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ON THE SPECTRUM OF A FAMILY OF PRECONDITIONED BLOCK TOEPLITZ MATRICES *

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Abstract. Research on preconditioning Toeplitz matrices with circulant matrices has been active recently. The preconditioning technique can be easily generalized to block Toeplitz matrices. That is, for a block Toeplitz matrix T consisting of $N \times N$ blocks with $M \times M$ elements per block, we can use a block circulant matrix R with the same block structure as its preconditioner. In this research, we examine the spectral clustering property of the preconditioned matrix $R^{-1}T$ with T generated by two-dimensional rational functions $T(z_x, z_y)$ of order (p_x, q_x, p_y, q_y). We show that the eigenvalues of $R^{-1}T$ are clustered around unity except at most $O(M\gamma_y + N\gamma_x)$ outliers, where $\gamma_x = \max(p_x, q_x)$ and $\gamma_y = \max(p_y, q_y)$. Furthermore, if T is separable, the outliers are clustered together such that $R^{-1}T$ has at most $(2\gamma_x + 1)(2\gamma_y + 1)$ asymptotic distinct eigenvalues. The superior convergence behavior of the preconditioned conjugate gradient (PCG) method over the conjugate gradient (CG) method is explained by a smaller condition number and a better clustering property of the spectrum of the preconditioned matrix $R^{-1}T$.

Key words. Block Toeplitz matrix, preconditioned conjugate gradient method

AMS(MOS) subject classifications. 65F10, 65F15

1. Introduction. The systems of linear equations associated with block Toeplitz matrices arise in many two-dimensional digital signal processing applications, such as linear prediction and estimation [8], [11], [12], image restoration [6] and the discretization of constant-coefficient partial differential equations. To solve the block Toeplitz system $T\mathbf{u} = \mathbf{b}$, where T is an $N \times N$-matrix with $M \times M$-blocks, by direct methods such as Levinson type algorithms requires $O(M^3N^2)$ operations [2], [13], [16]. Recently, there has been active research on the application of iterative methods such as the preconditioned conjugate gradient (PCG) method to the solution of Toeplitz systems. To accelerate the convergence rate, various preconditioners have been proposed for symmetric positive definite (SPD) Toeplitz matrices [5], [7], [9], [14]. The proposed preconditioning techniques can be easily generalized to block Toeplitz matrices. Simply speaking, we can construct the preconditioner with a block circulant matrix R which has the same block structure as T. Since both $R^{-1}\mathbf{w}$ and $T\mathbf{w}$, where $\mathbf{w}$ denotes an arbitray vector of length MN, can be performed with $O(MN \log MN)$ operations via two-dimensional fast Fourier transform (FFT), the computational complexity per PCG iteration is only $O(MN \log MN)$. The PCG method can be much more attractive than direct methods for solving block Toeplitz systems if it converges fast.

The convergence rate of the PCG method depends on the eigenvalue distribution of the preconditioned matrix $R^{-1}T$ [1]. Generally speaking, the PCG method converges faster if $R^{-1}T$ has clustered eigenvalues and/or a smaller condition number. The spectral properties of preconditioned point Toeplitz matrices have been extensively studied. Chan and Strang [3] [4] have proved that, for a Toeplitz matrix with a positive generating function in the Wiener class, the spectrum of the preconditioned matrix has eigenvalues clustered around unity except a finite number of outliers. If the Toeplitz matrix is generated by a positive *rational* function $A(z)/B(z) + A(z^{-1})/B(z^{-1})$ in the Wiener class, an even stronger result has been derived by Trefethen [15] and the authors [10]. That is, if $A(z)$ and $B(z)$ are polynominals in z of orders p and q without common roots, the number of outliers is equal to

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$2\max(p,q)$ and the PCG method converges in at most $2\max(p,q)+1$ iterations asymptotically. Therefore, an $N \times N$ preconditioned rational Toeplitz system can be solved with $\max(p,q) \times O(N \log N)$ operations.

The spectral properties of preconditioned block Toeplitz matrices have so far not yet been very well understood. In this research, we analyze the spectral *clustering* property for a class of preconditioned block Toeplitz matrices. The block Toeplitz matrices under our study has a two-dimensional quadrantal-symmetric generating sequence generated by a rational function in the Wiener class. We divide our discussion into two cases depending on whether the generating sequence is separable or not. When the block Toeplitz matrix has a separable generating sequence, the spectrum of the preconditioned block Toeplitz can be easily derived by using the preconditioned point Toeplitz result as given in [10]. However, we derive the preconditioned point Toeplitz result from a new viewpoint in this paper so that the same approach can be used for both separable and nonseparable cases. With this viewpoint, we interpret the operation $L\mathbf{w}$, where L is an $MN \times MN$ block matrix, as a two-dimensional constant-coefficient mask operating on a certain two-dimensional sequence constructed based on $\mathbf{w}$.

Our main results can be summarized as follows. Let T be an $MN \times MN$ block Toeplitz matrix generated by a rational function of order (p_x, q_x, p_y, q_y) (defined in §3), $\gamma_x = \max(p_x, q_x)$ and $\gamma_y = \max(p_y, q_y)$. For the separable generating sequence case, the eigenvalues of $R^{-1}T$ are clustered together so that it has asymptotically $(2\gamma_x+1)(2\gamma_y+1)$ distinct eigenvalues and the PCG method converges in at most $(2\gamma_x+1)(2\gamma_y+1)$ iterations. The complexity of the PCG method is therefore $O(MN \log MN)$. For the nonseparable generating sequence case, the eigenvalues of $R^{-1}T$ are clustered around unity except at most $O(M\gamma_y + N\gamma_x)$ outliers. Since the number of outliers is proportional to M and N, rather than being $O(1)$ as in the point Toeplitz case, the convergence rate of the PCG method cannot be completely characterized by the number of outliers. The condition number $\kappa(R^{-1}T)$ should also be taken into account. For example, when $T(z)$ is a positive function in the Wiener class, both $\kappa(T)$ and $\kappa(R^{-1}T)$ are $O(1)$. For this class of problems, the complexity of the PCG method is $O(MN \log MN)$. Then, the superior performance of the PCG method over the CG method is explained by a better spectral clustering property as well as a smaller condition number of the preconditioned matrix $R^{-1}T$.

The outline of this paper is as follows. The construction of the block circulant preconditioner R for block Toeplitz matrices T are presented in §2. In §3, we study the spectral clustering property of preconditioned block Toeplitz matrices $R^{-1}T$. Toeplitz matrices with separable and nonseparable generating sequences are examined, respectively, in §3.1 and §3.2. Numerical results are given in §4 to access the performance of the PCG method.

2. Construction of block Toeplitz preconditioners. Let T be a block Toeplitz matrix consisting of $N \times N$ blocks with $M \times M$ elements per block, which can be expressed as

$$T = \begin{bmatrix} T_0 & T_{-1} & \cdot & T_{2-N} & T_{1-N} \\ T_1 & T_0 & T_{-1} & \cdot & T_{2-N} \\ \cdot & T_1 & T_0 & \cdot & \cdot \\ T_{N-2} & \cdot & \cdot & \cdot & T_{-1} \\ T_{N-1} & T_{N-2} & \cdot & T_1 & T_0 \end{bmatrix}, \tag{2.1}$$

where T_n with $|n| \le N-1$ are $M \times M$ Toeplitz matrices with elements

$$[T_n]_{i,j} = t_{i-j,n} \quad \text{where} \quad 1 \le i,j \le M.$$

The $MN \times MN$ block Toeplitz matrix T is completely characterized by the two-dimensional sequence,

$$t_{m,n} \quad \text{where} \quad |m| \le M-1, \ |n| \le N-1, \tag{2.2}$$

known as the generating sequence of T. To construct the preconditioner for T, we generalize the idea in [4] [9] [14] and consider an $MN \times MN$ block circulant matrix of the form

$$R = \begin{bmatrix} R_0 & R_{N-1} & \cdot & R_2 & R_1 \\ R_1 & R_0 & R_{N-1} & \cdot & R_2 \\ \cdot & R_1 & R_0 & \cdot & \cdot \\ R_{N-2} & \cdot & \cdot & \cdot & R_{N-1} \\ R_{N-1} & R_{N-2} & \cdot & R_1 & R_0 \end{bmatrix}, \tag{2.3}$$

where R_n with $0 \leq n \leq N-1$ are $M \times M$ circulant matrices with elements

$$[R_n]_{i,j} = r_{(i-j) \bmod M, n} \quad \text{where} \quad 1 \leq i,j \leq M.$$

Thus, the block-circulant matrix R is completely characterized by the two-dimensional sequence,

$$r_{m,n} \quad \text{where} \quad 0 \leq m \leq M-1, \; 0 \leq n \leq N-1. \tag{2.4}$$

The construction of R based on T is described below.

