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Abstract

New methods for parameter estimation and blind system identification for impulsive signal environments are presented. The data are modeled as stable processes. First, methods for estimating the parameters (characteristic exponent and dispersion) of a symmetric stable distribution are presented. The fractional lower-order moments, both positive and negative order, and their applications are introduced. Then a new algorithm for blind channel identification based on fractional lower-order moments is proposed. Alpha-Spectrum, a spectral representation for impulsive environments, is developed. Conditions for blind identifiability of non-minimum phase FIR channels are established using the properties of the Alpha-Spectrum.

1 Introduction

The statistical signal processing framework is incomplete without the study on α-stable ($0 < \alpha \leq 2$) distributions. By the Generalized Central Limit Theorem, they are the *only* class of distributions that can be the limiting distributions for sums of i.i.d random variables. Familiar members of the family are Gaussian ($\alpha = 2$) and Cauchy ($\alpha = 1$) distributions. Many signal/noise processes are impulsive in nature and can be best modeled as α-stable processes [Shao and Nikias, 1993] . Unlike most statistical models, α-stable distributions (except Gaussian) have infinite second- or higher-order moments. With this unique property, many fundamental theories in signal processing have to be modified. For a comprehensive introduction of α-stable distributions and their applications to signal processing, see the first engineering textbook by Nikias and Shao [1995] and the references therein. When all the second- and higher-order statistics fail for the impulsive environments, an alternative tool that is robust against outliers is the fractional lower-order moments (FLOM). It is known that the p^{th} order FLOM for a symmetric α-stable ($S\alpha S$) random variable is finite for $0 \leq p < \alpha$. [Samorodnitsky and Taqqu, 1994].

Most algorithms for blind identification of a finite impulse response (FIR) channel with non-Gaussian input are based on second- or higher-order statistics [Nikias and Petropulu, 1993]. However, the theoretical basis of higher-order moment estimators is the asymptotic normality [Van Ness, 1966] , i.e., the estimation error has a normal distribution. When the input is impulsive in nature and modeled as α-stable process, the asymptotic normality of higher-order moment estimators no longer holds. Therefore, fractional lower-order statistics are the most appropriate tools for analysis. In the first part of the paper, we present several new algorithms for parameter estimation from both independent and dependent data time series. In the second part, we propose a robust blind identification method based on a new spectral representation for impulsive environments: α-Spectrum, which is completely determined by the output covariations and characteristic exponent α.

With α-Spectrum, we prove the blind identifiability of any FIR channel (mixed-phase in general, with unknown order) driven by white $S\alpha S$ ($\alpha > 1$) processes. Simulation results for both deterministic and stochastic signals are presented.

2 Estimation of Characteristic Exponent and Dispersion

The most important parameters of a $S\alpha S$ distribution are characteristic exponent α and dispersion γ. Several estimation methods have been introduced in the literature [Fama and Roll, 1968; DuMouchel, 1971; McCulloch, 1986; Brorsen, 1990]. We present an alternative method based on the negative-order moment concept.

2.1 Fractional Lower-Order Moments: Positive-Order and Negative-Order

It is known that a real non-Gaussian $S\alpha S$ random variable X with zero location parameter, has finite fractional lower-order moment [Zolotarev, 1986]:

$$\mathbf{E}(|X|^p) = C_1(p,\alpha)\gamma^{p/\alpha}, \text{ for } 0 < p < \alpha, \tag{1}$$

where $C_1(p,\alpha) = \frac{2^p\Gamma(\frac{p+1}{2})\Gamma(1-p/\alpha)}{\sqrt{\pi}\Gamma(1-p/2)}$. α is the characteristic exponent ($0 < \alpha < 2$), γ is the dispersion and $\Gamma(\cdot)$ is the Gamma function.

