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PRECONDITIONED ITERATIVE METHODS FOR SOLVING TOEPLITZ-PLUS-HANKEL SYSTEMS *

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Abstract. The use of preconditioned iterative methods to solve a system of equations with a Toeplitz-plus-Hankel coefficient matrix is studied. We propose a new preconditioner suitable for Toeplitz-plus-Hankel matrices, and examine the spectral properties of preconditioned rational Toeplitz-plus-Hankel matrices. We show that the eigenvalues of the preconditioned matrix are clustered around unity except a finite number of outliers depending on the orders of the rational generating functions, and the clustering radius is proportional to the magnitude of the last elements in Toeplitz and Hankel matrices. With the spectral regularities, an $N \times N$ rational Toeplitz-plus-Hankel system can be solved by preconditioned iterative methods with $O(N \log N)$ operations. Numerical experiments are given to demonstrate the efficiency of the proposed preconditioner.

1. Introduction. The systems of linear equations with Toeplitz, Hankel and Toeplitz-plus-Hankel coefficient matrix arise in many signal processing applications. For example, the inverse scattering problem can be formulated as Toeplitz, Hankel and Toeplitz-plus-Hankel systems of equations, which were done by Krein [22], Agranovich and Marchenko [1] and Gelfand and Levitan [16], respectively. (For more recent work, we refer to [3], [4].) By exploiting the special structures of Toeplitz or Hankel matrices, an $N \times N$ system of equations can be solved by fast direct methods based on the Levinson or Schur algorithm with $O(N^2)$ operations [12], [13], [14], [23]. Direct algorithms for inverting $N \times N$ Toeplitz-plus-Hankel matrices with $O(N^2)$ complexity have also been derived [18], [19], [30]. Although the computational complexity of these fast algorithms is lower than that of the Gaussian elimination with pivoting, i.e. $O(N^3)$, their stability is not guaranteed when applied to indefinite or nonsymmetric matrices [5], [11]. In this research, we propose to use preconditioned iterative methods to solve Toeplitz-plus-Hankel systems, which have a low computational complexity and a stable convergence performance.

Toeplitz preconditioners in circulant or skew-circulant matrix form have been proposed and analyzed by many researchers [8], [10], [21], [25], [33]. It was shown by Chan and Strang [9] that, for a large class of Toeplitz matrices, the eigenvalues of preconditioned matrices are clustered around unity and the preconditioned iterative method converges at a superlinear rate. For Toeplitz matrices generated by rational functions, an even stronger convergence result was obtained by Trefethen [34] and the authors [26], [28]. That is, the preconditioned iterative method converges in a finite number of iterations independent of the problem size N. Consequently, a rational Toeplitz system can be solved with $O(N \log N)$ operations. In addition to low computational complexity, preconditioned iterative methods demonstrate a very stable convergence behavior [27]. Since a Hankel system can be transformed to a Toeplitz system by reversing the order of the linear equations, the same results also hold for Hankel systems.

The inverse scattering problem is often formulated as two waves propagating in the opposite directions. The discrete version of the formulation can be naturally expressed as an $N \times N$ Toeplitz-plus-Hankel system $A\mathbf{x} = \mathbf{b}$, where A is the sum of a Toeplitz matrix T and a Hankel matrix H with elements $T_{i,j} = t_{i-j}$ and $H_{i,j} = h_{N+1-(i+j)}$ [2], [3], [4], [7], [29]. The idea to construct a Toeplitz-plus-Hankel preconditioner can be simply stated as follows. Let J be an $N \times N$ matrix which has ones along the secondary diagonal and zeros elsewhere (i.e. $J_{i,j} = 1$ if $i + j = N + 1$ and $J_{i,j} = 0$ if $i + j \neq N + 1$). One can easily verify that the product of J and H gives a Toeplitz matrix

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$T_H = JH$ with elements $[T_H]_{i,j} = h_{i-j}$, and that the Toeplitz-plus-Hankel matrix can be expressed as $A = T + H = T + JT_H$. Now, given preconditioners K_T and K_H for Toeplitz matrices T and T_H, we propose to use $P = K_T + JK_H$ as a preconditioner for A.

To solve the Toeplitz-plus-Hankel system $A\mathbf{x} = \mathbf{b}$ with preconditioner P, two major computations required at each iteration are the matrix-vector products $A\mathbf{v}$ and $P^{-1}\mathbf{v}$ with an arbitrary vector $\mathbf{v}$. The operation $A\mathbf{v}$ can be performed effectively via fast Fourier transform (FFT) with $O(N \log N)$ operations, since both $T\mathbf{v}$ and $H\mathbf{v}$ can be embedded in a $2N \times 2N$ circulant matrix-vector product. To implement the preconditioning step $P\mathbf{z} = \mathbf{v}$, we relate it to an equivalent $N \times N$ circulant system which can be inverted via FFT with $O(N \log N)$ operations. Consequently, the computational complexity for each iteration is $O(N \log N)$.

In the context of inverse scattering, the generating sequences of Toeplitz and Hankel matrices can be selected with great flexibility. Thus, we focus on the case that the sequences $\{t_n\}$ and $\{h_n\}$ are generated, respectively, by rational functions of orders $(\alpha_T, \beta_T, \gamma_T, \delta_T)$ and $(\alpha_H, \beta_H, \gamma_H, \delta_H)$, (see the definition in §4) and study the spectral properties of preconditioned matrices. The eigenvalues of $P^{-1}A$ can be classified into two classes, i.e. the outliers and the clustered eigenvalues, depending on whether they converge to unity asymptotically. Then, the preconditioned matrix has the following two spectral properties: (1) the number of outliers is bounded by a constant which depends on the orders of the rational generating functions; and (2) the clustered eigenvalues are confined in a disk centered at unity with radius ϵ proportional to $O(|t_N| + |t_{-N}| + |h_N| + |h_{-N}|)$. With the above spectral properties, various preconditioned iterative methods, including CGN (the Conjugate Gradient iteration applied to the Normal equations) [20], GMRES (the Generalized Minimal Residual) [31], and CGS (the Conjugate Gradient Square) [32], can be effectively applied. It turns out that a rational Toeplitz-plus-Hankel system can be solved in a finite number of iterations independent of the problem size N so that the total operations required are $O(N \log N)$. Besides, the preconditioned iterative methods are highly parallelizable due to the parallelism provided by FFT. The time complexity can be reduced to $O(\log N)$ if $O(N)$ processors are used.

This paper is organized as follows. We discuss the construction of a preconditioner for Toeplitz-plus-Hankel matrices in §2. The computational complexity per iteration is discussed in §3. The spectral properties of the preconditioner and the preconditioned rational matrix are examined in §4. Numerical experiments are given in §5 to illustrate our theoretical study.

2. Construction of the preconditioner. Consider the $N \times N$ Toeplitz-plus-Hankel system

$$A\mathbf{x} = \mathbf{b}, \qquad A = T + H, \tag{2.1}$$

where T and H are given Toeplitz and Hankel matrices with elements $T_{i,j} = t_{i-j}$ and $H_{i,j} = h_{N+1-(i+j)}$. With the special structures of Toeplitz and Hankel matrices, a Hankel matrix H can be transformed to a Toeplitz matrix by premultiplying (or postmultiplying) it with

$$J = \begin{bmatrix} 0 & 0 & \cdot & 0 & 1 \\ 0 & \cdot & 0 & 1 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 1 & 0 & \cdot & 0 \\ 1 & 0 & \cdot & 0 & 0 \end{bmatrix},$$

which is known as the time-reversal operator. It is clear that $T_H = JH$ is a Toeplitz matrix with elements $[T_H]_{i,j} = h_{i-j}$ and that $JT_H = J^2H = H$. Thus, (2.1) can also be written as

$$(T + JT_H)\mathbf{x} = \mathbf{b}, \tag{2.2}$$

where T and T_H are $N \times N$ Toeplitz matrices generated by the sequences $\{t_n\}$ and $\{h_n\}$, respectively.

The procedure to construct a circulant preconditioner for Toeplitz matrix

$$T = \begin{bmatrix} t_0 & t_{-1} & \cdot & t_{-(N-2)} & t_{-(N-1)} \\ t_1 & t_0 & t_{-1} & \cdot & t_{-(N-2)} \\ \cdot & t_1 & t_0 & \cdot & \cdot \\ t_{N-2} & \cdot & \cdot & \cdot & t_{-1} \\ t_{N-1} & t_{N-2} & \cdot & t_1 & t_0 \end{bmatrix}$$

is summarized as follows [25]. Motivated by the observation that we can solve the $2N \times 2N$ circulant system effectively,

$$R_T \begin{bmatrix} \mathbf{x} \\ \mathbf{x} \end{bmatrix} = \begin{bmatrix} \mathbf{b} \\ \mathbf{b} \end{bmatrix}, \qquad R_T = \begin{bmatrix} T & \Delta T \\ \Delta T & T \end{bmatrix}, \tag{2.3}$$

where ΔT is determined by the elements of T to make R_T circulant, i.e.,

$$\Delta T = \begin{bmatrix} 0 & t_{N-1} & \cdot & t_2 & t_1 \\ t_{-(N-1)} & 0 & t_{N-1} & \cdot & t_2 \\ \cdot & t_{-(N-1)} & 0 & \cdot & \cdot \\ t_{-2} & \cdot & \cdot & \cdot & t_{N-1} \\ t_{-1} & t_{-2} & \cdot & t_{-(N-1)} & 0 \end{bmatrix}.$$

Since (2.3) is equivalent to

$$K_T \mathbf{x} = \mathbf{b}, \qquad K_T = T + \Delta T, \tag{2.4}$$

we choose the circulant matrix K_T as a preconditioner for T. Similarly, we have the circulant preconditioner $K_H = T_H + \Delta T_H$ for Toeplitz matrix T_H. Then, with respect to (2.2), we propose to use

$$P = K_T + JK_H, \tag{2.5}$$

as a preconditioner for the Toeplitz-plus-Hankel matrix A.

3. Computational complexity per iteration. To solve the Toeplitz-plus-Hankel system $A\mathbf{x} = \mathbf{b}$ with preconditioner P, two major computations required at each iteration are the matrix-vector products $A\mathbf{v}$ and $P^{-1}\mathbf{v}$ with an arbitrary vector $\mathbf{v}$. We show below that they both can be achieved with $O(N \log N)$ operations. Since other operations involved are vector additions or inner products whose complexity is proportional to $O(N)$, the total computational complexity per iteration is $O(N \log N)$.