In (2.1) and (2.3), linear operators T and R are expressed in matrix form. However, it is more convenient for our discussion to characterize T and R in terms of the relationship between input and output vectors. Consider two arbitrary MN-dimensional vectors $\mathbf{w}$ and $\mathbf{v}$ related via $\mathbf{v} = T\mathbf{w}$. By using the natural rowwise ordering, we can rearrange elements of these vectors into two-dimensional sequences

$$w_{m,n} \text{ and } v_{m,n} \quad \text{where} \quad 0 \leq m \leq M-1, \; 0 \leq n \leq N-1. \tag{2.5}$$

Then, $\mathbf{v} = T\mathbf{w}$ can be interpreted as a linear operator characterized by the two-dimensional mask

$$\begin{bmatrix} t_{M-1,-N+1} & t_{M-2,-N+1} & \cdots & t_{0,-N+1} & \cdots & t_{-M+2,-N+1} & t_{-M+1,-N+1} \\ t_{M-1,-N+2} & t_{M-2,-N+2} & \cdots & t_{0,-N+2} & \cdots & t_{-M+2,-N+2} & t_{-M+1,-N+2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ t_{M-1,0} & t_{M-2,0} & \cdots & t_{0,0} & \cdots & t_{-M+2,0} & t_{-M+1,0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ t_{M-1,N-2} & t_{M-2,N-2} & \cdots & t_{0,N-2} & \cdots & t_{-M+2,N-2} & t_{-M+1,N-2} \\ t_{M-1,N-1} & t_{M-2,N-1} & \cdots & t_{0,N-1} & \cdots & t_{-M+2,N-1} & t_{-M+1,N-1} \end{bmatrix} \tag{2.6}$$

operating on a extended sequence

$$\bar{w}_{m,n} = \begin{cases} w_{m,n}, & 0 \leq m \leq M-1, \; 0 \leq n \leq N-1, \\ 0, & \text{otherwise.} \end{cases} \tag{2.7}$$

To compute the output element v_{m_0,n_0}, $0 \leq m_0 \leq M-1$, $0 \leq n_0 \leq N-1$, we put the center of the mask (i.e. $t_{0,0}$) on $\bar{w}_{m_0,n_0}$, multiply $\bar{w}_{m,n}$ with the corresponding coefficients t_{m_0-m,n_0-n} and sum the resulting products. Now, let us use the mask (2.6) to operate on a periodic sequence

$$\tilde{w}_{m,n} = w_{m \bmod M, n \bmod N}, \qquad -\infty < m < \infty, \; -\infty < n < \infty. \tag{2.8}$$

This defines a block circulant matrix-vector product $R\mathbf{w}$, which is close to the operation $T\mathbf{w}$. Since $R^{-1}\mathbf{v}$ can be computed efficiently with two-dimensional fast Fourier transform (FFT), it is natural to use R as a preconditioner for T.

The above construction can be described in matrix notation. First, for every point Toeplitz matrix T_n, $|n| \leq N-1$, we construct a circulant preconditioner $\mathcal{R}_n$ with

$$t_{0,n}, \; t_{-1,n} + t_{M-1,n}, \; t_{-2,n} + t_{M-2,n}, \; \cdots, \; t_{1-M,n} + t_{1,n}. \tag{2.9}$$

as the first row [9]. Then, we use $\mathcal{R}_n$ to construct R_n according to the linear combination,

$$R_n = \begin{cases} \mathcal{R}_n, & n = 0, \\ \mathcal{R}_n + \mathcal{R}_{N-n}, & 1 \leq n \leq N-1. \end{cases} \tag{2.10}$$

It is worthwhile to point out that it is possible to design different preconditioners by considering different periodic extension to form $\tilde{w}_{m,n}$. For readers interested in the design of preconditioners, we refer to [9].

3. The spectral clustering property of preconditioned block Toeplitz matrices. Let us consider a family of block Toeplitz matrices T whose generating sequences $t_{m,n}$ are quadrantally symmetric,

$$t_{m,n} = t_{|m|,|n|} \quad |m| \le M-1, \ \ |n| \le N-1, \tag{3.1}$$

and absolutely summable (i.e. T is in the Wiener class),

$$\sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} |t_{m,n}| \le K < \infty, \tag{3.2}$$

and whose generating functions are of the form

$$\begin{aligned} T(z_x, z_y) &\equiv \sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} t_{i,j} z_x^{-i} z_y^{-j} \\ &= \frac{A(z_x, z_y)}{B(z_x, z_y)} + \frac{A(z_x^{-1}, z_y)}{B(z_x^{-1}, z_y)} + \frac{A(z_x, z_y^{-1})}{B(z_x, z_y^{-1})} + \frac{A(z_x^{-1}, z_y^{-1})}{B(z_x^{-1}, z_y^{-1})}, \end{aligned} \tag{3.3a}$$

where

$$A(z_x, z_y) = \sum_{i=0}^{p_x}\sum_{j=0}^{p_y} a_{i,j} z_x^{-i} z_y^{-j}, \qquad B(z_x, z_y) = \sum_{i=0}^{q_x}\sum_{j=0}^{q_y} b_{i,j} z_x^{-i} z_y^{-j}. \tag{3.3b}$$

We also assume that T has a nonsingular preconditioner R so that $R^{-1}T$ is well defined. We call T satisfying (3.1)-(3.3) the $MN \times MN$ block Toeplitz matrix generated by a quadrantally-symmetric rational function of order (p_x, q_x, p_y, q_y). For convenience, we use the notation

$$\gamma_x = \max(p_x, q_x), \qquad \gamma_y = \max(p_y, q_y).$$

The following discussion focuses on the spectral clustering property of the preconditioned matrix $R^{-1}T$, namely, a bound on the number of eigenvalues clustered around unity. Note that the positive-definiteness of T, R or $R^{-1}T$ is not our concern. Even though $R^{-1}T$ is indefinite, we can form a positive-definite matrix $(R^{-1}T)^2$ whose spectral clustering property can still be characterized by that of $R^{-1}T$.

3.1. Separable generating sequences. One special case of block Toeplitz matrices described by (3.1) - (3.3) is that $T(z_x, z_y)$ is separable, i.e.

$$T(z_x, z_y) = T_x(z_x) T_y(z_y), \tag{3.4a}$$

where

$$T_x(z_x) = \frac{A_x(z_x)}{B_x(z_x)} + \frac{A_x(z_x^{-1})}{B_x(z_x^{-1})}, \quad T_y(z_y) = \frac{A_y(z_y)}{B_y(z_y)} + \frac{A_y(z_y^{-1})}{B_y(z_y^{-1})}, \tag{3.4b}$$

and where

$$A_x(z_x) = \sum_{i=0}^{p_x} a_i z_x^{-i}, \ \ B_x(z_x) = \sum_{i=0}^{q_x} b_i z_x^{-i}, \ \ A_y(z_y) = \sum_{i=0}^{p_y} c_i z_y^{-i}, \ \ B_y(z_y) = \sum_{i=0}^{q_y} d_i z_y^{-i}. \tag{3.4c}$$

Note that the separability of $T(z_x, z_y)$ implies the separability of the generating sequence $t_{m,n}$, i.e.

$$t_{m,n} = t_{x,m} t_{y,n}.$$

Based on $t_{x,m}$ (or $t_{y,n}$), we can construct Toeplitz matrix T_x (or T_y) and the corresponding preconditioner R_x (or R_y), where T_x and R_x (or T_y and R_y) are of dimension $M \times M$ (or $N \times N$). It is easy to see that the preconditioner R is also separable, and the eigenvalues of $R^{-1}T$ are obtained from the products of the eigenvalues of $R_x^{-1}T_x$ and $R_y^{-1}T_y$. Thus, to understand the spectral properties of $R^{-1}T$, we only have to examine those of preconditioned (point) Toeplitz matrices $R_x^{-1}T_x$ and $R_y^{-1}T_y$.

According to the construction (2.9) and the symmetric property of $t_{x,m}$, we know that R_x is a circulant matrix with the first row

$$t_{x,0},\ t_{x,1}+t_{x,M-1},\ t_{x,2}+t_{x,M-2},\ \cdots,\ t_{x,M-1}+t_{x,1}.$$

When $B_x(z_x)=1$ (i.e. $q_x=0$), T_x is banded with bandwidth p_x, and R_x is almost the same as T_x except the addition of elements in the northeast and southwest corners to make R_x circulant. Thus, the elements of $\Delta T_x = R_x - T_x$ are all zeros except the first and the last p_x rows. Consequently, ΔT_x has at least $M-2p_x (= M-2\gamma_x)$ eigenvalues at zero and $R_x^{-1}T_x = (T_x+\Delta T_x)^{-1}T_x$ has at least $M-2p_x$ eigenvalues at one. This result can also be obtained by using the operator-mask interpretation. That is, the products $T_x\mathbf{w}$ and $R_x\mathbf{w}$, for arbitrary $\mathbf{w}=(w_0,\cdots,w_m,\cdots,w_{M-1})^T$, can be viewed a linear operator characterized by the mask

$$[t_{x,p_x}\ \ t_{x,p_x-1}\ \ \cdots\ \ t_{x,1}\ \ t_{x,0}\ \ t_{x,1}\ \ \cdots\ \ t_{x,p_x-1}\ \ t_{x,p_x}]$$

operating, respectively, on two extended sequences

$$\bar{w}_m = \begin{cases} w_m, & 0 \le m \le M-1, \\ 0, & \text{otherwise,} \end{cases} \qquad \text{and} \qquad \tilde{w}_m = w_{m \bmod M}.$$

It is clear that $T_x\mathbf{w}$ and $R_x\mathbf{w}$ give the same output elements if the center of the mask is located at $p_x \le m \le M-p_x-1$. There are $M-2p_x$ such elements and, as a consequence, the dimension of the null space of ΔT_x is at least $M-2p_x$. The operator-mask viewpoint will be generalized to the case of higher dimensional generating sequences (see §3.2).

