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by

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Abstract

Impulsive noise bursts in communications systems are traditionally handled by incorporating in the receiver a limiter which clips the received signal before integration. An empirical justification for this procedure is that it generally causes the signal-to-noise ratio to increase. Recently, very accurate models of impulsive noise were presented, based on the theory of symmetric α-stable probability density functions. In this paper, we examine the performance of optimum receivers, designed to detect signals embedded in impulsive noise which is modeled as an infinite variance symmetric alpha-stable process, and compare it against the performance of several suboptimum receivers. As a measure of receiver performance, we compute an asymptotic expression for the probability of error for each receiver and compare it to the probability of error calculated by extensive Monte-Carlo simulations.

Key words: Impulsive noise, α-stable distribution, optimum receiver, suboptimum receivers.

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1. Introduction

There exist several communication links for which the conventional additive Gaussian noise assumption is quite inadequate and the dominant source of interference contains a significant noise component that is termed "impulsive" to indicate the probability of large interference levels. The sources of impulsive noise may be either natural or man made and may include atmospheric noise in radio links, ambient acoustic noise due to ice cracking in the arctic region in underwater sonar and submarine communications, and lightning, switching transients, and accidental hits in telephone lines [1]. The effects of impulsive noise in the operation of conventional communication systems, which are built upon the assumption of additive Gaussian noise, are drastic, significantly degrading the overall performance of the system, and giving rise to the need for redesigning communication systems to effectively filter out the impulsive interference.

For this purpose, a quantitative description of impulsive noise is needed. Towards this goal, several models of non-Gaussian random processes have been considered (for example, see [2, 3, 4, 5, 6, 7, 8, 9, 10] and references therein). On many occasions [1], the empirical data indicate that the probability density functions (pdfs) of the associated noise processes maintain a similarity to the Gaussian pdf, being bell-shaped, smooth, and symmetric, but at the same time have significantly heavier tails. For example, atmospheric noise may be considered as arising from a superposition of many statistically independent sources so that central limit theorems are applicable in the evaluation of its pdf. The empirical fact [1] of algebraic (inverse power) tails in the pdf of atmospheric noise naturally leads to the assumption of a stable pdf. On the other hand, a certain class of non-Gaussian pdfs was considered in [11], which was parameterized in such a way that the Gaussian pdf was obtained at a certain limit. A similar parameterization of the class of stable random processes, in the characteristic function domain, is possible [12] in a manner that the Gaussian pdf is again included as a certain special case. The above evidence, combined with a recent, increasing interest in the application of the theory of stable random variables and processes in statistical signal processing [13, 14], clearly suggests that possible quite accurate models for large classes of impulsive noise in communication links may be the stable pdfs [15]. In fact, the Cauchy distribution, which belongs to the class of stable distributions, has already been studied as a model for severe impulsive noise. In particular, the form and the performance of optimum receivers designed to operate in Cauchy distributed noise environments has been investigated in [16]. Also, it was empirically found by Stuck and Kleiner [17] that the noise over certain telephone lines was best described by almost Gaussian stable pdfs. Very recently, it was also theoretically shown that, under general assumptions, a broad class of impulsive noise, including Poisson models and generalized Gaussian models, has, indeed, a stable pdf [15].

On the hand of handling impulsive noise, communication receivers traditionally consist of a limiting device, which limits the amplitude of the incoming signal plus noise, followed by a linear receiver (integrator). The justification for this structure is that it results in an increased signal-to-noise ratio at the input of the linear receiver, thus reducing the probability of error of the overall detector. At the same time, no excessive computational or economic cost is added to the operation of the receiver. Certainly, the assumption is intuitively valid; however, it is not based on any optimality criterion and there is no rigorous guarantee of best performance in any sense [1]. Moreover, the limiter may introduce undesirable distortion to the signal of interest [19]. With today's availability of inexpensive computer hardware and software, the trend has become not to sacrifice performance over simplicity and ease of implementation of the mathematical models and the corresponding algorithms employed in the processing of signals [20] and it is now possible to economically and efficiently build receivers of more complex structure, the performance of which is optimum or, at least, very close to optimum.

In this paper, we examine the structure and the performance of optimum receivers designed to operate in environments of impulsive noise modeled as a stable random process. As a criterion of performance, we compute and plot an asymptotic expression for the probability of erroneous decision as a function of the dispersion of the noise for various values of the characteristic exponent α of the interfering noise. Then, we compare this probability of error to the error rate obtained via extensive Monte-Carlo simulation. We compare the performance of the optimum receiver to that of the conventional linear (Gaussian) receiver appropriate for detection of signals in additive Gaussian noise; to the performance of the traditional receiver consisting of a limiter followed by an integrator; and to the performance of the Cauchy receiver appropriate for detection of signals in Cauchy noise. The paper is organized as follows: In section 2, we present the basic definitions and

[1] Except that this receiver is the local minimax robust receiver for detection of a weak signal of known shape in additive noise modeled as a mixture of a Gaussian and a heavily tailed process [18].

properties of the symmetric alpha-stable (SαS) random variables and processes. We also discuss methods for computation of a SαS pdf in real time and ways to generate SαS pseudorandom deviates. In section 3, we analyze theoretically the performance of four types of receivers: (i) the maximum likelihood (ML) receiver for detecting signals embedded in independent SαS noise, (ii) the conventional linear receiver (integrator), optimum for detecting signals embedded in white Gaussian noise, (iii) a limiter followed by the linear receiver, and (iv) the Cauchy receiver. In section 4, we present plots of the theoretical expressions of section 3 for a range of values of the characteristic exponent and the dispersion of the interefering SαS noise, as well plots of the observed performance of the various receivers in extensive Monte–Carlo simulations. Finally, in section 5 we summarize the key observations of this paper and suggest possible related research topics to be addressed in the future.

2. SαS Distributions, Random Variables, and Processes

2.1 The class of SαS pdfs

A SαS pdf $f_\alpha(\gamma, \delta; \cdot)$ is best defined [12] via its corresponding characteristic function as

$$f_\alpha(\gamma, \delta; \xi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(i\delta\omega - \gamma|\omega|^\alpha) e^{-i\omega\xi} \, d\omega, \qquad (2-1)$$

where the *characteristic exponent* α is restricted to the values $0 < \alpha \leq 2$. In this equation, δ ($-\infty < \delta < \infty$) is the *location parameter* and γ ($\gamma > 0$) is the *dispersion* of the SαS pdf. When $1 < \alpha \leq 2$, the location parameter δ corresponds to the mean of the SαS pdf, while for $0 < \alpha \leq 1$, when the SαS pdf does not have a finite mean, δ corresponds to its median. The dispersion parameter γ is a measure of the spread of the pdf around its location parameter δ, similar to the variance of a Gaussian pdf. From now on, unless explicitly stated otherwise, we are going to assume that $\delta = 0$. This assumption does not restrict the generality of our results and is similar to the usual zero mean assumption for Gaussian pdfs. Two well known classes of SαS pdfs are the Gaussian distribution with mean δ and variance 2γ, arising from Eq.(2-1) when $\alpha = 2$, and the Cauchy distribution with dispersion γ and median δ, corresponding to $\alpha = 1$:

$$f_2(\gamma, \delta; \xi) = \frac{1}{\sqrt{4\pi\gamma}} \exp[-\frac{(\xi - \delta)^2}{4\gamma}] \qquad (2\text{-}2)$$

$$f_1(\gamma, \delta; \xi) = \frac{1}{\pi} \frac{\gamma}{\gamma^2 + (\xi - \delta)^2}. \qquad (2\text{-}3)$$

No closed form expressions exist for SαS pdfs other than the Gaussian and the Cauchy; however, asymptotic expansions for all SαS pdfs are known, valid for either small (i.e., $|\xi-\delta|\to 0$) or large (i.e., $|\xi-\delta|\to\infty$) argument ξ. In particular [21]:

$$f_\alpha(\gamma,\delta;\xi)=\sum_{k=0}^{n} a_k(\xi-\delta)^{2k}+O(|\xi-\delta|^{2n+1}), \tag{2-4}$$

as $|\xi-\delta|\to 0$, and

$$f_\alpha(\gamma,\delta;\xi)=\sum_{k=1}^{n}\frac{b_k}{(\xi-\delta)|\xi-\delta|^{\alpha k}}+O(|\xi-\delta|^{-\alpha(n+1)-1}), \tag{2-5}$$

as $|\xi-\delta|\to\infty$.[2] In the above Eqs.(2-4) and (2-5), we have

$$a_k = \frac{1}{\pi\alpha}\frac{(-1)^k}{(2k)!}\gamma^{-\frac{2k+1}{\alpha}}\Gamma(\frac{2k+1}{\alpha}) \tag{2-6}$$

$$b_k = -\frac{1}{\pi}\frac{(-1)^k}{k!}\gamma^k\Gamma(\alpha k+1)\sin(\frac{k\alpha\pi}{2}). \tag{2-7}$$

If the asymptotic series (2-4) or (2-5) are to be computed for a large number n of terms, inaccuracies may arise from the computation for large argument of the gamma functions appearing in Eqs.(2-6) and (2-7). These difficulties can be minimized, however, by the following procedure. We observe that:

$$a_k(\xi-\delta)^{2k}=-[a_{k-1}(\xi-\delta)^{2(k-1)}]\frac{\Gamma(\frac{2k+1}{\alpha})}{\Gamma(\frac{2k-1}{\alpha})}\frac{(\xi-\delta)^2}{2k(2k-1)},\quad k=1,2,3,\ldots \tag{2-8}$$

with

$$a_0=\frac{1}{\pi\alpha}\gamma^{-\frac{1}{\alpha}}\Gamma(\frac{1}{\alpha}). \tag{2-9}$$