The $N \times N$ Toeplitz matrix-vector product $T\mathbf{v}$ can be embedded in the $2N \times 2N$ circulant matrix-vector product

$$\begin{bmatrix} T & \Delta T \\ \Delta T & T \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} T\mathbf{v} \\ \Delta T\mathbf{v} \end{bmatrix},$$

so that it can be effectively computed via FFT with $O(N \log N)$ operations. Premultiplying J to a vector $\mathbf{v}$ corresponds to the reverse of the order of the elements in $\mathbf{v}$. Thus, the Toeplitz-plus-Hankel matrix-vector product

$$A\mathbf{v} = T\mathbf{v} + JT_H\mathbf{v}$$

can also be achieved with $O(N \log N)$ operations.

With the equality $J^2 = I$ and (2.5), we have

$$P\mathbf{z} = K_T\mathbf{z} + JK_HJJ\mathbf{z} = \mathbf{v}, \tag{3.1}$$

which is equivalent to

$$JP\mathbf{z} = JK_TJJ\mathbf{z} + K_H\mathbf{z} = J\mathbf{v}. \tag{3.2}$$

Since K_T and K_H are circulant, $JK_TJ = K_T^T$, $JK_HJ = K_H^T$ and K_T and K_H commute. By multiplying (3.1) with K_T^T and (3.2) with K_H^T, we can write the difference between the two resulting equations as

$$(K_T^T K_T - K_H^T K_H)\mathbf{z} = K_T^T\mathbf{v} - K_H^T J\mathbf{v}. \tag{3.3}$$

Thus, the solution of $\mathbf{z} = P^{-1}\mathbf{v}$ can also be determined from (3.3). It is easy to verify that $K_T^T K_T - K_H^T K_H$ is circulant and can be diagonalized with Fourier functions. Consequently, $P^{-1}\mathbf{v}$ can be solved effectively via FFT with $O(N\log N)$ complexity. Note also that the Fourier coefficients $\hat{v}_k$ and $\hat{w}_k$ of the vectors $\mathbf{v}$ and $\mathbf{w} = J\mathbf{v}$ are related via

$$\hat{w}_k = \hat{v}_{-k \ mod \ N}, \qquad 0 \le k \le N-1,$$

which means that only an FFT and an inverse FFT are needed in solving (3.3), given the eigenvalues of K_T and K_H.

In conclusion, the total computational complexity inside each iteration of the preconditioned iterative methods is $O(N\log N)$ only. In addition to the low complexity, the preconditioned iterative methods are highly parallelizable due to the parallelism provided by FFT, and the time complexity can be reduced to $O(\log N)$ if $O(N)$ processors are used.

4. Analysis of the preconditioner. We assume that Toeplitz matrix T and Hankel matrix H are generated by real sequences $\{t_n\}$ and $\{h_n\}$ satisfying

$$\sum_{-\infty}^{\infty} |t_n| \le B_T < \infty, \qquad \sum_{-\infty}^{\infty} |h_n| \le B_H < \infty. \tag{4.1}$$

The circulant matrices K_T, K_T^T, K_H and K_H^T share the same Fourier functions as their eigenvectors with eigenvalues

$$\lambda_k(K_T) = \lambda_k^*(K_T^T) = \sum_{n=-(N-1)}^{N-1} t_n e^{-j\frac{2\pi kn}{N}}, \qquad \lambda_k(K_H) = \lambda_k^*(K_H^T) = \sum_{n=-(N-1)}^{N-1} h_n e^{-j\frac{2\pi kn}{N}},$$

where $k = 0, 1, \cdots, N-1$ and λ_k^* denotes the complex conjugate of λ_k. We also assume that

$$|\ |\lambda_k(K_T)|^2 - |\lambda_k(K_H)|^2\ | \ge \mu > 0, \qquad 0 \le k \le N-1, \tag{4.2}$$

which implies the invertibility of $K_T^T K_T - K_H^T K_H$.

The generating functions of T and T_H are defined as

$$T(z) = \sum_{n=-\infty}^{\infty} t_n z^{-n}, \qquad H(z) = \sum_{n=-\infty}^{\infty} h_n z^{-n}.$$

We focus on the case

$$T(z) = \frac{A_T(z^{-1})}{B_T(z^{-1})} + \frac{C_T(z)}{D_T(z)}, \qquad H(z) = \frac{A_H(z^{-1})}{B_H(z^{-1})} + \frac{C_H(z)}{D_H(z)}, \tag{4.3}$$

where $T(z)$ is a rational function of order $(\alpha_T, \beta_T, \gamma_T, \delta_T)$, i.e.

$$\begin{array}{ll} A_T(z^{-1}) = \sum_{i=0}^{\alpha_T} a_{T,i} z^{-i}, & C_T(z) = \sum_{i=0}^{\gamma_T} c_{T,i} z^{i}, \\ B_T(z^{-1}) = \sum_{i=0}^{\beta_T} b_{T,i} z^{-i}, & D_T(z) = \sum_{i=0}^{\delta_T} d_{T,i} z^{i}, \end{array} \tag{4.4}$$

with $a_{T,\alpha_T} b_{T,\beta_T} c_{T,\gamma_T} d_{T,\delta_T} \ne 0$, $b_{T,0} = 1$, $d_{T,0} = 1$, and polynomials $A_T(z^{-1})$ and $B_T(z^{-1})$ (or $C_T(z^{-1})$ and $D_T(z^{-1})$) have no common factors. Similarly, $H(z)$ is a rational function of order $(\alpha_H, \beta_H, \gamma_H, \delta_H)$. For convenience, we let

$$\begin{array}{ll} r_T = \max(\alpha_T, \beta_T), & s_T = \max(\gamma_T, \delta_T), \\ r_H = \max(\alpha_H, \beta_H), & s_H = \max(\gamma_H, \delta_H). \end{array} \tag{4.5}$$

4.1. Invertibility of the preconditioner. With conditions (4.1) and (4.2), we prove that P^{-1} exists in the following theorem.

THEOREM 1. *Let T and H be $N \times N$ Toeplitz and Hankel matrices satisfying (4.1) and (4.2). Then, the preconditioning matrix $P = K_T + JK_H$ is invertible.*

Proof. Recall that (3.1) is equivalent to

$$(K_T^T K_T - K_H^T K_H)\mathbf{z} = (K_T^T - K_H^T J)\mathbf{v}. \tag{4.6}$$

With the definition of K_T in (2.4) and assumption (4.1), it is easy to see that

$$\|K_T^T\|_1 = \|K_T^T\|_\infty \le 2 \sum_{n=-N-1}^{N-1} |t_n| \le 2B_T.$$

As a consequence, we have

$$\|K_T^T\|_2 \le (\|K_T^T\|_1 \|K_T^T\|_\infty)^{1/2} \le 2B_T.$$

Similar results hold for K_H^T. Thus, the right-hand-side of (4.6) is bounded. With condition (4.2), the magnitude of any eigenvalue of $K_T^T K_T - K_H^T K_H$ is also bounded by

$$|\lambda_k(K_T^T K_T - K_H^T K_H)| = \big|\, |\lambda_k(K_T)|^2 - |\lambda_k(K_H)|^2 \,\big| \ge \mu > 0.$$

Therefore, (4.6) is nonsingular and the preconditioner P is invertible. □

4.2. The number of outliers of $P^{-1}A$. Given rational generating functions as specified by (4.3), the eigenvalue of $P^{-1}A$ is called an outlier if it does not converge to one for asymptotically large N. According to the definitions of K_T and K_H, the difference matrix $\Delta A = P - A$ can be written as

$$\Delta A = \Delta T + J \Delta T_H. \tag{4.7}$$

Since

$$\lambda(P^{-1}A) = 1 - \lambda(P^{-1}\Delta A),$$

the eigenvalues of $P^{-1}A$ clustered around 1 correspond to those of $P^{-1}\Delta A$ clustered around 0, and the number of outliers equals the number of the asymptotic nonzero eigenvalues of $P^{-1}\Delta A$.

Our analysis for the number of outliers proceeds as follows.

Step 1: Construct a low rank matrix ΔF based on the recursion in t_n and h_n described in Lemma 1.
Step 2: Show that ΔF is asymptotically equivalent to ΔA (Lemma 2).
Step 3: Establish an upper bound for the rank of ΔF, which is equal to the number of outliers of $P^{-1}A$ (Theorem 2).

LEMMA 1. *The sequences t_n and h_n generated by (4.3) follow the recursions,*

$$\begin{aligned}
t_{n+1} &= -(b_{T,1}t_n + b_{T,2}t_{n-1} + \cdots + b_{T,\beta_T}t_{n-\beta_T+1}), & n &\ge r_T,\\
t_{n-1} &= -(d_{T,1}t_n + d_{T,2}t_{n+1} + \cdots + d_{T,\delta_T}t_{n+\delta_T-1}), & n &\le -s_T,\\
h_{n+1} &= -(b_{H,1}h_n + b_{H,2}h_{n-1} + \cdots + b_{H,\beta_H}h_{n-\beta_H+1}), & n &\ge r_H,\\
h_{n-1} &= -(d_{H,1}h_n + d_{H,2}h_{n+1} + \cdots + d_{H,\delta}h_{n+\delta_H-1}), & n &\le -s_H,
\end{aligned} \tag{4.8}$$

where $b_{T,i}$ $d_{T,i}$, $b_{H,i}$ and $d_{H,i}$ are given in (4.4) and r_T, s_T, r_H and s_H are defined in (4.5).

