes rr^T][\Phi \otimes I]^T \\
& + \sigma_w^2(k)[I \otimes \Phi][rr^T \otimes E\{x(k+m)x^T(k+m)\}][\Phi^m \otimes \Phi]^T \\
& + \sigma_w^2(k)\sigma_w^2(k+m)[rr^T \otimes rr^T][\Phi^m \otimes I]^T
\end{aligned} \tag{G.22}$$

When $m = 0$:

$$\begin{aligned}
& E\{x(k+1)x^T(k+m+1)\} \otimes E\{x(k+m+1)x^T(k+m+1)\} \\
= \; & [\Phi \otimes \Phi][E\{x(k)x^T(k)\} \otimes E\{x(k)x^T(k)\}][\Phi \otimes \Phi]^T \\
& + \sigma_w^2(k)[\Phi \otimes I][E\{x(k)x^T(k)\} \otimes rr^T][\Phi \otimes I]^T \\
& + \sigma_w^2(k)[I \otimes \Phi][rr^T \otimes E\{x(k)x^T(k)\}][I \otimes \Phi]^T \\
& + [\sigma_w^2(k)]^2[rr^T \otimes rr^T]
\end{aligned} \tag{G.23}$$

When $m < 0$:

$$\begin{aligned}
& E\{x(k+1)x^T(k+m+1)\} \otimes E\{x(k+m+1)x^T(k+m+1)\} \\
= \; & [\Phi \otimes \Phi][E\{x(k)x^T(k+m)\} \otimes E\{x(k+m)x^T(k+m)\}][\Phi \otimes \Phi]^T \\
& + \sigma_w^2(k+m)[\Phi \otimes I][E\{x(k)x^T(k+m)\} \otimes rr^T][\Phi \otimes I]^T \\
& + \sigma_w^2(k+m)[\Phi^{-m} \otimes \Phi][rr^T \otimes E\{x(k+m)x^T(k+m)\}][I \otimes \Phi]^T \\
& + [\sigma_w^2(k+m)]^2[\Phi^{-m} \otimes I][rr^T \otimes rr^T]
\end{aligned} \tag{G.24}$$

Next, from Eq.(G.2), we find that

$$\begin{aligned}
E\{x(k+1) \otimes x(k+m+1)\} \; = \; & [\Phi \otimes \Phi]E\{x(k) \otimes x(k+m)\} \\
& + H2(k;m)
\end{aligned} \tag{G.25}$$

where

$$H2(k;m) = \begin{cases} \sigma_w^2(k)[I \otimes \Phi^m][r \otimes r] & \text{when } m > 0 \\ \sigma_w^2(k)[r \otimes r] & \text{when } m = 0 \\ \sigma_w^2(k+m)[\Phi^{-m} \otimes I][r \otimes r] & \text{when } m < 0 \end{cases}$$

From Eq.(F.4), we have

$$E\{x(k+m+1) \otimes x(k+m+1)\} = [\Phi \otimes \Phi]E\{x(k+m) \otimes x(k+m)\} + \sigma_w^2(k+m)[r \otimes r] \tag{G.26}$$

Combining Eqs.(G.25) and (G.26), we obtain the following representation for the third term of Eq.(G.3):

When $m > 0$:

$$\begin{aligned}
& [E\{x(k+1)\otimes x(k+m+1)\}][E\{x(k+m+1)\otimes x(k+m+1)\}]^T \\
= \; & [\Phi\otimes\Phi][E\{x(k)\otimes x(k+m)\}][E\{x(k+m)\otimes x(k+m)\}]^T[\Phi\otimes\Phi]^T \\
& +\sigma_w^2(k+m)[\Phi\otimes\Phi][E\{x(k)\otimes x(k+m)\}][r\otimes r]^T \\
& +\sigma_w^2(k)[I\otimes\Phi^m][r\otimes r][E\{x(k+m)\otimes x(k+m)\}]^T[\Phi\otimes\Phi]^T \\
& +\sigma_w^2(k)\sigma_w^2(k+m)[I\otimes\Phi^m][r\otimes r][r\otimes r]^T
\end{aligned} \tag{G.27}$$

When $m = 0$:

$$\begin{aligned}
& [E\{x(k+1)\otimes x(k+m+1)\}][E\{x(k+m+1)\otimes x(k+m+1)\}]^T \\
= \; & [\Phi\otimes\Phi][E\{x(k)\otimes x(k)\}][E\{x(k)\otimes x(k)\}]^T[\Phi\otimes\Phi]^T \\
& +\sigma_w^2(k)[\Phi\otimes\Phi][E\{x(k)\otimes x(k)\}][r\otimes r]^T \\
& +\sigma_w^2(k)[r\otimes r][E\{x(k)\otimes x(k)\}]^T[\Phi\otimes\Phi]^T \\
& +[\sigma_w^2(k)]^2[r\otimes r][r\otimes r]^T
\end{aligned} \tag{G.28}$$

When $m < 0$:

$$\begin{aligned}
& [E\{x(k+1)\otimes x(k+m+1)\}][E\{x(k+m+1)\otimes x(k+m+1)\}]^T \\
= \; & [\Phi\otimes\Phi][E\{x(k)\otimes x(k+m)\}][E\{x(k+m)\otimes x(k+m)\}]^T[\Phi\otimes\Phi]^T \\
& +\sigma_w^2(k+m)[\Phi\otimes\Phi][E\{x(k)\otimes x(k+m)\}][r\otimes r]^T \\
& +\sigma_w^2(k+m)[\Phi^{-m}\otimes I][r\otimes r][E\{x(k+m)\otimes x(k+m)\}]^T[\Phi\otimes\Phi]^T \\
& +[\sigma_w^2(k+m)]^2[\Phi^{-m}\otimes I][r\otimes r][r\otimes r]^T
\end{aligned} \tag{G.29}$$