Our analysis shows that finite $\mathbf{E}(|X|^p)$ also exists for $p < 0$. The proof is straightforward. Assume X is a real $S\alpha S$ random variable with p.d.f:

$$f_X(x) = \frac{1}{\pi}\int_0^\infty \cos(\omega x)\exp(-\gamma\omega^\alpha)d\omega. \tag{2}$$

Define a new random variable $Y = |X|$; the p.d.f for Y is:

$$f_Y(y) = 2f_X(y) = \frac{2}{\pi}\int_0^\infty \cos(\omega y)\exp(-\gamma\omega^\alpha)d\omega, 0 \le y < \infty. \tag{3}$$

From the definition, we have:

$$\mathbf{E}(|X|^p) = \mathbf{E}(Y^p) = \frac{2}{\pi}\int_0^\infty (\int_0^\infty y^p \cos(\omega y) dy) \exp(-\gamma\omega^\alpha) d\omega. \tag{4}$$

The following identities can be used to establish the existence of the negative-order moments:

$$\int_0^\infty x^p \cos(ax) dx = -\frac{1}{a^{p+1}}\Gamma(1+p)\sin(\frac{\pi p}{2}), \text{ for } a > 0, -1 < p < 0, \tag{5}$$

and

$$\int_0^\infty x^{\nu-1}\exp(-\mu x^\alpha) dx = \frac{1}{|\alpha|}\mu^{-\nu/\alpha}\Gamma(\nu/\alpha), \text{ for } \mathrm{Re}\,\mu > 0, \mathrm{Re}\,\nu > 0. \tag{6}$$

Then, we have

$$\int_0^\infty y^p \cos(\omega y) dy = -\frac{\Gamma(p+1)}{\omega^{p+1}}\sin(\frac{\pi p}{2}), \text{ for } \omega > 0, -1 < p < 0, \tag{7}$$

and

$$\int_0^\infty \omega^{-p-1}\exp(-\gamma\omega^\alpha) d\omega = \frac{1}{\alpha}\gamma^{p/\alpha}\Gamma(-p/\alpha), \text{ for } \gamma > 0, p < 0. \tag{8}$$

Therefore,

$$\begin{aligned} \mathbf{E}(|X|^p) &= -\frac{2}{\alpha\pi}\gamma^{p/\alpha}\Gamma(p+1)\Gamma(-p/\alpha)\sin(p\pi/2) \\ &= \frac{2^{p+1}\Gamma(\frac{p+1}{2})\Gamma(-p/\alpha)}{\alpha\sqrt{\pi}\Gamma(-p/2)}\gamma^{p/\alpha}, \text{ for } -1 < p < 0. \end{aligned} \tag{9}$$

Together with the positive order moments, we have:

$$\mathbf{E}(|X|^p) = C_1(p,\alpha)\gamma^{p/\alpha}, \text{ for } -1 < p < \alpha. \tag{10}$$

When X is an n dimensional spherically symmetric $S\alpha S$ random variable, a similar expression is:

$$\mathbf{E}(|X|^p) = 2^p\frac{\Gamma(\frac{p+n}{2})\Gamma(1-\frac{p}{\alpha})}{\Gamma(1-\frac{p}{2})\Gamma(\frac{n}{2})}\gamma^{p/\alpha}, \text{ for } -n < p < \alpha. \tag{11}$$

Order of Moment ($-1 < p < \alpha = 1.5$)	p=-0.5	p=-0.25	p=0.25	p=1.0
Computed ($\mathbf{E}\{\lvert X\rvert^p\}$)	1.425	1.1078	0.997	1.7055
Estimate of $\mathbf{E}(\lvert X\rvert^p)$ with n samples				
n=100	1.4291	1.1062	1.0062	1.5141
n=1000	1.3213	1.0816	1.0168	1.7463
n=10000	1.4155	1.1055	0.9988	1.7297
n=100000	1.4334	1.108	0.9977	1.6696
n=1000000	1.424	1.1077	0.9972	1.7797

Table 1: Comparison of theoretical values of $\mathbf{E}(|X|^p)$ and simulations.

Especially, when X is an isotropic bivariate $S\alpha S$ random variable, we have:

$$\mathbf{E}(|X|^p) = C_2(p,\alpha)\gamma^{p/\alpha}, \text{ for } \; -2 < p < \alpha, \tag{12}$$

where $C_2(p,\alpha) = 2^p \frac{\Gamma(1+\frac{p}{2})\Gamma(1-\frac{p}{\alpha})}{\Gamma(1-\frac{p}{2})}\gamma^{p/\alpha}$. The simplest yet most important complex $S\alpha S$ random variables are isotropic bivariate.

Table (1) shows simulation results that compare estimated moments (of positive and negative orders) with the theoretical ones generated by Eq.(10). The estimate of $\mathbf{E}(|X|^p)$ is $\frac{\sum_{i=1}^n |x_i|^p}{n}$.