Proof. Similar to the proof of Lemma 1 in [26]. □

Based on the recursion (4.8), we construct two low rank Toeplitz matrices ΔF_T and ΔH_T as

$$\Delta F_T = F_{T,1} + F_{T,2}, \quad \text{and} \quad \Delta F_H = F_{H,1} + F_{H,2}, \tag{4.9}$$

where

$$F_{T,1} = \begin{bmatrix} t_N & t_{N-1} & \cdot & t_2 & t_1 \\ t_{N+1} & t_N & t_{N-1} & \cdot & t_2 \\ \cdot & t_{N+1} & t_N & \cdot & \cdot \\ t_{2N-2} & \cdot & \cdot & \cdot & t_{N-1} \\ t_{2N-1} & t_{2N-2} & \cdot & t_{N+1} & t_N \end{bmatrix},$$

$$F_{T,2} = \begin{bmatrix} t_{-N} & t_{-(N+1)} & \cdot & t_{-(2N-2)} & t_{-(2N-1)} \\ t_{-(N-1)} & t_{-N} & t_{-(N+1)} & \cdot & t_{-(2N-2)} \\ \cdot & t_{-(N-1)} & t_{-N} & \cdot & \cdot \\ t_{-2} & \cdot & \cdot & \cdot & t_{-(N+1)} \\ t_{-1} & t_{-2} & \cdot & t_{-(N-1)} & t_{-N} \end{bmatrix},$$

and $F_{H,1}$ and $F_{H,2}$ are similarly constructed by changing elements of $F_{T,1}$ and $F_{T,2}$ from t_n to h_n. The ΔF_T and ΔF_H are asymptotically equivalent to ΔT and ΔT_H, respectively, and we have the following lemma.

LEMMA 2. *Let T and H be $N \times N$ Toeplitz and Hankel matrices generated by $T(z)$ and $H(z)$ in (4.3) with the corresponding generating sequences satisfying (4.1) and (4.2). Then,*

$$\|\Delta A - \Delta F\|_2 \leq O(|t_N| + |t_{-N}| + |h_N| + |h_{-N}|),$$

where ΔA is defined in (4.7) and

$$\Delta F = \Delta F_T + J \Delta F_H$$

with ΔF_T and ΔF_H given by (4.9). Consequently, ΔA is asymptotically equivalent to ΔF.

Proof. The difference between ΔF_T and ΔT is

$$\Delta F_T - \Delta T = \begin{bmatrix} t_N + t_{-N} & t_{-(N+1)} & \cdot & t_{-(2N-2)} & t_{-(2N-1)} \\ t_{N+1} & t_N + t_{-N} & t_{-(N+1)} & \cdot & t_{-(2N-2)} \\ \cdot & t_{N+1} & t_N + t_{-N} & \cdot & \cdot \\ t_{2N-2} & \cdot & \cdot & \cdot & t_{-(N+1)} \\ t_{2N-1} & t_{2N-2} & \cdot & t_{N+1} & t_N + t_{-N} \end{bmatrix},$$

whose l_1- and l_∞-norms are bounded by

$$\tau_T = \sum_{n=N}^{2N-1} |t_n| + \sum_{n=-N}^{-(2N-1)} |t_n| = O(|t_N| + |t_{-N}|).$$

The last equality is based on the fact that t_n, $n > 0$ or $n < 0$, can be decomposed into exponentially decaying sequences. With the property $\|B\|_2 \leq (\|B\|_1 \|B\|_\infty)^{1/2}$ for arbitrary matrix B, we know that the l_2-norm of $\Delta F_N - \Delta T_N$ is also bounded by τ_T. Similarly, the l_1-, l_2- and l_∞-norms of $\Delta T_H - \Delta F_H$ are all bounded by

$$\tau_H = O(|h_N| + |h_{-N}|).$$

Consequently, we have, for $p = 1, 2, \infty$,

$$\|\Delta A - \Delta F\|_p \leq \|\Delta T - \Delta F_T\|_p + \|J\|_p \|\Delta T_H - \Delta F_H\|_p \leq \tau_T + \tau_H.$$

With condition (4.1), $\tau_T + \tau_H$ converges to zero as N goes to infinity so that ΔA and ΔF are asymptotically equivalent. □

With recursion (4.8), one can easily determine upper bounds for the ranks of ΔF_T and ΔF_H, i.e.

$$\text{rank}(\Delta F_T) \leq r_T + s_T \quad \text{and} \quad \text{rank}(\Delta F_H) \leq r_H + s_H.$$

Since both J and P are full rank matrices, the rank of $P^{-1}\Delta F$ (or equivalently the number of outliers of $P^{-1}A$) is bounded above by

$$\tilde{\eta} = r_T + s_T + r_H + s_H. \tag{4.10}$$

However, the bound $\tilde{\eta}$ is not tight. A tighter bound is available according to the following theorem.

THEOREM 2. *Let T and H be $N \times N$ Toeplitz and Hankel matrices generated by $T(z)$ and $H(z)$ in (4.3) with the corresponding generating sequences satisfying (4.1) and (4.2). The number of outliers of $P^{-1}A$ is bounded by*

$$\eta = \beta_T + \beta_H + \delta_T + \delta_H + \max(\alpha_T - \beta_T, \alpha_H - \beta_H, 0) + \max(\gamma_T - \delta_T, \gamma_H - \delta_H, 0) - \eta_c, \tag{4.11}$$

where η_c is the number of the common roots in $B_T(z)D_T(z)$ and $B_H(z)D_H(z)$.

Proof. We first focus on the case that all roots of $B_T(z)$, $D_T(z)$, $B_H(z)$ and $D_H(z)$ are simple. By applying the partial fractional expansion to $A_T(z^{-1})/B_T(z^{-1})$ and $C_T(z)/D_T(z)$ and determining the corresponding Toeplitz matrix for each term, we obtain a decomposition for T, i.e.

$$T = T_{1,0} + \sum_{i=1}^{\beta_T} T_{1,i} + T_{2,0} + \sum_{i=1}^{\delta_T} T_{2,i}.$$

In the above expression, $T_{1,i}$ and $T_{2,i}$ are, respectively, lower and upper triangular Toeplitz matrices. If $\alpha_T - \beta_T \geq 0$, $T_{1,0}$ has a finite lower bandwidth $\alpha_T - \beta_T$. Otherwise, it is equal to zero. The $T_{1,i}$, $i \neq 0$, corresponds to the Toeplitz matrix generated by a root of $B_T(z^{-1})$. The $T_{2,i}$, $0 \leq i \leq \delta_T$ can be similarly defined.

We construct $F_{T,1,i}$ and $F_{T,2,i}$ for $T_{1,i}$ and $T_{2,i}$ based on (4.9). Since the construction is linear, we have

$$\Delta F_T = F_{T,1,0} + \sum_{i=1}^{\beta_T} F_{T,1,i} + F_{T,2,0} + \sum_{i=1}^{\delta_T} F_{T,2,i}.$$

It is easy to verify that the elements of $F_{T,1,0}$ and $F_{T,2,0}$ are zeros except the northeast $\max(\alpha_T - \beta_T, 0)$ and the southwest $\max(\gamma_T - \delta_T, 0)$ diagonals, respectively. All remaining terms in ΔF_T are rank one matrices. The ΔF_H can be similarly decomposed as

$$\Delta F_H = F_{H,1,0} + \sum_{i=1}^{\beta_H} F_{H,1,i} + F_{H,2,0} + \sum_{i=1}^{\delta_H} F_{H,2,i},$$

where the elements of $F_{H,1,0}$ and $F_{H,2,0}$ are zeros except the northeast $\max(\alpha_H - \beta_H, 0)$ and the southwest $\max(\gamma_H - \delta_H, 0)$ diagonals, respectively, and all other terms are rank one matrices.

Let us examine the rank of $\Delta F = \Delta F_T + J\Delta F_H$. The ranks of $F_{T,1,0} + JF_{H,1,0}$ and $F_{T,2,0} + JF_{H,2,0}$ are clearly bounded by $\max(\alpha_T - \beta_T, \alpha_H - \beta_H, 0)$ and $\max(\gamma_T - \delta_T, \gamma_H - \delta_H, 0)$, respectively. Since all other terms in ΔF_T and ΔF_H are rank one matrices, the rank of ΔF is bounded above by

$$\beta_T + \beta_H + \delta_T + \delta_H + \max(\alpha_T - \beta_T, \alpha_H - \beta_H, 0) + \max(\gamma_T - \delta_T, \gamma_H - \delta_H, 0) \tag{4.12}$$

which is the same as η given in (4.11) with $\eta_c = 0$.

Now, suppose that $B_T(z)D_T(z)$ and $B_H(z)D_H(z)$ have η_c common roots. This implies four combinations. That is, $B_T(z)$ and $B_H(z)$, $B_T(z)$ and $D_H(z)$, $D_T(z)$ and $B_H(z)$, or $D_T(z)$ and $D_H(z)$ have common roots. Without loss of generality, we assume that $B_T(z)$ and $B_H(z)$ have a common root μ and that Toeplitz matrices $T_{1,k}$ and $JH_{1,k}$ are generated by this root. By postmultiply $F_{T,1,k}$, $F_{H,1,k}$, $F_{T,1,k} + JF_{H,1,k}$ with the lower triangular Toeplitz matrix L_μ which has

$$[1, -\mu, 0, \cdots, 0]^T$$

as the first column, the resulting matrices have only one nonzero column. Thus, the rank of $\Delta F = \Delta F_T + J\Delta F_H$ is lower than (4.12) by one due to the root μ shared by $B_T(z)$ and $B_H(z)$.

As to the other three combinations, similar arguments hold with the following modifications. When $D_T(z)$ and $D_H(z)$ have common roots, we postmultiply $F_{T,2,k} + JF_{H,2,k}$ by the corresponding upper triangular banded Toeplitz matrix constructed with respect to the common root. When $B_T(z)$ and $D_H(z)$ (or $B_H(z)$ and $D_T(z)$) have common roots, we examine the rank of

$$\Delta F_{T,1,k} + J\Delta F_{H,2,k}JJ = \Delta F_{T.1.k} + \Delta F_{H,2,k}^T J$$

by premultiplying an appropriate lower (or upper) triangular Toeplitz banded matrix.