Let $Z = [z^T \otimes z]$, where z is a deterministic vector with dimension equal to $x(k)$. Then,

$$\begin{aligned}
& E\{[x(k+1)\otimes I]Z[I\otimes x^T(k+m+1)]\} \\
= \; & E\{[x(k+1)z^T\otimes zx^T(k+m+1)]\} \\
= \; & [\Phi\otimes I][E\{x(k)z^T\otimes zx^T(k+m)\}][I\otimes\Phi]^T \\
& +[\Phi\otimes I][E\{x(k)z^T\otimes zr^T w(k+m)\}] \\
& +[E\{rw(k)z^T\otimes zx^T(k+m)\}][I\otimes\Phi]^T \\
& +E\{rw(k)z^T\otimes zr^T w(k+m)\} \\
= \; & [\Phi\otimes I]E\{[x(k)\otimes I]Z[Ix^T(k+m)]\}[I\otimes\Phi]^T \\
& +H_3(k;m)
\end{aligned} \tag{G.30}$$

where

$$H_3(k;m) = \begin{cases} \sigma_w^2(k)[rz^T\otimes zr^T][I\otimes\Phi^m]^T & \text{when } m>0 \\ \sigma_w^2(k)[rz^T\otimes zr^T] & \text{when } m=0 \\ \sigma_w^2(k)[\Phi^{-m}\otimes I][E\{rz^T\otimes zr^T] & \text{when } m<0 \end{cases}$$

i.e.

$$H_3(k;m) = \begin{cases} \sigma_w^2(k)[r \otimes I]Z[I \otimes r]^T[I \otimes \Phi^m]^T & \text{when } m > 0 \\ \sigma_w^2(k)[r \otimes I]Z[I \otimes r]^T & \text{when } m = 0 \\ \sigma_w^2(k+m)[\Phi^{-m} \otimes I][r \otimes I]Z[I \otimes r]^T & \text{when } m < 0 \end{cases}$$

Additionally, from Eq.(F.5), we have

$$E\{x^T(k+m+1) \otimes x^T(k+m+1)\} = \Phi E\{x^T(k+m) \otimes x(k+m)\}\Phi^T + \sigma_w^2(k+m)[r^T \otimes r] \qquad (G.31)$$

Using Eqs.(F.18) and (F.19), and letting Z in Eq.(G.30) equal the right-hand-side of Eq.(G.31), we obtain the following expansion of the fourth term of Eq.(G.3):

When $m > 0$

$$\begin{aligned} & E\{[x(k+1) \otimes I][E\{x^T(k+m+1) \otimes x(k+m+1)\}][I \otimes x^T(k+m+1)]\} \\ = \; & [\Phi \otimes I]E\{[x(k) \otimes I]\Phi E\{x^T(k+m) \otimes x(k+m)\}\Phi^T[Ix^T(k+m)]\}[I \otimes \Phi]^T \\ & +[\Phi \otimes I]E\{[x(k) \otimes I]\sigma_w^2(k+m)[r^T \otimes r][Ix^T(k+m)]\}[I \otimes \Phi]^T \\ & +\sigma_w^2(k)[r \otimes I]\Phi E\{x^T(k+m) \otimes x(k+m)\}\Phi^T[I \otimes r]^T[I \otimes \Phi^m]^T \\ & +\sigma_w^2(k)[r \otimes I]\sigma_w^2(k+m)[r^T \otimes r][I \otimes r]^T[I \otimes \Phi^m]^T \qquad (G.32) \\ = \; & [\Phi \otimes \Phi]E\{[x(k) \otimes I]E\{x^T(k+m) \otimes x(k+m)\}[Ix^T(k+m)]\}[\Phi \otimes \Phi]^T \\ & +\sigma_w^2(k+m)[\Phi \otimes I]E\{[x(k) \otimes I][r^T \otimes r][Ix^T(k+m)]\}[I \otimes \Phi]^T \\ & +\sigma_w^2(k)[I \otimes \Phi][r \otimes I]E\{x^T(k+m) \otimes x(k+m)\}[I \otimes r]^T[\Phi \otimes \Phi^m]^T \\ & +\sigma_w^2(k)\sigma_w^2(k+m)[r \otimes I][r^T \otimes r][I \otimes r]^T[I \otimes \Phi^m]^T \end{aligned}$$