For large k, we have [22, p. 257]

$$\frac{\Gamma(\frac{2k+1}{\alpha})}{\Gamma(\frac{2k-1}{\alpha})}\approx(\frac{2k}{\alpha})^{\frac{2}{\alpha}}[1-\frac{1}{2k}+\frac{1}{3}\frac{\alpha+2}{(2k)^2}], \tag{2-10}$$

which, if substituted back in Eq.(2-8), gives a recursive formula for the accurate evaluation of the asymptotic series (2-4). Similarly, we use the formulae

$$\frac{b_k}{(\xi-\delta)|\xi-\delta|^{\alpha k}}=-[\frac{b_{k-1}}{(\xi-\delta)|\xi-\delta|^{\alpha(k-1)}}]\frac{(\alpha k+1)^\alpha}{k|\xi-\delta|^\alpha}\frac{\sin(\frac{k\pi\alpha}{2})}{\sin(\frac{(k-1)\pi\alpha}{2})}\cdot$$
$$\cdot[1-\frac{\alpha(\alpha+1)}{2}\frac{1}{\alpha k+1}+\frac{1}{24}\frac{\alpha(\alpha-1)(\alpha+1)(3\alpha+4)}{(\alpha k+1)^2}],\quad k=2,3,4,\ldots \tag{2-11}$$

[2]Eq.(2-4) is an asymptotic expansion for all $0<\alpha<2$, $|\xi-\delta|\to 0$, and becomes a convergent series for $1<\alpha<2$. Eq.(2-5) is an asymptotic expansion for all $0<\alpha<2$, $|\xi-\delta|\to\infty$, and becomes a convergent series for $0<\alpha<1$.

with

$$\frac{b_1}{(\xi-\delta)|\xi-\delta|^\alpha} = \frac{1}{\pi}\frac{\gamma\Gamma(\alpha+1)\sin(\frac{\alpha\pi}{2})}{(\xi-\delta)|\xi-\delta|^\alpha}, \tag{2-12}$$

to recursively compute the asymptotic series (2-5).

The SαS pdfs present several similarities to the Gaussian pdfs, such as the stability property and a generalized form of the central limit theorem; however, they also differ from the Gaussian pdfs in many ways. For example, the SαS pdfs have maxima more peaked than those of the Gaussian pdfs and algebraic (inverse power) tails in opposition to the exponential tails of the Gaussian pdfs. As a result, the pth order moments of SαS pdfs are finite only for $0 < p < \alpha$. These properties of the SαS pdfs have allowed more accurate modeling of certain economical, physical, biological, and hydrological phenomena and may also indicate applications in statistical signal processing and communications [13]. For illustration purposes, we show in Fig. 1 plots of the SαS pdfs for location parameter $\delta = 0$ and for characteristic exponents $\alpha = 0.5, 1, 1.5, 1.99$, and 2. The curves in Fig. 1 have been produced by calculation of the inverse Fourier transform integral in Eq.(2-1). This procedure is quite accurate for computation of SαS pdfs; however, it is not readily applicable for real time calculations because of the extensive numerical integrations it requires. Therefore, alternative expressions need to be used for real time computation of a SαS pdf. As a first try, we use the asymptotic series in Eqs.(2-4) and (2-5). In particular, Eq.(2-4) is expected to give a good approximation to a SαS pdf $f_\alpha(\gamma, \delta; \cdot)$ as long as its argument is small, while Eq.(2-5) will give a good approximation to $f_\alpha(\gamma, \delta; \cdot)$ for large argument. We have found, however, that there exists an interval of values of the argument of $f_\alpha(\gamma, \delta; \cdot)$ for which none of the asymptotic series in Eqs.(2-4) and (2-5) is sufficiently accurate. This is clearly illustrated in Fig. 2, where we show plots of the S($\alpha = 1.5$)S pdf (with $\delta = 0$ and $\gamma = 1$) obtained with the following three different methods: (i) by numerical computation of the inverse Fourier transform in Eq.(2-1), (ii) via the asymptotic series expansions in Eqs.(2-4) and (2-5), in which the first $n = 100$ terms of the Taylor series expansion have been computed for $|\xi| \leq 3$ and the first $n = 9$ terms of the asymptotic expansion have been computed for $|\xi| > 3$, and (iii) by the method described in the next paragraph. Clearly, the asymptotic series approximation of Eq.(2-4) is accurate only for very small values of the argument of the pdf and deteriorates very fast for larger values. On the other hand, the asymptotic series of Eq.(2-5) is not accurate for sufficiently small arguments for which the series (2-4) is not valid.

The method we found very accurate and very efficient to implement relies on the fact [23] that the SαS pdfs are entire analytic functions, thus having bounded derivatives of all orders. Our approach, therefore, for the real time computation of a SαS pdf f_α consists of three steps: (i) choice of a cutoff argument, such that the pdf at larger values can be accurately computed via the first few terms of Eq.(2-5), (ii) evaluation of the pdf at a number of points in the interval (0, cutoff) via numerical computation of the Fourier integral in Eq.(2-1), and (iii) evaluation of the coefficients of an interpolating polynomial of small degree from the values of the pdf at the selected points of the previous step. This method is, indeed, very accurate and is illustrated in Fig. 2, for the computation of the S($\alpha = 1.5$)S pdf with $\gamma = 1$ and $\delta = 0$. We have performed this procedure for the SαS pdfs with $\gamma = 1$, $\delta = 0$, [3] and $\alpha = 0.5, 1.5$, and 1.99 and interpolating polynomials of appropriate degree and asymptotic series of appropriate number of terms. The resulting coefficients in the asymptotic series and in the interpolating polynomials are shown in Table 1:

Computation of SαS pdfs		
$\alpha =$	$f_\alpha(1,0;x) =$	
0.5	$-0.3238x^5 + 1.7166x^4 - 3.6100x^3 + 3.8470x^2 - 2.2125x + 0.6709$	if $\|x\| \leq 1.5$
	$-\frac{0.0062}{x^{3.5}} - \frac{9.522\cdot 10^{-11}}{x^3} + \frac{0.0499}{x^{2.5}} - \frac{0.1592}{x^2} + \frac{0.1995}{x^{1.5}}$	if $\|x\| > 1.5$
1.5	$0.0001x^9 - 0.0008x^8 + 0.0048x^7 - 0.0125x^6 + 0.0083x^5$ $+0.0196x^4 + 0.0007x^3 - 0.1058x^2 + 0.0016x + 0.2874$	if $\|x\| \leq 3$
	$\frac{1.4323\cdot 10^4}{x^{14.5}} + \frac{8.1449\cdot 10^{-5}}{x^{13}} - \frac{0.0531\cdot 10^4}{x^{11.5}} - \frac{0.0160\cdot 10^4}{x^{10}} - \frac{0.0026\cdot 10^4}{x^{8.5}}$ $-\frac{1.0284\cdot 10^{-7}}{x^7} + \frac{0.0002\cdot 10^4}{x^{5.5}} \frac{0.0001\cdot 10^4}{x^4} + \frac{8.1449\cdot 10^{-5}}{x^{2.5}}$	if $\|x\| > 3$
1.99	$1.442\cdot 10^{-8}x^{11} - 8.171\cdot 10^{-7}x^{10} + 1.778\cdot 10^{-5}x^9 - 0.0002x^8 + 0.0013x^7$ $-0.0046x^6 + 0.0070x^5 + 0.0011x^4 + 0.0049x^3 - 0.0725x^2 + 0.0014x$ $+0.2821$	if $\|x\| \leq 6$
	$\frac{1.0944\cdot 10^{12}}{x^{22.89}} + \frac{0.0247\cdot 10^{12}}{x^{20.9}} + \frac{0.0006\cdot 10^{12}}{x^{18.91}} + \frac{1.6546\cdot 10^7}{x^{16.92}} + \frac{5.0112\cdot 10^5}{x^{14.93}} + \frac{1.7129\cdot 10^4}{x^{12.94}}$ $+\frac{671.5207}{x^{10.95}} + \frac{30.8253}{x^{8.96}} + \frac{1.7011}{x^{6.97}} + \frac{0.1164}{x^{4.98}} + \frac{0.0099}{x^{2.99}}$	if $\|x\| > 6$.

[3] We need only obtain an expression for $f_\alpha(\gamma = 1, \delta = 0; \xi)$ since $f_\alpha(\gamma, \delta; \xi) = \gamma^{-\frac{1}{\alpha}} f_\alpha(1, 0; (\xi - \delta)\gamma^{-\frac{1}{\alpha}})$, as can be seen from Eq.(2-1).

2.2 Generation of SαS random deviates

The lack of a closed form expression for the general SαS pdf and, in particular, for the inverse of the corresponding cumulative distribution function (cdf) renders the usual transformation method [24, chapter 7] inappropriate for the generation of pseudorandom SαS deviates. Fortunately, alternative procedures have been devised, originally restricted to certain values of the characteristic exponent α and later extended to arbitrary $\alpha \in (0,2]$. Without getting into the details of the method, we present the result and refer the reader to [25] and [23, pp. 248-251] for its derivation and further details.

Let U be uniform in $(-\frac{\pi}{2}, \frac{\pi}{2})$ and E standard exponential (see for example [24, chapter 7] for definitions and algorithms that can generate these deviates). Then, the random variables:

$$Y_1(U) = \tan(U) \tag{2-13}$$

$$Y_\alpha(U) = \frac{\sin(\alpha U)}{(\cos U)^{\frac{1}{\alpha}}}\{\frac{\cos[(1-\alpha)U]}{E}\}^{\frac{1-\alpha}{\alpha}}, \quad 0 < \alpha \leq 2, \quad \alpha \neq 1 \tag{2-14}$$

are Cauchy and S($\alpha \neq 1$)S distributed, respectively.

For illustration and comparison purposes, we show in Figs. 3 SαS deviates generated with this method. In particular, we draw the reader's attention to Figs. 3a and 3b, where independent Gaussian and independent S($\alpha = 1.99$)S noise are shown. Clearly, several outliers can be observed in the second case, despite the fact that the characteristic exponent $\alpha = 1.99$ is so close to the Gaussian value $\alpha = 2$. As the characteristic exponent gets smaller, both the number and the strength of the outliers increase, resulting to the extremely impulsive process of Figs. 3e and 3f in the case of $\alpha = 0.5$.