Let us now focus on the case that $B_T(z)$, $D_T(z)$, $B_H(z)$ or $D_H(z)$ has repeated roots which are common for both $B_T(z)D_T(z)$ and $B_H(z)D_H(z)$. Without loss of generality, we assume that $B_T(z)$

has k_T roots at μ and that $B_H(z)$ has k_H roots at μ. Let $F_{T,1,i}$, $1 \le i \le k_T$, and $F_{H,1,j}$, $1 \le j \le k_H$, be constructed with respect to the repeated roots μ. It can be shown that

$$(\sum_{i=1}^{k_T} F_{T,1,i} + \sum_{j=1}^{k_H} F_{H,1,i})L_\mu^k, \qquad k = \max(k_T, k_H),$$

has at most k nonzero columns. Therefore, the rank of $\Delta F = \Delta F_T + J\Delta F_H$ is lower than (4.12) by $\min(k_T, k_H)$ due to the repeated roots shared by $B_T(z)$ and $B_H(z)$. □

4.3. The clustering radius of $P^{-1}A$. The following analysis is performed under the assumptions that T is symmetric, and that H is symmetric with respect to the secondary diagonal. In other words, both T and H are symmetric centrosymmetric matrices, i.e. $T = T^T$ and $JT = (JT)^T$ [6]. Besides, we assume that $B_T(z)D_T(z)$ and $B_H(z)D_H(z)$ have no common roots ($\eta_c = 0$). Since

$$[\lambda(P^{-1}A)]^{-1} = \lambda(A^{-1}(A + \Delta A)) = 1 + \lambda(A^{-1}\Delta A), \tag{4.13}$$

the eigenvalues of $P^{-1}A$ clustered around unity is equivalent to the eigenvalues of $A^{-1}\Delta A$ clustered around zero. To study the clustering radius of the eigenvalues of $A^{-1}\Delta A$, we divide the discussion into four steps.

Step 1: Relate the generalized eigenvalue problem of $A^{-1}\Delta A$ to another generalized eigenvalue problem $B_A^{-1}\Delta B_A$ where B_A and ΔB_A are $2N \times 2N$ block Toeplitz matrices (Lemma 3), and show that ΔB_A is asymptotically equivalent to a low rank matrix ΔB_F.

Step 2: Transform B_A and ΔB_F into matrices Q_A and ΔQ_F whose eigenstructures are easier to understand (Lemma 4).

Step 3: Extract the invariant part $\Delta\overline{Q}_F$ from ΔQ_F such that $\Delta\overline{Q}_F$ is asymptotically equivalent to ΔQ_F and that the nonzero elements of $\Delta\overline{Q}_F$ do not change with N (Lemma 5).

Step 4: Use the perturbation theory of eigenvalues to determine the spectral clustering radius of $A^{-1}\Delta A$ or $P^{-1}A$ (Lemmas 6 and 7, Theorem 3).

The system (2.1) can be augmented into

$$\begin{bmatrix} T & T_H \\ T_H & T \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ J\mathbf{x} \end{bmatrix} = \begin{bmatrix} \mathbf{b} \\ J\mathbf{b} \end{bmatrix},$$

where the property $JW = WJ$ for any symmetric Toeplitz W and the symmetric assumptions for T and T_H are used. Let us define

$$B_A = \begin{bmatrix} T & T_H \\ T_H & T \end{bmatrix}, \qquad \Delta B_A = \begin{bmatrix} \Delta T & \Delta T_H \\ \Delta T_H & \Delta T \end{bmatrix}. \tag{4.14}$$

Since T, T_H, ΔT and ΔT_H are symmetric centrosymmetric matrices, B_A, ΔB_A are also symmetric centrosymmetric. It is straightforward to verify that $B_A^{-1}\Delta B_A$ has the symmetric and skew-symmetric vectors

$$\mathbf{y}_+ = \begin{bmatrix} \mathbf{x}_+ \\ J\mathbf{x}_+ \end{bmatrix}, \qquad \mathbf{y}_- = \begin{bmatrix} \mathbf{x}_- \\ -J\mathbf{x}_- \end{bmatrix}, \tag{4.15}$$

as its eigenvectors, where their associated eigenvalues λ_+ and λ_- and the vectors $\mathbf{x}_+$ and $\mathbf{x}_-$ can be determined by considering two decoupled subproblems [25], i.e.

$$\begin{aligned} (\Delta T + J\Delta T_H)\mathbf{x}_+ &= \lambda_+(T + JT_H)\mathbf{x}_+, \\ (\Delta T - J\Delta T_H)\mathbf{x}_- &= \lambda_-(T - JT_H)\mathbf{x}_-. \end{aligned} \tag{4.16}$$

The first equation of (4.16) turns out to be the same as the original generalized eigenvalue problem $A^{-1}\Delta A$. Hence, we obtain the following lemma.

LEMMA 3. *Let T and H be $N \times N$ symmetric centrosymmetric Toeplitz and Hankel matrices generated by $T(z)$ and $H(z)$ in (4.3) with the corresponding generating sequences satisfying (4.1) and (4.2).*

The eigenvalues of $A^{-1}\Delta A$ are those of $B_A^{-1}\Delta B_A$, defined in (4.14), associated with the symmetric eigenvectors $\mathbf{y}_+$ given in (4.15).

It is worthwhile to point out that the reason to examine the eigenvalues of $B_A^{-1}\Delta B_A$, rather than those of $A^{-1}\Delta A$ directly, is that the difficulty caused by the time-reversal operator J can be avoided. As given in the proof of Lemma 2, ΔT and ΔT_H are asymptotically equivalent to ΔF_T and ΔF_H, respectively. Therefore, ΔB_A is asymptotically equivalent to

$$\Delta B_F = \begin{bmatrix} \Delta F_T & \Delta F_H \\ \Delta F_H & \Delta F_T \end{bmatrix}.$$

Consider the transformation for B_A and ΔB_F by premultiplying $B_L^T B_L$ and postmultiplying $B_L^T B_L$, where

$$B_L = \begin{bmatrix} L_{B,T}L_{B,H} & 0 \\ 0 & L_{B,T}L_{B,H} \end{bmatrix},$$

and where $L_{B,T}$ and $L_{B,H}$ are $N \times N$ lower triangular Toeplitz matrices with the first N coefficients in $B_T(z^{-1})$ and $B_H(z^{-1})$ as the first columns, respectively. Note that $\|B_L\|_p$, $p = 1, 2, \infty$, is bounded by a constant independent of N due to condition (4.1). Thus, the generalized eigenvalue problem

$$\Delta B_F \mathbf{y} = \lambda B_A \mathbf{y}$$

is transformed to another generalized eigenvalue problem

$$\Delta Q_F \tilde{\mathbf{y}} = \lambda Q_A \tilde{\mathbf{y}}, \qquad \mathbf{y} = B_L^T B_L \tilde{\mathbf{y}}, \tag{4.17}$$

where

$$\Delta Q_F = B_L^T B_L \Delta B_F B_L^T B_L = \begin{bmatrix} \Delta \mathcal{T} & \Delta \mathcal{H} \\ \Delta \mathcal{H} & \Delta \mathcal{T} \end{bmatrix}, \tag{4.18}$$

$$Q_A = B_L^T B_L B_A B_L^T B_L = \begin{bmatrix} \mathcal{T} & \mathcal{H} \\ \mathcal{H} & \mathcal{T} \end{bmatrix}, \tag{4.19}$$

and where

$$\begin{aligned}
\Delta \mathcal{T} &= (L_{B,T}^T L_{B,H}^T L_{B,T} L_{B,H})\, \Delta F_T\, (L_{B,T}^T L_{B,H}^T L_{B,T} L_{B,H}), \\
\Delta \mathcal{H} &= (L_{B,T}^T L_{B,H}^T L_{B,T} L_{B,H})\, \Delta F_H\, (L_{B,T}^T L_{B,H}^T L_{B,T} L_{B,H}), \\
\mathcal{T} &= (L_{B,T}^T L_{B,H}^T L_{B,T} L_{B,H})\, T\, (L_{B,T}^T L_{B,H}^T L_{B,T} L_{B,H}), \\
\mathcal{H} &= (L_{B,T}^T L_{B,H}^T L_{B,T} L_{B,H})\, T_H\, (L_{B,T}^T L_{B,H}^T L_{B,T} L_{B,H}).
\end{aligned}$$

The motivation for the transformation (4.17) is that ΔQ_F and Q_A have some special structures so that the eigenstructure of $Q_A^{-1}\Delta Q_F$ can be understood more easily. The special structures of ΔQ_F are described in Lemma 4.

LEMMA 4. *Let T and H be $N \times N$ symmetric centrosymmetric Toeplitz and Hankel matrices generated by $T(z)$ and $H(z)$ in (4.3) with the corresponding generating sequences satisfying (4.1) and (4.2). The elements of $\Delta \mathcal{T}$ are zeros except the four $(r_T + \beta_H) \times (r_T + \beta_H)$ corner blocks, and the elements of $\Delta \mathcal{H}$ are zeros except the four $(r_H + \beta_T) \times (r_H + \beta_T)$ corner blocks.*

Proof. Recall from (4.9) that $\Delta F_T = F_{T,1} + F_{T,2}$ and

$$(F_{T,1})_{i,j} = t_{N+i-j}, \qquad (F_{T,2})_{i,j} = t_{-N+i-j},$$

The (i, j) element of $F_{T,1} L_{B,T}^T L_{B,H}^T L_{B,T} L_{B,H}$ is

$$\sum_{k=1}^{N}\sum_{l=1}^{N}\sum_{m=1}^{N}\sum_{n=1}^{N} t_{N+i-k}\, b_{T,l-k}\, b_{H,m-l}\, b_{T,m-n}\, b_{H,n-j}, \tag{4.20}$$

where $b_{T,i}=0$ if $i>\beta_T$ and $b_{H,i}=0$ if $i>\beta_H$. If $j\le N-(r_T+\beta_H)$, the above summation can be simplified as

$$\sum_{k'=0}^{\beta_T}\sum_{l'=0}^{\beta_H}\sum_{n'=0}^{\beta_H}\left(\sum_{m'=0}^{\beta_T} t_{N+i+k'+l'-m'-n'-j}b_{T,m'}\right)b_{T,k'}b_{H,l'}b_{H,n'}=0, \tag{4.21}$$

where $k'=l-k$, $l'=m-l$, $m'=m-n$, $n'=n-j$, and the equality is due to (4.8). Similarly, the (i,j) element of $F_{T,2}L_{B,T}^TL_{B,H}^TL_{B,T}L_{B,H}$ can be simplified as

$$\sum_{l'=0}^{\beta_H}\sum_{m'=0}^{\beta_T}\sum_{n'=0}^{\beta_H}\left(\sum_{k'=0}^{\beta_T} t_{-N+i+k'+l'-m'-n'-j}b_{T,k'}\right)b_{H,l'}b_{T,m'}b_{H,n'}=0,$$

for $j>s_T+\beta_H$. Thus, the elements of $\Delta F_TL_{B,T}^TL_{B,H}^TL_{B,T}L_{B,H}$ are zeros except the first and last $r_T+\beta_H$ columns since T is symmetric ($s_T=r_T$). We can argue in a similar fashion that $L_{B,T}^TL_{B,H}^TL_{B,T}L_{B,H}\Delta F_T$ are zeros except the first and last $r_T+\beta_H$ rows. Since all the nonzero elements of $\Delta\mathcal{T}$ are in the intersections of the above nonzero columns and rows, the elements of $\Delta\mathcal{T}$ are zeros except the four $(r_T+\beta_H)\times(r_T+\beta_H)$ corner blocks. The structure of $\Delta\mathcal{H}$ can be proved by the same approach. □