When $m = 0$

$$\begin{aligned} & E\{[x(k+1) \otimes I][E\{x^T(k+m+1) \otimes x(k+m+1)\}][I \otimes x^T(k+m+1)]\} \\ = \; & [\Phi \otimes I]E\{[x(k) \otimes I]\Phi E\{x^T(k+m) \otimes x(k+m)\}\Phi^T[Ix^T(k+m)]\}[I \otimes \Phi]^T \\ & +[\Phi \otimes I]E\{[x(k) \otimes I]\sigma_w^2(k+m)[r^T \otimes r][Ix^T(k+m)]\}[I \otimes \Phi]^T \\ & +\sigma_w^2(k)[r \otimes I]\Phi E\{x^T(k+m) \otimes x(k+m)\}\Phi^T[I \otimes r]^T \\ & +\sigma_w^2(k)[r \otimes I]\sigma_w^2(k+m)[r^T \otimes r][I \otimes r]^T \qquad (G.33) \\ = \; & [\Phi \otimes \Phi]E\{[x(k) \otimes I]E\{x^T(k+m) \otimes x(k+m)\}[Ix^T(k+m)]\}[\Phi \otimes \Phi]^T \\ & +\sigma_w^2(k+m)[\Phi \otimes I]E\{[x(k) \otimes I][r^T \otimes r][Ix^T(k+m)]\}[I \otimes \Phi]^T \\ & +\sigma_w^2(k)[I \otimes \Phi][r \otimes I]E\{x^T(k+m) \otimes x(k+m)\}[I \otimes r]^T[I \otimes \Phi]^T \\ & +[\sigma_w^2(k)]^2[r \otimes I][r^T \otimes r][I \otimes r]^T \end{aligned}$$

When $m < 0$

$$
\begin{aligned}
& E\{[x(k+1)\otimes I][E\{x^T(k+m+1)\otimes x(k+m+1)\}][I\otimes x^T(k+m+1)]\} \\
= \; & [\Phi\otimes I]E\{[x(k)\otimes I]\Phi E\{x^T(k+m)\otimes x(k+m)\}\Phi^T[Ix^T(k+m)]\}[I\otimes\Phi]^T \\
& +[\Phi\otimes I]E\{[x(k)\otimes I]\sigma_w^2(k+m)[r^T\otimes r][Ix^T(k+m)]\}[I\otimes\Phi]^T \\
& +\sigma_w^2(k+m)[\Phi^{-m}\otimes I][r\otimes I]\Phi E\{x^T(k+m)\otimes x(k+m)\}\Phi^T[I\otimes r]^T \\
& +\sigma_w^2(k+m)[\Phi^{-m}\otimes I][r\otimes I]\sigma_w^2(k+m)[r^T\otimes r][I\otimes r]^T \qquad (G.34) \\
= \; & [\Phi\otimes\Phi]E\{[x(k)\otimes I]E\{x^T(k+m)\otimes x(k+m)\}[Ix^T(k+m)]\}[\Phi\otimes\Phi]^T \\
& +\sigma_w^2(k+m)[\Phi\otimes I]E\{[x(k)\otimes I][r^T\otimes r][Ix^T(k+m)]\}[I\otimes\Phi]^T \\
& +\sigma_w^2(k+m)[\Phi^{-m}\otimes\Phi][r\otimes I]E\{x^T(k+m)\otimes x(k+m)\}[I\otimes r]^T[I\otimes\Phi]^T \\
& +[\sigma_w^2(k+m)]^2[\Phi^{-m}\otimes I][r\otimes I][r^T\otimes r][I\otimes r]^T
\end{aligned}
$$

Combining Eqs.(G.17), (G.22), (G.27) and (G.32), we find the following expansion of $C_{4x}(k+1;m)$; for $m > 0$:

$$
\begin{aligned}
C_{4x}(k+1;m) \; = \; & [\Phi\otimes\Phi][E\{x(k)x^T(k+m)\otimes x(k+m)x^T(k+m)\}][\Phi\otimes\Phi]^T \\
& +[\Phi\otimes I][E\{x(k)x^T(k+m)\}\otimes\sigma_w^2(k+m)rr^T][\Phi\otimes I]^T \\
& +[\Phi\otimes\Phi][E\{x(k)\otimes x(k+m)\}][\sigma_w^2(k+m)r^T\otimes r^T] \\
& +[\Phi\otimes I][E\{[x(k)\otimes I][\sigma_w^2(k+m)r^T\otimes r][I\otimes x^T(k+m)]\}][I\otimes\Phi]^T \\
& +\sigma_w^2(k)[I\otimes\Phi][rr^T\otimes E\{x(k+m)x^T(k+m)\}][\Phi^m\otimes\Phi]^T \\
& +\sigma_w^2(k)[I\otimes\Phi^m]E\{[r\otimes r][x(k+m)\otimes x(k+m)]^T\}[\Phi\otimes\Phi]^T \\
& +\sigma_w^2(k)[I\otimes\Phi]E\{[r\otimes I][x^T(k+m)\otimes x(k+m)][I\otimes r^T][\Phi\otimes\Phi^m]^T\} \\
& -3[\sigma_w^2(k)]^2[I\otimes\Phi^{m-1}][rr^T\otimes rr^T][\Phi^{m-1}\otimes\Phi^{m-1}]^T \\
& +\gamma_w(k)[I\otimes\Phi^m][rr^T\otimes rr^T][\Phi^m\otimes\Phi^m]^T \\
& +\sigma_w^2(k)\sigma_w^2(k+m)[rr^T\otimes rr^T][\Phi^m\otimes I]^T \\
& +\sigma_w^2(k)\sigma_w^2(k+m)[I\otimes\Phi^m][rr^T\otimes rr^T] \\
& +\sigma_w^2(k)\sigma_w^2(k+m)[rr^T\otimes rr^T][I\otimes\Phi^m]^T \\
& -[\Phi\otimes\Phi][E\{x(k)x^T(k+m)\}\otimes E\{x(k+m)x^T(k+m)\}][\Phi\otimes\Phi]^T \\
& -\sigma_w^2(k+m)[\Phi\otimes I][E\{x(k)x^T(k+m)\}\otimes rr^T][\Phi\otimes I]^T \\
& -\sigma_w^2(k)[I\otimes\Phi][rr^T\otimes E\{x(k+m)x^T(k+m)\}][\Phi^m\otimes\Phi]^T \\
& -\sigma_w^2(k)\sigma_w^2(k+m)[rr^T\otimes rr^T][\Phi^m\otimes I]^T \\
& -[\Phi\otimes\Phi][E\{x(k)\otimes x(k+m)\}][E\{x(k+m)\otimes x(k+m)\}]^T[\Phi\otimes\Phi]^T \\
& -\sigma_w^2(k+m)[\Phi\otimes\Phi][E\{x(k)\otimes x(k+m)\}][r\otimes r]^T \\
& -\sigma_w^2(k)[I\otimes\Phi^m][r\otimes r][E\{x(k+m)\otimes x(k+m)\}]^T[\Phi\otimes\Phi]^T \\
& -\sigma_w^2(k)\sigma_w^2(k+m)[I\otimes\Phi^m][r\otimes r][r\otimes r]^T
\end{aligned}
$$

