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ON THE ACCURACY OF COMPUTED WAVELET COEFFICIENTS*

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Abstract. Several types of wavelet transforms such as the continuous wavelet transform (CWT), wavelet series transform (WST) and discrete wavelet transform (DWT) have been considered by researchers. Only the DWT coefficients can be computed numerically, and they are often used to approximate the WST and CWT coefficients. Numerical algorithms have been proposed to compute the DWT coefficients such as the Mallat and the Shensa algorithms. In this research, we study the accuracy of the computed DWT coefficients obtained from these algorithms as an approximate of the WST coefficients. Our analysis indicates that the accuracy of the Shensa algorithm can be adjusted by choosing an appropriate prefilter, and the corresponding result is better than that of the Mallat algorithm.

Key words. wavelet, discrete wavelet transform, wavelet series transform.

AMS(MOS) subject classifications. 41, 42.

1. Introduction. Wavelet transforms have recently attracted a lot of attention as a useful tool for signal and image processing applications [1], [2], [3], [5], [6], [8], [9], [11], [14], [15], [19], [20], [23], [25], [27], [28]. Several different types of wavelet transforms have been examined by researchers, including the continuous wavelet transform (CWT) [11], wavelet series transform (WST) [7], [8], [12], [15], [16], [17], [18], and discrete wavelet transform (DWT) [21], [25]. Let $\mathbf{Z}$, $\mathbf{R}$, $L^2(\mathbf{R})$ denote the sets of integers and real numbers and the space of all square-integrable functions, respectively. The definitions of these transforms can be briefly summarized as follows. Consider a suitable function $\psi(t)$ called the mother wavelet whose dilations and translations

$$\{\psi_{jk}(t) \triangleq 2^{j/2}\psi(2^j t - k)\}_{j,k\in\mathbf{Z}} \tag{1.1}$$

form an orthonormal basis of $L^2(\mathbf{R})$ usually known as the wavelet basis. For $f(t) \in L^2(\mathbf{R})$, its CWT with respect to the mother wavelet $\psi(t)$ is defined as

$$CWT\{f(t); a, b\} = \int f(t)\psi_{a,b}(t)dt\,,$$

where

$$\psi_{a,b}(t) = \frac{1}{\sqrt{|a|}}\psi(\frac{t-b}{a}),\ a, b \in \mathbf{R},\ a \neq 0\,,$$

and where a and b are called the scale and time parameters, respectively. The WST of $f(t)$ is obtained by sampling its CWT on the scale-time plane (a, b) with the so-called "dyadic" grid, i.e.

$$WST\{f(t); j, k\} = CWT\{f(t); a = 2^{-j}, b = k2^{-j}\}\,,\ j, k \in \mathbf{Z}\,.$$

Thus, the WST coefficients, also denoted by $b_{j,k}$, can be determined by

$$WST\{f(t); j, k\} \triangleq b_{j,k} = \int f(t)\psi_{jk}(t)dt\,,\ j, k \in \mathbf{Z}\,. \tag{1.2}$$

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Since (1.1) is a basis, the WST coefficients provide a complete representation of $f(t)$. In contrast, the set of CWT coefficients contains redundant information. Furthermore, if the t as well as parameters (a, b) all take discrete values, which is recognized as a natural wavelet transform for the discrete-time signal $f(m\Delta t)$ with $m \in \mathbf{Z}$, the resulting transform is called the DWT. It is clear that only the DWT coefficients can be computed numerically, and the CWT and WST coefficients have to be approximated by those of DWT in practice.

Several numerical algorithms have been examined to compute the DWT coefficients such as the Mallat algorithm [7], [15], [16], [17], [22], the "à trous" algorithm of Holschneider et al. [13] and the Shensa algorithm [21] as a unified approach for the former two. The relationship of the Mallat and Shensa algorithms can be simply explained below. Let the sequence $x[n]$ be obtained by sampling a function $f(t) \in L^2(\mathbf{R})$. Mallat [17] proposed an effective recursive algorithm to compute the DWT of $x[n]$ by implementing the recursion with cascaded quadrature mirror filter banks [24]. Shensa [19], [21] suggested a prefiltering process, in which we convolve $x[n]$ with a sequence $q[n]$ to obtain a new sequence $x'[n]$ and then apply the same recursion to $x'[n]$. Thus, the Mallat algorithm can be viewed as a special case of the Shensa algorithm with $q[n]$ chosen to be the unit impulse sequence $\delta[n]$.

