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Signal Detection in Incompletely Characterized Impulsive Noise using Lower-Order Statistics

by

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Abstract

We address the problem of detection of signals of known shape but unknown strength in impulsive noise using lower-order statistics. We form a generalized likelihood ratio test which is based on moment, rather than maximum likelihood, estimates of the unknown parameters of the detection problem. We show that the moment estimates we propose are both asymptotically consistent and that the proposed generalized likelihood ratio test is asymptotically equivalent to the optimum likelihood ratio test corresponding to completely known signal and noise parameters (clairvoyant test). The proposed detection schemes can be very useful not only in the detection of sonar/radar/communication signals in impulsive interference, but also in other dual-use commercial applications.

Key words: Impulsive Noise, Stable Distribution, Optimum Detection, Generalized Likelihood Ratio.

1. Introduction

The detection of deterministic signals in additive noise is a very important problem that often arises in sonar, radar, communications, and other areas of application of modern signal processing. The appropriate algorithm to perform the detection task depends significantly on the degree of knowledge of the signal and noise characteristics. For detection of a completely known signal in completely characterized noise, a likelihood ratio test constitutes the optimum detection algorithm in either the Neyman–Pearson or the Bayes sense [1]. Likelihood ratio tests have a very high level of performance; however, their performance may drop significantly when the signal and noise characteristics depart from the assumed ones. Moreover, the test statistics implied by the likelihood ratio consist usually of very complicated, nonlinear mathematical expressions which are difficult to implement in real world applications.

The difficulties associated with likelihood ratio tests can be overcome with use of detection algorithms based on other (suboptimum) test statistics. On several occasions, simplifying assumptions are made regarding the signal and noise characteristics. For example, on the basis of the central limit theorem, the additive noise may be assumed to have a *Gaussian* distribution, even though this assumption rarely holds true. A different approach consists of constructing a detection test the minimum performance of which in an entire class of signal and noise characteristics is maximized. Such a detector is called *robust* in the class of signals and noises considered [2]. Similarly, a *nonparametric* detector is one, the performance of which is constant over an entire class of signals and noises [3]. We will collectively refer to these detectors as *suboptimum.*

The performance of the suboptimum detectors is, as expected, significantly inferior to the performance of optimum detectors. On one hand, the optimum detectors assume very detailed knowledge, while, on the other hand, suboptimum detectors assume very little knowledge of the detection problem characteristics. Therefore, in cases where the characteristics of the detection problem are known up to a certain (incomplete) degree, other types of tests are expected to exist, which have a performance closer to that of the optimum tests than the performance of the suboptimum tests. One very important such case arises when the signal and noise processes can be satisfactorily modeled as belonging to classes of processes completely characterized except for a finite number of parameters. Then, one approach would be to compute the optimum detection test, as if the signal and noise characterizing parameters were completely known, and to substitute in the resulting expression *estimates* of the unknown parameters directly derived from the available data. This test is referred to as the *generalized likelihood ratio* test and, of course, is optimum under none of the usual criteria [1]. However, it is quite popular and often results in detection performance quite close to the performance of the optimum test [4, chapter 6]. The most commonly employed estimates for the unknown signal and noise parameters are the *maximum likelihood* estimates, obtained via maximization of the corresponding likelihood function. In fact, the performance of generalized likelihood ratio tests for large numbers of data (asymptotic performance) can be theoretically computed under general assumptions.

In this paper, we are concerned with the problem of detection of signals of known shape but unknown strength embedded in impulsive noise modeled as an independent stable process. Stable processes have been established as appropriate models for intereference which, for small intervals of time, attains very large values (impulsive noise) [5, 6]. The performance of optimum receivers, relative to the performance of several suboptimum receivers, was examined in [7]. The approach in [7] assumed, however, that a complete characterization of the signal and the noise processes were available. Here, we relax this assumption and allow the signal level, as well as the noise characteristic exponent and dispersion, to remain unknown. We develop a generalized likelihood ratio test and compare its performance with that of the corresponding optimum test, as well as that of the popular Student's