When $B_x(z_x)\ne 1$, we approximate the block matrix $\Delta T_x = R_x - T_x$ with an asymptotically equivalent block matrix ΔE_x, and then use the recursive property of $t_{x,m}$ to show that ΔE_x has eigenvalues repeated at zero. The recursive property of $t_{x,m}$ is stated in the following lemma.

LEMMA 1. *The sequence $t_{x,m}$ generated by $T_x(z_x)$ in (3.4b) follows the recursion,*

$$t_{x,m+1} = -(b_1 t_{x,m} + b_2 t_{x,m-1} + \cdots + b_{q_x} t_{x,m-q_x+1})/b_0, \qquad m \ge \gamma_x = \max(p_x, q_x). \tag{3.5}$$

Proof. The generating sequence associated with $A_x(z_x)/B_x(z_x)$ given by (3.4c) is

$$\frac{1}{2}t_{x,0}, t_{x,1}, t_{x,2}, \cdots, t_{x,m}, \cdots.$$

Thus, we have

$$\left(\frac{t_{x,0}}{2} + \sum_{m=1}^{\infty} t_{x,m} z^{-m}\right)(b_0 + b_1 z^{-1} + \cdots + b_{q_x} z^{-q_x}) = a_0 + a_1 z^{-1} + \cdots + a_{p_x} z^{-p_x}.$$

The proof is completed by comparing the coefficients of the above equation. □

Consider the approximation of $\Delta T_x = R_x - T_x$ with

$$\Delta E_x = F_x + F_x^T,$$

where

$$F_x = \begin{bmatrix} t_M & t_{M-1} & \cdot & \cdot & t_2 & t_1 \\ t_{M+1} & t_M & t_{M-1} & \cdot & \cdot & t_2 \\ \cdot & t_{M+1} & t_M & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & t_{M-1} & \cdot \\ t_{2M-2} & \cdot & \cdot & t_{M+1} & t_M & t_{M-1} \\ t_{2M-1} & t_{2M-2} & \cdot & \cdot & t_{M+1} & t_M \end{bmatrix},$$

and where the subscript x is omitted for simplicity and t_m with $m > M-1$ is recursively constructed from (3.5). Since elements $t_{x,m}$ in F_x satisfies (3.5), the rank of F_x or F_x^T is at most γ_x. Consequently, ΔE_x has at least $M - 2\gamma_x$ eigenvalues at zero.

Then, we examine the difference between ΔT_x and ΔE_x. Consider

$$\mathbf{v} = (\Delta T_x - \Delta E_x)\mathbf{w}, \tag{3.6}$$

for nonzero $\mathbf{w}$. The mth, $1 \leq m \leq M$, elements of $\Delta T_x \mathbf{w}$ and $\Delta E_x \mathbf{w}$ can be written, respectively, as

$$[\Delta T_x \mathbf{w}]_m = [R_x \mathbf{w}]_m - [T_x \mathbf{w}]_m = \sum_{i=m+1}^{M} t_{x,M+m-i} w_i + \sum_{i=1}^{m-1} t_{x,M-m+i} w_i, \tag{3.7}$$

and

$$[\Delta E_x \mathbf{w}]_m = [F_x \mathbf{w}]_m + [F_x^T \mathbf{w}]_m = \sum_{i=1}^{M} t_{x,M+m-i} w_i + \sum_{i=1}^{M} t_{x,M-m+i} w_i. \tag{3.8}$$

Therefore, $\mathbf{v}$ is bounded above by

$$\begin{aligned} ||\mathbf{v}||_1 &= ||\Delta E_x \mathbf{w} - \Delta T_x \mathbf{w}||_1 \leq \max_{1 \leq m \leq M} |\sum_{i=1}^{m} t_{x,M+m-i} w_i + \sum_{i=m}^{M} t_{x,M-m+i} w_i| \\ &\leq 2 \sum_{m=M}^{2M-1} |t_{x,m}| ||\mathbf{w}||_1. \end{aligned}$$

We have

$$||\Delta T_x - \Delta E_x||_1 \equiv \max_{\mathbf{w}} \frac{||\mathbf{v}||_1}{||\mathbf{w}||_1} \leq 2 \sum_{m=M}^{2M-1} |t_{x,m}|, \tag{3.9}$$

which converges to zero as M goes to infinity, since $\sum_{m=0}^{\infty} |t_{x,m}|$ converges and $S_m = \sum_{m'=0}^{m} |t_{x,m'}|$ is a Cauchy sequence. Thus, ΔT_x is asymptotically equivalent to ΔE_x. It also follows that ΔT_x has at least $M - 2\gamma_x$ eigenvalues asymptotically converging to 0, or $R_x^{-1} T_x = (T_x + \Delta T_x)^{-1} T_x$ has at least $M - 2\gamma_x$ eigenvalues asymptotically converging to 1.

The ΔT_x and ΔE_x above are amenable to the operator-mask interpretation (see Fig. 1 with $M = 8$). One can easily verify that $\Delta T_x \mathbf{w}$ and $\Delta E_x \mathbf{w}$ correspond, respectively, to the use of the two masks

$$\begin{aligned} \Delta T_x &: [t_{x,M-1}\ t_{x,M-2}\ \cdots\ t_{x,1}\ t_{x,0}\ t_{x,1}\ \cdots\ t_{x,M-2}\ t_{x,M-1}], \\ \Delta E_x &: [\cdots\ t_{x,M}\ t_{x,M-1}\ \cdots\ t_{x,1}\ t_{x,0}\ t_{x,1}\ \cdots\ t_{x,M-1}\ t_{x,M}\ \cdots], \end{aligned}$$

operating on the same sequence

$$\breve{w}_m = \begin{cases} w_{m \bmod M}, & -M \leq m \leq -1 \text{ or } M \leq m \leq 2M-1, \\ 0, & \text{elswhere.} \end{cases}$$

Note that the mask for ΔE_x is of infinite length. The corresponding mask for $\Delta E_x - \Delta T_x$ is

$$\Delta E_x - \Delta T_x \ : \ [\cdots\ t_{x,M+1}\ t_{x,M}\ 0\ \cdots\ 0\ 0\ 0\ \cdots\ 0\ t_{x,M}\ t_{x,M+1}\ \cdots].$$

It is easy to derive (3.9) from the operator-mask viewpoint. Note that, for larger M, although there are more terms contributing to the 1-norm of $\Delta E_x - \Delta T_x$, the weighting coefficient $t_{x,m}$, $m \geq M$, decays more rapidly. The resulting 1-norm of $\Delta E_x - \Delta T_x$ asymptotically converges to zero.

We conclude the above discussion as follows.

(a) When $B(z_x, z_y) = 1$, T_x (or T_y) is banded and $R_x^{-1} T_x$ (or $R_y^{-1} T_y$) has at most $2p_x + 1$ (or $2p_y + 1$) distinct eigenvalues so that $R^{-1} T$ has at most $(2p_x + 1)(2p_y + 1)$ distinct eigenvalues.

(b) When $B(z_x, z_y) \neq 1$, $R_x^{-1}T_x$ (or $R_y^{-1}T_y$) has at most $2\gamma_x$ (or $2\gamma_y$) outliers not converging to unity and other eigenvalues are clustered between $(1-\epsilon_x, 1+\epsilon_x)$ (or $(1-\epsilon_y, 1+\epsilon_y)$). Thus, the eigenvalues of $R^{-1}T$ can be grouped into several clusters. The centers and clustering radii of these clusters, and the numbers of eigenvalues contained are listed in Table 1, where $\lambda_{x,i}$ (or $\lambda_{y,j}$) denotes a typical outlier for $R_x^{-1}T_x$ (or $R_y^{-1}T_y$). Since ϵ_x and ϵ_y converge to zero as M and N become large, $R^{-1}T$ has asymptotically at most $(2\gamma_x+1)(2\gamma_y+1)$ distinct eigenvalues.

Table 1. Eigenvalues of $R^{-1}T$

Center	$\lambda_{x,i}\lambda_{y,j}$	$\lambda_{x,i}$	$\lambda_{y,j}$	1
Radius	0	$O(\epsilon_y)$	$O(\epsilon_x)$	$O(\epsilon_x+\epsilon_y)$
Number	1	$N-2\gamma_y$	$M-2\gamma_x$	$(M-2\gamma_x)(N-2\gamma_y)$

As a consequence, the PCG method converges in at most $(2\gamma_x+1)(2\gamma_y+1)$ iterations for positive definite T with sufficiently large M and N for both cases (a) and (b). This is confirmed numerically in §4.

3.2. Nonseparable generating sequences. For nonseparable generating functions $T(z_x, z_y)$ given by (3.3), we examine two typical cases, i.e. $B(z_x, z_y) = 1$ and $B(z_x, z_y) \neq 1$ with $q_x > 0$ and $q_y > 0$.