In order to apply the perturbation theory, we have to search an asymptotically equivalent matrix of $\Delta\mathcal{T}$ with invariant nonzero elements. Let us examine the elements in the four corner blocks of $\Delta F_TL_{B,T}^TL_{B,H}^TL_{B,T}L_{B,H}$, or equivalently, those of

$$F_{T,1}L_{B,T}^TL_{B,H}^TL_{B,T}L_{B,H} \quad\text{and}\quad F_{T,2}L_{B,T}^TL_{B,H}^TL_{B,T}L_{B,H}.$$

According to (4.20) and (4.21), the magnitude of the (i,j) element of $F_{T,1}L_{B,T}^TL_{B,H}^TL_{B,T}L_{B,H}$ is bounded by

$$\begin{aligned}&\sum_{k'=0}^{\beta_T}\sum_{l'=0}^{\beta_H}\sum_{m'=0}^{\beta_T}\sum_{n'=0}^{\beta_H}|t_{N+i+k'+l'-m'-n'-j}|\,|b_{T,k'}|\,|b_{H,l'}|\,|b_{T,m'}|\,|b_{H,n'}|\\ \le\ &\sum_{k'=0}^{\beta_T}|b_{T,k'}|\sum_{l'=0}^{\beta_H}|b_{H,l'}|\sum_{m'=0}^{\beta_T}|b_{T,m'}|\sum_{n'=0}^{\beta_H}|b_{H,n'}|\max_{-(\beta_T+\beta_H)\le n\le(\beta_T+\beta_H)}|t_{N+i-j+n}|.\end{aligned}$$

To determine an upper bound for $\sum_{k=0}^{\beta_T}|b_{T,k}|$ (or $\sum_{l=0}^{\beta_H}|b_{H,l}|$), we factorize $B_T(z^{-1})$ as

$$B_T(z^{-1})=(1-r_1z^{-1})(1-r_2z^{-1})\cdots(1-r_{\beta_T}z^{-1}).$$

A direct consequence of (4.1) is that $|r_i|<1$, $1\le i\le\beta_T$, so that

$$|b_{T,k}|\le\binom{\beta_T}{k}(\max|r_i|)^k\le\binom{\beta_T}{k},\quad\text{where}\quad\binom{\beta_T}{k}\equiv\frac{\beta_T!}{(\beta_T-k)!k!}.$$

Therefore, we obtain

$$\sum_{k=0}^{\beta_T}|b_{T,k}|\le\sum_{k=0}^{\beta_T}\binom{\beta_T}{k}=2^{\beta_T}.$$

Similarly, $\sum_{l=0}^{\beta_H}|b_{H,l}|\le 2^{\beta_H}$. Thus, the magnitude of the (i,j) element of $F_{T,1}L_{B,T}^TL_{B,H}^TL_{B,T}L_{B,H}$ is bounded by

$$4^{\beta_T+\beta_H}\max_{-(\beta_T+\beta_H)\le n\le(\beta_T+\beta_H)}|t_{N+i-j+n}|.$$

It is straightforward to show that the (i,j) element of

$$\mathcal{T}_1=(L_{B,T}^TL_{B,H}^TL_{B,T}L_{B,H})\,F_{T,1}\,(L_{B,T}^TL_{B,H}^TL_{B,T}L_{B,H}),$$

is bounded by

$$16^{\beta_T+\beta_H} \max_{-2(\beta_T+\beta_H)\le n\le 2(\beta_T+\beta_H)} |t_{N+i-j+n}| = O(|t_{N+i-j+n}|). \tag{4.22}$$

We can see from (4.22) that the nonzero elements in the northeast, northwest, southeast and southwest $(r_T+\beta_H)\times(r_T+\beta_H)$ blocks of $\mathcal{T}_1$ are bounded by $O(|t_0|)$, $O(|t_N|)$, $O(|t_N|)$ and $O(|t_{2N}|)$, respectively. In addition, one can verify that the elements in the northeast $(r_T+\beta_H)\times(r_T+\beta_H)$ block of $\mathcal{T}_1$ are fixed for sufficiently large N, which is due to the banded structures of $L_{B,T}^T L_{B,H}^T L_{B,T} L_{B,H}$ [15], [26]. Since T is symmetric, $F_{T,2} = F_{T,1}^T$ and

$$\mathcal{T}_2 = (L_{B,T}^T L_{B,H}^T L_{B,T} L_{B,H})\ F_{T,2}\ (L_{B,T}^T L_{B,H}^T L_{B,T} L_{B,H}) = \mathcal{T}_1^T.$$

To conclude, the matrix $\Delta\mathcal{T}$ asymptotically converges to

$$\Delta\overline{\mathcal{T}} = \begin{bmatrix} 0 & 0 & \Delta\overline{Q}_T \\ 0 & 0 & 0 \\ \Delta\overline{Q}_T^T & 0 & 0 \end{bmatrix}$$

where $\Delta\overline{Q}_T$ is an $(r_T+\beta_H)\times(r_T+\beta_H)$ block whose elements do not change with N. Clearly, similar arguments apply to $\Delta\mathcal{H}$. The above discussion is summarized in the following lemma.

LEMMA 5. *Let T and H be $N\times N$ symmetric centrosymmetric Toeplitz and Hankel matrices generated by $T(z)$ and $H(z)$ in (4.3) with the corresponding generating sequences satisfying (4.1) and (4.2). We can split ΔQ_F into two parts*

$$\Delta Q_F = \Delta\overline{Q}_F + (\Delta Q_F - \Delta\overline{Q}_F)$$

such that the $2N\times 2N$ matrix $\Delta\overline{Q}_F$ can be expressed in block matrix form as

$$\Delta\overline{Q}_F = \begin{bmatrix} 0 & 0 & \Delta\overline{Q}_T & \vdots & 0 & 0 & \Delta\overline{Q}_H \\ 0 & 0 & 0 & \vdots & 0 & 0 & 0 \\ \Delta\overline{Q}_T^T & 0 & 0 & \vdots & \Delta\overline{Q}_H^T & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \Delta\overline{Q}_H & \vdots & 0 & 0 & \Delta\overline{Q}_T \\ 0 & 0 & 0 & \vdots & 0 & 0 & 0 \\ \Delta\overline{Q}_H^T & 0 & 0 & \vdots & \Delta\overline{Q}_T^T & 0 & 0 \end{bmatrix}, \tag{4.23}$$

where $\Delta\overline{Q}_T$ is the the northeast $(r_T+\beta_H)\times(r_T+\beta_H)$ block of

$$\mathcal{T}_1 = (L_{B,T}^T L_{B,H}^T L_{B,T} L_{B,H})\ F_{T,1}\ (L_{B,T}^T L_{B,H}^T L_{B,T} L_{B,H}),$$

and $\Delta\overline{Q}_H$ is the the northeast $(r_H+\beta_T)\times(r_H+\beta_T)$ block of

$$\mathcal{H}_1 = (L_{B,T}^T L_{B,H}^T L_{B,T} L_{B,H})\ F_{H,1}\ (L_{B,T}^T L_{B,H}^T L_{B,T} L_{B,H}).$$

The nonzero elements of $\Delta\overline{Q}_T$ and $\Delta\overline{Q}_H$ do not change with the problem size N. The matrix $\Delta Q_F - \Delta\overline{Q}_F$ asymptotically converges to the zero matrix at a rate bounded by

$$||\Delta Q_F - \Delta\overline{Q}_F||_2 \le O(|t_N| + |h_N|). \tag{4.24}$$

Note that $\Delta\overline{Q}_F$ has at most

$$4\max(r_T+\beta_H, r_H+\beta_T)$$

nonzero columns (or rows), which is also equal to 2η for symmetric centrosymmetric T and H with $\eta_c = 0$. The rank η_r of $\Delta\overline{Q}_F$ is therefore bounded by

$$\eta_r \le 2\eta \le 2\tilde{\eta}, \tag{4.25}$$

where $\tilde{\eta}$ and η are defined in (4.10) and (4.11). Combining Lemmas 3, 4 and 5, we obtain another proof for the number of outliers as stated in Theorem 2 with symmetric centrosymmetric T and H. However, our focus here is the perturbation effect on the other $2(N-\eta)$ eigenvalues repeated exactly at zero. For the case that $B_T(z)D_T(z)$ and $B_H(z)D_H(z)$ share common roots, i.e. $\eta_c > 0$, we can modify the premultiplying and the postmultiplying matrices as discussed in the proof of Theorem 2, namely, to delete one of the two repeated triangular banded Toeplitz matrices corresponding to the common roots in $L_{B,T}$ and $L_{B,H}$ for transformation (4.17).

Similar to (4.18) and (4.19), we transform ΔB_A defined in (4.14) to

$$\Delta Q_A = B_L^T B_L \Delta B_A B_L^T B_L,$$

and view ΔQ_A as the sum of $\Delta\overline{Q}_F$ and a perturbation matrix

$$\Delta Q_E \equiv \Delta Q_A - \Delta\overline{Q}_F = (\Delta Q_A - \Delta Q_F) + (\Delta Q_F - \Delta\overline{Q}_F). \tag{4.26}$$

To set up the framework for the perturbation analysis for eigenvalues, two more lemmas (Lemmas 6 and 7) are needed.