An immediate application of the negative-order moments is for estimating the characteristic exponent α and the dispersion γ of $S\alpha S$ random processes. Assuming X is a real $S\alpha S$ random variable, with zero location parameter, then its positive order FLOM is given by

$$\mathbf{E}(|X|^p) = C_1(p,\alpha)\gamma^{p/\alpha}, \text{ for } 0 < p < \alpha. \tag{13}$$

and its negative order FLOM is given by

$$\mathbf{E}(|X|^q) = C_1(q,\alpha)\gamma^{q/\alpha}, \text{ for } \; -1 < q < 0. \tag{14}$$

Number of Samples	$p = 0.1$		$p = 0.2$		$p = 0.5$	
	$\hat{\alpha}$	$\hat{\gamma}$	$\hat{\alpha}$	$\hat{\gamma}$	$\hat{\alpha}$	$\hat{\gamma}$
1000	1.0038 (0.0463)	1.0050 (0.0590)	1.0042 (0.0488)	1.0045 (0.0612)	1.0085 (0.0712)	1.0115 (0.1064)
2000	1.0033 (0.0343)	1.0046 (0.0411)	1.0016 (0.0353)	1.0058 (0.0427)	1.0077 (0.0523)	1.0082 (0.0787)
5000	0.9997 (0.0210)	1.0021 (0.0259)	0.9994 (0.0227)	1.0015 (0.0268)	1.0029 (0.0354)	1.0057 (0.0531)

Table 2: New estimator performance (true $\alpha = 1, \gamma = 1$).

Choosing $p = -q$ (since $-1 < q < 0$, then $0 < p < \min(\alpha, 1)$ such that both positive- and negative-order moments are finite), then we have:

$$\mathbf{E}(|X|^p)\mathbf{E}(|X|^{-p}) = \frac{2\tan(p\pi/2)}{\alpha\sin(p\pi/\alpha)}, \tag{15}$$

i.e., α can be found by the solving the following sinc function equation:

$$\text{sinc}(\frac{p\pi}{\alpha}) = \frac{\sin(\frac{p\pi}{\alpha})}{(\frac{p\pi}{\alpha})} = \frac{2\tan(p\pi/2)}{p\pi\mathbf{E}(|X|^p)\mathbf{E}(|X|^{-p})}, 0 < p < \min(\alpha, 1). \tag{16}$$

The above equation does not involve γ. Once α is estimated, γ can obtained from Eq.(1):

$$\gamma = \left(\frac{\mathbf{E}(|X|^p)}{C_1(p, \alpha)}\right)^{\alpha/p}. \tag{17}$$

Table (2) illustrates the average and standard deviation values (in parentheses) of Monte-Carlo simulation results based on the proposed estimator. We generated different numbers of samples from a standard (zero location parameter, unit dispersion) $S\alpha S$ random number generator. The experiment was repeated independently 1000 times. Values of order p were chosen to be 0.1, 0.2 and 0.5, respectively.

Results in Table (2) suggest that the order of moments p should be kept small to achieve better performance. The asymptotic property of the proposed estimator is shown

NS	200	500	1000	2000	5000	10000	25000	50000
$\hat{\alpha}$	1.5852 (0.3274)	1.5284 (0.1735)	1.5132 (0.1257)	1.5049 (0.0849)	1.5027 (0.0536)	1.5009 (0.0406)	1.5009 (0.0234)	1.50126 (0.0167)
$\hat{\gamma}$	1.0703 (0.3012)	1.0252 (0.1325)	1.0101 (0.1014)	1.0076 (0.0669)	1.0023 (0.0423)	1.0014 (0.0300)	1.0008 (0.0176)	1.0005 (0.0133)

Table 3: Asymptotic property of the new estimator (true $\alpha = 1.5, \gamma = 1.0$, order $p = 0.2$.)

in Table (3), where the true value of α is 1.5 and p is fixed at 0.2. The number of samples (NS) ranges from 200 to 50000. Fig.(1) shows the average and standard deviation values of both characteristic exponent and dispersion versus the number of samples used in the simulations.