When $B(z_x, z_y) = 1$, we have a corresponding generating sequence of finite duration. As described in §2, the products Tw and Rw correspond to a linear operator characterized by the mask

$$\begin{bmatrix} t_{p_x,p_y} & t_{p_x-1,p_y} & \cdots & t_{0,p_y} & \cdots & t_{p_x-1,p_y} & t_{p_x,p_y} \\ t_{p_x,p_y-1} & t_{p_x-1,p_y-1} & \cdots & t_{0,p_y-1} & \cdots & t_{p_x-1,p_y-1} & t_{p_x,p_y-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ t_{p_x,0} & t_{p_x-1,0} & \cdots & t_{0,0} & \cdots & t_{p_x-1,0} & t_{p_x,0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ t_{p_x,p_y-1} & t_{p_x-1,p_y-1} & \cdots & t_{0,p_y-1} & \cdots & t_{p_x-1,p_y-1} & t_{p_x,p_y-1} \\ t_{p_x,p_y} & t_{p_x-1,p_y} & \cdots & t_{0,p_y} & \cdots & t_{p_x-1,p_y} & t_{p_x,p_y} \end{bmatrix}$$

operating, respectively, on $\bar{w}_{m,n}$ and $\tilde{w}_{m,n}$ given by (2.7) and (2.8). The output elements of Tw and Rw are identical if the center of the mask is located at

$$p_x \le m \le M - p_x - 1, \qquad p_y \le n \le N - p_y - 1.$$

The dimension of the null space of $\Delta T = R - T$ is at least $(M-2p_x)(N-2p_y)$. Consequently, $R^{-1}T$ has at least $(M-2p_x)(N-2p_y)$ eigenvalues repeated at 1 or, equivalently, there are at most $2(Mp_y + Np_x) - 4p_xp_y$ outliers. This result is summarized in the following theorem.

THEOREM 1. *Let T be an $MN \times MN$ block-Toeplitz matrices characterized by (3.1) - (3.3) with $B(z_x, z_y) = 1$. The preconditioned matrix $R^{-1}T$ has at least $MN - 2(Mp_y + Np_x) + 4p_xp_y$ eigenvalues repeated at one.*

When $B(z_x, z_y) \neq 1$ with $q_x > 0$ and $q_y > 0$, the products Tw and Rw correspond to a linear operator characterized by the mask

$$\begin{bmatrix} t_{M-1,N-1} & t_{M-2,N-1} & \cdots & t_{0,N-1} & \cdots & t_{M-2,N-1} & t_{M-1,N-1} \\ t_{M-1,N-2} & t_{M-2,N-2} & \cdots & t_{0,N-2} & \cdots & t_{M-2,N-2} & t_{M-1,N-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ t_{M-1,0} & t_{M-2,0} & \cdots & t_{0,0} & \cdots & t_{M-2,0} & t_{M-1,0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ t_{M-1,N-2} & t_{M-2,N-2} & \cdots & t_{0,N-2} & \cdots & t_{M-2,N-2} & t_{M-1,N-2} \\ t_{M-1,N-1} & t_{M-2,N-1} & \cdots & t_{0,N-1} & \cdots & t_{M-2,N-1)} & t_{M-1,N-1} \end{bmatrix} \tag{3.10}$$

operating on $\bar{w}_{m,n}$ and $\tilde{w}_{m,n}$, respectively. Let us exploit the quadrantally symmetric property (3.1) and decompose $t_{m,n}$ into four sequences

$$t_{m,n} = \sum_{k=1}^{4} t_{k,m,n}$$

where $t_{k,m,n}$ is called the kth quadrant-support sequence and defined as

(3.11a)
$$t_{1,m,n} = \begin{cases} t_{0,0}/4, & m = n = 0, \\ t_{m,0}/2, & 1 \le m \le M-1,\ n = 0, \\ t_{0,n}/2, & m = 0,\ 1 \le n \le N-1, \\ t_{m,n}, & 1 \le m \le M-1 \text{ and } 1 \le n \le N-1, \\ 0, & m < 0 \text{ or } n < 0, \end{cases}$$

and

(3.11b)
$$t_{2,m,n} = t_{1,-m,n}, \quad t_{3,m,n} = t_{1,-m,-n}, \quad t_{4,m,n} = t_{1,m,-n}.$$

The following lemma is on the recursive property of $t_{1,m,n}$.

LEMMA 2. *Let $t_{m,n}$ be a quadrantal symmetric sequence generated by the two-dimensional rational function $T(z_x, z_y)$ given in (3.3), and $t_{1,m,n}$ is the first-quadrant support sequence defined by (3.11a). Then,*

(3.12)
$$\sum_{i=0}^{q_x}\sum_{j=0}^{q_y} b_{i,j} t_{1,m-i,n-j} = 0 \qquad \text{for} \qquad m > \gamma_x \ \text{ or } \ n > \gamma_y.$$

Proof. It is clear that $A(z_x, z_y)/B(z_x, z_y)$ is the generating function for $t_{1,m,n}$. Therefore,

$$\frac{A(z_x, z_y)}{B(z_x, z_y)} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} t_{1,i,j} z_x^{-i} z_y^{-j}.$$

We multiply both sides of the above equation with $B(z_x, z_y)$, substitute (3.3b) for $A(z_x, z_y)$ and $B(z_x, z_y)$, and compare the corresponding coefficients. This gives the desired equation (3.12). □
Thus, we can use (3.12) to recursively define $t_{1,m,n}$ with $m \ge M$ or $n \ge N$ in the first quadrant, and the corresponding $t_{k,m,n}$ with $k = 2, 3, 4$ can be obtained from $t_{1,m,n}$ through the symmetry (3.11b).

As a generalization of the one-dimensional case, we define

$$\breve{w}_{m,n} = \begin{cases} w_{m \bmod M, n \bmod N}, & (m,n) \in [\cup_{-1 \le i,j \le 1} Q_{i,j}] - Q_{0,0}, \\ 0, & \text{elswhere}, \end{cases}$$

where

$$Q_{i,j} = \{(m,n): \ iM \le m \le (i+1)M - 1, \ jN \le n \le (j+1)N - 1\}.$$

Then, the operation $\Delta T\mathrm{w} = (R - T)\mathrm{w}$ corresponds to the mask (3.10) operating on $\breve{w}_{m,n}$ We choose the approximation $\Delta E\mathrm{w}$ to be an extended infinite mask, with recursively defined $t_{m,n}$ via (3.12), operating on $\breve{w}_{m,n}$. This is illustrated in Fig. 2, where we only show the first quadrant of the mask for ΔE. For the rest of this subsection, we are concerned with two issues: (i) the asymptotic equivalence of ΔT and ΔE and (ii) the number of eigenvalues of ΔE repeated at zero.

The operation $(\Delta E - \Delta T)\mathrm{w}$ corresponds to the difference between the extended mask and the original mask (3.10) operating on $\breve{w}_{m,n}$. The 1-norm of the first quadrant of the difference mask $\Delta E - \Delta T$ operating on $\breve{w}_{m,n}$ in regions $Q_{1,0}$, $Q_{0,1}$ and $Q_{1,1}$ are bounded, respectively, by

$$K_{1,1,0} = \sum_{m=M}^{2M-1}\sum_{n=0}^{N-1} |t_{1,m,n}|, \quad K_{1,0,1} = \sum_{m=0}^{M-1}\sum_{n=N}^{2N-1} |t_{1,m,n}|, \quad K_{1,1,1} = \sum_{m=M}^{2M-1}\sum_{n=N}^{2N-1} |t_{1,m,n}|.$$

By exploiting the symmetry, we have the bound for $\Delta E - \Delta T$,

$$||\Delta E - \Delta T||_1 \le 4(K_{1,1,0} + K_{1,0,1} + K_{1,1,1}) = K.$$

With the assumption (3.2), we can order $t_{m,n}$ appropriately to be a Cauchy sequence and argue that K converges to zero for asymptotically large M and N. This establishes the asymptotic equivalence of ΔE and ΔT.

The operator ΔE can be expressed as a superposition of 12 operators,

$$\begin{aligned}\Delta E &= F_{1,1,0}+F_{1,1,1}+F_{1,0,1}+F_{2,0,1}+F_{2,-1,1}+F_{2,-1,0},\\ &+ F_{3,-1,0}+F_{3,-1,-1}+F_{3,0,-1}+F_{4,0,-1}+F_{4,1,-1}+F_{4,1,0},\end{aligned} \tag{3.13}$$

where $F_{k,i,j}$ denotes the $k$the quadrant of the extended mask operating on sequences defined on $Q_{i,j}$. Consider operators $F_{1,1,0}$, $F_{1,0,1}$ and $F_{1,1,1}$. Their operations on $\mathbf{w}$ can be written as

$$\begin{aligned} F_{1,1,0} &: \quad v_{1,i,j} = \sum_{m=0}^{M-1}\sum_{n=j}^{N-1} t_{1,M+m-i,n-j}w_{m,n},\\ F_{1,0,1} &: \quad v_{2,i,j} = \sum_{m=i}^{M-1}\sum_{n=0}^{N-1} t_{1,m-i,N+n-j}w_{m,n},\\ F_{1,1,1} &: \quad v_{3,i,j} = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} t_{1,M+m-i,N+n-j}w_{m,n}.\end{aligned}$$

With (3.12), we have

$$\sum_{i=i_0}^{i_0+q_x}\sum_{j=j_0}^{j_0+q_y} b_{i-i_0,j-j_0}v_{1,i,j} = \sum_{m=0}^{M-1}\sum_{n=j+j_0}^{N-1}\left[\sum_{i=0}^{q_x}\sum_{j=0}^{q_y} b_{i,j}t_{1,M+m-i_0-i,n-j_0-j}\right]w_{m,n}=0,$$

for $0\le i_0\le M-1-\gamma_x$,

$$\sum_{i=i_0}^{i_0+q_x}\sum_{j=j_0}^{j_0+q_y} b_{i-i_0,j-j_0}v_{2,i,j} = \sum_{m=i+i_0}^{M-1}\sum_{n=0}^{N-1}\left[\sum_{i=0}^{q_x}\sum_{j=0}^{q_y} b_{i,j}t_{1,m-i_0-i,N+n-j_0-j}\right]w_{m,n}=0,$$

for $0\le j_0\le N-1-\gamma_y$, and

$$\sum_{i=i_0}^{i_0+q_x}\sum_{j=j_0}^{j_0+q_y} b_{i-i_0,j-j_0}v_{3,i,j} = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1}\left[\sum_{i=0}^{q_x}\sum_{j=0}^{q_y} b_{i,j}t_{1,M+m-i_0-i,N+n-j_0-j}\right]w_{m,n}=0,$$

for $0\le i_0\le M-1-\gamma_x$ or $0\le j_0\le N-1-\gamma_y$.