t-test [3, chapter 3]. Our test is shown to outperform Student's t-test and, in fact, to have a performance which asymptotically approaches that of the corresponding optimum test. In particular, the paper is organized as follows: In Section 2, we formulate the detection problem of interest as a hypothesis testing problem and propose a generalized likelihood ratio test for its solution. In Section 3, we examine the performance of the estimates that we have chosen for the generalized likelihood ratio test. In Section 4, we study the performance of the test relative to the performance of other tests. In Section 5, we generalize the concepts of the first four sections of the papers to consider the detection of multiple signals in incompletely characterized impulsive noise. Finally in section 6, we draw conclusions and suggest possible future extensions of this research.

2. Formulation of the Detection Problem

2.1 Hypothesis testing problem

We consider the following hypothesis testing problem:

$$\begin{aligned} x(k) &= n_{1,\gamma}(k) && \text{under hypothesis } H_0 \\ x(k) &= A + n_{1,\gamma}(k) && \text{under hypothesis } H_1, \end{aligned} \tag{1}$$

where $k = 1, 2 \ldots, N$ and $\{x(k)\}$ and $\{n_{1,\gamma}(k)\}$ are the observation sequence and a sequence of independent, identically distributed Cauchy random variables of unknown dispersion γ, respectively. A is the unknown signal amplitude. The detection problem, therefore, consists of deciding whether the observed data sequence $\{x(k)\}$ contains noise only or if a constant signal is also present.

2.2 Clairvoyant test

If the signal level A and the noise dispersion γ were known exactly, the optimum test for the hypothesis testing problem in Eqs.(1) would employ the likelihood ratio:

$$t_c[x(1), x(2), \ldots, x(N)] = \sum_{k=1}^{N} \log\{\frac{\gamma^2 + x^2(k)}{\gamma^2 + [x(k) - A]^2}\}. \tag{2}$$

This form of the likelihood ratio is attained by considering the Cauchy probability density function (pdf). We will refer to the optimum test as the *clairvoyant* test, since it assumes that the receiver has complete knowledge of the signal and noise parameters.

2.3 Proposed generalized likelihood ratio test

The test we propose, uses estimates of the unknown signal and noise parameters obtained from the observed data under the hypotheses H_0 and H_1. In particular, the test computes the statistic

$$t_g[x(1), x(2), \ldots, x(N)] = \sum_{k=1}^{N} \log\{\frac{\frac{\hat{\hat{\gamma}}}{\hat{\hat{\gamma}}^2 + [x(k) - \hat{\hat{A}}]^2}}{\frac{\hat{\gamma}}{\hat{\gamma}^2 + x^2(k)}}\}, \tag{3}$$

where the estimates $\hat{\gamma}$, $\hat{\hat{\gamma}}$, and $\hat{\hat{A}}$ are defined as:

$$\begin{aligned}
\hat{\gamma} &= [\frac{\frac{1}{N}\sum_{k=1}^{N} |x(k)|^p}{C(p,1)}]^{\frac{1}{p}} && (4)\\
\hat{\hat{A}} &= \text{median}[x(1), x(2), \ldots, x(N)] && (5)\\
\hat{\hat{\gamma}} &= [\frac{\frac{1}{N}\sum_{k=1}^{N} |x(k) - \hat{\hat{A}}|^p}{C(p,1)}]^{\frac{1}{p}}. && (6)
\end{aligned}$$

In the above, $C(p, \alpha)$ denotes

$$C(p, \alpha) = \frac{1}{\cos(\frac{\pi}{2}p)} \frac{\Gamma(1 - p/\alpha)}{\Gamma(1 - p)}$$

for $0 < p < \alpha$.

2.4 Student's t–test

A test of interest to the presented problem is Student's t–test. This test arises in a detection problem similar to that of Eqs.(1), but with the noise process assumed to follow a Gaussian distribution of unknown variance. In particular, the test attains the form:

$$t_S[x(1), x(2), \ldots, x(N)] = \frac{\sqrt{N}\overline{x}}{\{\frac{1}{N-1}\sum_{k=1}^{N}[x(k) - \overline{x}]^2\}^{\frac{1}{2}}}, \tag{7}$$

where

$$\overline{x} = \frac{1}{N}\sum_{k=1}^{N} x(k). \tag{8}$$

It has been shown that, assuming $A > 0$, Student's t–test is a uniformly most powerful invariant test [3, chapter 3] for Gaussian noise.