2.2 Parameter Estimation with $\log|S\alpha S|$ process

Assuming X is a real $S\alpha S$ random variable, recall that its p^{th}-order moment is $\mathbf{E}\{|X|^p\} = C_1(p,\alpha)\gamma^{p/\alpha}, \forall p : -1 < p < \alpha$, We can rewrite $\mathbf{E}(|X|^p)$ as $\mathbf{E}(e^{p\log|X|})$, (note that $\log|X|$ is bounded because the p.d.f of X: $f(x)$ is bounded at $x = 0$, i.e., the probability of $x = 0$ is 0). Define a new random variable $Y = \log|X|$, therefore:

$$\mathbf{E}(|X|^p) = \mathbf{E}(e^{p\log|X|}) = \mathbf{E}(e^{pY}). \tag{18}$$

Notice the last term in the above equation $\mathbf{E}(e^{pY})$ is the moment-generating function of Y. We can expand it into a power series:

$$\mathbf{E}(e^{pY}) = \sum_{k=0}^{\infty} \mathbf{E}(Y^k)\frac{p^k}{k!}. \tag{19}$$

On the other hand,

$$\mathbf{E}(e^{pY}) = C_1(p,\alpha)\gamma^{p/\alpha}, \forall p : -1 < p < \alpha. \tag{20}$$