$$-[\Phi \otimes \Phi]E\{[x(k) \otimes I]E\{x^T(k+m) \otimes x(k+m)\}[Ix^T(k+m)]\}[\Phi \otimes \Phi]^T$$
$$-\sigma_w^2(k+m)[\Phi \otimes I]E\{[x(k) \otimes I][r^T \otimes r][Ix^T(k+m)]\}[I \otimes \Phi]^T$$
$$-\sigma_w^2(k)[I \otimes \Phi][r \otimes I]E\{x^T(k+m) \otimes x(k+m)\}[I \otimes r]^T[\Phi \otimes \Phi^m]^T$$
$$-\sigma_w^2(k)\sigma_w^2(k+m)[r \otimes I][r^T \otimes r][I \otimes r]^T[I \otimes \Phi^m]^T$$

After some cancelations, we obtain

$$\begin{aligned}
C_{4x}(k+1;m) &= [\Phi \otimes \Phi][E\{x(k)x^T(k+m) \otimes x(k+m)x^T(k+m)\}][\Phi \otimes \Phi]^T \\
&\quad -3[\sigma_w^2(k)]^2[I \otimes \Phi^m][rr^T \otimes rr^T][\Phi^m \otimes \Phi^m]^T \\
&\quad +\gamma_w(k)[I \otimes \Phi^m][rr^T \otimes rr^T][\Phi^m \otimes \Phi^m]^T \\
&\quad -[\Phi \otimes \Phi][E\{x(k)x^T(k+m)\} \otimes E\{x(k+m)x^T(k+m)\}][\Phi \otimes \Phi]^T \\
&\quad -[\Phi \otimes \Phi][E\{x(k) \otimes x(k+m)\}][E\{x(k+m) \otimes x(k+m)\}]^T[\Phi \otimes \Phi]^T \\
&\quad -[\Phi \otimes \Phi]E\{[x(k) \otimes I]E\{x^T(k+m) \otimes x(k+m)\}[I \otimes x^T(k+m)]\}[\Phi \otimes \Phi]^T
\end{aligned}$$

i.e.

$$\begin{aligned}
C_{4x}(k+1;m) &= [\Phi \otimes \Phi]C_{4x}(k;m)[\Phi \otimes \Phi]^T \\
&\quad +c_{4w}(k)[I \otimes \Phi^m][rr^T \otimes rr^T][\Phi^m \otimes \Phi^m]^T \\
&= [\Phi \otimes \Phi]C_{4x}(k;m)[\Phi \otimes \Phi]^T \\
&\quad +c_{4w}(k)[I \otimes \Phi]^m[rr^T \otimes rr^T][\Phi^T \otimes \Phi^T]^m
\end{aligned} \tag{G.35}$$

Combining Eqs.(G.18), (G.23), (G.28) and (G.33), we find the following expansion of $C_{4x}(k+1;m)$ for $m=0$:

$$\begin{aligned}
C_{4x}(k+1;0) &= [\Phi \otimes \Phi][E\{x(k)x^T(k) \otimes x(k)x^T(k)\}][\Phi \otimes \Phi]^T \\
&\quad +[\Phi \otimes I][E\{x(k)x^T(k)\} \otimes \sigma_w^2(k)rr^T][\Phi \otimes I]^T \\
&\quad +[\Phi \otimes \Phi][E\{x(k) \otimes x(k)\}][\sigma_w^2(k)r^T \otimes r^T] \\
&\quad +[\Phi \otimes I][E\{[x(k) \otimes I][\sigma_w^2(k)r^T \otimes r][I \otimes x^T(k)]\}][I \otimes \Phi]^T \\
&\quad +\sigma_w^2(k)[I \otimes \Phi][r \otimes I][E\{x^T(k) \otimes x(k)][I \otimes r^T][\Phi \otimes I]^T \\
&\quad +\sigma_w^2(k)[r \otimes r][E\{x(k) \otimes x(k)]^T[\Phi \otimes \Phi]^T \\
&\quad +\sigma_w^2(k)[I \otimes \Phi][rr^T \otimes E\{x(k)x^T(k)\}][I \otimes \Phi]^T \\
&\quad +\gamma_{4w}(k)[rr^T \otimes rr^T] \\
&\quad -[\Phi \otimes \Phi][E\{x(k)x^T(k)\} \otimes E\{x(k)x^T(k)\}][\Phi \otimes \Phi]^T \\
&\quad -\sigma_w^2(k)[\Phi \otimes I][E\{x(k)x^T(k)\} \otimes rr^T][\Phi \otimes I]^T \\
&\quad -\sigma_w^2(k)[I \otimes \Phi][rr^T \otimes E\{x(k)x^T(k)\}][I \otimes \Phi]^T \\
&\quad -[\sigma_w^2(k)]^2[rr^T \otimes rr^T] \\
&\quad -[\Phi \otimes \Phi][E\{x(k) \otimes x(k)\}][E\{x(k) \otimes x(k)\}]^T[\Phi \otimes \Phi]^T \\
&\quad -\sigma_w^2(k)[\Phi \otimes \Phi][E\{x(k) \otimes x(k)\}][r \otimes r]^T
\end{aligned}$$