3. Performance of the Proposed Estimators

In this section, we analyze the performance of the estimators of Eqs.(4), (5), (6), and (8) via Monte–Carlo simulation. A (asymptotic) theoretical analysis is also possible; however, space limitations do not allow its inclusion in here and, therefore, we postpone the presentation of the theoretical results to a later time. Our simulation consisted of 10,000 Monte–Carlo runs of the estimation procedure. We examined several cases, namely we assumed a signal amplitude $A = 1$, a noise dispersion $\gamma = 1$, and the values $N = 10, 20, 50$, and 100 for the number of observations. The following table shows the results of our simulation and, in particular, the sample average and the standard deviation (in parentheses) in the estimates returned via application of Eqs.(4), (5), (6), and (8).

Performance of Estimators				
	$N = 10$	$N = 20$	$N = 50$	$N = 100$
$\hat{\gamma}$	1.5148 (6.6245)	1.3696 (3.6381)	1.2665 (1.0657)	1.2420 (0.8337)
$\hat{A}$	1.0099 (0.5779)	1.0008 (0.3693)	1.0005 (0.2281)	1.0004 (0.1578)
$\hat{\hat{\gamma}}$	1.2601 (6.5452)	1.1314 (3.5396)	1.0448 (1.0101)	1.0286 (0.8071)
$\bar{x}$	2.6898 (107.7242)	2.5347 (118.8331)	2.2215 (63.1576)	2.5850 (103.5716)

From this table, we clearly see the asymptotic consistency, as well as efficiency, of the estimators (5) and (6) for the signal amplitude and the noise dispersion, respectively. On the other hand, the usual sample average of Eq.(8) clearly performs poorly as an estimator of the signal amplitude. This is easily understood if one considers the fact that this estimator has exactly the same distribution as any individual observation.

4. Performance of the Proposed Test

In this section, we examine the performance of the proposed test (3) and compare it to the performance of the clairvoyant test (2) and Student's t-test (7). The performance is, again, evaluated via Monte-Carlo simulation, even though a theoretical analysis is possible.

We ran 10,000 Monte–Carlo runs of the detection tests (2), (3), and (7) for the same values of the problem parameters, namely $A = 1$, $\gamma = 1$, and $N = 10, 20, 50$, and 100. The results are shown in Fig. 1, in which the receiver operating characteristics of the clairvoyant, the proposed, and Student's t–test are drawn in solid, dashed, and dash–dotted line, respectively. It is clear that the test we propose outperforms Student's t–test for all values of the number N of observations. Moreover, as this number increases, Student's t–test maintains the same performance while, on the other hand, our test very fast reaches in performance the clairvoyant test and, thus, is shown to asymptotically attain the maximum achievable performance in the Neyman–Pearson sense.

5. Detection of Multiple Signals using an Array of Sensors

In this section, we generalize the material of the previous four sections to consider the detection of multiple signals embedded in incompletely characterized impulsive noise of unknown level. This problem is very significant in underwater sonar, where detection of the presence/absense of multiple signals and estimation of their parameters needs to be made, as well as in radar and communication channels. In all these applications, the noise component contains an impulsive term, causing degradation in the performance of algorithms designed on a Gaussianity assumption. In underwater sonar, for example, impulsive interference may be due to ice cracking in the arctic region or random signal reflections from the sea bed [8]. Similarly, in radar and communication channels, impulsive interference may be caused by lightning in the atmosphere, switching transients, and accidental hits.

For the detection of multiple signals, we need to use an array of sensors, rather than a single sensor, measuring the received signal over a number of snapshots. In particular, let us consider the receiving configuration of Fig. 2, where an array of N elements observes an incoming waveform $x(k, l)$, $k = 0, 1, \ldots, K-1$, $l = 0, 1, \ldots, L-1$, over K snapshots.