Efficient implementations and detailed computational complexity analysis for these algorithms were discussed by Rioul and Duhamel [19]. However, an important issue which has not yet been addressed is the numerical accuracy of the computed DWT coefficients $\tilde{b}_{j,k}$ with respect to the true WST coefficients $b_{j,k}$ as defined in (1.2). Two different types of errors occur in numerical computation: errors due to discrete approximation and errors due to finite-precision computation. In this research, we focus on the analysis of approximation errors. The error formulas of computed wavelet coefficients by the Mallat and Shensa algorithms are derived (see Theorems 1 and 2 in §3 and 4). With the error formula for the Shensa algorithm, we can determine a criterion for choosing the prefilter $q[n]$ to reduce the error as much as possible. Another interesting related problem is to determine the condition under which the WST coefficients of a function $f(t)$ can be exactly computed with the Shensa algorithm. In [26], Walter proposed one condition which guarantees an exact reconstruction of the WST coefficients. However, the proposed condition is difficult to verify in practice. A more general and easily verifiable sufficient condition is considered in this work (see Theorem 3 in §5). Finally, although all wavelet coefficients can be efficiently computed by the Mallat or Shensa algorithm, we may sometimes be interested in only a few coefficients. Direct evaluation of the integral may be preferred in this case. An error formula with direct integral evaluation is also examined (see Theorem 4 in §6).

This paper is organized as follows. We first briefly review the Mallat algorithm and the Shensa algorithm in §2. Then, the accuracy of the computed coefficients is analyzed in §3 and 4. In §5, we describe a sufficient condition which guarantees the exact computation of WST coefficients with the Shensa algorithm. It is also possible to compute the DWT coefficients via a direct evaluation of the integral of the product of a given function and the wavelet basis numerically. The error resulted from the direct integration is studied in §6. Finally, some concluding remarks are given in §7.

2. Review of computational algorithms for wavelet coefficients. Let us first introduce some notation and present several well known results. The $\hat{f}(\omega)$ is the Fourier transform of $f(t) \in L^2(\mathbf{R})$, i.e.

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt .$$

Let $\psi(t)$ be a wavelet mother function with its associated scaling function $\phi(t)$. It can be shown that they satisfy the two-scale difference equations [7], [15], [16], [17], [22],

$$\phi(t) = \sum_k h_k \phi(2t + k) , \tag{2.1}$$

and

$$\psi(t) = \sum_k g_k \phi(2t + k) , \tag{2.2}$$

with sequences h_k and g_k related via

$$g_k = (-1)^k h_{1-k}.$$

In order for (1.1) to provide an orthonomal basis, sequence h_k has to satisfy some conditions, i.e. the uniqueness, orthogonality and approximation conditions (see [22] for details). Similar to (1.1), we define

$$\phi_{jk}(t) \triangleq 2^{j/2}\phi(2^j t - k), \ j,k \in \mathbf{Z}.$$

Besides, the sampled signal of $f(t)$ is denoted by

$$x[n] \triangleq f(\frac{n}{2^J}) \tag{2.3}$$

for some integer $J > 0$.

Due to the complexity in direct evaluation of the integral (1.2) several fast algorithms have been proposed to compute $b_{j,k}$ [4], [10], [13], [17], [19]. In the following, we will review the well known Mallat algorithm and the Shensa algorithm which is a generalization of the "à trous" algorithm.

2.1. Mallat algorithm. Given an integer $J > 0$, the Jth resolution $f_J(t)$ of $f(t)$ can be represented by

$$f_J(t) = \sum_k c_{J,k}\phi_{Jk}(t),$$

where

$$c_{J,k} \triangleq \int f(t)\phi_{Jk}(t)dt, \ k \in \mathbf{Z}, \tag{2.4}$$

or

$$f_J(t) = \sum_{j<J}\sum_k b_{j,k}\psi_{jk}(t). \tag{2.5}$$

The Mallat algorithm specifies a systematic procedure to obtain approximations $c_{J,K}^{(M)}$ and $b_{j,k}^{(M)}$ for $c_{J,K}$ and $b_{j,k}$, respectively. Based on the algorithm [7], [17], [22], we first let

$$c_{J,k}^{(M)} \triangleq x[k] = f(\frac{k}{2^J}).$$

Then, $c_{j,k}^{(M)}$ and $b_{j,k}^{(M)}$ for $j < J, k \in \mathbf{Z}$ are computed recursively via

$$c_{j-1,k}^{(M)} = \frac{1}{2}\sum_n h_{n-2k}c_{j,n}^{(M)}, \tag{2.6}$$

$$b_{j-1,k}^{(M)} = \frac{1}{2}\sum_n g_{n-2k}c_{j,n}^{(M)}, \tag{2.7}$$

for $j = J, J-1, \cdots, J_c$ and $J_c < J$. We call $b_{j,k}^{(M)}$ in (2.7) the wavelet coefficients obtained from the Mallat algorithm. The following synthesis formula

$$c_{j,n}^{(M)} = \frac{1}{2}\sum_k h_{n-2k}c_{j-1,k}^{(M)} + \frac{1}{2}\sum_k g_{n-2k}b_{j-1,k}^{(M)}, \tag{2.8}$$

with $j = J_c, J_c+1, \cdots, J$ can be used to reconstruct $c_{J,k}^{(M)}$.

The recursion in (2.6) and (2.7) stops at $j = J_c$ corresponding to the coarsest resolution appropriate for a certain application. Without loss of generality, we let J_c approach to $-\infty$ and study the error of $b_{j,k}^{(M)}$ for all $j < J$. Note also that the Mallat algorithm defines the discrete wavelet transform (DWT) for the sequence $x[n]$ [19]. Thus, we have

$$DWT\{x[n]; j,k\} \equiv b_{j,k}^{(M)}, \ j < J, k \in \mathbf{Z}.$$

The computed coefficients $b_{j,k}^{(M)}$ are in general not equal to the WST coefficients $b_{j,k}$ appearing in (2.5). Their difference will be examined in §3.