3. Asymptotic Probability of Error for Four Classes of Receivers in Independent SαS Noise

In this paper, we examine the performance of coherent receivers in independent SαS noise. For simplicity, we restrict the presentation to binary signaling; however, the generalization to arbitrary M-ary signaling and to incoherent reception is straightforward and the findings of this paper will also hold for those cases.

Our mathematical model is, therefore, the following hypothesis testing problem:

$$\begin{aligned} H_0 : x(k) &= s_0(k) + n_\alpha(k), \quad k = 1, 2, \ldots, N \\ H_1 : x(k) &= s_1(k) + n_\alpha(k), \quad k = 1, 2, \ldots, N, \end{aligned}$$

where $s_i(\cdot)$, $i = 0, 1$, is one of two possible transmitted signals and $n_\alpha(\cdot)$ is a realization of a sequence of N independent, identically distributed SαS random variables of characteristic exponent α ($0 < \alpha \leq 2$) and dispersion γ. The receiver needs to make a decision on which hypothesis is true (i.e., which signal $s_i(\cdot)$ was sent) on the basis of the observed data $x(\cdot)$. We will assume that $s_1(k) = -s_0(k)$, $k = 1, 2, \ldots, N$ (antipodal signaling), as is for example the case of a BPSK communications system. The following analysis is, however, valid for arbitrary signaling waveforms and is, thus, applicable to all the communications systems in use.

3.1 Optimum Receiver

To decide between the two hypotheses H_0 and H_1, the optimum (in the maximum likelihood (ML) sense) receiver computes the test statistic

$$\begin{aligned} \Lambda &= \log\{\frac{\prod_{k=1}^{N} f_\alpha[x(k) - s_0(k)]}{\prod_{k=1}^{N} f_\alpha[x(k) - s_1(k)]}\} \\ &= \sum_{k=1}^{N} \log\{\frac{f_\alpha[x(k) - s_0(k)]}{f_\alpha[x(k) - s_1(k)]}\} \end{aligned} \tag{3-1}$$

and compares it to a preset threshold η. When $\Lambda \geq \eta$, the receiver decides that $s_0(\cdot)$ was sent, otherwise that $s_1(\cdot)$ was sent. We are going to assume that $\eta = 0$, a threshold setting which minimizes the probability of error. The following analysis is valid, however, for arbitrary threshold η.

Clearly, the test statistic Λ is a random variable with finite mean and variance, even when the noise process n_α is a SαS process with infinite variance [16].[4] The mean of the test statistic Λ, assuming $s_0(\cdot)$ was sent, can then be computed as

$$\begin{aligned} \mu_0 &= \mathcal{E}\{\Lambda | s_0(\cdot) \text{ sent}\} \\ &= \sum_{k=1}^{N} \int_{-\infty}^{\infty} f_\alpha(\xi - s_0(k)) \log\{\frac{f_\alpha[\xi - s_0(k)]}{f_\alpha[\xi - s_1(k)]}\} \, d\xi \end{aligned}$$

[4]One way to understand this assertion is to consider the asymptotic behavior of the integrands in Eqs.(3-2) and (3-3) as $\xi \to \infty$.

$$
\begin{aligned}
&= \sum_{k=1}^{N} \int_{-\infty}^{\infty} f_\alpha(\xi) \log\{\frac{f_\alpha(\xi)}{f_\alpha[\xi + 2s_0(k)]}\} \, d\xi \\
&= -\mathcal{E}\{\Lambda | s_1(\cdot) \text{ sent}\} = -\mu_1 \qquad \text{(3-2)}
\end{aligned}
$$

and its finite variance as

$$
\begin{aligned}
\sigma^2 &= \text{var}\{\Lambda | s_0(\cdot) \text{ or } s_1(\cdot) \text{ sent}\} \\
&= \sum_{k=1}^{N} \int_{-\infty}^{\infty} f_\alpha(\xi) \log^2\{\frac{f_\alpha(\xi)}{f_\alpha[\xi + 2s_0(k)]}\} \, d\xi - \sum_{k=1}^{N} [\int_{-\infty}^{\infty} f_\alpha(\xi) \log\{\frac{f_\alpha(\xi)}{f_\alpha[\xi + 2s_0(k)]}\} \, d\xi]^2. \qquad \text{(3-3)}
\end{aligned}
$$

The test statistic Λ being a superposition of N independent random variables satisfying the assumptions of the Central Limit Theorem, its asymptotic (for large N) distribution is Gaussian with mean and variance given by Eqs.(3-2) and (3-3), respectively. Assuming that the signals $s_0(\cdot)$ and $s_1(\cdot)$ are sent with equal probability (equal to $\frac{1}{2}$), the average probability of an erroneous decision will be

$$
P_e = \Pr\{\Lambda < 0 | s_0(\cdot) \text{ sent}\} = \frac{1}{2} \text{erfc}(\frac{\mu_0}{\sqrt{2\sigma^2}}), \qquad (3-4)
$$

where erfc is the complimentary error function: $\text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-\xi^2} \, d\xi$.

3.2 Linear (Gaussian) Receiver

The linear receiver is derived by assuming the characteristic exponent of the noise pdf $f_\alpha(\cdot)$ in the test statistic Λ of Eq.(3-1) to equal $\alpha = 2$. Thus the linear receiver is the optimum receiver in the ML sense when the interfering noise is Gaussian. We will now examine its performance when the incoming signal is corrupted by SαS noise of characteristic exponent α not necessarily equal to 2 and compare it to the performance of the corresponding optimum receiver of the previous subsection.

After setting $\alpha = 2$ and carrying out the algebra in Eq.(3-1), we end up with the expression

$$
\Lambda_l = \frac{1}{\gamma} \sum_{k=1}^{N} x(k) s_0(k) \qquad (3-5)
$$

for the test statistic of the linear receiver. In Eq.(3-5) above, γ is the dispersion in the interfering SαS noise. Since the test statistic Λ_l is a linear superposition of N independent SαS random variables, it is itself a SαS random variable. Its location parameter, assuming $s_0(\cdot)$ was sent, is

$$
\delta_0 = \frac{1}{\gamma} \sum_{k=1}^{N} s_0^2(k) = -\delta_1 \qquad (3-6)
$$

and its dispersion

$$\gamma_l = \gamma^{1-\alpha} \sum_{k=1}^{N} |s_0(k)|^\alpha. \qquad (3-7)$$

Thus, assuming again equiprobable signals $s_0(\cdot)$ and $s_1(\cdot)$, the average probability of erroneous decision at the linear receiver becomes

$$P_e^l = \Pr\{\Lambda_l < 0 | s_0(\cdot) \text{ sent}\} = \int_{-\infty}^{0} f_\alpha(\gamma_l, \delta_0; \xi)\, d\xi, \qquad (3-8)$$

where $f_\alpha(\gamma_l, \delta_0; \cdot)$ is the SαS pdf with location parameter δ_0 and dispersion γ_l.

3.3 Limiter plus Integrator

Let us consider a limiting device with input-output characteristic

$$g(\xi) = \begin{cases} \xi, & \text{if } |\xi| < \kappa \\ 0, & \text{otherwise.} \end{cases}$$

Let the input to this limiter be a random variable x of pdf $f_{in}(\cdot)$ and cumulative distribution function $F_{in}(\cdot)$. At the output of the limiter, a random variable y will then be observed with pdf [26, pp. 123-124]

$$f_{out}(\xi) = [1 - F_{in}(\kappa)]\delta(\xi - \kappa) + F_{in}(-\kappa)\delta(\xi + \kappa) + I_{(-\kappa,\kappa)}(\xi) f_{in}(\xi), \qquad (3-9)$$

where $I_{(-\kappa,\kappa)}(\xi) = 1$, when $|\xi| < \kappa$, and zero, otherwise.

In our case, let $y(\cdot)$ be the output of the limiter when the data $x(\cdot)$ is observed, i.e., $y(k) = g[x(k)]$, $k = 1, 2, \ldots, N$. Then, $y(\cdot)$ is a sequence of independent, identically distributed random variables, each having a pdf given by Eq.(3-9) above with $f_{in}(\cdot)$ ($F_{in}(\cdot)$) being a SαS pdf $f_\alpha(\gamma, \delta_i; \cdot)$ (cumulative distribution function $F_\alpha(\gamma, \delta_i; \cdot)$) of dispersion γ and location parameter δ_i. Here, $i = 0$ or 1, depending on whether $s_0(\cdot)$ or $s_1(\cdot)$ was sent. The receiver bases its decision on the test statistic

$$\Lambda_{LI} = \sum_{k=1}^{N} y(k) s_0(k) \qquad (3-10)$$

and decides that $s_0(\cdot)$ was sent if $\Lambda_{LI} \geq 0$, otherwise $s_1(\cdot)$ was sent. Assuming that $s_0(\cdot)$ was sent, the test statistic has mean

$$\mu_{0,LI} \;=\; \mathcal{E}\{\Lambda_{LI} | s_0(\cdot) \text{ sent}\}$$

$$
\begin{aligned}
&= \sum_{k=1}^{N} s_0(k)\mathcal{E}\{y(k)|s_0(\cdot) \text{ sent}\} \\
&= -\mathcal{E}\{\Lambda_{LI}|s_1(\cdot) \text{ sent}\} \\
&= -\mu_{1,LI} \qquad (3\text{-}11)
\end{aligned}
$$

and finite variance

$$
\begin{aligned}
\sigma^2_{LI} &= \text{var}\{\Lambda_{LI}|s_0(\cdot) \text{ or } s_1(\cdot) \text{ sent}\} \\
&= \sum_{k=1}^{N} |s_0(k)|^2 \text{ var}\{y(k)|s_0(\cdot) \text{ or } s_1(\cdot) \text{ sent}\}.
\end{aligned} \qquad (3\text{-}12)
$$