The detection problem consists of deciding whether the observed data consist of noise only or they contain a signal. In mathematical terms, the detection algorithm needs to decide between the two possible hypotheses:

$$\begin{aligned} x(k,l) &= n_{1,\gamma}(k,l) && \text{under hypothesis } H_0 \\ x(k,l) &= s(k,l) + n_{1,\gamma}(k,l) && \text{under hypothesis } H_1, \end{aligned} \tag{9}$$

where $k = 0, 1, \ldots, K-1$ and $l = 0, 1, \ldots, L-1$ and $\{n_{1,\gamma}(k,l)\}$ is a sequence of i.i.d. Cauchy noise of zero location parameter and unknown dispersion γ. For the incoming signal, we assume that it consists of the superposition of P independent continuous–time signals $s_j(t)$, $j = 1, 2, \ldots, P$, each of unknown amplitude A_j. Therefore, we have

$$s(k,l) = \sum_{j=1}^{P} A_j s_j([k + \beta l]T), \tag{10}$$

where T is the temporal spacing between successive snapshots. Assuming the element spacing in the array to be d and the wave velocity of the incoming signal to be c, we have that the constant $\beta = \frac{d}{cT}\cos\theta$ in the above equation, where θ is the direction of arrival of the incoming signal.

It is more convenient to introduce the following vector/matrix notation

$$\mathbf{x}(k) = \begin{bmatrix} x(k,0) \\ x(k,1) \\ \cdots \\ x(k,L-1) \end{bmatrix} \tag{11}$$

$$\mathbf{n}(k) = \begin{bmatrix} n(k,0) \\ n(k,1) \\ \cdots \\ n(k,L-1) \end{bmatrix} \tag{12}$$

$$\mathbf{A} = \begin{bmatrix} A_1 \\ A_2 \\ \cdots \\ A_P \end{bmatrix} \tag{13}$$

and

$$\mathbf{S}^H(k) = \begin{bmatrix} s_1(kT) & s_2(kT) & \cdots & s_P(kT) \\ s_1([k+\beta]T) & s_2([k+\beta]T) & \cdots & s_P([k+\beta]T) \\ & & \cdots & \\ s_1([k+(L-1)\beta]T) & s_2([k+(L-1)\beta]T) & \cdots & s_P([k+(L-1)\beta]T) \end{bmatrix}, \tag{14}$$

where $k = 0, 1, \ldots, K-1$ and the superscript ^{H} denotes the complex conjugate transpose.

With this notation in mind, the observations in one snapshot can be collectively rewritten as

$$\mathbf{x}(k) = \mathbf{S}^H(k) \cdot \mathbf{A} + \mathbf{n}(k), \quad k = 0, 1, \ldots, K-1 \tag{15}$$

and the detection problem can be rephrased as that of deciding between the following two hypotheses

$$\mathbf{A} = 0 \qquad \text{under hypothesis } H_0 \tag{16}$$

$$\mathbf{A} \neq 0 \qquad \text{under hypothesis } H_1, \tag{17}$$

on the basis of the observations $\mathbf{x}(k)$, $k = 0, 1, \ldots, K-1$. For simplicity, we are going to assume that the direction of arrival θ of the incoming signals is known. If this is not the case, a slight modification of our algorithm is needed as discussed later in this section of the report.

Our detection algorithm will be of the type of a generalized likelihood ratio, in which appropriate estimates of the signal and noise parameters are used. Let us begin by forming the least–square estimates of the vector of the signal amplitudes from each snapshot

$$\hat{\hat{\mathbf{A}}}(k) = [\mathbf{S}(k)\mathbf{S}^H(k)]^{-1}\mathbf{S}(k), \quad k = 0, 1 \ldots, K-1. \tag{18}$$

It is clear from Eq.(15) that the estimates $\hat{\hat{\mathbf{A}}}(k)$ are i.i.d. Cauchy random variables with location parameter equal to the true vector $\mathbf{A}$ of signal amplitudes. We, therefore, propose to estimate the vector $\mathbf{A}$ of signal amplitudes from the entire observed data as

$$\hat{\hat{\mathbf{A}}} = \text{ median } \{\hat{\hat{\mathbf{A}}}(1), \hat{\hat{\mathbf{A}}}(2), \ldots, \hat{\hat{\mathbf{A}}}(K-1)\}. \tag{19}$$