2.2. Shensa algorithm. In the Shensa algorithm, we form a new function $\tilde{f}(t)$ from the sequence $x[n]$ by using an interpolation

$$\tilde{f}(t) = \sum_n x[n]\chi(2^J t - n) \tag{2.9}$$

such that the WST coefficients of $\tilde{f}(t)$ are exactly the same as the DWT coefficients of a new sequence $x'[n]$ which is obtained by prefiltering $x[n]$ with a sequence $q[n]$, i.e.

$$x'[n] = \sum_m x[m]q[n-m] \tag{2.10}$$

and where

$$q[n] = 2^{-J/2} \int \chi(t)\phi(t-n)dt\,. \tag{2.11}$$

The DWT coefficients of $x'[n]$, denoted by

$$DWT\{x'[n]; j, k\} \triangleq b_{j,k}^{(S)}, \ j < J, k \in \mathbf{Z},$$

are called the wavelet coefficients obtained from the Shensa algorithm. The fact stated above can be proved as follows.

PROPOSITION 1. *For* $j < J$, $k \in \mathbf{Z}$,

$$b_{j,k}^{(S)} = WST\{\tilde{f}(t); j, k\}\,. \tag{2.12}$$

Proof. Let

$$b_{j,k}^{(1)} = WST\{\tilde{f}(t); j, k\}\,, \ j < J,\ k \in \mathbf{Z}\,,$$

and

$$c_{J,k}^{(1)} = \int \tilde{f}(t)\phi_{Jk}(t)dt\,.$$

It is a well known fact that $\{b_{j,k}^{(1)}\}_{j<J,k\in\mathbf{Z}}$ can be obtained from $\{c_{Jk}^{(1)}\}_k$ by the recursion (2.6) and (2.7) [7], [16]. Also, by the definition of DWT, $\{b_{j,k}^{(S)}\}_{j<J,k\in\mathbf{Z}}$ can be obtained from $\{x'[k]\}_k$ using the same recursion (2.6) and (2.7). To prove $b_{j,k}^{(S)} = b_{j,k}^{(1)}$, $j < J$, $k \in \mathbf{Z}$, we only need to prove that these two recursion procedures share the same initial values, i.e. $c_{J,k}^{(1)} = x'[k]$ for all $k \in \mathbf{Z}$. This can be easily verified as,

$$\begin{aligned} c_{J,k}^{(1)} &= \int \tilde{f}(t)\phi_{Jk}(t)dt \\ &= \int \sum_n x[n]\chi(2^J t - n)2^{J/2}\phi(2^J t - k)dt \\ &= \sum_n x[n]2^{-J/2} \int \chi(t-n)\phi(t-k)dt \\ &= \sum_n x[n]q[k-n] = x'[k]\,. \end{aligned}$$

Thus, the proof is completed. □

Note that the only difference between the Mallat and Shensa algorithms is the prefiltering (2.10) adopted in the Shensa algorithm. Once sequences $x[n]$ and $x'[n]$ are given, the same recursion formulas (2.6) and (2.7) are applied for both algorithms. The Mallat algorithm can therefore be viewed as a special case of the Shensa algorithm where $q[n]$ is chosen to be the unit impulse sequence $\delta[n]$. It is worthwhile to point out that equations (2.6) and (2.7) can be implemented as cascaded quadrature mirror filters banks which are well studied in signal processing [17], [21], [24].

3. Accuracy of computed wavelet coefficients by Mallat algorithm. In this section, we estimate the difference between $\{b_{j,k}\}$ and $\{b_{j,k}^{(M)}\}$. To do so, we first estimate the difference between $\{c_{J,k}\}$ and $\{2^{-J/2}f(k/2^J)\}$, where $c_{J,k}$ is defined in (2.4).

LEMMA 1. *For $k \in \mathbf{Z}$,*

$$c_{J,k} - 2^{-J/2}f(k/2^J) = \frac{2^{-J/2}}{2\pi}\int \hat{f}(-\omega)(\phi(\omega/2^J) - 1)e^{-ik\omega/2^J}d\omega. \tag{3.1}$$

Proof. We have

$$\begin{aligned} c_{J,k} &= 2^{J/2}\int f(t)\phi(2^Jt-k)dt \\ &= 2^{J/2}\int f(t)\frac{1}{2\pi}\int \hat{\phi}(\omega)e^{i\omega(2^Jt-k)}d\omega dt \\ &= \frac{2^{J/2}}{2\pi}\int (\int f(t)e^{i\omega 2^J t}dt)\hat{\phi}(\omega)e^{-i\omega k}d\omega \\ &= \frac{2^{J/2}}{2\pi}\int \hat{f}(-2^J\omega)\hat{\phi}(\omega)e^{-i\omega k}d\omega \\ &= \frac{2^{-J/2}}{2\pi}\int \hat{f}(-\omega)\hat{\phi}(\omega/2^J)e^{-i\omega k/2^J}d\omega. \end{aligned} \tag{3.2}$$

Besides, it is known that

$$2^{-J/2}f(k/2^J) = \frac{2^{-J/2}}{2\pi}\int \hat{f}(-\omega)e^{-i\omega k/2^J}d\omega.$$

This proves (3.1). □

We have the straightforward corollary.