The expected mean and expected variance, $\mathcal{E}\{y(k)|s_0(\cdot) \text{ sent}\}$ and $\text{var}\{y(k)|s_0(\cdot) \text{ or } s_1(\cdot) \text{ sent}\}$, can be directly computed using Eq.(3-9) as

$$
\mathcal{E}\{y(k)|s_0(\cdot) \text{ sent}\} = \int_{-\infty}^{\infty} \xi f_{out}(\xi|s_0(\cdot) \text{ sent})\, d\xi = \kappa - \int_{-\kappa}^{\kappa} F_\alpha(\gamma, s_0(k); \xi)\, d\xi \qquad (3-13)
$$

$$
\begin{aligned}
\text{var}\{y(k)|s_0(\cdot) \text{ or } s_1(\cdot) \text{ sent}\} &= \int_{-\infty}^{\infty} \xi^2 f_{out}(\xi|s_0(\cdot) \text{ sent})\, d\xi - \mathcal{E}^2\{y(k)|s_0(\cdot) \text{ sent}\} \\
&= \kappa^2 - 2\int_{-\kappa}^{\kappa} \xi F_\alpha(\gamma, s_0(k); \xi)\, d\xi - \mathcal{E}^2\{y(k)|s_0(\cdot) \text{ sent}\}. \qquad (3\text{-}14)
\end{aligned}
$$

Substituting Eqs.(3-13) and (3-14) into Eqs.(3-11) and (3-12), we get

$$
\mu_{0,LI} = \sum_{k=1}^{N} s_0(k)[\kappa - \int_{-\kappa}^{\kappa} F_\alpha(\gamma, s_0(k); \xi)\, d\xi] \qquad (3\text{-}15)
$$

$$
\sigma^2_{LI} = \sum_{k=1}^{N} |s_0(k)|^2[\kappa^2 - 2\int_{-\kappa}^{\kappa} \xi F_\alpha(\gamma, s_0(k); \xi)\, d\xi] - \mu^2_{0,LI}. \qquad (3\text{-}16)
$$

Assuming again equiprobable signals $s_0(\cdot)$ and $s_1(\cdot)$, the asymptotic (for large N) probability of error for this receiver will be

$$
P_e^{LI} = \frac{1}{2}\text{erfc}(\frac{\mu_{0,LI}}{\sqrt{2\sigma^2_{LI}}}). \qquad (3-17)
$$

3.4 Cauchy Receiver

By Cauchy receiver we mean the receiver that employs as a test the statistic Λ_C derived from Eq.(3-1) under the assumption that $\alpha = 1$ (Cauchy noise). Given this definition and the expression in Eq.(2-3) for the Cauchy distribution, we get

$$
\Lambda_C = \sum_{k=1}^{N} \log\{\frac{f_1[x(k) - s_0(k)]}{f_1[x(k) - s_1(k)]}\} = \sum_{k=1}^{N} \log\{\frac{\gamma^2 + [x(k) + s_0(k)]^2}{\gamma^2 + [x(k) - s_0(k)]^2}\}. \qquad (3-18)
$$

The performance of the Cauchy receiver can be analyzed following similar steps to those followed in the analysis of the optimum receiver. In fact, the same approach can be taken to analyze the performance of a general SαS receiver constructed with a characteristic exponent mismatched to the actual characteristic exponent of the interfering noise. Following steps similar to those of section 3.1, we end up with the expression for the probability of error of the Cauchy receiver:

$$P_e^C = \frac{1}{2}\text{erfc}(\frac{\mu_{0,C}}{\sqrt{2\sigma_c^2}}), \qquad (3-19)$$

where

$$\mu_{0,C} = \sum_{k=1}^{N}\int_{-\infty}^{\infty} f_\alpha(\xi)\log\{\frac{f_1(\xi)}{f_1[\xi+2s_0(k)]}\}\,d\xi \qquad (3-20)$$

and

$$\sigma_C^2 = \sum_{k=1}^{N}\int_{-\infty}^{\infty} f_\alpha(\xi)\log^2\{\frac{f_1(\xi)}{f_1[\xi+2s_0(k)]}\}\,d\xi - \sum_{k=1}^{N}[\int_{-\infty}^{\infty} f_\alpha(\xi)\log\{\frac{f_1(\xi)}{f_1[\xi+2s_0(k)]}\}\,d\xi]^2. \qquad (3-21)$$

3.5 A Note on the Exact Computation of the Probability of Error

With the exception of the linear receiver of section 3.2, the expressions for the probability of error that we derived in this section of the paper are only asymptotically valid, i.e., they hold true only when the length N of the data sequence is large enough for the true distribution of the several test statistics to be well approximated by a Gaussian distribution. Thus a minimum number N of data samples may be needed for the expressions in Eqs.(3-4), (3-17), and (3-19) to be meaningful (see for example [26, pp. 266-268] for an illustration of the convergence rate of the central limit theorem). In general, we expect an error in the predicted values for the probabilities of error of the different receivers to arise from the sensitivity of these probabilities to the far tails of the true pdf of the corresponding test statistics, which are only poorly approximated by the Gaussian pdf [27]. A better estimate of the probabilities of error of the several receivers can be obtained either by performing extensive Monte-Carlo simulations or by the characteristic function based method that we describe next (see also [28] for similar computations).

Let t be any test statistic of the set $\{\Lambda, \Lambda_l, \Lambda_{LI}, \Lambda_C\}$. Since t is a nonlinear transformation of the random vector $x(\cdot)$ of independent components, its corresponding characteristic function will then be

$$\mathcal{E}\{e^{i\omega t}|s_0(\cdot)\text{ sent}\} = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{i\omega t(\xi_1,\ldots,\xi_N)}[\prod_{k=1}^{N} f_\alpha(\gamma, s_0(k);\xi_k)]\,d\xi_1\ldots d\xi_k \equiv T(\omega). \qquad (3-22)$$

In general, the characteristic function $T(\cdot)$ can be computed numerically for a large number of values of the parameter ω and stored. After this task has been completed, the probability of error, when one uses the statistic t, is computed, again via numerical integration, as [29]:

$$P_e^t = \Pr\{t < 0 | s_0(\cdot) \text{ sent}\} = \frac{1}{2} - \frac{1}{\pi}\int_0^\infty \frac{d\omega}{\omega}\Im\{T(\omega)\}, \qquad (3-23)$$

where $\Im$ denotes the imaginary part. This procedure is extremely computer intensive, requiring extensive numerical integrations, and we have chosen not to follow it. Instead, we performed comprehensive Monte–Carlo simulations of all the receivers in various SαS noise environments and we discuss our findings in the following section.

4. Performance Evaluation

We have evaluated the expressions in Eqs.(3-4), (3-8), (3-17) and (3-19) for the probability of erroneous decision for $s_0(\cdot)$ a square pulse of unit height ($s_0(k) = 1$, $k = 1, 2, \ldots, N$) and for different values of the characteristic exponent α and the dispersion γ in the pdf of the additive SαS noise. In particular, we examined the cases of $\alpha = 0.5, 1, 1.5, 1.99$, and 2 and of $\gamma = 1, 2, \ldots, 10$. We have assumed that the decision is based on $N = 10$ samples of the incoming signal plus noise, a value which is high enough for validity of the asymptotic probabilities of error of the different receivers. The choice of an appropriate threshold for the limiter is not entirely straightforward. If this threshold is set to a very high value, then the limiter may not clip enough noise, especially in situations of very impulsive noise of high level, leading to a poor performance. On the other hand, if this threshold is set to a very low value, the limiter may over-distort the signal, especially in situations of high signal-to-noise ratio, leading again to poor performance. Thus, an optimization procedure has to be followed, in which threshold is found which minimizes the average probability of error of the receiver. The value of this "optimum" threshold will depend significantly on the impulsiveness of the additive noise, the signal-to-noise ratio, and the transmitted pulse shape. In our simulations, the threshold of the limiter was set to $\kappa = 1$, under the rationale that the signal does not exceed this value and is, therefore, not distorted as it goes through clipping.[5] In Figs. 4, we show the performance of the optimum, the linear, the limiter plus integrator, and the Cauchy

[5] A similar limited study of the selection of the limiter threshold, based on the processing of real data, can be found in [30].

receiver for the above values of α and γ. Clearly, the optimum receiver outperforms the other three types of receivers in providing a minimum probability of error for all different values of the parameters α and γ of the interfering noise. This result is, of course, compatible with the well known fact [31] that the ML receiver also minimizes the probability of erroneous decision. The limiter plus integrator receiver provides an improvement over the linear receiver; however, it is not an optimum receiver.

To verify the accuracy and the validity of the asymptotic expressions, we ran extensive Monte–Carlo simulations of all receivers for all the previous values of the parameters α and γ of the interfering noise. In particular, random sequences of transmitted symbols of total length of 20000 were generated, with each symbol in the sequence being with the same probability ($\frac{1}{2}$) equal to one of two possibilities. For each symbol transmitted, the corresponding pulse $s_0(\cdot)$ or $s_1(\cdot)$ was generated and independent SαS stable noise, generated with the techniques of section 2.2, was added to it. The output of the several receivers was then computed and, finally, the number of erroneous decisions for each receiver were counted to obtain an estimate of the performance of the receivers. Because a large number (20000) of different symbols were employed, the estimates of the average probabilities of error are quite reliable. In fact, a comparison of the theoretical probability of error of the linear receiver, the corresponding expression of which (Eq.(3-8)) is exact, with the estimated probability of error obtained via Monte–Carlo simulation demonstrates excellent agreement. In Figs. 5, we compare the performance of the optimum, the linear, the limiter plus integrator, and the Cauchy receivers operating in independent SαS noise with characteristic exponent α and dispersion γ as in the previous paragraph. To facilitate the comparison of the relative performance of the various receivers, we give in Figs. 6 plots of the average probability of error of all receivers in SαS noise environments of various values of the parameters α and γ.