For the dispersion parameter, we propose the estimates

$$\hat{\gamma} \; = \; [\frac{\frac{1}{K \cdot L}\sum_{k=0}^{K-1}\sum_{l=0}^{L-1}|x(k,l)|^p}{C(p,1)}]^{\frac{1}{p}} \tag{20}$$

$$\hat{\hat{\gamma}} \; = \; [\frac{\frac{1}{K \cdot L}\sum_{k=0}^{K-1}\sum_{l=0}^{L-1}|x(k,l) - \hat{s}(k,l)|^p}{C(p,1)}]^{\frac{1}{p}}, \tag{21}$$

where

$$\hat{s}(k,l) = \sum_{j=1}^{P} \hat{A}_j s_j([k + \beta l]T) \tag{22}$$

and

$$C(p,\alpha) = \frac{1}{\cos(\frac{\pi}{2}p)} \frac{\Gamma(1 - p/\alpha)}{\Gamma(1-p)}$$

for $0 < p < \alpha$.

The test statistic can now be formulated as

$$t[x(0,0), x(0,1), \ldots, x(K-1, L-1)] = \sum_{k=0}^{K-1}\sum_{l=0}^{L-1} \log\{\frac{\frac{\hat{\hat{\gamma}}}{\hat{\hat{\gamma}}^2 + [x(k,l) - \hat{s}(k,l)]^2}}{\frac{\hat{\gamma}}{\hat{\gamma}^2 + x^2(k,l)}}\}. \tag{23}$$

The receiver computes this statistic and compares it to a threshold. Whenever the threshold is exceeded, a signal is declared present. Otherwise, it is assumed that only noise is contained in the observed data.

If the direction of arrival of the incoming signals is not known, then the matrix $\mathbf{S}^H$ in Eq.(14) needs to be computed for every possible direction of arrival $\theta \in [0, 2\pi)$ or, equivalently, for every possible value of the parameter β. This will produce a set of θ–dependent estimates of the vector of signal amplitudes through Eq.(18), which if carried through Eq.(19), will produce a median estimate that depends on θ. The test statistic for our detection problem will be

$$t[x(0,0), x(0,1), \ldots, x(K-1, L-1)] = \max_{\theta \in [0,2\pi)} \sum_{k=0}^{K-1} \sum_{l=0}^{L-1} \log\left\{ \frac{\frac{\hat{\hat{\gamma}}}{\hat{\hat{\gamma}}^2 + [x(k,l) - \hat{\hat{s}}(k,l;\theta)]^2}}{\frac{\hat{\gamma}}{\hat{\gamma}^2 + x^2(k,l)}} \right\}. \tag{24}$$

6. Summary, Conclusions, and Possible Future Research

In this paper, we have examined the problem of detection of incompletely known signals (signals with unknown parameters) embedded in incompletely characterized impulsive noise modeled as an i.i.d. Cauchy sequence. We developed a generalized likelihood ratio test which uses estimates of the unknown parameters of the signal and the noise that are based on fractional lower order statistics and order statistics of the observations. We compared the proposed test to the corresponding optimum test which assumes complete knowledge of the signal and the noise (clairvoyant test) and showed, via Monte–Carlo simulation, that, in the limit of large observation lengths, the two tests are asymptotically equivalent. We also compared the proposed test to the existing Student's t–test and found that the performance of the latter is significantly inferior to the performance of the proposed test and remains constant as a function of the length of the observations. We, finally, extended our results to include the detection of multiple signals with unknown parameters embedded in incompletely characterized stable noise.

Future topics in the same area of research, that need to be addressed, include the detection of multiple signals with unknown parameters embedded in linearly dependent impulsive noise. Of relevance are also algorithms for parameter estimation from data corrupted by impulsive interference. Possible applications of this research can be found in the detection of low intercept (spread spectrum) communication signals and the identification of incompletely specified communication channels. This research is currently conducted and its results will be announced soon.