$$-\sigma_w^2(k)[r \otimes r][E\{x(k) \otimes x(k)\}]^T[\Phi \otimes \Phi]^T$$
$$-[\sigma_w^2(k)]^2[r \otimes r][r \otimes r]^T$$
$$-[\Phi \otimes \Phi]E\{[x(k) \otimes I]E\{x^T(k) \otimes x(k)\}[Ix^T(k)]\}[\Phi \otimes \Phi]^T$$
$$-\sigma_w^2(k)[\Phi \otimes I]E\{[x(k) \otimes I][r^T \otimes r][Ix^T(k)]\}[I \otimes \Phi]^T$$
$$-\sigma_w^2(k)[I \otimes \Phi][r \otimes I]E\{x^T(k) \otimes x(k)\}[I \otimes r]^T[I \otimes \Phi]^T$$
$$-[\sigma_w^2(k)]^2[r \otimes I][r^T \otimes r][I \otimes r]^T$$

Rearranging all the terms, we obtain:

$$\begin{aligned} C_{4x}(k+1,0) &= [\Phi \otimes \Phi][E\{x(k)x^T(k) \otimes x(k)x^T(k)\}][\Phi \otimes \Phi]^T \\ &\quad +\gamma_{4w}(k)[rr^T \otimes rr^T] \\ &\quad -[\Phi \otimes \Phi][E\{x(k)x^T(k)\} \otimes E\{x(k)x^T(k)\}][\Phi \otimes \Phi]^T \\ &\quad -[\sigma_w^2(k)]^2[rr^T \otimes rr^T] \\ &\quad -[\Phi \otimes \Phi][E\{x(k) \otimes x(k)\}][E\{x(k) \otimes x(k)\}]^T[\Phi \otimes \Phi]^T \\ &\quad -[\sigma_w^2(k)]^2[r \otimes r][r \otimes r]^T \\ &\quad -[\Phi \otimes \Phi]E\{[x(k) \otimes I]E\{x^T(k) \otimes x(k)\}[Ix^T(k)]\}[\Phi \otimes \Phi]^T \\ &\quad -[\sigma_w^2(k)]^2[r \otimes I][r^T \otimes r][I \otimes r]^T \end{aligned}$$

i.e.

$$\begin{aligned} C_{4x}(k+1;0) &= [\Phi \otimes \Phi]C_{4x}(k;0)[\Phi \otimes \Phi]^T \\ &\quad +c_{4w}(k)[rr^T \otimes rr^T] \end{aligned} \qquad (G.36)$$

Finally, the representation of $C_{4x}(k+1;m)$ for $m < 0$ is obtained by combining Eqs.(G.19), (G.24), (G.29) and (G.34):

$$\begin{aligned} C_{4x}(k+1;m) &= [\Phi \otimes \Phi][E\{x(k)x^T(k+m) \otimes x(k+m)x^T(k+m)\}][\Phi \otimes \Phi]^T \\ &\quad +[\Phi \otimes I][E\{x(k)x^T(k+m)\} \otimes \sigma_w^2(k+m)rr^T][\Phi \otimes I]^T \\ &\quad +[\Phi \otimes \Phi][E\{x(k) \otimes x(k+m)\}][\sigma_w^2(k+m)r^T \otimes r^T] \\ &\quad +[\Phi \otimes I][E\{[x(k) \otimes I][\sigma_w^2(k+m)r^T \otimes r][I \otimes x^T(k+m)]\}][I \otimes \Phi]^T \\ &\quad +\sigma_w^2(k+m)[\Phi^{-m} \otimes \Phi][r \otimes I]E\{x^T(k+m) \otimes x(k+m)\}[I \otimes r^T][\Phi \otimes I]^T \\ &\quad +[\Phi^{-m} \otimes I][r \otimes r][E\{x(k+m) \otimes x(k+m)\}]^T[\Phi \otimes \Phi]^T \\ &\quad +[\Phi^{-m} \otimes \Phi][\sigma_w^2(k+m)rr^T \otimes E\{x(k+m)x^T(k+m)\}][I \otimes \Phi]^T \\ &\quad +\gamma_{4w}(k+m)[\Phi^{-m} \otimes I][rr^T \otimes rr^T] \\ &\quad -[\Phi \otimes \Phi][E\{x(k)x^T(k+m)\} \otimes E\{x(k+m)x^T(k+m)\}][\Phi \otimes \Phi]^T \\ &\quad -\sigma_w^2(k+m)[\Phi \otimes I][E\{x(k)x^T(k+m)\} \otimes rr^T][\Phi \otimes I]^T \\ &\quad -\sigma_w^2(k+m)[\Phi^{-m} \otimes \Phi][rr^T \otimes E\{x(k+m)x^T(k+m)\}][I \otimes \Phi]^T \\ &\quad -[\sigma_w^2(k+m)]^2[\Phi^{-m} \otimes I][rr^T \otimes rr^T] \\ &\quad -[\Phi \otimes \Phi][E\{x(k) \otimes x(k+m)\}][E\{x(k+m) \otimes x(k+m)\}]^T[\Phi \otimes \Phi]^T \\ &\quad -\sigma_w^2(k+m)[\Phi \otimes \Phi][E\{x(k) \otimes x(k+m)\}][r \otimes r]^T \end{aligned}$$