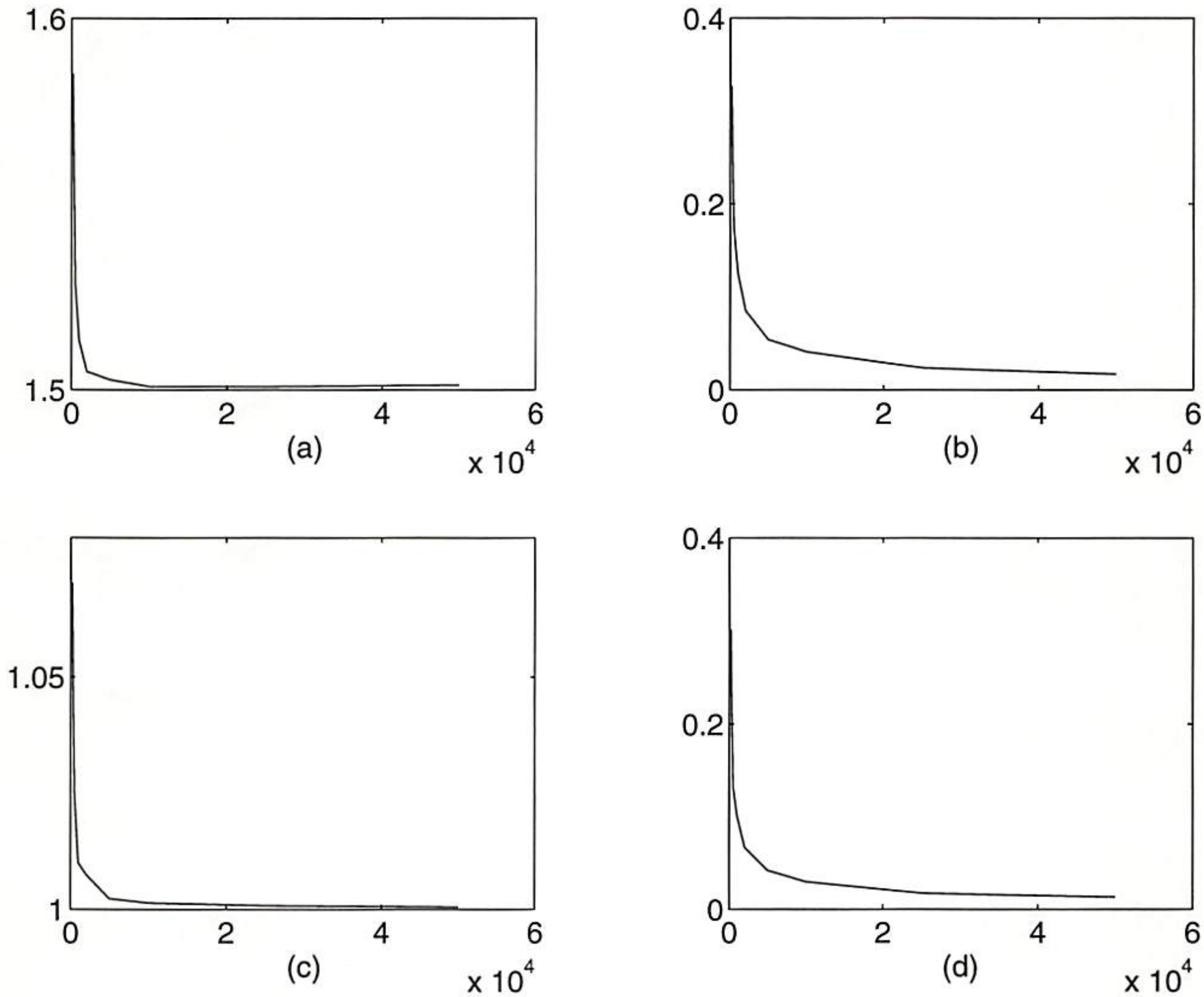

Figure 1: Estimates of the parameters v.s. number of samples. (a). Average of $\hat{\alpha}$ over 1000 realizations with true $\alpha = 1.5$. (b). Standard deviation of $\hat{\alpha}$ over 1000 realizations. (c). Average of $\hat{\gamma}$ over 1000 realizations with true $\gamma = 1$. (d). Standard deviation of $\hat{\gamma}$ over 1000 realizations.

Therefore, moments of Y of any order must be finite and they satisfy:

$$\mathbf{E}(Y^k) = \frac{d^k}{dp^k}\left(C_1(p,\alpha)\gamma^{p/\alpha}\right)|_{p=0}. \tag{21}$$

Simplifying the above equation, we have:

$$\mathbf{E}(Y) = \mathbf{C}_e(\frac{1}{\alpha} - 1) + \frac{1}{\alpha}\log\gamma, \tag{22}$$

where $\mathbf{C}_e = 0.57721566\cdots$ is the Euler constant, α is the characteristic exponent, γ is the dispersion, and

$$\mathbf{Var}(Y) = \mathbf{E}\{(Y - \mathbf{E}\{Y\})^2\} = \frac{\pi^2}{6}\left(\frac{1}{\alpha^2} + \frac{1}{2}\right), \tag{23}$$

$$\mathbf{E}\{(Y - \mathbf{E}\{Y\})^3\} = 2\zeta(3)\left(\frac{1}{\alpha^3} - 1\right), \tag{24}$$

where $\zeta(\cdot)$ is the Riemann Zeta function, and $\zeta(3)$ is a constant: $\zeta(3) = 1.2020569\cdots$. Also,

$$\mathbf{E}\{(Y - \mathbf{E}\{Y\})^4\} = \pi^4\left(\frac{3}{20\alpha^4} + \frac{1}{12\alpha^2} + \frac{19}{240}\right) \tag{25}$$

The higher-order moments of Y always exist and from the second-order moment and above, the equations only involve α. This property provides a simple method to estimate the parameters α and γ of a $S\alpha S$ random process. Since we can estimate the mean and variance of Y by:

$$\bar{Y} = \frac{\sum_{i=1}^{N} Y_i}{N}, \quad \hat{\sigma}_Y^2 = \frac{\sum_{i=1}^{N}(Y_i - \bar{Y})^2}{N-1}, \tag{26}$$

where N is the number of samples and Y_i are i.i.d observations, then by solving Eq.(23), we can obtain an estimate of α and substitute into Eq.(22) to get an estimate of γ. We should mention that similar results were obtained by Zolotarev [1986] with a transformation of the characteristic function of $S\alpha S$ random variables.

We can also use higher-order moments of Y to estimate α, but it is known that the variance of the estimator of higher-order moments tends to increase as the order goes

Order Moment of Y	Second-Order	Third-Order	Fourth-Order
$\hat{\alpha}$	1.5139 (0.1167)	1.4837 (0.2578)	1.3564 (0.3226)
$\hat{\gamma}$	1.0195 (0.0909)	1.0079 (0.1032)	0.9549 (0.1365)

Table 4: Estimator Performance v.s. Order of Moment of Y.

Estimation Method	$\hat{\alpha}$	$\hat{\gamma}$
$\log\|S\alpha S\|$ Approach	1.4969 (0.0522)	0.9989 (0.0385)
Negative-order Moment Approach	1.5027 (0.0536)	1.0023 (0.0423)

Table 5: Performance comparison of the $\log|S\alpha S|$ approach v.s. the negative-order moment approach.

higher. Table (4) shows a comparison of results obtained by using different orders of moments. The true values for the parameters are $\alpha = 1.5$ and $\gamma = 1.0$. Y is the $\log|S\alpha S|$ process, the sample size is 1000, and the experiment is repeated 1000 times independently. As we can see from Table (4), the standard deviation of the estimators (α-estimator and γ-estimator) increases as the order of the moment of Y increases for a fixed sample size.