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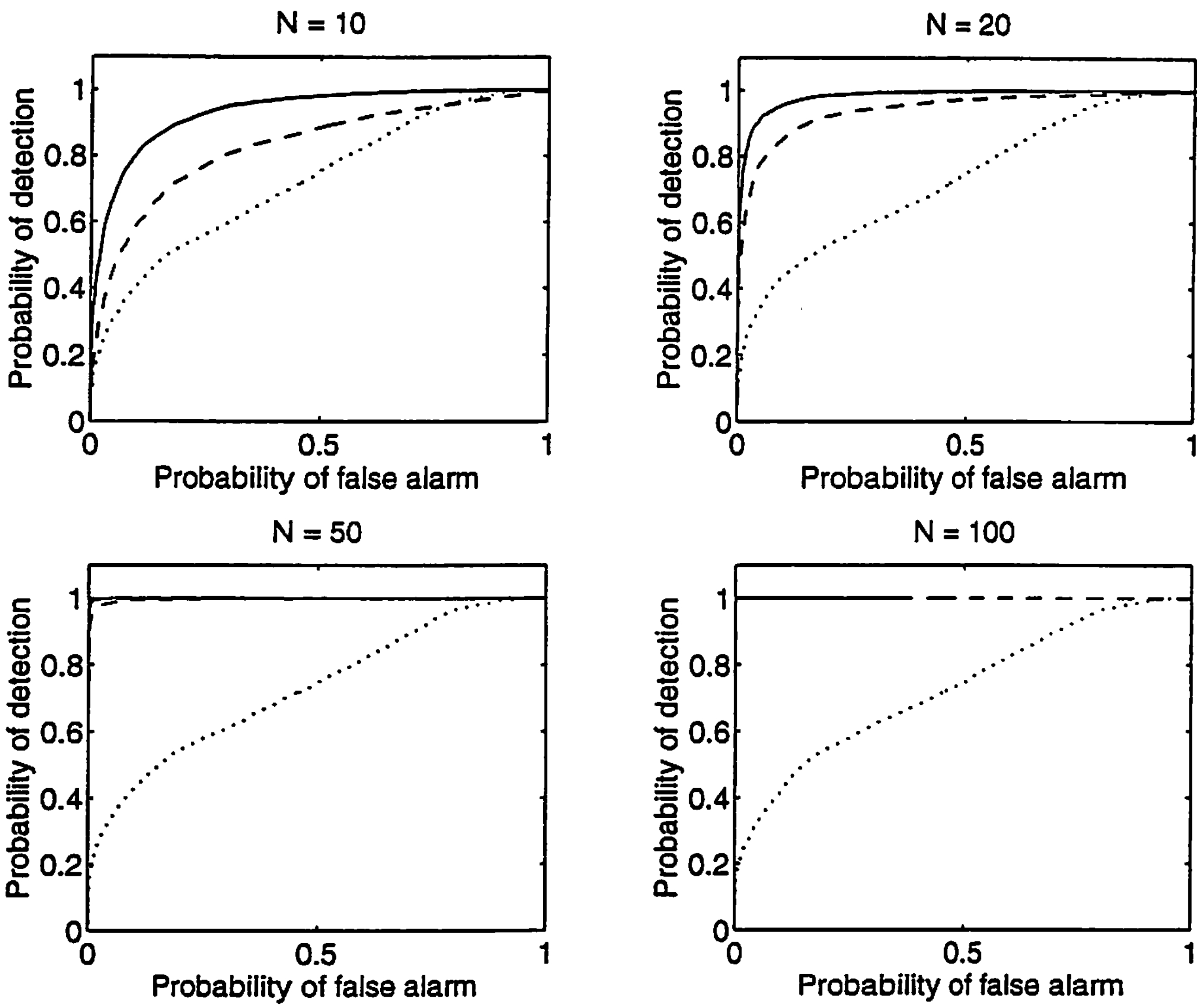

FIGURE 1: Receiver operating characteristics: Continuous line: Clairvoyant receiver, Dashed line: Proposed receiver, Dotted line: Student's t-test.