$$
\begin{aligned}
&-\sigma_w^2(k+m)[\Phi^{-m}\otimes I][r\otimes r][E\{x(k+m)\otimes x(k+m)\}]^T[\Phi\otimes\Phi]^T \\
&-[\sigma_w^2(k+m)]^2[\Phi^{-m}\otimes I][r\otimes r][r\otimes r]^T \\
&-[\Phi\otimes\Phi]E\{[x(k)\otimes I]E\{x^T(k+m)\otimes x(k+m)\}[Ix^T(k+m)]\}[\Phi\otimes\Phi]^T \\
&-\sigma_w^2(k+m)[\Phi\otimes I]E\{[x(k)\otimes I][r^T\otimes r][Ix^T(k+m)]\}[I\otimes\Phi]^T \\
&-\sigma_w^2(k+m)[\Phi^{-m}\otimes\Phi][r\otimes I]E\{x^T(k+m)\otimes x(k+m)\}[I\otimes r]^T[I\otimes\Phi]^T \\
&-[\sigma_w^2(k+m)]^2[\Phi^{-m}\otimes I][r\otimes I][r^T\otimes r][I\otimes r]^T
\end{aligned}
$$

Combining terms, we obtain

$$
\begin{aligned}
C_{4x}(k+1,m) \;=\; & [\Phi\otimes\Phi][E\{x(k)x^T(k+m)\otimes x(k+m)x^T(k+m)\}][\Phi\otimes\Phi]^T \\
&+\gamma_{4w}(k+m)[\Phi^{-m}\otimes I][rr^T\otimes rr^T] \\
&-[\Phi\otimes\Phi][E\{x(k)x^T(k+m)\}\otimes E\{x(k+m)x^T(k+m)\}][\Phi\otimes\Phi]^T \\
&-[\sigma_w^2(k+m)]^2[\Phi^{-m}\otimes I][rr^T\otimes rr^T] \\
&-[\Phi\otimes\Phi][E\{x(k)\otimes x(k+m)\}][E\{x(k+m)\otimes x(k+m)\}]^T[\Phi\otimes\Phi]^T \\
&-[\sigma_w^2(k+m)]^2[\Phi^{-m}\otimes I][r\otimes r][r\otimes r]^T \\
&-[\Phi\otimes\Phi]E\{[x(k)\otimes I]E\{x^T(k+m)\otimes x(k+m)\}[Ix^T(k+m)]\}[\Phi\otimes\Phi]^T \\
&-[\sigma_w^2(k+m)]^2[\Phi^{-m}\otimes I][r\otimes I][r^T\otimes r][I\otimes r]^T
\end{aligned}
$$

i.e.

$$
\begin{aligned}
C_{4x}(k+1;m) \;=\; & [\Phi\otimes\Phi]C_{4x}(k;m)[\Phi\otimes\Phi]^T \\
&+c_{4w}(k+m)[\Phi\otimes I]^{-m}[rr^T\otimes rr^T]
\end{aligned}
\qquad (G.37)
$$

Collecting Eqs.(G.35) , (G.36) and (G.37), we obtain Eqs.(32) and (33) in Proposition 5.