5. Conclusions and Future Research

We examined the performance of four different classes of receivers in the presence of impulsive noise modeled as a SαS random process. As measure of performance, we employed the average probability of erroneous decision as computed theoretically in the limit of a large number of samples of the incoming signal and also as obtained via extensive Monte-Carlo simulations. We found that, as expected, the optimum receiver outperforms the linear, the limiter plus integrator, and the Cauchy receiver in all cases in that it provides the lowest probability of error. The traditionally employed limiter plus integrator receiver was found to often perform below the Cauchy receiver or even the linear receiver in cases of very impulsive (i.e., of small characteristic exponent) noise of large dispersion. The Cauchy receiver was found to perform quite closely to, eventhough below, the optimum receiver for a wide range of characteristic exponents α and dispersion γ.

In the future, it seems interesting to examine the structure of detectors which are used for the detection of impulsive stochastic transients in a background of Gaussian noise. It is also interesting to examine the possibility of constructing non-parametric receivers for robust detection in stable noise of unknown characteristic exponent and/or dispersion. Such and related research is currently underway and the results of it are expected to be published in the near future.

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Fig. 1: The SaS Probability Density Functions

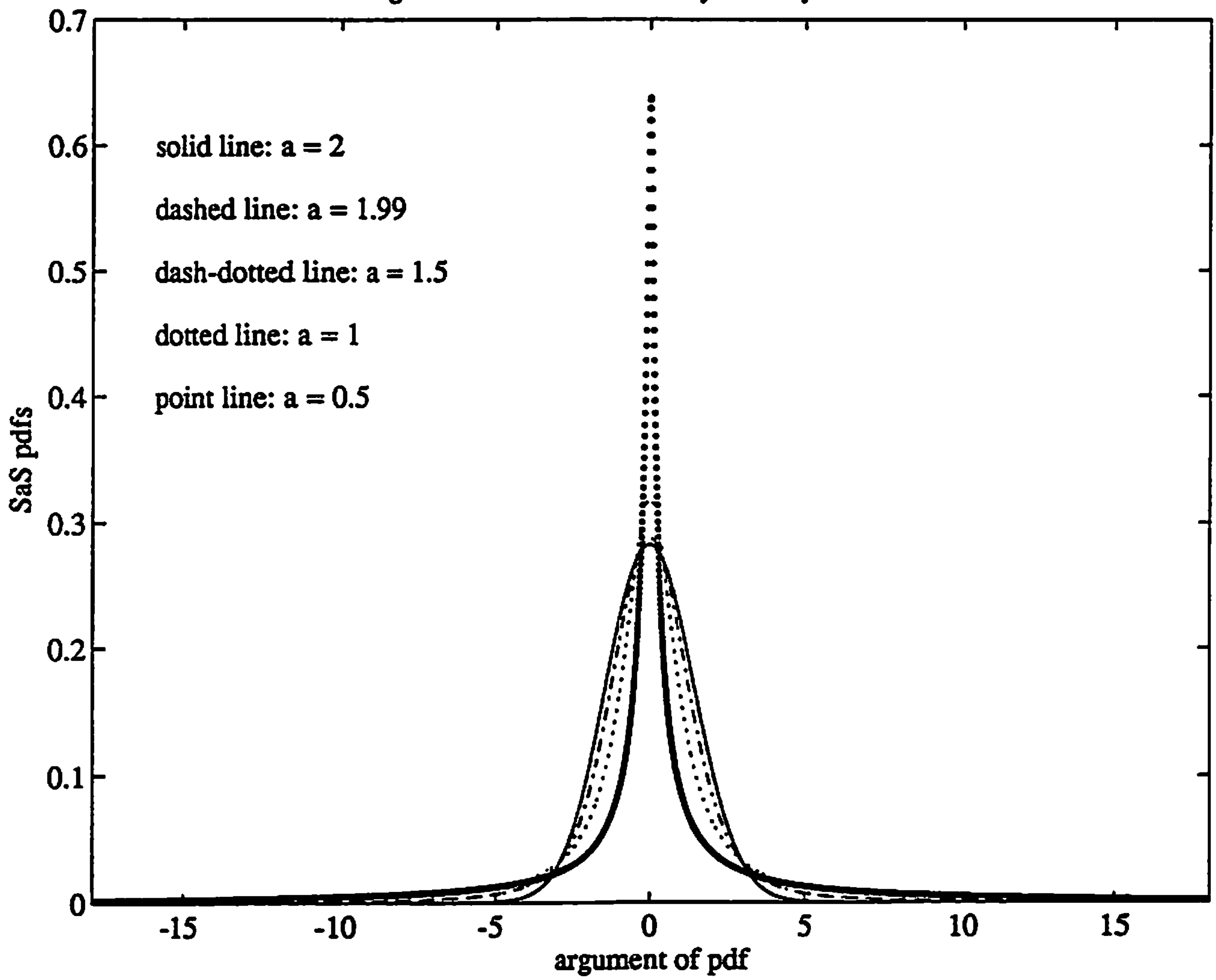


Fig. 2: Computation of the S(a=1.5)S Probability Density Function

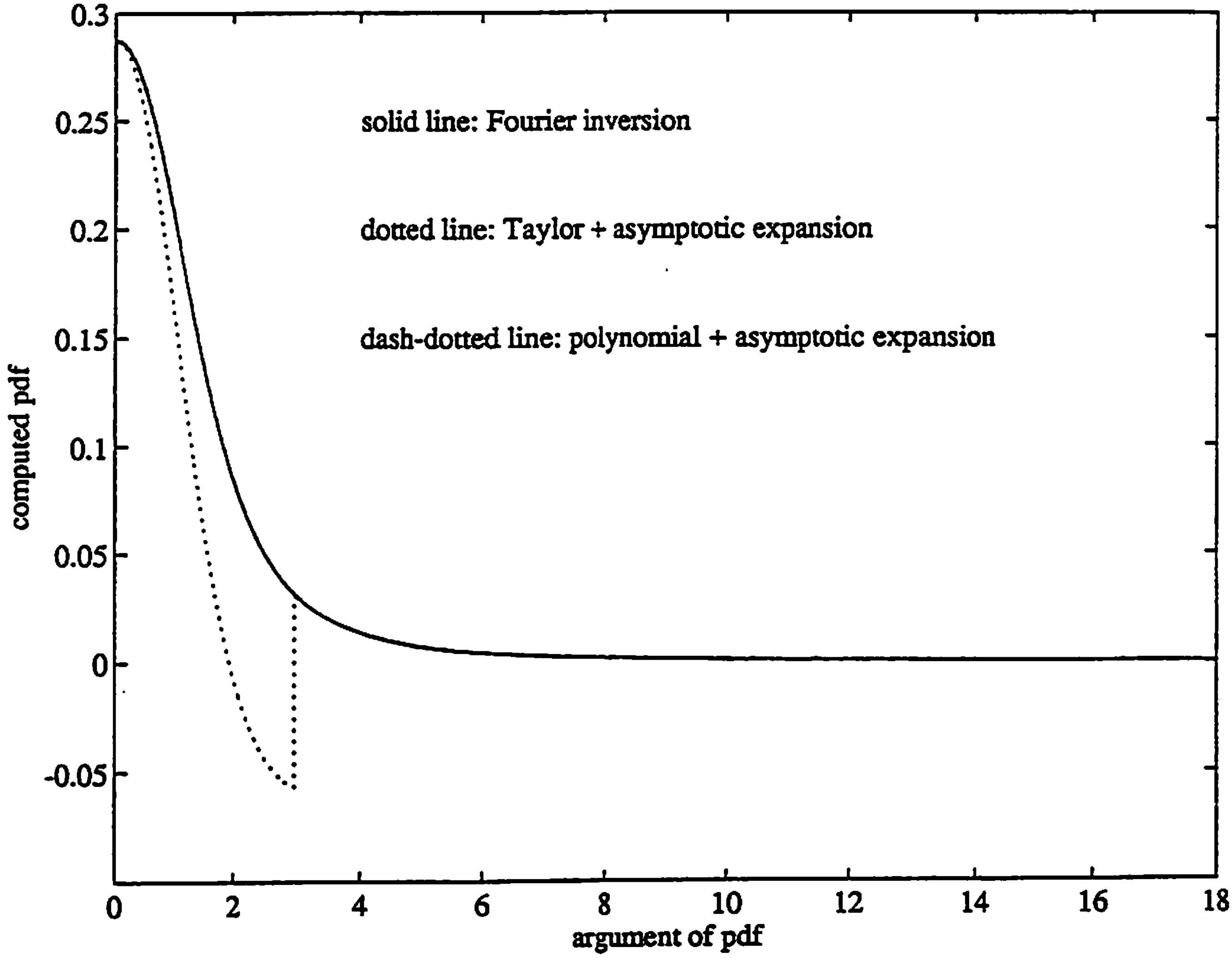


Fig. 3a: alpha = 2

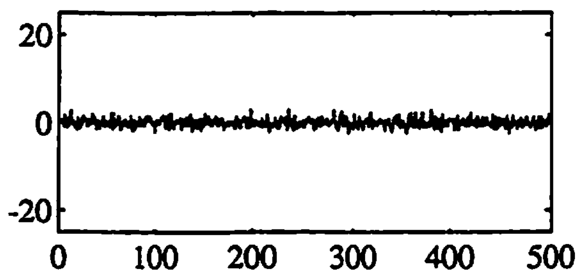


Fig. 3b: alpha = 1.99

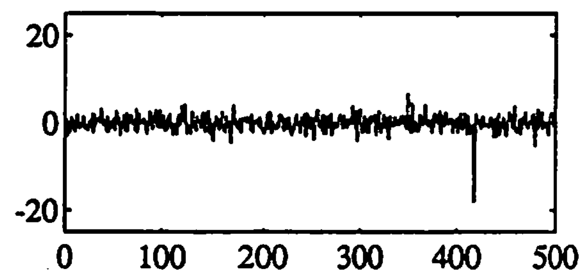


Fig. 3c: alpha = 1.5

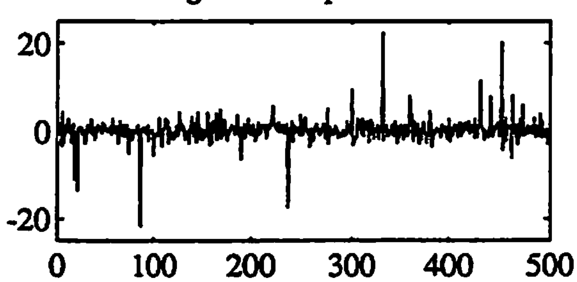


Fig. 3d: alpha = 1

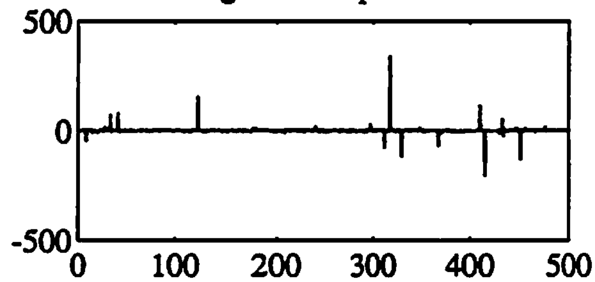


Fig. 3e: alpha = 0.5

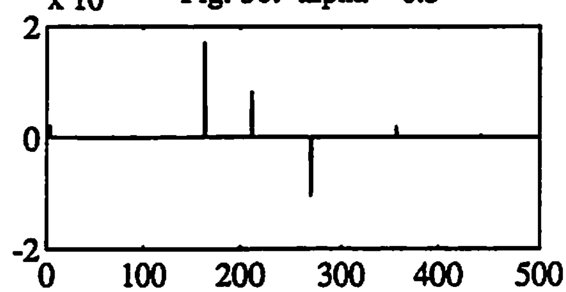
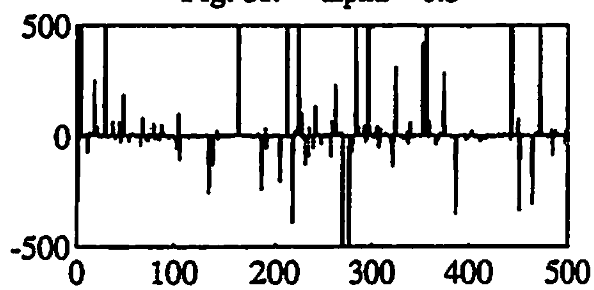


Fig. 3f: alpha = 0.5



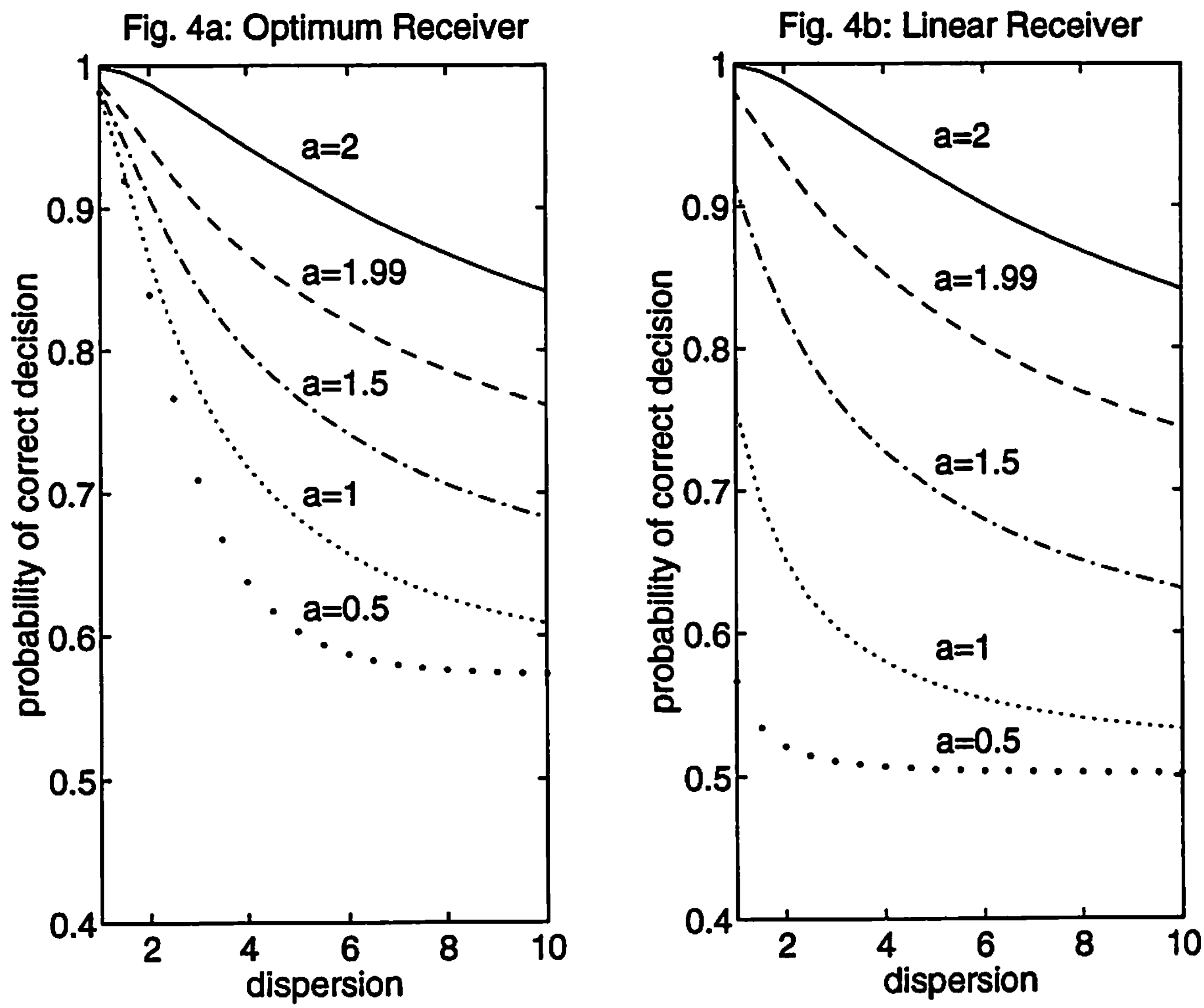


Fig. 4a: Optimum Receiver
Fig. 4b: Linear Receiver
probability of correct decision
dispersion
a=2
a=1.99
a=1.5
a=1
a=0.5
1
0.9
0.8
0.7
0.6
0.5
0.4
2
4
6
8
10