COROLLARY 1. *For $k \in \mathbf{Z}$,*

$$|c_{J,k} - 2^{-J/2}f(k/2^J)| \le \frac{2^{-J/2}}{2\pi}\int |\hat{f}(-\omega)||1-\hat{\phi}(\omega/2^J)|d\omega.$$

In what follows, we assume that $f(t)$ is $2^J\pi$ band-limited, that is, $\hat{f}(\omega) = 0$ for $|\omega| > 2^J\pi$. For such $f(t)$, (3.1) becomes

$$c_{J,k} - 2^{-J/2}f(k/2^J) = \frac{2^{-J/2}}{2\pi}\int_{-2^J\pi}^{2^J\pi} \hat{f}(-\omega)(\phi(\omega/2^J) - 1)e^{-ik\omega/2^J}d\omega.$$

By using the Parseval equality and $c_{J,k}^{(M)} = f(k/2^J)$, we can derive another corollary.

COROLLARY 2. *If $f(t)$ is $2^J\pi$ band-limited,*

$$\sum_k |c_{J,k} - 2^{-J/2}f(k/2^J)|^2 = \frac{2^{J-1}}{\pi}\int_{-\pi}^{\pi}|\hat{f}(-2^J\omega)|^2|1-\hat{\phi}(\omega)|^2d\omega. \tag{3.3}$$

The following theorem gives an estimate on the distance between the actual wavelet coefficients $\{b_{j,k}\}_{j<J,k\in\mathbf{Z}}$ and the computed wavelet coefficients $\{2^{-J/2}b_{j,k}^{(M)}\}_{j<J,k\in\mathbf{Z}}$ obtained by the Mallat algorithm.

THEOREM 1. *If $f(t)$ is $2^J\pi$ band-limited, then*

$$\sum_{j\le J-1}\sum_k |b_{j,k} - 2^{-J/2}b_{j,k}^{(M)}|^2 = \frac{2^{J-1}}{\pi}\int_{-\pi}^{\pi}|\hat{f}(-2^J\omega)|^2|1-\hat{\phi}(\omega)|^2d\omega. \tag{3.4}$$

Proof. For $j < J$, $k \in \mathbf{Z}$, since $2^{-J/2}c_{j,k}^{(M)}$ and $2^{-J/2}b_{j,k}^{(M)}$ are obtained from $2^{-J/2}x[k]$ by (2.6)-(2.7) and $c_{j,k}$ and $b_{j,k}$ are obtained from $c_{J,k}$ also by (2.6)-(2.7), it is clear that $2^{-J/2}c_{j,k}^{(M)} - c_{j,k}$ and $2^{-J/2}b_{j,k}^{(M)} - b_{j,k}$, can be obtained from $2^{-J/2}x[k] - c_{J,k}$ by the same recursion:

$$2^{-J/2}c_{j-1,k}^{(M)} - c_{j-1,k} = \frac{1}{2}\sum_n h_{n-2k}(2^{-J/2}c_{j,n}^{(M)} - c_{j,n}),$$

and

$$2^{-J/2}b_{j-1,k}^{(M)} - b_{j-1,k} = \frac{1}{2}\sum_n g_{n-2k}(2^{-J/2}c_{j,n}^{(M)} - c_{j,n}),$$

where $j \leq J, k \in \mathbf{Z}$. It is known that the filters h_k and g_k in (2.1)-(2.2) have the following property

$$|H(\omega)|^2 + |G(\omega)|^2 \equiv 1,$$

where

$$H(\omega) = \frac{1}{2}\sum_k h_k e^{ik\omega}, \quad \text{and} \quad G(\omega) = \frac{1}{2}\sum_k g_k e^{ik\omega}.$$

Therefore, for $j \leq J$,

$$\sum_k |c_{j-1,k} - 2^{-J/2}c_{j-1,k}^{(M)}|^2 + \sum_k |b_{j-1,k} - 2^{-J/2}b_{j-1,k}^{(M)}|^2 = \sum_k |c_{j,k} - 2^{-J/2}c_{j,k}^{(M)}|^2.$$

By summing up the above equations with all $j \leq J$, we get

$$\sum_{j \leq J-1} \sum_k |b_{j,k} - 2^{-J/2}b_{j,k}^{(M)}|^2 = \sum_k |c_{J,k} - 2^{-J/2}c_{J,k}^{(M)}|^2.$$

By Corollary 2, the theorem is proved. □

The above theorem tells that the error bound depends on the function $f(t)$ as well as the scaling function $\phi(t)$ (or, equivalently, the wavelet basis). In Table 1, we compute numerically the error bound for a family of band-limited functions parameterized by σ, i.e.