Table (5) illustrates the comparison of the $\log|S\alpha S|$ approach with the method based on the negative-order moments X is the standard $S\alpha S$ random variable with $\alpha = 1.5$ and $Y = \log|X|$. We chose $p = 0.2$ in the negative-order moment method. The experiment was repeated 1000 times independently with 5000 i.i.d samples.

Table (5) shows that these two methods have similar performance. But the advantages of the $\log|S\alpha S|$ approach are:

1. It gives explicit closed form expressions of the unknown parameters; on the other

hand, in the negative-order moment method, α is the solution of a sinc function equation, which does not have a closed form expression.

2. The quality of the α-estimator and γ-estimator is completely determined by the sample size, whereas in the negative-order moment approach, the estimation results are also affected by the choice of value p, which is often empirical.

It is worth pointing out that the proposed estimators are not optimal in a maximum likelihood sense. However, maximum likelihood estimators for α and γ do not have simple closed form expressions; their estimates are obtained by solving nonlinear optimization equations [Brorsen, 1990].

Similarly, for an isotropic complex $S\alpha S$ random variable X, recall that its $p^t h$- order moment is: $\mathbf{E}\{|X|^p\} = C_2(p,\alpha)\gamma^{p/\alpha}, \forall p : -2 < p < \alpha$, let $Y = \log|X|$, the $\log|S\alpha S|$ approach yields:

$$\mathbf{E}(Y) = \mathbf{C}_e(\frac{1}{\alpha} - 1) + \log 2 + \frac{1}{\alpha}\log\gamma, \tag{27}$$

and

$$\mathbf{Var}(Y) = \frac{\pi^2}{6\alpha^2}. \tag{28}$$

2.3 Parameter Estimation From Data Dependent Time Series

Employing Eq.(23) and Eq.(22) to estimate the parameters α and γ, we assume that the data samples are i.i.d. If the samples are not i.i.d, we need the ergodic properties of the $\log|S\alpha S|$ process. The proof of ergodicity is based on the following properties of $S\alpha S$ processes:

1. A $S\alpha S$ random variable U has finite p-th order moments: $\mathbf{E}(|U|^p) = C_1(p,\alpha)\gamma^{p/\alpha}$. The $p^t h$-order moments are finite and continuous in the neighborhood of $p = 0$.

2. n jointly $S\alpha S$ random variables $U_1, U_2, \cdots U_n$ have finite joint moments with orders $p_1, p_2, \cdots p_n$: $\mathbf{E}(|U_1|^{p_1}|U_2|^{p_2}\cdots|U_n|^{p_n}) = G(\phi; p_1, p_2, \cdots p_n)$ (where $G(\cdot)$ is a function

of the joint characteristic function $\phi(\cdot)$ of the random variables U_1, U_2, $\cdots$ U_n and the constants p_1, p_2, $\cdots$ p_n). The joint fractional order moments are finite and continuous in the neighborhood of $p_1 = 0, p_2 = 0, \cdots p_n = 0$.

Consider the problem of estimating the parameters α and γ from an FIR channel output, which is an MA process. We have the following:

Theorem 1 *Let U be a $S\alpha S$ moving average ($MA(q)$) random process. Its corresponding* $\log|S\alpha S|$ *random process $V = \log|U|$ is stationary and mean-ergodic as well as correlation-ergodic.* ∎

Proof: Since U is a $MA(q)$ process, U_n and $U_{n\pm(q+1)}$ are independent, so are V_n and $V_{n\pm(q+1)}$. We now show that $\mathbf{E}(V_n V_{n-1} \cdots V_{n-q})$ is finite and independent of n. Since $V_i = \log|U_i|$, then

$$\begin{aligned} \mathbf{E}(e^{p_0 V_n + p_1 V_{n-1} + p_2 V_{n-2} + \cdots + p_q V_{n-q}}) &= \mathbf{E}(|U_n|^{p_0}|U_{n-1}|^{p_1}|U_{n-2}|^{p_2}\cdots|U_{n-q}|^{p_q}) \\ &= G(\phi; p_0, p_1, \cdots p_q), \end{aligned} \tag{29}$$