By combining the above three equations, we have

$$\sum_{i=i_0}^{i_0+q_x}\sum_{j=j_0}^{j_0+q_y} b_{i-i_0,j-j_0}(v_{1,i,j}+v_{2,i,j}+v_{3,i,j})=0,$$

for $0\le i_0\le M-1-\gamma_x$ and $0\le j_0\le N-1-\gamma_y$. Therefore, the rank of $F_1=F_{1,1,0}+F_{1,0,1}+F_{1,1,1}$ is at most $M\gamma_y+N\gamma_x-\gamma_x\gamma_y$. By using the symmetry, we can argue that the rank of $F_k=\sum_{i,j}F_{k,i,j}$, $k=2,3,4$, is also at most $M\gamma_y+N\gamma_x-\gamma_x\gamma_y$. Consequently, the rank of ΔE is at most $4(M\gamma_y+N\gamma_x-\gamma_x\gamma_y)$ or, equivalently, ΔE has at least $MN-4(M\gamma_y+N\gamma_x-\gamma_x\gamma_y)$ eigenvalues repeated at 0.

To conclude this section, we have the following theorem.

THEOREM 2. *Let T be an $MN\times MN$ block-Toeplitz matrix satisfying (3.1), (3.2) and (3.3) with $q_x>0$ and $q_y>0$. Then, the preconditioned matrix $R^{-1}T$ has at least $MN-4(M\gamma_x+N\gamma_y-\gamma_x\gamma_y)$ eigenvalues asymptotically converging to one.*

4. Numerical experiments. Numerical experiments are performed to illustrate the spectra of T and $R^{-1}T$ and the convergence behavior of the CG and PCG methods. Note that the spectral clustering property derived in §3 does not require T to be positive definite. However, we focus on positive-definite T in our experiments so that the CG and PCG methods can be conveniently applied. For all test problems below, we choose $M = N$ and use $(0, \cdots, 0)^T$ and $(1, \cdots, 1)^T$ as the initial and right-hand-side vectors, respectively.

The first three test problems have positive rational generating functions in the Wiener class.

Example 1: Rational Separable Toeplitz with $(p_x, q_x, p_y, q_y) = (0, 2, 0, 2)$.

The $T(z_x, z_y)$ is of the form (3.4) with $A_x(z_x) = A_y(z_y) = 1$,

$$B_x(z_x) = (1 + 0.8z_x^{-1})(1 - 0.7z_x^{-1}) \quad \text{and} \quad B_y(z_y) = (1 + 0.9z_y^{-1})(1 - 0.6z_y^{-1}).$$

Example 2: Rational Toeplitz with $(p_x, q_x, p_y, q_y) = (2, 0, 2, 0)$.

The $T(z_x, z_y)$ is of the form (3.3) with $B(z_x, z_y) = 1$ and

$$\begin{aligned} A(z_x, z_y) &= 0.25 - 0.02(z_x^{-1} + z_y^{-1}) + 0.015(z_x^{-2} + z_y^{-2}) + 0.03z_x^{-1}z_y^{-1} \\ &- 0.02z_x^{-1}z_y^{-1}(z_x^{-1} + z_y^{-1}) - 0.01z_x^{-2}z_y^{-2}. \end{aligned}$$

Example 3: Rational Toeplitz with $(p_x, q_x, p_y, q_y) = (0, 2, 0, 1)$.

The $T(z_x, z_y)$ is of the form (3.3) with $A(z_x, z_y) = 1$ and

$$B(z_x, z_y) = 1 + 0.5z_x^{-1} - 0.3z_y^{-1} - 0.2z_x^{-2} - 0.1z_x^{-1}z_y^{-1} + 0.2z_x^{-2}z_y^{-1}.$$

We plot the corresponding spectra of T and $R^{-1}T$ and the convergence history of the CG and PCG methods with different N in Figs. 3-5 (a) and (b), respectively. Since the eigenvalues of T for Problems 1-3 all satisfy

$$0 < \delta_1 \le \lambda(T) \le \delta_2 < \infty,$$

where δ_1 and δ_2 are constants independent of the dimensions M and N of the given block matrix, the condition number $\kappa(T)$ is bounded by $\delta_2/\delta_1 = O(1)$. For each problem, we observe that the condition number of T (or $R^{-1}T$) increases slowly with N, which should reach a constant for asymptotically large N. Clearly, $R^{-1}T$ has a smaller condition number and a better clustering feature than T. Consequently, the PCG method performs better than the CG method.

One important difference between the separable and nonseparable cases is that, as N becomes larger, the PCG method converges faster for the separable case but more slowly for the nonseparable case. This can be easily explained by the analysis given in §3. When T is separable, the number of clusters is fixed and the clustering radius ϵ becomes smaller as N becomes larger. According to the analysis, $R^{-1}T$ has asymptotically 25 ($= (2\gamma_x + 1)(2\gamma_y + 1)$) distinct eigenvalues, including isolated outliers, clustered outliers, and clustered eigenvalues converging to 1. This is consistent with the results given in Fig. 3(a). As given in Fig. 3(b), the 2-norm of the residual decreases rapidly in 5 iterations. We observe an empirical formula for the separable case, namely, the PCG method converges in $\gamma_x\gamma_y + 1$ iterations. This phenomenon is closely related to the point Toeplitz result [9], where we found that although there are asymptotically $2\gamma_x + 1$ distinct eigenvalues, the PCG method converges in $\gamma_x + 1$ iterations. When T is nonseparable, the number of outliers increases with N. Although the PCG method converges more slowly for larger N, the effect is not obvious until the 2-norm of residual is very small. Besides, the convergence history curves are getting closer for larger N. This indicates that the number of PCG (or CG) iterations required is $O(1)$, which is determined by the condition number rather than the number of outliers.

A block Toeplitz with a nonrational generating function is given in Example 4.

Example 4: Nonrational Toeplitz

The block Toeplitz matrix is generated by a spherically-symmetric sequence

$$t_{m,n} = 0.7^{\sqrt{m^2+n^2}} + 0.5^{\sqrt{m^2+n^2}} + 0.3^{\sqrt{m^2+n^2}}.$$

The spectra of T and $R^{-1}T$ and the convergence history of the CG and PCG methods are plotted in Fig. 6. Although our analysis in §3 is restricted to the rational generating function case, it appears that the nonrational case does not differ much from the nonseparable case. The PCG method converges faster than the CG method due to a smaller condition number and a better spectral clustering property of $R^{-1}T$. The condition numbers of T and $R^{-1}T$ are again $O(1)$ so that the number of PCG (or CG) iterations required is $O(1)$, which is consistent with the observation that the convergence history curves are getting closer for larger N.

For the above 4 problems, T is well conditioned, i.e. $\kappa(T) = O(1)$, so that the condition number reduction through preconditioning is just a constant factor. To see a more dramatic condition number improvement, let us consider an ill-conditioned block Toeplitz below.

Example 5: Ill-conditioned Toeplitz with $(p_x, q_x, p_y, q_y) = (2, 0, 2, 0)$.

The block Toeplitz is characterized by the two-dimensional mask:

$$(-1) \times \begin{bmatrix} 0.01 & 0.02 & 0.04 & 0.02 & 0.01 \\ 0.02 & 0.04 & 0.12 & 0.04 & 0.02 \\ 0.04 & 0.12 & -1 & 0.12 & 0.04 \\ 0.02 & 0.04 & 0.12 & 0.04 & 0.02 \\ 0.01 & 0.02 & 0.04 & 0.02 & 0.01 \end{bmatrix}.$$

Note that the sum of the coefficients of the mask is zero. Masks of this nature arise in the discretization of constant-coefficient elliptic partial differential equations. However, the preconditioning block circulant matrix R is singular for this problem, since $R\mathrm{w} = 0$ for $\mathrm{w} = (1, 1, \cdots, 1, 1)^T$. In order to perform the preconditioning properly, we modify the preconditioner R slightly by replacing the zero eigenvalue with the smallest nonzero eigenvalue of R in our experiment.

The spectra of T and $R^{-1}T$, the convergence history of the CG and PCG methods, and the number of iterations required for the 2-norm of the relative residual less than 10^{-12} are plotted in Fig. 7 (a)-(c). As shown in Fig. 7(a), the condition numbers of T and $R^{-1}T$ increase in the rates of $O(N^2)$ and $O(N)$, respectively. Thus, the preconditioning provides an order of condition number improvement, but we have so far no theory to explain it. Note also the difference in the eigenvalue distributions of $R^{-1}T$ and T. Eigenvalues of T are more uniformly distributed in the interval than those of $R^{-1}T$. We see from Fig. 7(c) that PCG and CG methods converge in $O(\sqrt{N})$ and $O(N)$ iterations, respectively.