$$\hat{f}(\omega) = e^{-\omega^2/\sigma^2}, \tag{3.5}$$

for $|\omega| < 2^6\pi$ (i.e. J=6) and $\hat{f}(\omega) = 0$ for $|\omega| \geq 2^6\pi$. The Haar, Daubechies D_4 and D_8 [7] wavelet bases are compared. Note that the scaling function $\phi(t)$ usually has the property [7], [16]

$$\sum_k |\hat{\phi}(\omega + 2k\pi)|^2 = 1, \quad \text{and} \quad \hat{\phi}(0) = 1, \tag{3.6}$$

so that $\hat{\phi}(\omega)$ behaves like a lowpass filter. Even though that $\hat{\phi}(0) = 1$, the phase of $\hat{\phi}(\omega)$ still plays an important role in error calculation in the neighborhood of $\omega = 0$.

TABLE 1
Errors on computed wavelet coefficients by the Mallat algorithm

σ	Haar	Daubechies D_4	Daubechies D_8
10^1	1.25235×10^{-3}	2.01734×10^{-3}	5.06789×10^{-3}
10^2	1.18350×10^0	2.16769×10^0	5.16201×10^0
10^3	1.72019×10^1	3.52467×10^1	7.69790×10^1

4. Accuracy of computed wavelet coefficients by Shensa algorithm. In this section, we estimate the difference between $\{b_{j,k}\}_{j<J,k\in\mathbf{Z}}$ and $\{b_{j,k}^{(S)}\}_{j<J,k\in\mathbf{Z}}$. Similar to the approach adopted in §3, we first estimate the difference between $\{c_{J,k}\}_{k\in\mathbf{Z}}$ and $\{x'[k]\}_{k\in\mathbf{Z}}$.

LEMMA 2. *For $k \in \mathbf{Z}$,*

$$x'[k] = \frac{2^{J/2}}{2\pi} \int \hat{f}(-2^J\omega)A(\omega)e^{-ik\omega}d\omega\,, \tag{4.1}$$

where

$$A(\omega) \triangleq \sum_m \hat{\phi}(\omega + 2m\pi)\hat{\chi}(-\omega - 2m\pi) \tag{4.2}$$

is a periodic function with period 2π.

Proof. We have

$$\begin{aligned}
x'[k] &= \sum_m x[m]q[k-m] \\
&= 2^{-J/2}\int \chi(t)\sum_m \phi(t-(k-m))x[m]dt \\
&= 2^{-J/2}\int \chi(t)\sum_m \phi(t-(k-m))\frac{1}{2\pi}\int \hat{f}(-\omega)e^{-i\omega m/2^J}d\omega dt \\
&= 2^{J/2}\int \chi(t)\sum_m \frac{1}{2\pi}\int \hat{\phi}(\omega_1)e^{i\omega_1(t-(k-m))}d\omega_1\frac{1}{2\pi}\int \hat{f}(-2^J\omega)e^{-i\omega m}d\omega dt \\
&= 2^{J/2}\frac{1}{2\pi}\int \hat{f}(-2^J\omega)\int \chi(t)\int \hat{\phi}(\omega_1)e^{i\omega_1(t-k)}\frac{1}{2\pi}\sum_m e^{im(\omega_1-\omega)}d\omega_1 dt d\omega \\
&= \frac{2^{J/2}}{2\pi}\int \hat{f}(-2^J\omega)\int \chi(t)\int \hat{\phi}(\omega_1)e^{i\omega_1(t-k)}\sum_m \delta(\omega_1-(\omega-2m\pi))d\omega_1 dt d\omega \\
&= \frac{2^{J/2}}{2\pi}\int \hat{f}(-2^J\omega)\int \chi(t)\sum_m \hat{\phi}(\omega+2m\pi)e^{i(\omega+2m\pi)(t-k)}dt d\omega \\
&= \frac{2^{J/2}}{2\pi}\int \hat{f}(-2^J\omega)\sum_m \hat{\phi}(\omega+2m\pi)\int \chi(t)e^{i(\omega+2m\pi)t}dt e^{-ik\omega}d\omega \\
&= \frac{2^{J/2}}{2\pi}\int \hat{f}(-2^J\omega)\left[\sum_m \hat{\phi}(\omega+2m\pi)\hat{\chi}(-\omega-2m\pi)\right]e^{-ik\omega}d\omega\,.
\end{aligned}$$

This proves Lemma 2. □

Combining (3.2) and (4.1), we obtain the following corollary.

COROLLARY 3. *For $k \in \mathbf{Z}$,*

$$c_{J,k} - x'[k] = \frac{2^{J/2}}{2\pi}\int \hat{f}(-2^J\omega)(\hat{\phi}(\omega) - A(\omega))e^{-ik\omega}d\omega\,.$$

By choosing the interpolant $\chi(t) = \phi(t)$ and using (3.6), we know that $A(\omega) = 1$. Besides, from Lemma 1 and Corollary 3 or directly from (2.10)-(2.11) we have $x'[k] = 2^{-J/2}x[k]$. Thus, we are lead to the following corollary.