Fig. 4c: Limiter plus Integrator

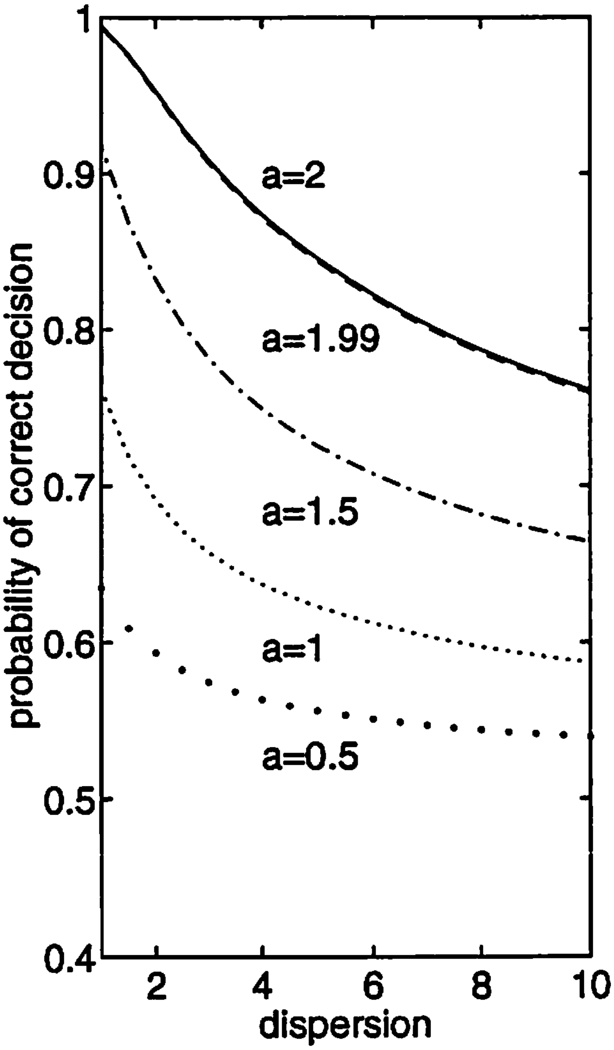


Fig. 4d: Cauchy Receiver

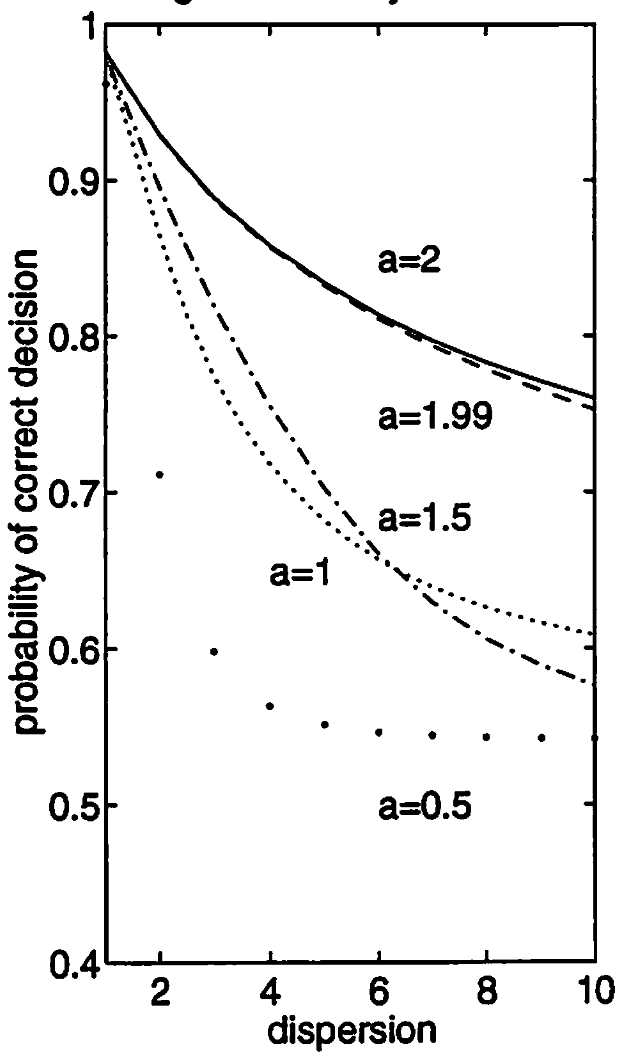


Fig. 5a: Optimum Receiver

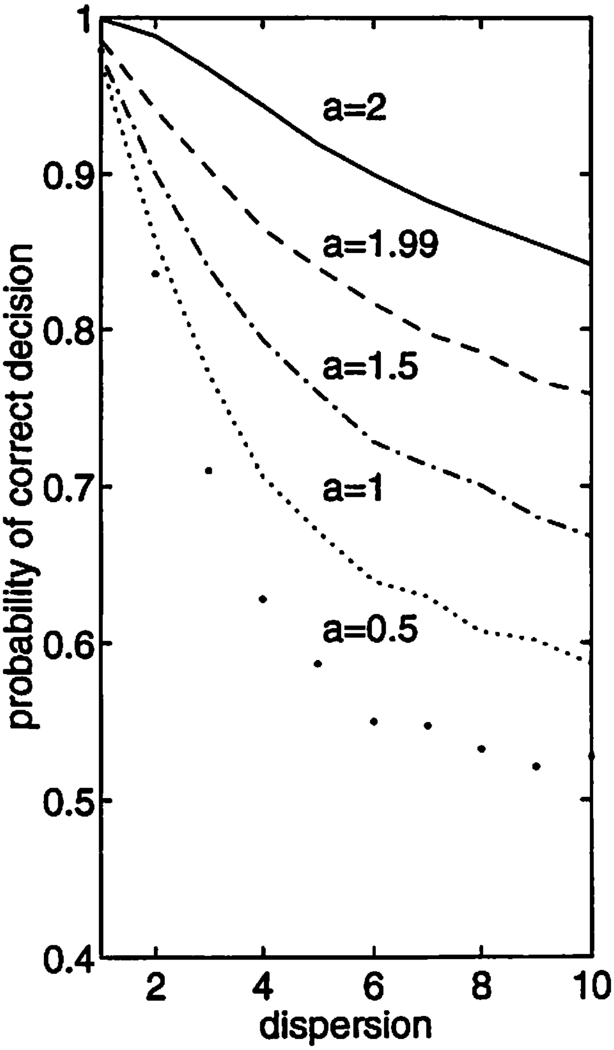
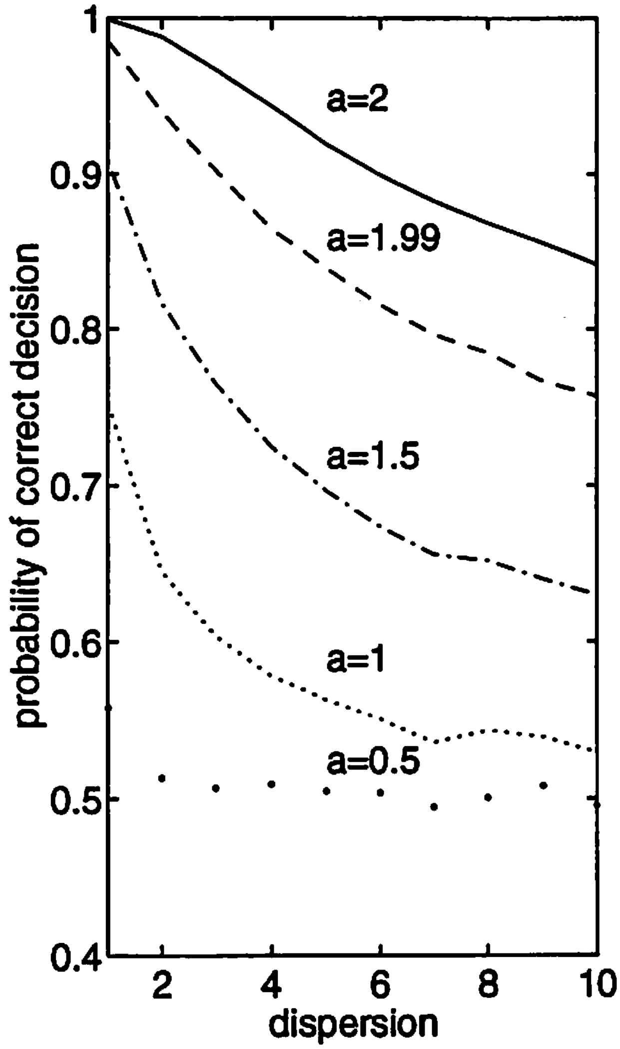


Fig. 5b: Linear Receiver



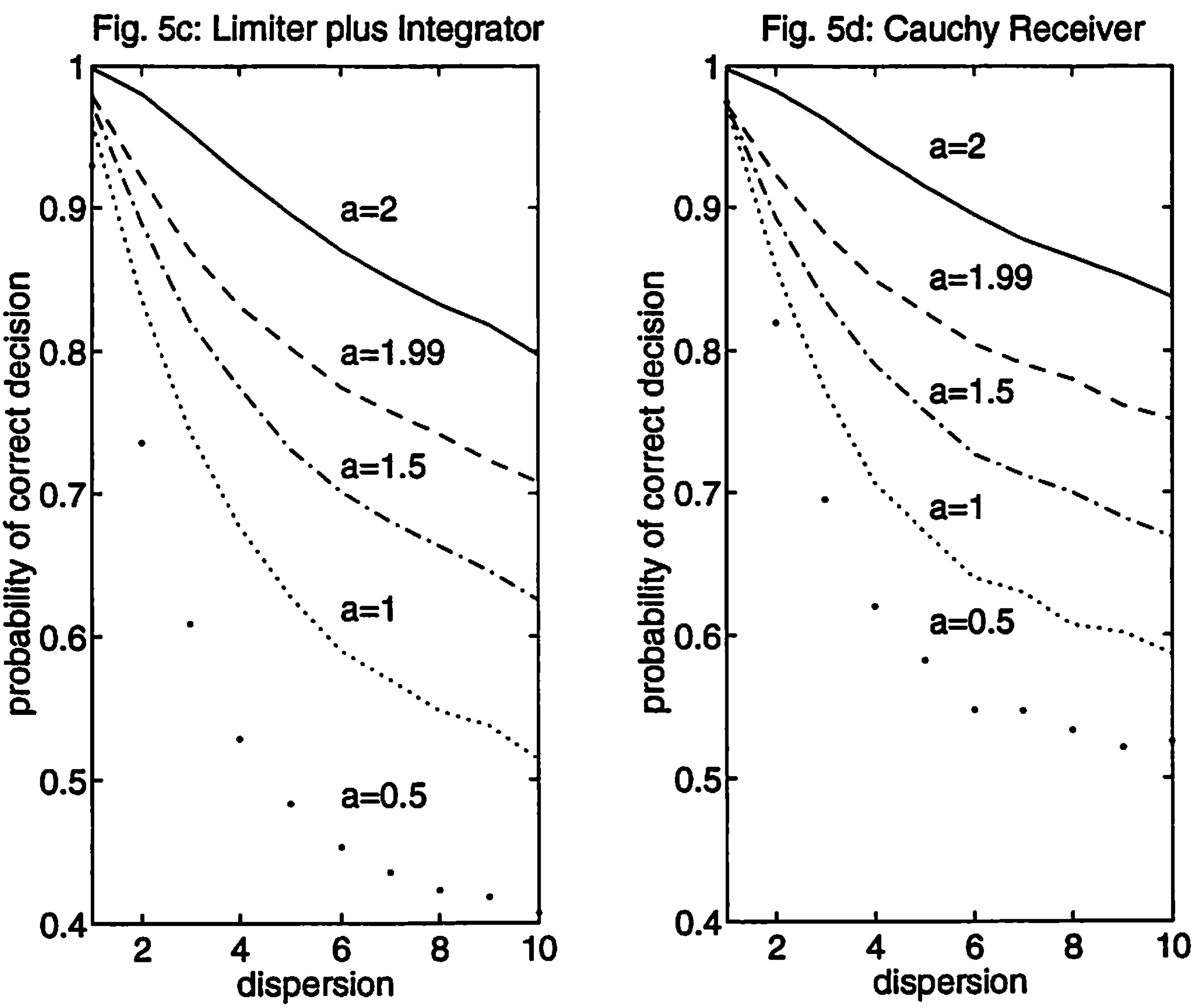


Fig. 5c: Limiter plus Integrator
probability of correct decision
dispersion
a=2
a=1.99
a=1.5
a=1
a=0.5
1
0.9
0.8
0.7
0.6
0.5
0.4
2
4
6
8
10
Fig. 5d: Cauchy Receiver
probability of correct decision
dispersion
a=2
a=1.99
a=1.5
a=1
a=0.5
1
0.9
0.8
0.7
0.6
0.5
0.4
2
4
6
8
10