where $G(\phi; p_0, p_1, \cdots, p_q)$ is a function of the joint characteristic function $\phi(\cdot)$ of the random variables U_n, U_{n-1}, U_{n-2}, $\cdots$ U_{n-q} and the constants p_0, p_1, $\cdots$, p_q. The above equation holds for p_i in the neighborhood of $p_0 = 0$, $p_1 = 0$, $\cdots$, $p_q = 0$. Therefore, $\mathbf{E}(e^{p_0 V_n + p_1 V_{n-1} + \cdots + p_q V_{n-q}})$ can be treated as the *joint moment-generating function* of V_n, V_{n-1}, $\cdots$, V_{n-q}, and we have:

$$\begin{aligned} \mathbf{E}(V_n V_{n-1}\cdots V_{n-q}) &= \left.\frac{\partial^{q+1}\mathbf{E}(e^{p_0 V_n + p_1 V_{n-1} + \cdots + p_q V_{n-q}})}{\partial p_0 \partial p_1 \cdots \partial p_q}\right|_{p_0=0, p_1=0, \cdots, p_q=0} \\ &= \left.\frac{\partial^{q+1}\mathbf{E}(|U_n|^{p_0}|U_{n-1}|^{p_1}\cdots|U_{n-q}|^{p_q})}{\partial p_0 \partial p_1 \cdots \partial p_q}\right|_{p_0=0, p_1=0, \cdots, p_q=0} \\ &= \left.\frac{\partial^{q+1} G(\phi; p_0, p_1, \cdots p_q)}{\partial p_0 \partial p_1 \cdots \partial p_q}\right|_{p_0=0, p_1=0, \cdots p_q=0}. \end{aligned} \tag{30}$$

Since the joint characteristic function $\phi(\cdot)$ of U_n, U_{n-1}, U_{n-2}, $\cdots$, U_{n-q} is independent of n (U is a $MA(q)$ process), $\mathbf{E}(V_n V_{n-1} V_{n-2} \cdots V_{n-q})$ is not only finite, but also independent of n. Therefore, V_n is stationary.

The sufficient condition for V_n to be mean-ergodic is:

$$\sum_{l=-q}^{l=q} |\mathbf{E}(V_n V_{n-l})| < \infty. \tag{31}$$

Since

$$\mathbf{E}(V_n V_{n-l}) = \left. \frac{\partial^2 \mathbf{E}(|U_n|^{p_0} |U_{n-l}|^{p_1})}{\partial p_0 \partial p_1} \right|_{p_0=0, p_1=0}, \tag{32}$$

we can show that $\mathbf{E}(V_n V_{n-l})$ is finite and independent of n. Mean-ergodicity of V_n is therefore proved.

The sufficient condition for V_n to be correlation-ergodic is that its fourth-order moment of V_n be finite and independent of n. This can be proved similarly. ∎

With the mean- and correlation-ergodicities of V_n, Eq.s (22), (23) and (26) can still be used to estimate $\mathbf{E}(V)$ and $\mathbf{Var}(V)$ even though V_i are dependent.

We now use the above properties to estimate the characteristic exponent and dispersion from *dependent* samples in the output of an FIR channel. Consider an FIR channel with standard i.i.d $S\alpha S$ input (characteristic exponent α and dispersion $\gamma_x = 1$),

$$Y_n = h_0 X_n + h_1 X_{n-1} + \cdots + h_q X_{n-q}, \tag{33}$$

where h_i are the channel impulse response coefficients. The output Y_i are identically distributed but *dependent to q^{th} order $S\alpha S$* random variables with the same characteristic exponent α and dispersion $\gamma_y = \sum_{i=0}^{q} |h_i|^\alpha$. Let $V = \log|Y|$, then the characteristic exponent α is estimated by solving:

$$\frac{\sum_{i=1}^{N}(V_i - \bar{V})^2}{N-1} = \frac{\pi^2}{6}\left(\frac{1}{\hat{\alpha}^2} + \frac{1}{2}\right), \tag{34}$$

sample size	1000	2000	5000	10000
$\hat{\alpha}$ true $\alpha = 1.5$	1.5153 (0.1318)	1.5022 (8.8726E-2)	1.5023 (6.0811E-2)	1.5042 (3.9782E-2)
$\hat{\gamma}_y$ true $\gamma_y = 3.4215$	3.5309 (0.6728)	3.4524 (0.4440)	3.4441 (0.2825)	3.4514 (0.1832)

Table 6: Estimation of α, γ_y from FIR channel output.

and the output dispersion γ_y is then estimated by solving:

$$\bar{V} = \frac{\sum_{i=1}^{N} V_i}{N} = \mathbf{C}_e(\frac{1}{\hat{\alpha}} - 1) + \frac{1}{\hat{\alpha}} \log \hat{\gamma}_y. \tag{35}$$

Consider the following example:

$$Y_n = X_n + 0.5X_{n-1} - 1.3X_{n-2} + 0.7X_{n-3}, \tag{36}$$

We estimate α and γ_y from the output Y_n. Table (6) lists the results from Monte-Carlo simulations of 500 independent realizations, where X_n are i.i.d samples from a standard $S\alpha S$ process with $\alpha = 1.5$ and $\gamma_x = 1$; so the output of the channel is a $MA(3)$ $S\alpha S$ process with $\alpha = 1.5$ and $\gamma_y = 3.4215$.