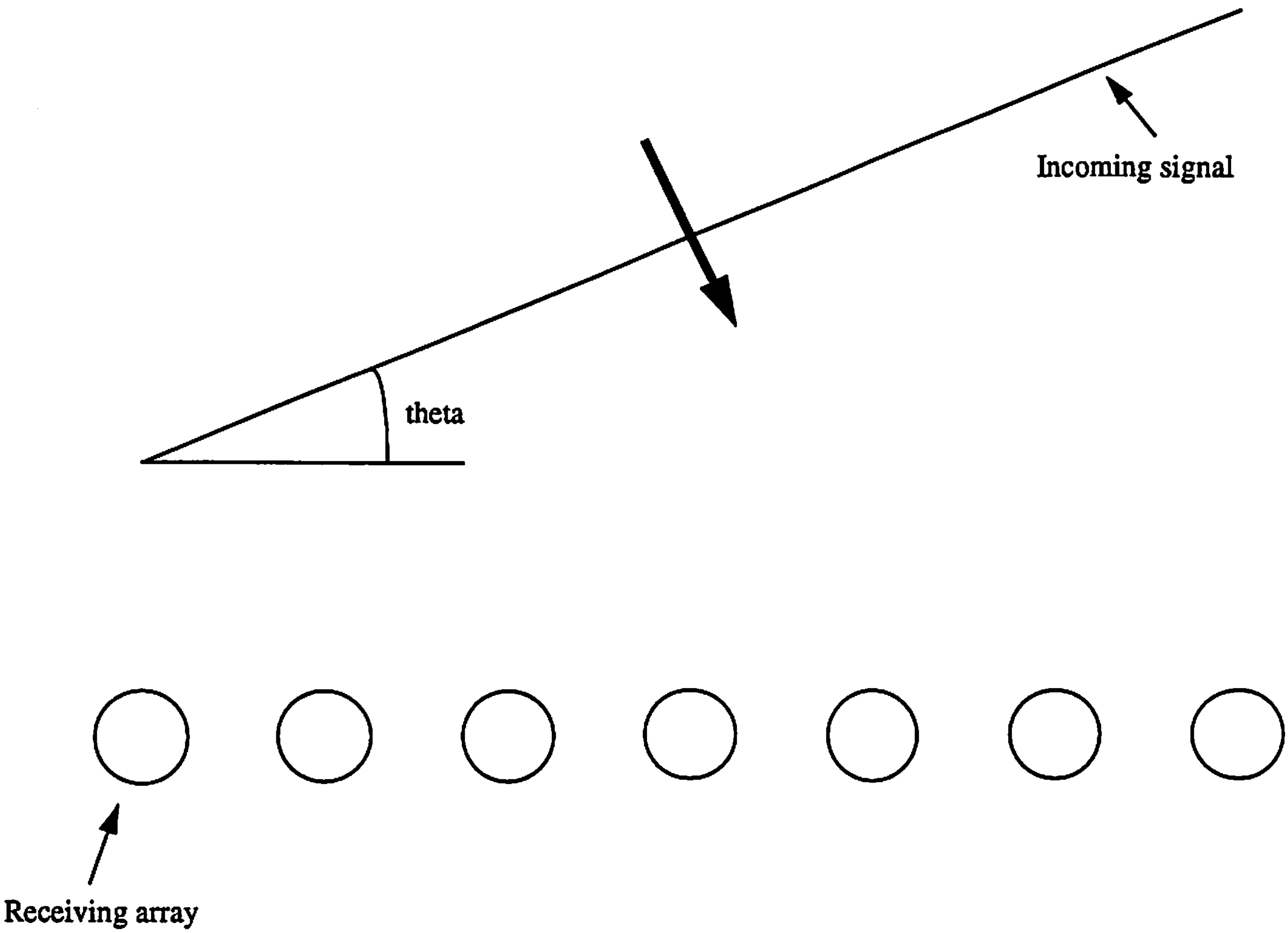

FIGURE 2.