solid line: optimum receiver

dashed line: Cauchy receiver

dash-dotted line: limiter plus
integrator

dotted line: linear receiver

Fig. 6a: a = 2

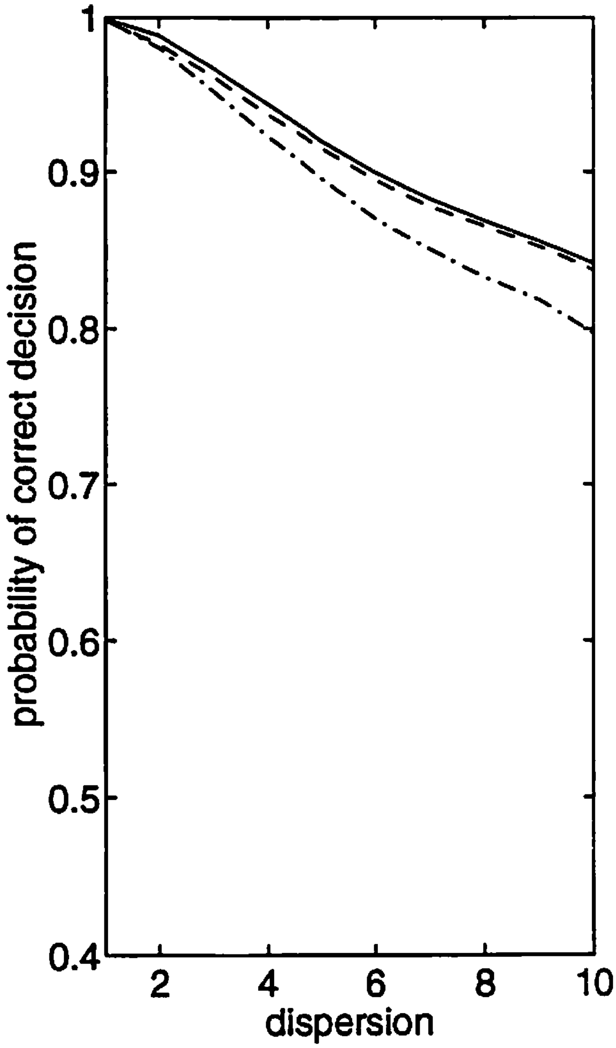


Fig. 6b: a = 1.99

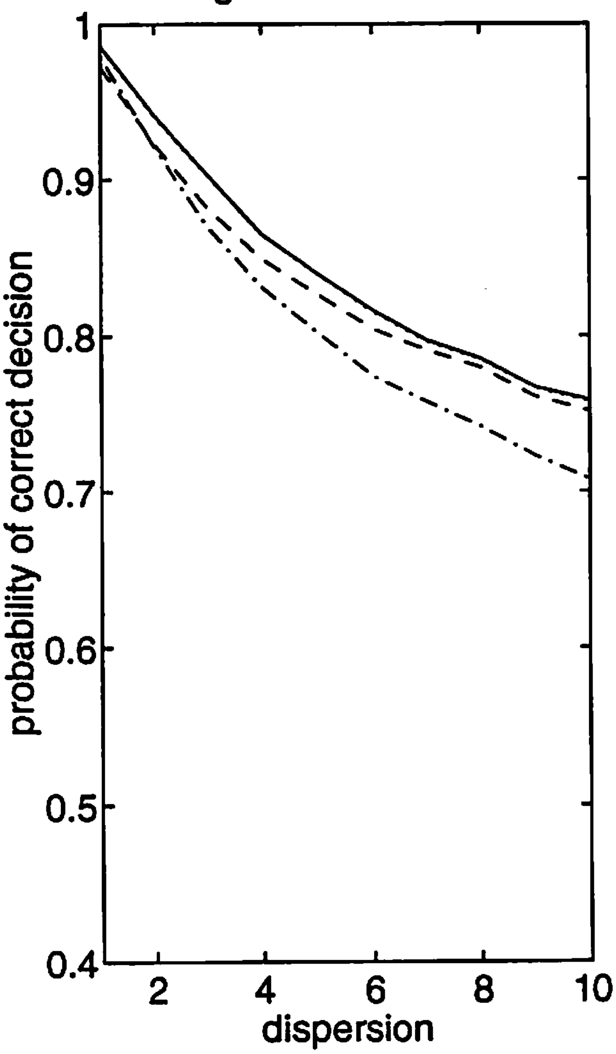
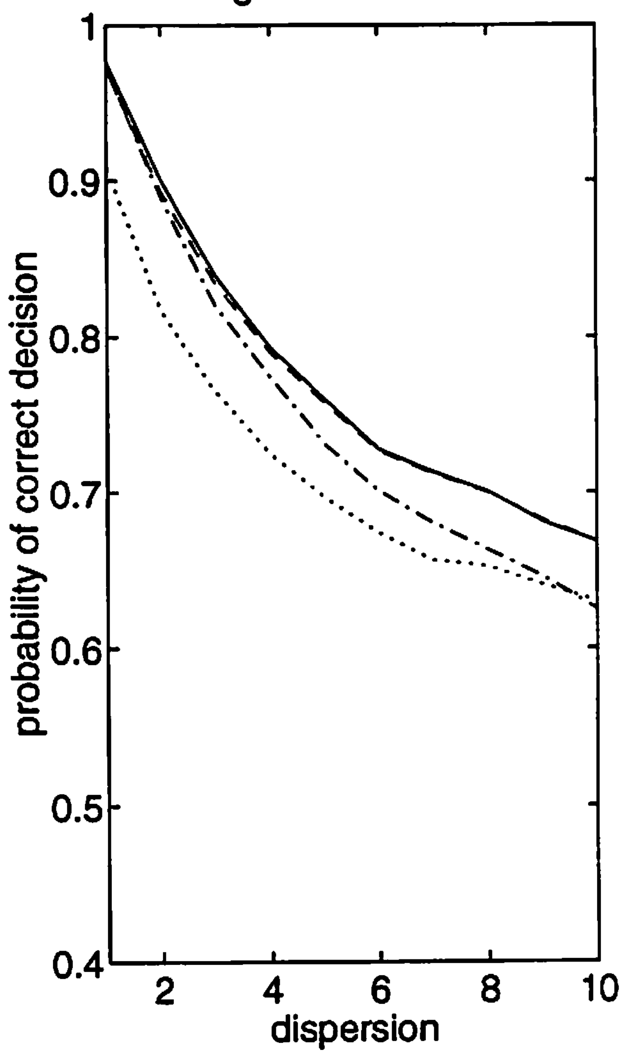


Fig. 6c: a = 1.5



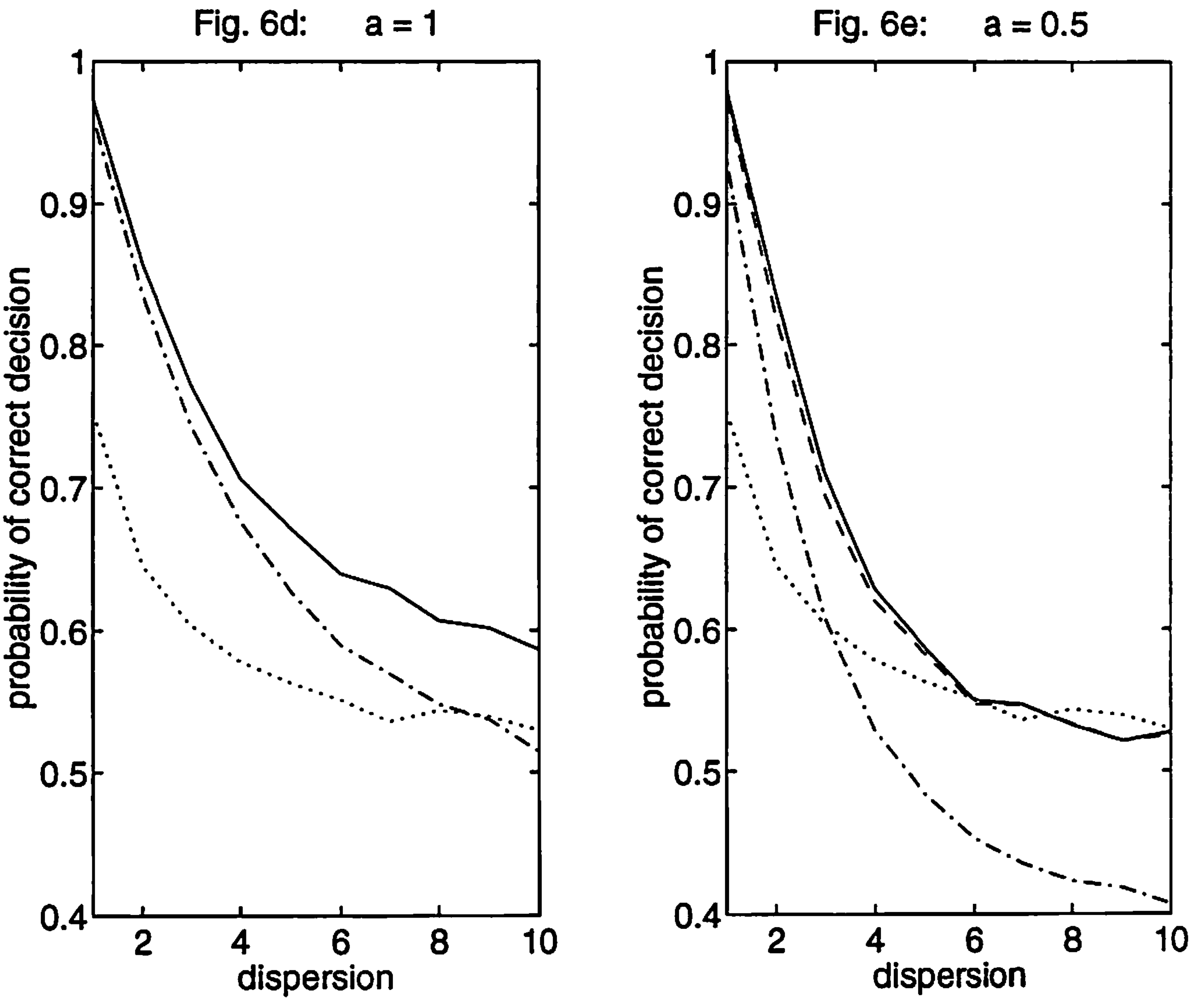


Fig. 6d: a = 1
Fig. 6e: a = 0.5
probability of correct decision
1
0.9
0.8
0.7
0.6
0.5
0.4
2
4
6
8
10
dispersion