As far as the computational complexity is concerned, both the PCG and CG methods require $O(N^2 \log N)$ operations for Problems 1-4, where the condition numbers of T and $R^{-1}T$ are $O(1)$. For Problem 5, the PCG and CG methods require, respectively, $O(N^{5/2} \log N)$ and $O(N^3 \log N)$ operations. For all above test problems, the PCG and CG methods require much lower computational complexity than the direct method, which requires $O(N^5)$ operations.

5. Conclusion. In this research, we extended the preconditioning technique from point Toeplitz matrices to block Toeplitz matrices. We interpreted the block matrix-vector $L\mathrm{w}$ in terms of a two-dimensional operator mask operating on a certain two-dimensional sequence constructed based on w. This viewpoint provides a natural way to analyze the spectral clustering property of $R^{-1}T$. For block Toeplitz matrices T generated by two-dimensional rational functions $T(z_x, z_y)$ of order (p_x, q_x, p_y, q_y), We showed that the eigenvalues of $R^{-1}T$ are clustered around unity except at most $O(M\gamma_y + N\gamma_x)$ outliers, where $\gamma_x = \max(p_x, q_x)$ and $\gamma_y = \max(p_y, q_y)$. Furthermore, if T is separable, the outliers are clustered together such that $R^{-1}T$ has at most $(2\gamma_x + 1)(2\gamma_y + 1)$ asymptotic distinct eigenvalues. Thus, $R^{-1}T$ has a better spectral clustering property than T. Additionally, it was shown numerically that $R^{-1}T$ in general has a smaller condition number than T. These two spectral properties explain the superior convergence behavior of the PCG method over the CG method.

For point rational Toeplitz matrices, the number of outliers is often small ($= 2\max(p, q)$) and independent of the size of the problem so that it can be used to characterize the convergence rate of the PCG method. However, for block rational Toeplitz matrices, the number of outliers is proportional to the size of the problem, and is often too many to be useful for characterizing the convergence behavior of the PCG method. Hence, we have to examine both the condition number improvement and the spectral clustering effect. More research on the adaptation of the preconditioning technique to more general classes of block Toeplitz matrices such as ill-conditioned and indefinite problems and the spectral analysis of the preconditioned matrices is expected in the future.

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Figure Captions

Figure 1: Operator-mask interpretations of (a) $\triangle T_x \mathbf{w}$, (b) $\triangle E_x \mathbf{w}$ and (c) $(\triangle T_x - \triangle E_x)\mathbf{w}$.

Figure 2: Operator-mask interpretations of (a) $\triangle T\mathbf{w}$ and (b) $\triangle E\mathbf{w}$.

Figure 3: (a) Eigenvalue distributions of $R^{-1}T$ and T and (b) convergence history for Problem 1.

Figure 4: (a) Eigenvalue distributions of $R^{-1}T$ and T and (b) convergence history for Problem 2.

Figure 5: (a) Eigenvalue distributions of $R^{-1}T$ and T and (b) convergence history for Problem 3.

Figure 6: (a) Eigenvalue distributions of $R^{-1}T$ and T and (b) convergence history for Problem 4.

Figure 7: (a) Eigenvalue distributions of $R^{-1}T$ and T, (b) convergence history and (c) convergence rate of CG and PCG method for Problem 5.

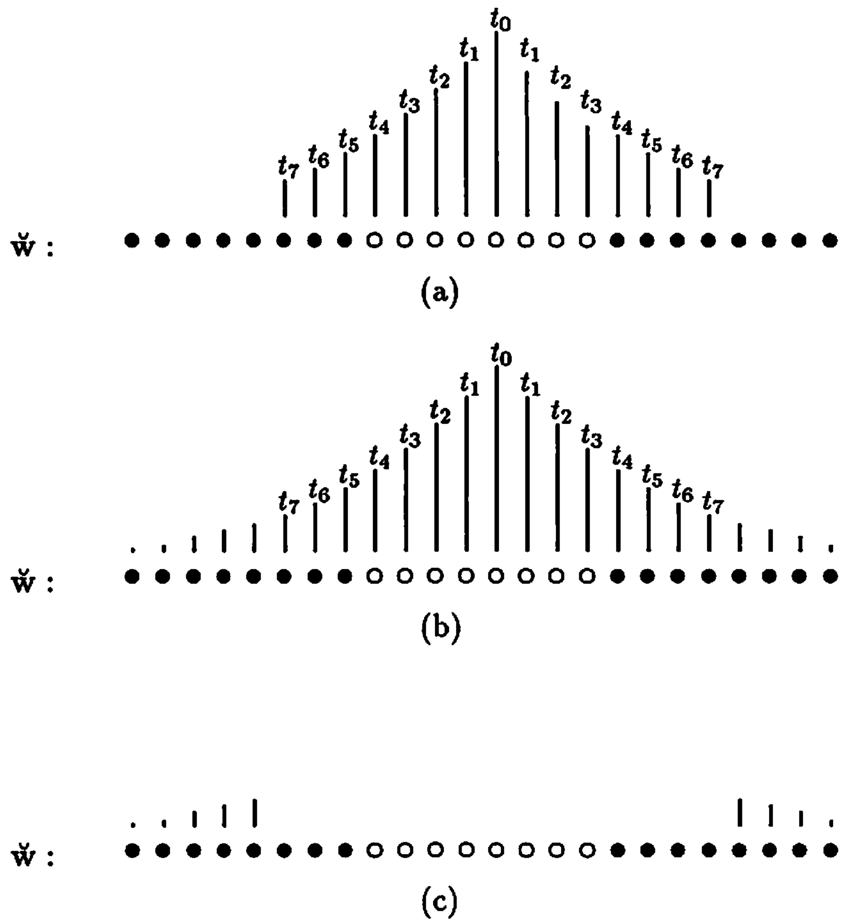

FIG. 1. *Operator-mask interpretations of* (a) $\Delta T_x \mathbf{w}$, (b) $\Delta E_x \mathbf{w}$ *and* (c) $(\Delta E_x - \Delta T_x)\mathbf{w}$.

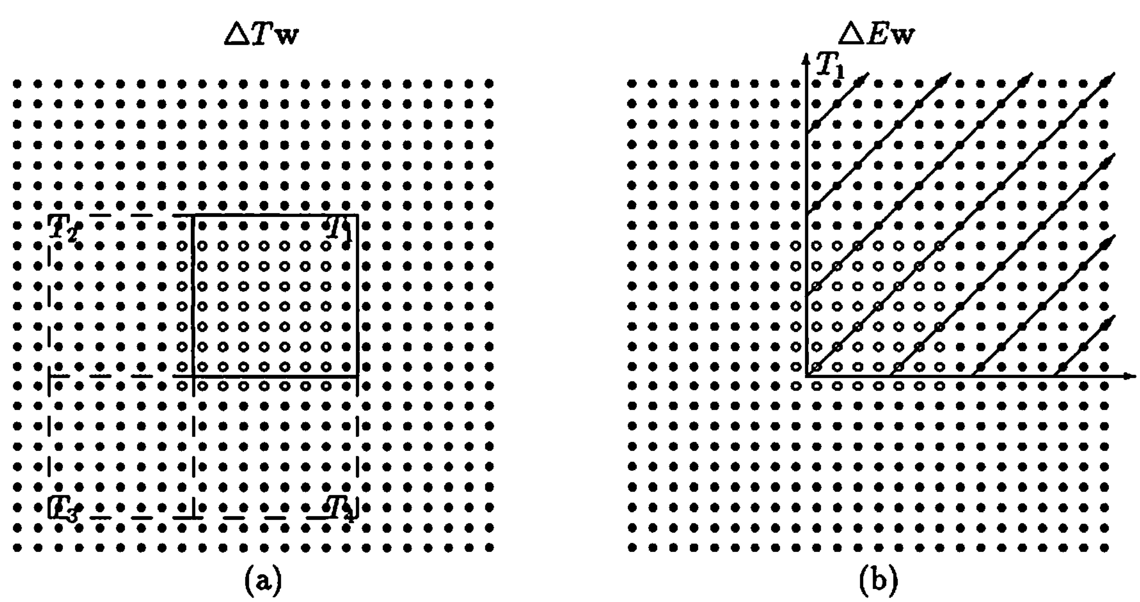

FIG. 2. *Operator-mask interpretations of* (a) $\Delta T\mathbf{w}$ *and* (b) $\Delta E\mathbf{w}$.