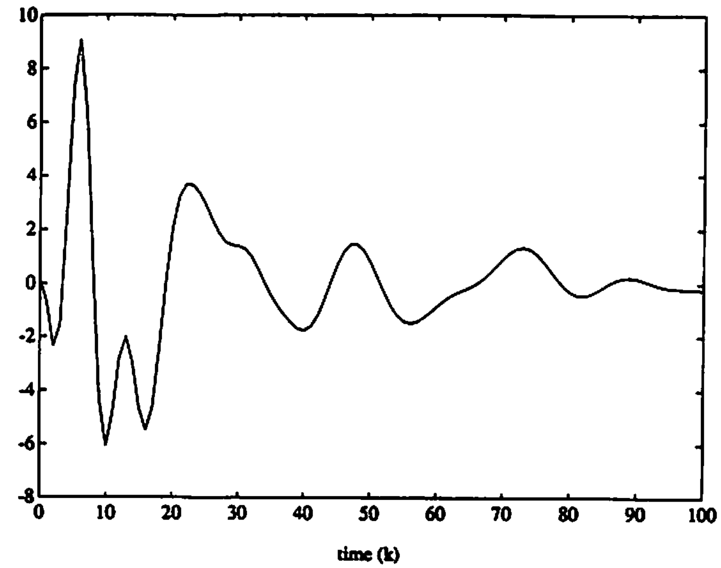

Figure 2. Impulse response of an air gun

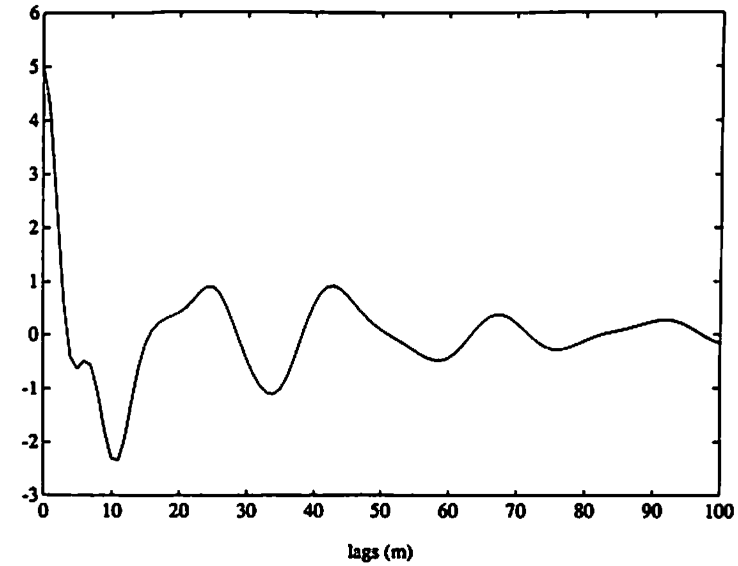

Figure 3. Correlation function

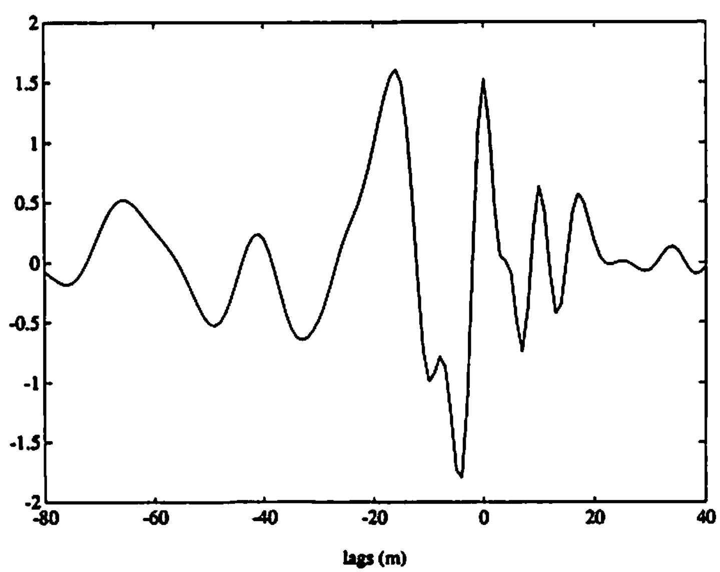

Figure 4. Third-order cumulants computed by recursive formulas

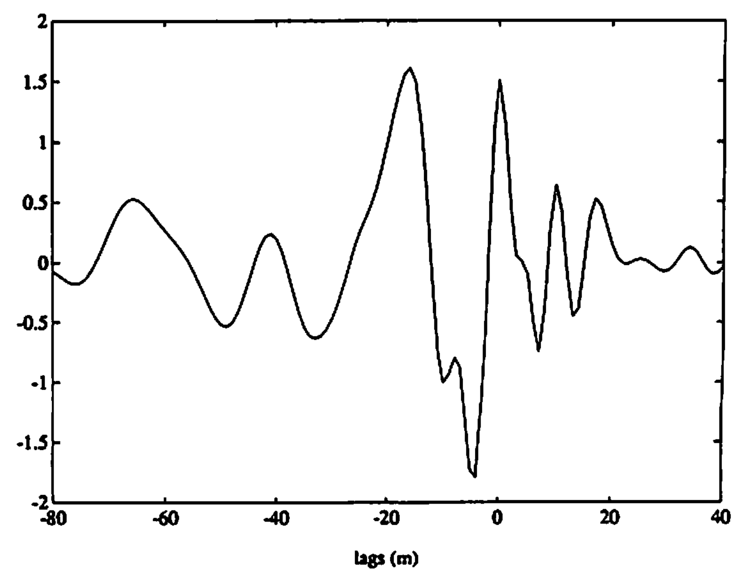

Figure 5. Third-order cumulants computed by impulse response method

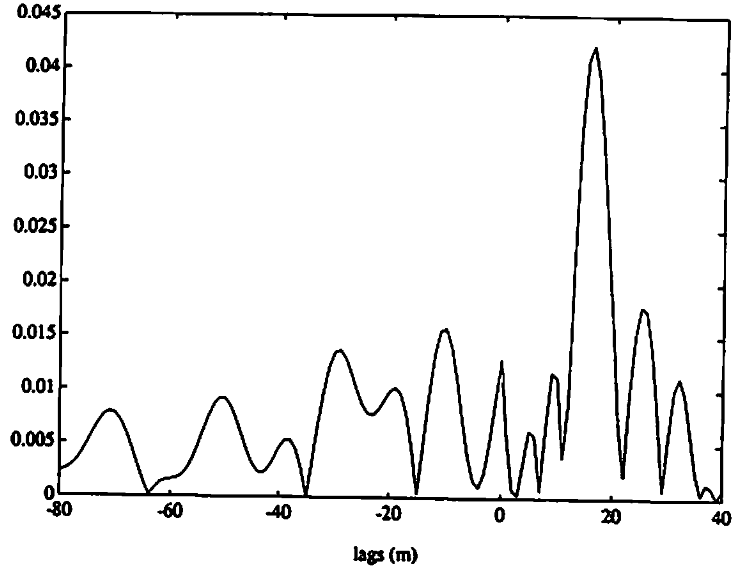

Figure 6. Difference between two third-order cumulant computations

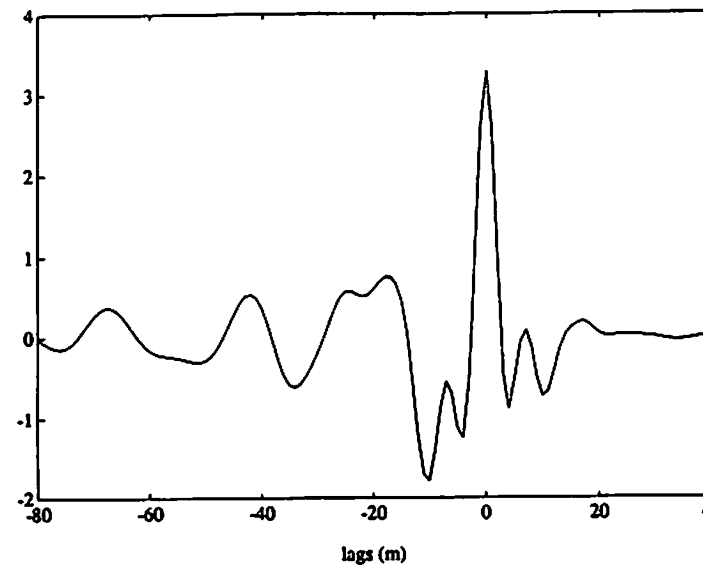

Figure 7. Fourth-order cumulants computed by recursive formulas

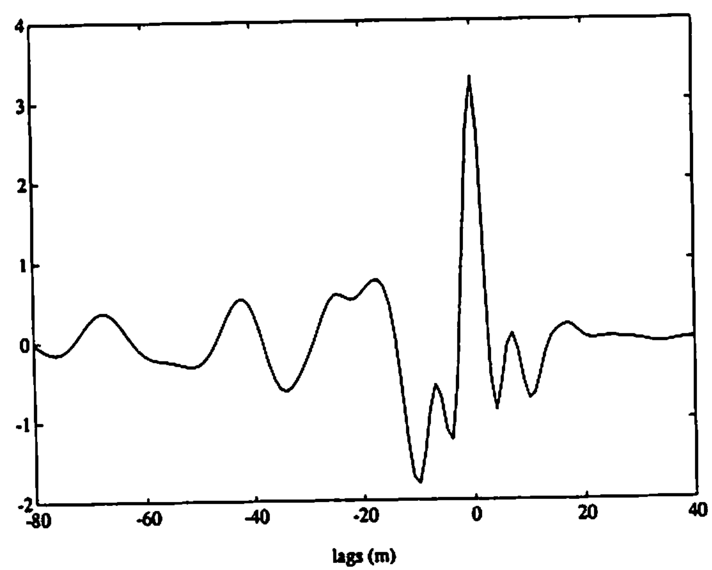

Figure 8. Fourth order cumulants computed by impulse response method

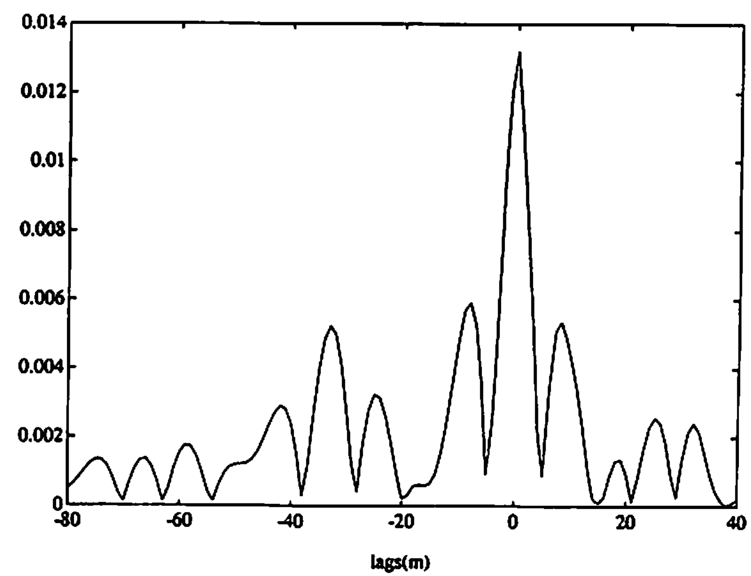

Figure 9. Difference between two fourth-order cumulant computations

Contents

Abstract

Recursive formulas for diagonal-slice cumulant computations of a state-variable model are developed. The computation formulas can be used for both stationary and non-stationary systems. The connection between stationary and non-stationary cumulant computations is also considered; it depends upon the assumption of an asympototic stable model. This report is divided into two parts.

In Part I, a matrix formulation for third-order diagonal-slice cumulant computation is discussed. An ARMA model is given as an example which shows computation aspects of the methods. Results are also verified by analysis of AR and MA models.

In Part II, the Kronecker product is used to derive third- and fourth-order cumulant recursive computation formulas. Vector representations for third- and fourth-order cumulants are given and elegant recursive cumulant computation formulas are obtained.

Examples and simulations are given in both parts.