COROLLARY 4. *If the interpolant $\chi(t)$ in (2.9) is equal to the scaling function $\phi(t)$, then*

$$b_{j,k}^{(M)} = b_{j,k}^{(S)}\,,\ j < J, k \in \mathbf{Z},$$

i.e. the Mallat and Shensa algorithms give the same results.

The main theorem on the error of computed coefficients obtained by the Shensa algorithm can be proved with a similar technique for proving Theorem 1.

THEOREM 2. *If $f(t)$ is $2^J\pi$ band-limited,*

$$(4.3)\qquad \sum_{j\le J-1}\sum_k |b_{j,k}-b^{(S)}_{j,k}|^2 = \frac{2^{J-1}}{\pi}\int_{-\pi}^{\pi} |\hat{f}(-2^J\omega)|^2|A(\omega)-\hat{\phi}(\omega)|^2 d\omega\,,$$

where $A(\omega)$ is given in (4.2).

An immediate consequence of Theorem 2 is:

COROLLARY 5. *If $f(t)$ is $2^J\pi$ band-limited and the interpolant $\chi(t)$ is the sinc function, i.e. $\frac{\sin\pi t}{\pi t}$, then $b^{(S)}_{j,k}=b_{j,k}$, or equivalently,*

$$WST\{f(t);j,k\} = DWT\{x'[n];j,k\}\,,\ j<J, k\in\mathbf{Z}.$$

The above corollary says that the wavelet series transform (WST) coefficients of a band-limited function can be computed exactly by the Shensa algorithm if the sinc interpolant is used in (2.9) and if the function is sampled at a rate higher than the Nyquist rate. We see from Theorem 2 that the accuracy of the Shensa algorithm can be adjusted by choosing the interpolant $\chi(t)$. The resulting coefficients should have a higher accuracy than those obtained by the Mallat algorithm, as long as $A(\omega)$ satisfies the condition:

$$|A(\omega)-\hat{\phi}(\omega)| < |1-\hat{\phi}(\omega)|\,.$$

Since $A(\omega)$ is periodic with period 2π, we also conclude from Corollary 3 that the aliasing error between the prefiltered $x'[n]$ and $c_{J,n}$ cannot be avoided for a function which is not band-limited.

Table 2 shows the numerical values of error bounds for computed wavelet coefficients using the Shensa algorithm with the same test functions and wavelet bases as given in Table 1. The function $\chi(t)$ is chosen to be the truncated sinc function for data in this table, i.e.

$$\chi(t) = \begin{cases} \frac{\sin(\pi t)}{\pi t}, & |t|\le 2\,,\\ 0 & \text{otherwise}\,. \end{cases}$$

By comparing Tables 1 and 2, we see clearly that the Shensa algorithm gives better results than the Mallat algorithm for non-narrow band signals.

TABLE 2

Errors on computed wavelet coefficients by the Shensa algorithm

σ	Haar	Daubechies D_4	Daubechies D_8
10^1	1.55512×10^{-2}	1.74750×10^{-2}	2.11439×10^{-2}
10^2	9.60729×10^{-2}	1.13722×10^{-1}	1.20147×10^{-1}
10^3	2.06311×10^{0}	2.92142×10^{0}	8.77725×10^{-1}

5. A sufficient condition for exact WST coefficient computation. We see from §4 that the WST coefficients of a $2^J\pi$ band-limited function can be computed from the Shensa algorithm exactly if the interpolant $\chi(t)$ is the sinc function. In this section, we examine another sufficient condition which guarantees exact reconstruction of the WST coefficients via the Shensa algorithm for a general class of functions which are not band-limited.

In this section, we assume that $\phi(t)$ is a real scaling function satisfying:

(a) $\phi(t)=O(|t|^{-1-\epsilon})$ as $t\to\pm\infty$, $t\in\mathbf{R}$,

(b) $\Phi(\omega)\triangleq\sum_n\phi(n)e^{-in\omega}\ne 0$, $\omega\in\mathbf{R}$.

Let $\{V_j(\phi)\}_{j\in\mathbf{Z}}$ be the multiresolution approximation of $L^2(\mathbf{R})$ corresponding to a scaling function $\phi(t)$ [17]. That is, for each $j\in\mathbf{Z}$,

$$(5.1)\qquad V_j(\phi) = \text{closure}\{\text{linear span}\{\phi_{jk}(t):k\in\mathbf{Z}\}\}\,.$$

Then, the following proposition was shown by Walter [26].

PROPOSITION 2. *If $f(t) \in V_J(\phi)$ for some integer $J > 0$, then*

$$f(t) = \sum_n f(\frac{n}{2^J})\chi(2^J t - n),$$

where the interpolant $\chi(t)$ satisfies

$$\hat{\chi}(\omega) = \frac{\hat{\phi}(\omega)}{\Phi(\omega)}. \tag{5.2}$$

This proposition implies that the WST coefficients of $f(t) \in V_J(\phi)$ can be exactly computed from the Shensa algorithm with the scaling function $\phi(t)$ satisfying (a) and (b) and the interpolant $\chi(t)$ satisfying (5.2) or, equivalently,

$$WST\{f(t); j, k\} = DWT\{x'[n]; j, k\}, \; j < J, k \in \mathbf{Z},$$

where $x'[n]$ is defined by (2.10)-(2.11). However, the condition $f(t) \in V_J(\phi)$ is in general difficult to verify and the applicability of the above proposition is therefore limited.