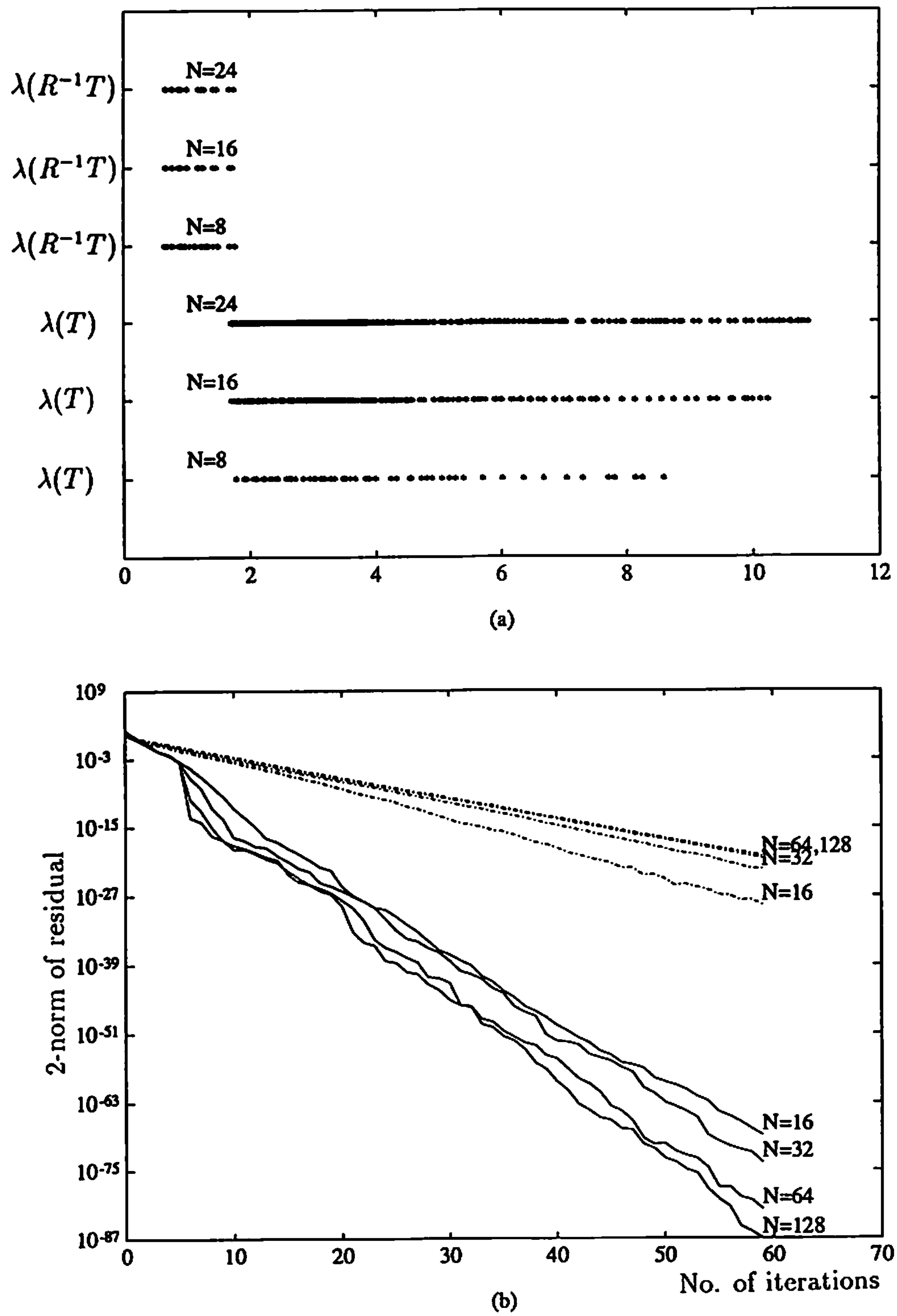

FIG. 3. (a) *Eigenvalue distribution of T and $R^{-1}T$ and* (b) *Convergence history for Problem 1.*

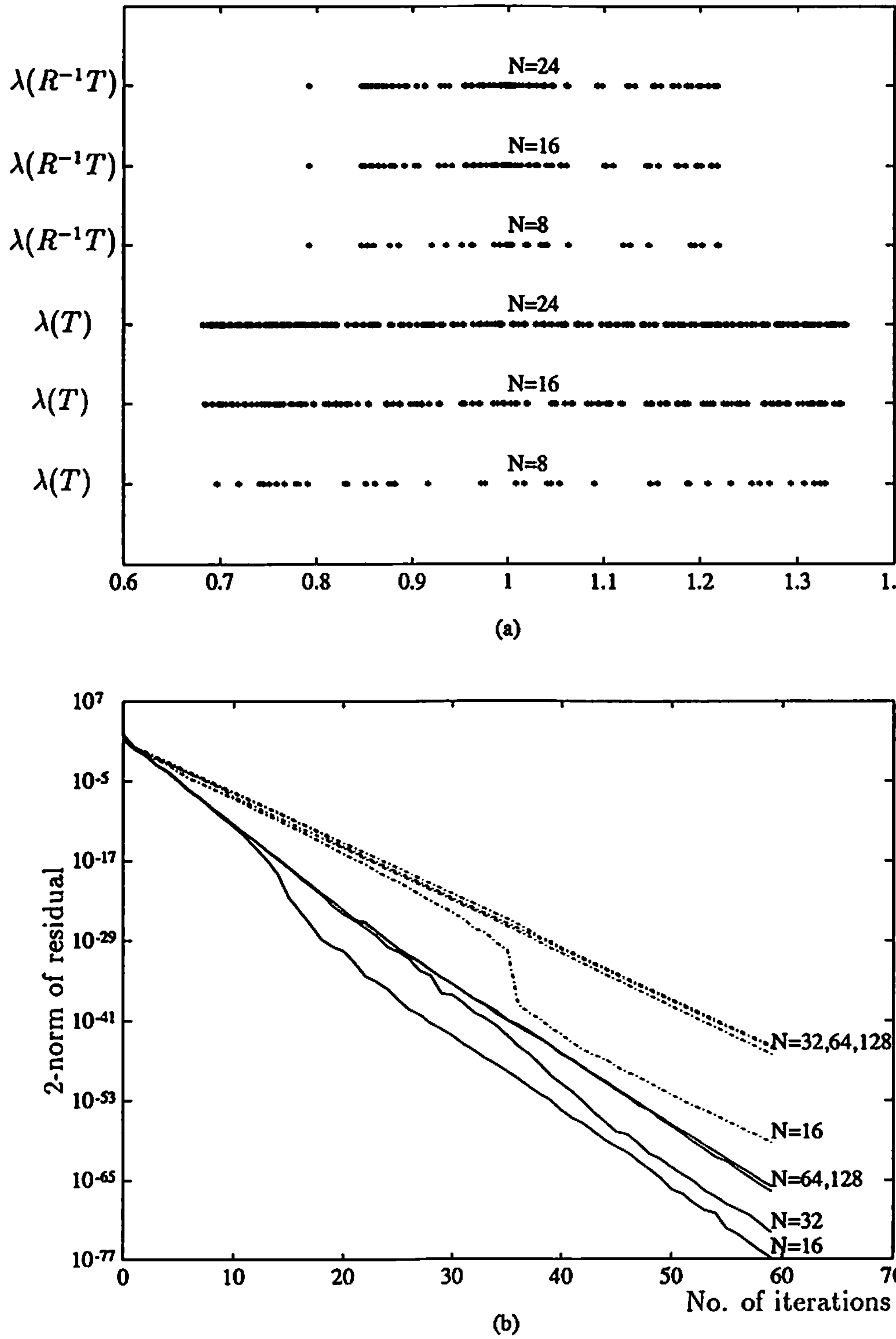

FIG. 4. (a) *Eigenvalue distribution of* T *and* $R^{-1}T$ *and* (b) *convergence history for Problem 2.*