According to Lemma 2, the boundness of $||B_L||_2$ and the symmetric properties of $\{t_n\}$ and $\{h_n\}$, we have

$$||\Delta Q_A - \Delta Q_F||_2 \le O(|t_N| + |h_N|).$$

The above equation, (4.24) and (4.26) give a bound on ΔQ_E, i.e.

$$||\Delta Q_E||_2 \le O(|t_N| + |h_N|).$$

In addition, we know from (4.19) that

$$||Q_A^{-1}||_2 \le ||(B_L^T B_L)^{-1}||_2 ||B_A^{-1}||_2 ||(B_L^T B_L)^{-1}||_2.$$

In above, $||(B_L^T B_L)^{-1}||_2$ is bounded by a constant independent of N, since both $||(B_{L,T}^T B_{L,T})^{-1}||_2$ and $||(B_{L,H}^T B_{L,H})^{-1}||_2$ are bounded by a constant independent of N due to (4.1). The matrix B_A is asymptotically equivalent to the block circulant matrix

$$B_K = \begin{bmatrix} K_T & K_H \\ K_H & K_T \end{bmatrix}.$$

Since $||B_K^{-1}||_2$ is bounded due to (4.2), $||B_A^{-1}||_2$ is also bounded by a constant independent of N. Therefore, $||Q_A^{-1}||_2 = O(1)$ and we obtain the following lemma.

LEMMA 6. *Let T and H be $N \times N$ symmetric centrosymmetric Toeplitz and Hankel matrices generated by $T(z)$ and $H(z)$ in (4.3) with the corresponding generating sequences satisfying (4.1) and (4.2). Then,*

$$||Q_A^{-1}\Delta Q_E||_2 \le \epsilon = O(|t_N| + |h_N|),$$

for sufficiently large N.

We arrange the eigenvalues of $Q_A^{-1}\Delta\overline{Q}_F$ in a descending order, i.e. $|\lambda_n| \ge |\lambda_{n+1}|$. and denote the corresponding normalized right-hand and left-hand eigenvectors by $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_{2N}$ and $\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_{2N}$, respectively. Since the rank of $\Delta\overline{Q}_F$ is η_r, $\lambda_n = 0$ for $\eta_r < n \le 2N$. We choose vectors $\mathbf{x}_n = \mathbf{y}_n$ with $\eta_r < n \le 2N$ to be othorgonal for different values of n. We also define

$$s_n = \mathbf{y}_n^H \mathbf{x}_n, \qquad 1 \le n \le 2N. \tag{4.27}$$

The reciprocal of s_n is usually known as the condition of the eigenvalue of λ_n [17], which is bounded uniformly by a constant independent of N as stated in Lemma 7.

LEMMA 7. *Let T and H be $N \times N$ symmetric centrosymmetric Toeplitz and Hankel matrices generated by $T(z)$ and $H(z)$ in (4.3) with the corresponding generating sequences satisfying (4.1) and (4.2). Then, the $|s_m^{-1}|, 1 \le m \le \eta_r$, of $Q_A^{-1}\Delta\overline{Q}_F$ is bounded by a constant independent of N.*

Proof. According to (4.23), it is clear that the rank space of $Q_A^{-1}\Delta\overline{Q}_F$ is contained in

$$\mathcal{R} = \{\mathbf{v} \in R^{2N} \mid v_n = 0,\ n_0 < n \le N - n_0 \text{ or } N + n_0 < n \le 2N - n_0\},$$

where $n_0 = \max(r_T + \beta_H, r_H + \beta_T)$. All nonzero eigenvalues of $Q_A^{-1}\Delta\overline{Q}_F$ can be determined by considering the generalized eigenvalue problem in the subspace $\mathcal{R}$. Since all the nonzero elements of $\Delta\overline{Q}_F$ in (4.23) do not change with N, the boundness of $|s_m^{-1}|$, $1 \le m \le \eta_r$, is guaranteed if the four $n_0 \times n_0$ corner blocks of $\mathcal{T}$ and $\mathcal{H}$ do not change with N. By applying the isomorphism between the ring of the power series and the ring of semi-infinite lower (or upper) triangular Toeplitz matrices with respect to $T(z)$ and $H(z)$ given in (4.3), we have

$$T = L_{A,T}L_{B,T}^{-1} + (L_{A,T}L_{B,T}^{-1})^T, \qquad T_H = L_{A,H}L_{B,H}^{-1} + (L_{A,H}L_{B,H}^{-1})^T,$$

where $L_{A,T}$ and $L_{A,H}$ are $N \times N$ lower triangular Toeplitz matrices with the first N coefficients in $A_T(z^{-1})$ $(A_H(z^{-1}))$ as the first columns, respectively. By using the commutative property of the matrix product among lower (or upper) triangular Toeplitz matrices, $\mathcal{T}$ and $\mathcal{H}$ defined in (4.19) are products of lower and upper triangular banded Toeplitz matrices. It is then straightforward to show that elements in the corresponding four corner blocks of $\mathcal{T}$ (and $\mathcal{H}$) do not change with N, and the proof is completed. □

The eigenvalues of

$$Q_A^{-1}\Delta Q_A = Q_A^{-1}\Delta\overline{Q}_F + Q_A^{-1}\Delta Q_E$$

can be estimated from those of $Q_A^{-1}\Delta\overline{Q}_F$ through perturbation theory. Since the norm of the perturbation matrix is equal to ε as given in Lemma 6, we denote the eigenvalues and the right-hand eigenvectors of the perturbed matrix $Q_A^{-1}\Delta Q_A$ by $\lambda_n(\varepsilon)$ and $\mathbf{x}_n(\varepsilon)$, respectively. According to perturbation theory for repeated eigenvalues [35], the eigenvectors $\mathbf{x}_n(\varepsilon)$ with $\eta_r < n \le 2N$ must take the form

$$\mathbf{x}_n(\varepsilon) = \sum_{m=1}^{\eta_r} \frac{\xi_{mn}}{(\lambda_n - \lambda_m)s_m}\mathbf{x}_m + \sum_{m=\eta_r+1}^{N} g_{mn}\mathbf{x}_m + O(\varepsilon^2),$$

where $\xi_{mn} = \mathbf{y}_m^H Q_A^{-1}\Delta Q_E \mathbf{x}_n$, $\lambda_n = 0$, $g_{nn} = 1$ and s_m defined in (4.27). Due to the construction, we know that

$$\|\mathbf{x}_n(\varepsilon)\|_2 \ge \|\mathbf{x}_n\|_2 = 1.$$

The factor $|\xi_{mn}|$ is bounded by

$$|\xi_{mn}| = |\mathbf{y}_m^H Q_A^{-1}\Delta Q_E \mathbf{x}_n| \le \|\mathbf{y}_m\|_2\|Q_A^{-1}\Delta Q_E\|_2\|\mathbf{x}_n\|_2 \le \varepsilon.$$

The $|s_m^{-1}|$, $1 \le m \le \eta_r$, is also bounded due to Lemma 7. The magnitude of $\lambda_n(\varepsilon)$, $\eta_r < n \le 2N$, of $Q_A^{-1}\Delta Q_A$ (or $B_A^{-1}\Delta B_A$) is approximated by

$$\begin{aligned} |\lambda_k(\varepsilon)| &= \|Q_A^{-1}(\Delta\overline{Q}_F + \Delta Q_E)\mathbf{x}_n(\varepsilon)\|_2 \\ &\le \frac{\|\xi_{mn}Q_A^{-1}\Delta\overline{Q}_F\mathbf{x}_m\|_2}{|\lambda_m s_m|\ \|\mathbf{x}_n(\varepsilon)\|_2} + \|Q_A^{-1}\Delta Q_E\|_2 \\ &\le \sum_{m=1}^{\eta_r} \frac{\varepsilon}{|s_m|} + \varepsilon. \end{aligned}$$

Since $\eta_r \leq 2\eta$ as given by (4.25), we conclude that $Q_A^{-1}\Delta Q_A$ has at least $2(N-\eta)$ eigenvalues confined in the interval $(-\epsilon_Q, +\epsilon_Q)$ with

$$\epsilon_Q = O(|t_N| + |h_N|)$$

for sufficiently large N. Finally, the eigenvalues of $A^{-1}\Delta A$ are eigenvalues of $B_A^{-1}\Delta B_A$ (or $Q_A^{-1}\Delta Q_A$) with symmetric eigenvectors (Lemma 3) so that $A^{-1}\Delta A$ has at least $N-\eta$ eigenvalues confined in the interval $(-\epsilon_Q, +\epsilon_Q)$. With (4.13), the above results are summarized in the following theorem.

THEOREM 3. *Let T and H be $N \times N$ symmetric centrosymmetric Toeplitz and Hankel matrices generated by $T(z)$ and $H(z)$ in (4.3) with the corresponding generating sequences satisfying (4.1) and (4.2). For sufficiently large N, there are at least $N-\eta$ eigenvalues of $P^{-1}A$ confined in the interval $(1-\epsilon, 1+\epsilon)$ where*

$$\epsilon = O(|t_N| + |h_N|).$$

It is possible to obtain an estimate for the spectral clustering radius for nonsymmetric T and H. That is,

$$\epsilon = O(|t_N| + |t_{-N}| + |h_N| + |h_{-N}|). \tag{4.28}$$

However, the proof is much more involved. We refer to [24] for details, and use numerical results to demonstrate the estimate (4.28) in this paper.

5. Numerical experiments.

We use four test problems, including symmetric and nonsymmetric T and T_H, to illustrate the analysis in §4. For all Toeplitz-plus-Hankel systems $A\mathbf{x} = \mathbf{b}$ to be solved in the experiments, we choose $\mathbf{b} = (1, \cdots, 1)^T$ and the zero initial guess.

Test Problem1. Symmetric T and T_H with $(\alpha_T, \beta_T, \gamma_T, \delta_T) = (1,1,1,1)$ and $(\alpha_H, \beta_H, \gamma_H, \delta_H) = (1,2,1,2)$.