In the following, we present another sufficient condition which is easier to verify.

THEOREM 3. *If $f(t) \in L^2(\mathbf{R})$ satisfies the condition*

$$\int \hat{f}(-2^J\omega)(\hat{\phi}(\omega) - \frac{1}{\Phi(-\omega)})e^{-ik\omega}d\omega = 0, \; \forall k \in \mathbf{Z}, \tag{5.3}$$

then the WST coefficients of $f(t)$ can be exactly computed from the Shensa algorithm, i.e.

$$b_{j,k} = b_{j,k}^{(S)}, \; j < J, k \in \mathbf{Z},$$

with the scaling function $\phi(t)$ given in (5.3) and the interpolant $\chi(t)$ satisfying (5.2).

Proof. Let $f(t) \in L^2(\mathbf{R})$ satisfy (5.3). Then,

$$\int \hat{f}(-2^J\omega)\hat{\phi}(\omega)e^{-ik\omega}d\omega = \int \frac{\hat{f}(-2^J\omega)}{\Phi(-\omega)}e^{-ik\omega}d\omega, \; \forall k \in \mathbf{Z}.$$

To prove Theorem 3, we only have to prove

$$c_{J,k} = x'[k], \; \forall k \in \mathbf{Z}. \tag{5.4}$$

It is clear that

$$\begin{aligned} A(\omega) &= \sum_k \hat{\phi}(\omega + 2k\pi)\hat{\chi}(-\omega - 2k\pi) \\ &= \sum_k \hat{\phi}(\omega + 2k\pi)\frac{\hat{\phi}(-\omega - 2k\pi)}{\Phi(-\omega - 2k\pi)} \\ &= \sum_k \frac{|\hat{\phi}(\omega + 2k\pi)|^2}{\Phi(-\omega - 2k\pi)}. \end{aligned}$$

According to the assumption (b) given in the beginning in this section, we know that $\Phi(\omega)$ is periodic with period 2π. By using this fact and the identity (3.6), we obtain

$$A(\omega) = \frac{1}{\Phi(-\omega)}.$$

Thus, it is concluded that

$$\int \hat{f}(-2^J\omega)\hat{\phi}(\omega)e^{-ik\omega}d\omega = \int \hat{f}(-2^J\omega)A(\omega)e^{-ik\omega}d\omega, \; \forall k \in \mathbf{Z}.$$

By combining (3.2), (4.1) and the above result, we have (5.4). This completes the proof of Theorem 3. □

In fact, Theorem 3 is a generalization of Walter's result due to the following corollary.

COROLLARY 6. *If $f(t) \in V_J(\phi)$, then $f(t)$ satisfies (5.3).*

Proof. By Proposition 2, we know that the condition $f(t) \in V_J$ implies

$$f(t) = \sum_n f(\frac{n}{2^J})\chi(2^J t - n).$$

Then, one can prove (5.4) using the same technique as given in Proposition 1. The condition (5.3) is a direct consequence of (5.4), which can be proved by a technique similar to that in proving Theorem 3. □

Generally, the condition (5.3) does not imply $f(t) \in V_J(\phi)$ due to the following arguments. Recall that

$$\tilde{f}(t) = \sum_n f(\frac{n}{2^J})\chi(2^J t - n),$$

as given in (2.9) and

$$x'[k] = \int \tilde{f}(t)\phi_{Jk}(t)dt, \tag{5.5}$$

as given in the proof of Proposition 1. Besides, (5.3) implies (5.4). With (5.4) and (5.5), one can conclude that the Jth resolution $f_J(t)$ of $f(t)$ is equal to the Jth resolution $\tilde{f}_J(t)$ of $\tilde{f}(t)$. However, $f_J(t) = \tilde{f}_J(t)$ does not imply $f(t) = \tilde{f}(t)$ or $f(t) \in V_J$ in general.

One might ask if there exists a scaling function $\phi(t)$ such that $\phi(t)$ satisfies conditions (a) and (b) given in the beginning of this section. It has been shown by Walter [26] that many familiar wavelets do satisfy these conditions such as the Haar, Franklin, spline, Meyer and Daubechies wavelets (see [26] for details).