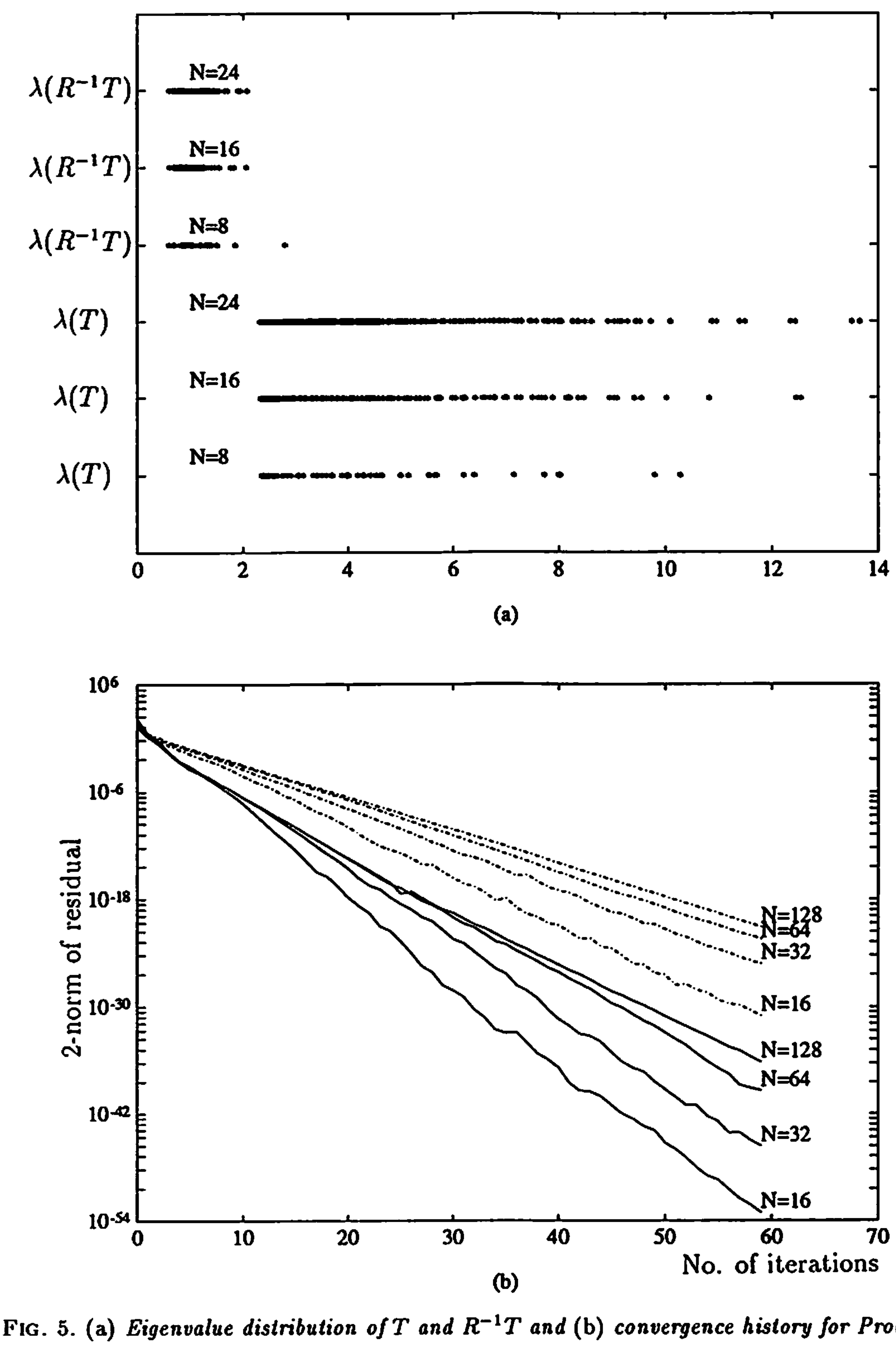

FIG. 5. (a) *Eigenvalue distribution of* T *and* $R^{-1}T$ *and* (b) *convergence history for Problem 3.*

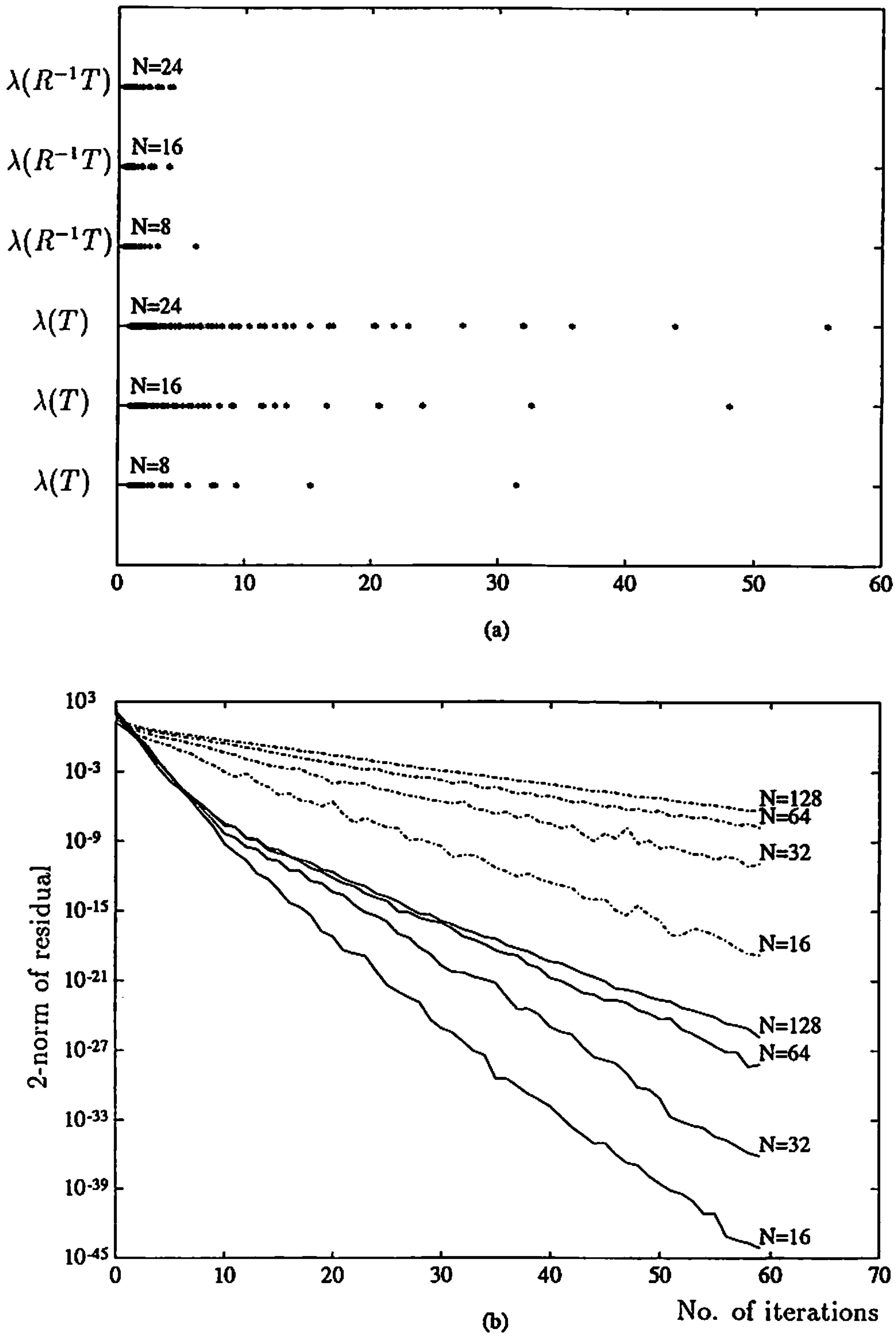

FIG. 6. *(a) Eigenvalue distribution of T and $R^{-1}T$ and (b) convergence history for Problem 4.*

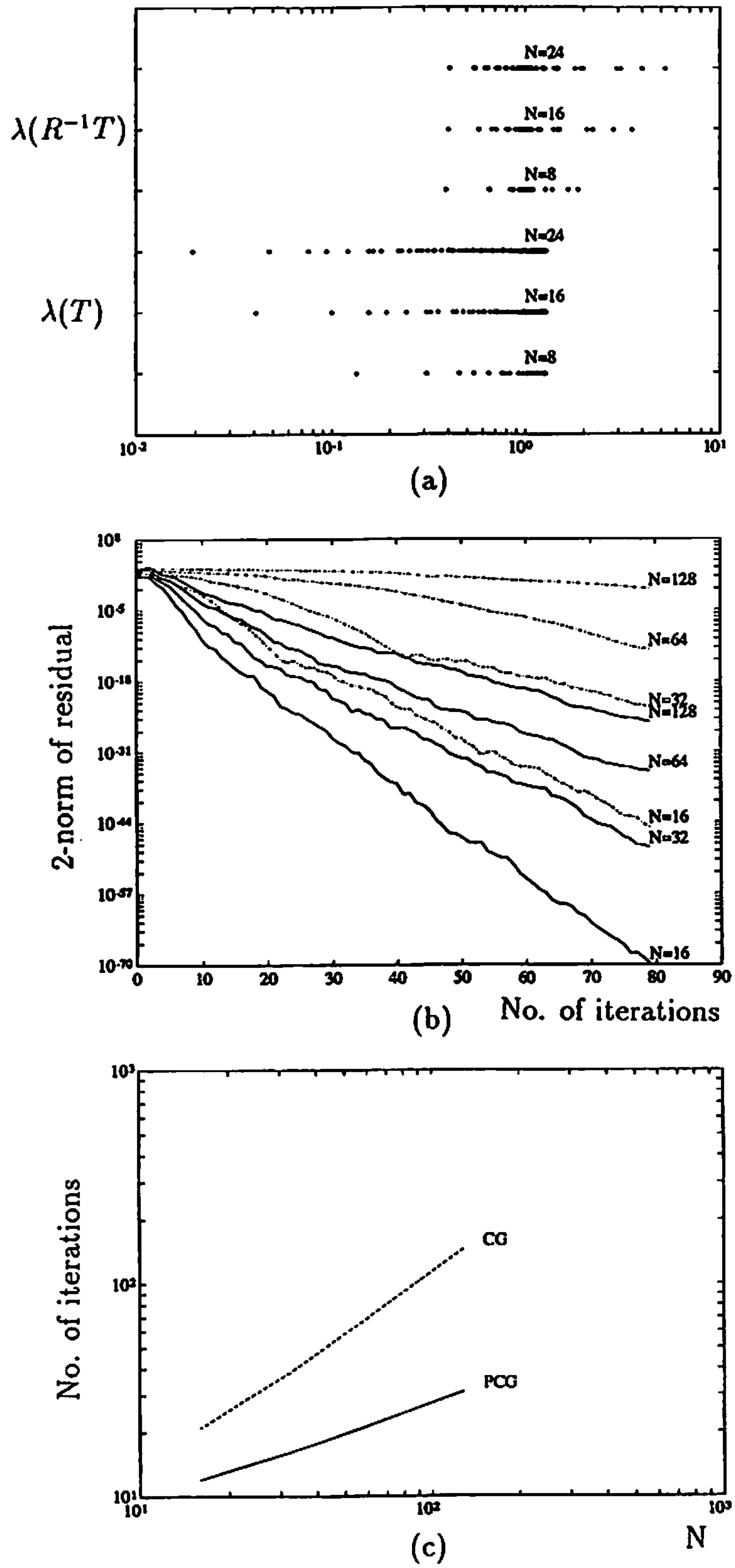

FIG. 7. *(a) Eigenvalue distribution of T and $R^{-1}T$, and (b) convergence history and (c) convergence rate of CG and PCG method for Problem 5.*