The generating functions of T and T_H are chosen to be

$$T(z) = \frac{0.5 + 0.7z^{-1}}{1 + 0.7z^{-1}} + \frac{0.5 + 0.7z}{1 + 0.7z}, \qquad H(z) = \frac{0.5 - 0.4z^{-1}}{(1 - 0.7z^{-1})(1 - 0.9z^{-1})} + \frac{0.5 - 0.4z}{(1 - 0.7z)(1 - 0.9z)}.$$

The eigenvalues of $A^{-1}\Delta A$, except those with magnitude less than 10^{-6}, are plotted in Fig. 1. Although it is difficult to distinguish the outliers from the clustered eigenvalues for $N \leq 64$, we can see 6 outliers more easily for the case $N = 128$. The number of outliers is consistent with (4.11), where the last three terms are all equal to zero. The clustering radii ϵ, $|t_N|$ and $|h_N|$ for different N are listed in Table 1. The values of ϵ decrease at a rate of $O(|t_N| + |h_N|)$. The convergence history of the CG and PCG methods are plotted in Fig. 2. The upper four curves are those of the CG method whereas the lower four curves correspond to those of the PCG method. The preconditioning does accelerate the convergence rate of the CG method significantly. The convergence rate of the CG method becomes slower for larger N. In contrast, the PCG method converges faster as N becomes larger. It in fact converges in four $(= \eta/2 + 1)$ iterations asymptotically. The reason that it takes only $\eta/2 + 1$ iterations for PCG method to converge can be explained by that the outliers are related in pairs such that only $\eta/2$ iterations are needed to eliminate the effects of all outliers. A similar phenomenon has been reported in [25] for solving symmetric positive-definite Toeplitz systems with the PCG method.

Test Problem 2. Symmetric T with $(\alpha_T, \beta_T, \gamma_T, \delta_T) = (1,1,1,1)$ and nonsymmetric T_H with $(\alpha_H, \beta_H, \gamma_H, \delta_H) = (0,0,1,3)$.

The generating functions of T and T_H are chosen to be

$$T(z) = \frac{0.5 + 0.3z^{-1}}{1 + 0.8z^{-1}} + \frac{0.5 + 0.3z}{1 + 0.8z}, \qquad H(z) = \frac{0.5 - 0.4z}{(1 - 0.5z)(1 + 0.8z)^2}.$$

TABLE 1
The clustering radius ϵ of $P^{-1}A$ for Test Problem 1.

N	$\|t_N\| = \|t_{-N}\|$	$\|h_N\| = \|h_{-N}\|$	ϵ
16	1.6×10^{-3}	4.7×10^{-2}	1.5×10^{-1}
32	5.5×10^{-6}	8.6×10^{-3}	8.8×10^{-2}
64	6.1×10^{-11}	2.9×10^{-4}	1.3×10^{-2}
128	7.0×10^{-21}	3.5×10^{-7}	1.4×10^{-5}

TABLE 2
The clustering radius ϵ of $P^{-1}A$ for Test Problem 2.

N	$\|t_N\| = \|t_{-N}\|$	$\|h_{-N}\|$	ϵ
16	3.5×10^{-3}	2.9×10^{-1}	8.9×10^{-1}
32	9.9×10^{-5}	1.6×10^{-2}	4.1×10^{-2}
64	7.8×10^{-8}	2.5×10^{-5}	3.1×10^{-4}
128	4.9×10^{-14}	3.1×10^{-11}	8.2×10^{-10}

Note that $B_T(z^{-1}) = 1 + 0.8z^{-1}$ and $B_T(z)D_T(z) = (1+0.8z)^2$. Since $B_T(z)D_T(z)$ and $D_H(z)$ have two common roots ($\eta_c = 2$), we know that there are at most 3 outliers according to (4.11), which is confirmed numerically. The other eigenvalues of $P^{-1}A$ are confined in the disk centered at unity with radius ϵ. Since $|h_{-N}| >> |t_N| = |t_{-N}|$, $\epsilon = O(|h_{-N}|)$. The clustering radii ϵ, $|t_N|$ and $|h_{-N}|$ are listed in Table 2. We apply the CGS method to solve the Toeplitz-plus-Hankel system and plot the convergence history in Fig. 3. The lower and upper four curves are those of the CGS method with and without preconditioning, respectively. The preconditioned CGS method converges faster as N becomes larger, and converges in 4 iterations asymptotically. In contrast, the CGS method without preconditioning does not converge at all.

Test Problem 3. Nonsymmetric T with $(\alpha_T, \beta_T, \gamma_T, \delta_T) = (1,2,0,0)$ and nonsymmetric T_H with $(\alpha_H, \beta_H, \gamma_H, \delta_H) = (1,1,0,0)$.

The generating functions of T and T_H are chosen to be

$$T(z) = \frac{1 - 0.9z^{-1}}{(1+0.5z^{-1})(1+0.8z^{-1})}, \qquad H(z) = \frac{1+0.5z^{-1}}{1-0.7z^{-1}}.$$

The theory predicts 3 outliers, which is confirmed by the experiment for large N. All other eigenvalues of $P^{-1}A$ are confined in the disk centered at unity with radius ϵ. The values of ϵ, $|t_N|$ and $|h_{-N}|$ are listed in Table 3. Since $|t_N| >> |h_N|$, ϵ decreases at a rate of $O(|t_N|)$. The convergence history of the CGS method with and without preconditioner is plotted in Fig. 4. They correspond to the lower and upper four curves, respectively. Again, we observe that the CGS method converges in 4 iterations for large N whereas the CGS method without preconditioning does not converge at all.

Test Problem 4. Symmetric nonrational T and T_H.

The preconditioner P is applied to nonrational Toeplitz-plus-Hankel matrices, where T and T_H are symmetric Toeplitz matrices with generating sequences

$$t_n = \frac{1}{|n|+1} \quad \text{and} \quad h_n = \frac{1}{(|n|+1)^{1.1}},$$

respectively. The eigenvalues of A and $P^{-1}A$ are plotted in Fig. 5. Although the spectral properties of $P^{-1}A$ are beyond our analysis in §3 and §4, the preconditioner P still provides a good spectral clustering property. We apply the CGN method to solve the preconditioned syetem $P^{-1}A\mathbf{x} = P^{-1}\mathbf{b}$, and plot the convergence history of this method with and without preconditioning in Fig. 6, where they correspond to the lower and upper three curves, respectively. It is clear that the CGN method with preconditioner K converges faster than that without preconditioning.

TABLE 3
The clustering radius ϵ of $P^{-1}A$ for Test Problem 3.

N	$\|t_N\|$	$\|h_N\|$	ϵ
16	1.6×10^{-1}	5.7×10^{-3}	1.5×10^{-1}
32	4.5×10^{-3}	1.9×10^{-5}	1.0×10^{-2}
64	3.6×10^{-6}	2.1×10^{-10}	8.1×10^{-6}
128	2.2×10^{-12}	2.6×10^{-20}	1.1×10^{-11}

6. Conclusion. We generalized the circulant preconditioning technique from Toeplitz to Toeplitz-plus-Hankel matrices in this research. When the Toeplitz and Hankel matrices are both generated by rational functions, we proved that the eigenvalues of the preconditioned matrix are clustered around 1 except a finite number of outliers depending on the order of the generating functions, and that the clustering radius is proportional to the magnitudes of the last elements in Toeplitz and Hankel matrices. With the spectral properties, an $N \times N$ rational Toeplitz-plus-Hankel systems can be solved by preconditioned iterative methods with $O(N \log N)$ operations. Although our discussion has focused on real Toeplitz-plus-Hankel systems, the generalization to complex Toeplitz-plus-Hankel systems can be done in a straightforward way.

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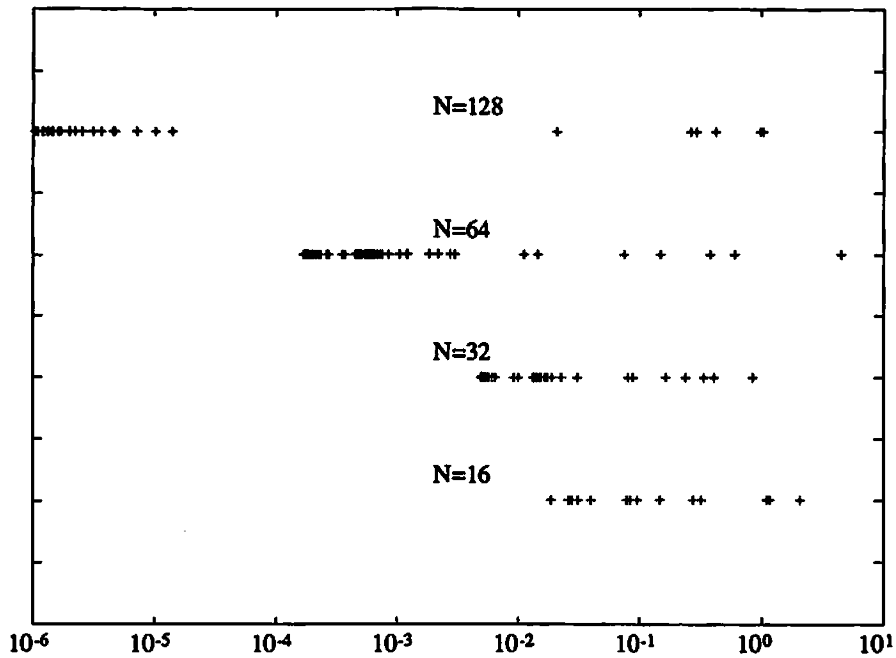