6. Accuracy of computed wavelet coefficients by direct integration.

Although all wavelet coefficients can be efficiently computed by the Mallat or Shensa algorithm, we may sometimes be interested in only a few coefficients. Direct evaluation of the integral

$$b_{j,k} = \int f(t)\psi_{jk}(t)dt, \quad j < J, \ k \in \mathbf{Z},$$

may be preferred in this case. By assuming only the knowledge of $x[n]$ given by (2.3), we introduce the discretization error into the integral evaluation. We analyze the discretization error in this section. Consider the following discrete form of the above integration,

$$\tilde{b}_{j,k}^{(I)} = 2^{-J}\sum_n x[n]\psi_{jk}(\frac{n}{2^J}), \quad j < J, \ k \in \mathbf{Z}. \tag{6.1}$$

We will estimate the difference between $\tilde{b}_{j,k}^{(I)}$ and $b_{j,k}$. First, we have a lemma.

LEMMA 3. *For $j < J$, $k \in \mathbf{Z}$,*

$$\tilde{b}_{j,k}^{(I)} = \frac{2^{j/2}}{2\pi}\int \hat{f}(-2^j\omega)B(\omega)e^{-ik\omega}d\omega,$$

where

$$B(\omega) = \sum_n \hat{\psi}(\omega + 2^{J-j+1}n\pi). \tag{6.2}$$

Proof. We have

$$\begin{aligned}
\tilde{b}_{j,k}^{(I)} &= 2^{-J}\sum_n \frac{1}{2\pi}\int \hat{f}(-\omega)e^{-i\omega n/2^J}d\omega 2^{j/2}\psi(2^j\frac{n}{2^J}-k) \\
&= 2^{j/2}\sum_n \frac{1}{2\pi}\int \hat{f}(-2^J\omega)e^{-in\omega}d\omega\frac{1}{2\pi}\int \hat{\psi}(\omega_1)e^{i(\frac{n}{2^{J-j}}-k)\omega_1}d\omega_1 \\
&= \frac{2^{J-j/2}}{2\pi}\int \hat{f}(-2^J\omega)\int \hat{\psi}(2^{J-j}\omega_1)e^{-ik\omega_1 2^{J-j}}\frac{1}{2\pi}\sum_n e^{in(\omega_1-\omega)}d\omega_1 d\omega \\
&= \frac{2^{J-j/2}}{2\pi}\int \hat{f}(-2^J\omega)\int \hat{\psi}(2^{J-j}\omega_1)e^{-ik\omega_1 2^{J-j}}\sum_n \delta(\omega_1-\omega-2n\pi)d\omega_1 d\omega \\
&= \frac{2^{J-j/2}}{2\pi}\int \hat{f}(-2^J\omega)\sum_n \hat{\psi}(2^{J-j}(\omega+2n\pi))e^{-ik(\omega+2n\pi)2^{J-j}}d\omega \\
&= \frac{2^{J-j/2}}{2\pi}\int \hat{f}(-2^J\omega)\sum_n \hat{\psi}(2^{J-j}(\omega+2n\pi))e^{-ik\omega 2^{J-j}}d\omega \\
&= \frac{2^{j/2}}{2\pi}\int \hat{f}(-2^j\omega)\left[\sum_n \hat{\psi}(\omega+2^{J-j+1}n\pi)\right]e^{-ik\omega}d\omega .
\end{aligned}$$

This proves Lemma 3. □
Since

$$b_{j,k} = \int f(t)\psi_{jk}(t)dt = \frac{2^{j/2}}{2\pi}\int \hat{f}(-2^j\omega)\hat{\psi}(\omega)e^{-jk\omega}d\omega ,$$

we obtain

$$\tilde{b}_{j,k}^{(I)} - b_{j,k} = \frac{2^{j/2}}{2\pi}\int \hat{f}(-2^j\omega)(B(\omega)-\hat{\psi}(\omega))e^{-ik\omega}d\omega .$$

Finally, we get the following theorem.

THEOREM 4. *For $j < J$, $k \in \mathbf{Z}$,*

$$|\tilde{b}_{j,k}^{(I)} - b_{j,k}| \le \frac{2^{j/2}}{2\pi}\int |\hat{f}(-2^j\omega)||B(\omega)-\hat{\psi}(\omega)|d\omega ,$$

where $B(\omega)$ is given in (6.2) and, moreover, if $f(t)$ is $2^j\pi$ band-limited,

$$\sum_k |\tilde{b}_{j,k}^{(I)} - b_{j,k}|^2 = \frac{2^{j-1}}{\pi}\int_{-\pi}^{\pi} |\hat{f}(-2^j\omega)|^2|B(\omega)-\hat{\psi}(\omega)|^2 d\omega .$$

7. Concluding remarks. In this research, we discussed the accuracy of computed wavelet coefficients based on Mallat, Shensa and direct integration algorithms. Exact error expressions for these methods have been derived for band-limited functions. It was also shown that the WST coefficients of a band-limited function can be exactly computed by using the Shensa algorithm with an ideal prefilering operation. However, the ideal prefiler is an infinite impulse response (IIR) filter and has to be replaced by a finite impulse response (FIR) filter in practice. Then, the error expressions provide us a clear guidance for choosing a wavelet basis $\phi(t)$ and the interpolant $\chi(t)$ (or the prefilter $q[n]$) to make the approximation error small. The determination of the optimal basis $\phi(t)$ and the optimal interpolant $\chi(t)$ which minimize the approximation error with the FIR filter constraint is under our current investigation.

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