FIG. 1. *The eigenvalue distribution of $A^{-1}\Delta A$ for Test Problem 1.*

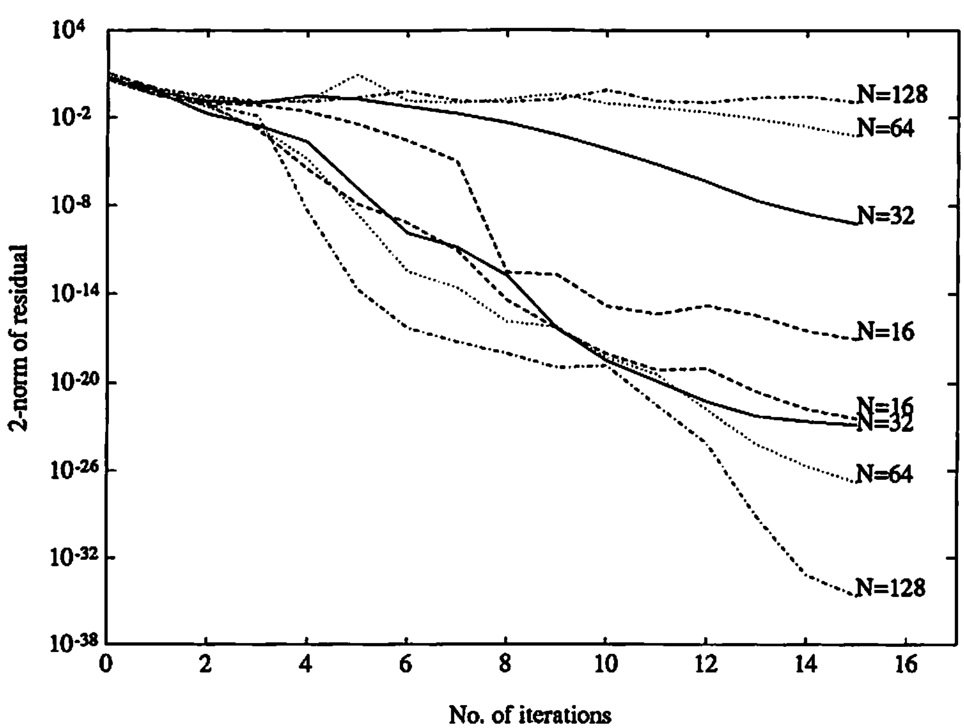

FIG. 2. *The convergence history of the PCG method for Test Problem 1.*

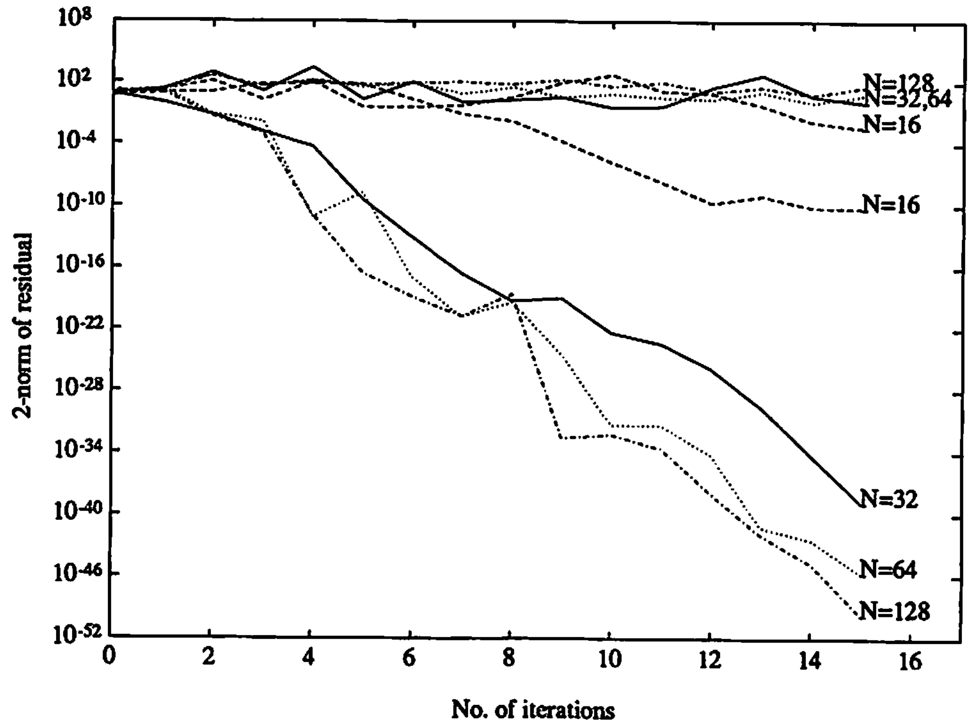

FIG. 3. *The convergence history of the CGS method for Test Problem 2.*

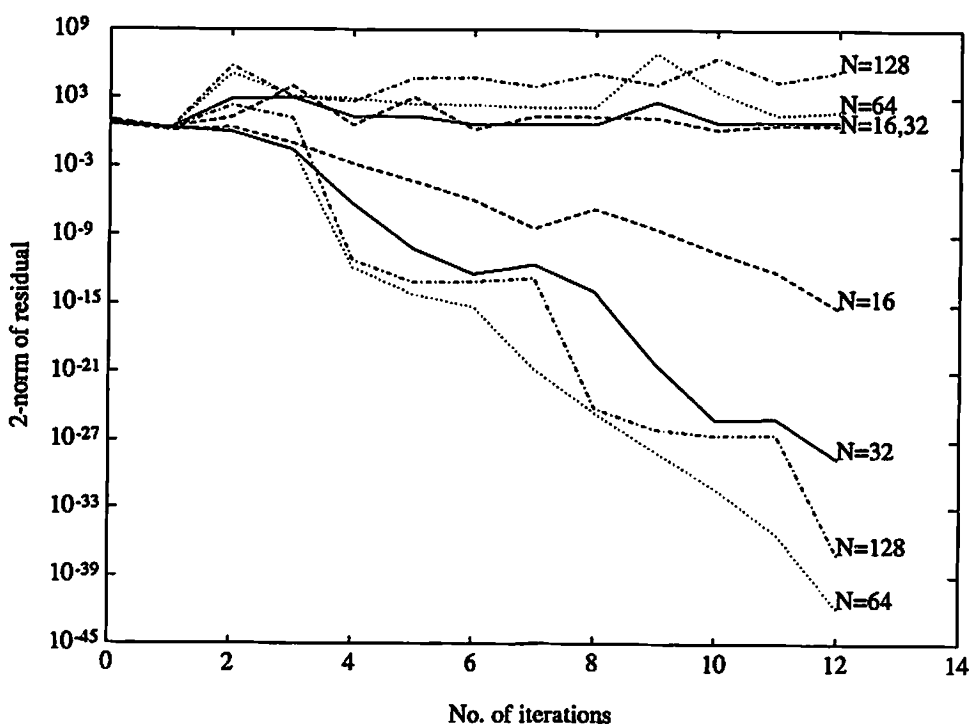

FIG. 4. *The convergence history of the CGS method for Test Problem 3.*

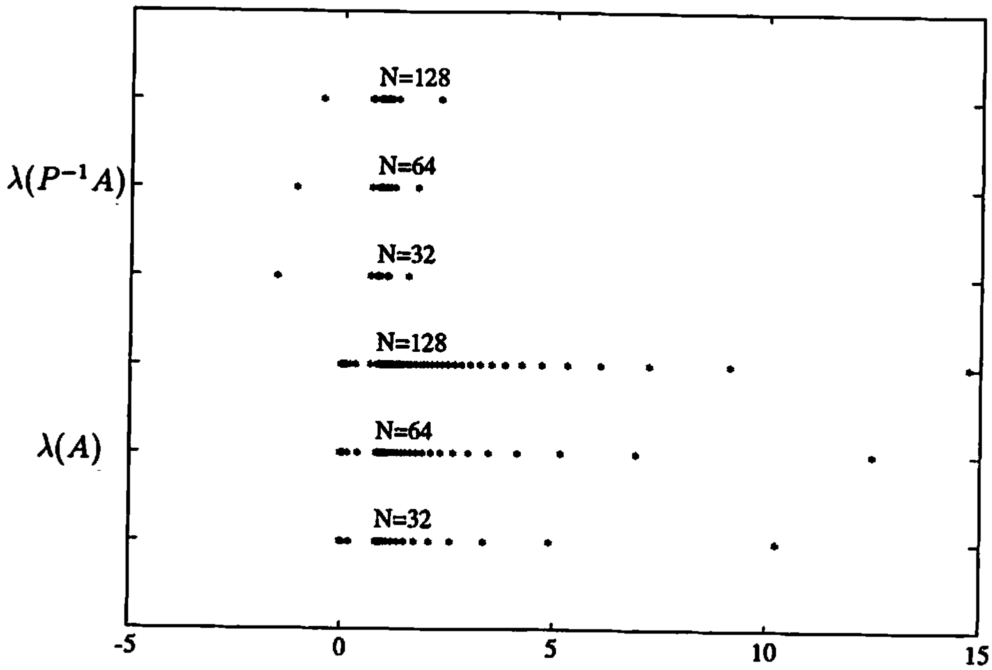

FIG. 5. *The eigenvalue distribution of A and $P^{-1}A$ for Test Problem 4.*

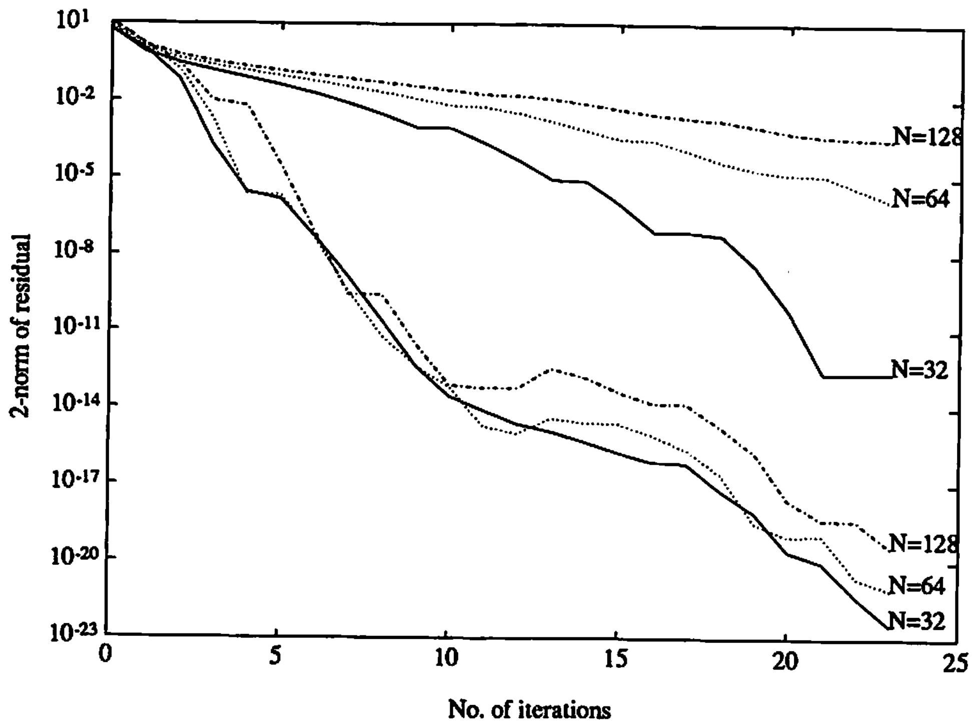

FIG. 6. *The convergence history of the CGN method for Test Problem 4.*