2.4 A New Iterative Parameter Estimation Method

Estimation accuracy increases as the sample size increases, which results in the increase of memory size of the estimator. Memory efficiency can be achieved by updating the estimation results iteratively. More specifically, upon observing k^{th} block of data (one block of data contains M samples, M is the memory size of the estimator), we update $\alpha(k)$ and $\gamma_y(k)$ from the previous results: $\alpha(k-1)$ and $\gamma_y(k-1)$. More specifically, assuming Y is a $S\alpha S$ data sequence consisting of dependent or independent samples, $V = \log|Y|$ is the corresponding $\log|S\alpha S|$ process. Assume k blocks of samples with

M samples in each block have been collected. Let Avg(k) and Var(k) denote the average and the standard deviation of k^{th} block of data, i.e.,

$$\text{Avg}(k) = \frac{\sum_{n=(k-1)M+1}^{kM} V_n}{M}, \tag{37}$$

$$\text{Var}(k) = \frac{\sum_{n=(k-1)M+1}^{kM} (V_n - \text{Avg}(k))^2}{M}, \tag{38}$$

From Eq.(34), we have:

$$\frac{\sum_{i=1}^{kM} V_i}{kM} = \mathbf{C}_e(\frac{1}{\alpha(k)} - 1) + \frac{1}{\alpha(k)} \log \gamma_y(k). \tag{39}$$

On the other hand,

$$\begin{aligned}
\frac{\sum_{i=1}^{kM} V_i}{kM} &= \frac{\sum_{i=1}^{(k-1)M} V_i + \sum_{i=(k-1)M+1}^{kM} V_i}{kM} \\
&= \frac{k-1}{k} \frac{\sum_{i=1}^{(k-1)M} V_i}{(k-1)M} + \frac{1}{k} \frac{\sum_{i=(k-1)M+1}^{kM} V_i}{M} \\
&= \frac{k-1}{k} \left(\mathbf{C}_e(\frac{1}{\alpha(k-1)} - 1) + \frac{1}{\alpha(k-1)} \log \gamma_y(k-1) \right) + \frac{1}{k} \text{Avg}(k). \quad (40)
\end{aligned}$$

Therefore, γ_y can be iteratively updated by:

$$\begin{aligned}
\frac{1}{\alpha(k)} \log \gamma_y(k) &= \frac{k-1}{k} \left[\mathbf{C}_e(\frac{1}{\alpha(k-1)} - 1) + \frac{1}{\alpha(k-1)} \log \gamma_y(k-1) \right] \\
&+ \frac{1}{k} \text{Avg}(k) + \mathbf{C}_e(1 - \frac{1}{\alpha(k)}). \quad (41)
\end{aligned}$$

Similarly, it is not difficult to show that $\alpha(k)$ can be updated by:

$$\begin{aligned}
\frac{\pi^2}{6}(\frac{1}{\alpha^2(k)} + \frac{1}{2}) &= \frac{k-1}{k^2} \left[\mathbf{C}_e(\frac{1}{\alpha(k-1)} - 1) + \frac{1}{\alpha(k-1)} \log \gamma_y(k-1) - \text{Avg}(k) \right]^2 \\
&+ \frac{k-1}{k} \frac{\pi^2}{6} (\frac{1}{\alpha^2(k-1)} + \frac{1}{2}) + \frac{1}{k} \text{Var}(k). \quad (42)
\end{aligned}$$

Let us now consider the same example

$$Y_n = X_n + 0.5X_{n-1} - 1.3X_{n-2} + 0.7X_{n-3}, \tag{43}$$

where the impulse response coefficients are $h_0 = 1, h_1 = 0.5, h_2 = -1.3, h_3 = 0.7$. We demonstrate the performance via Monte-Carlo simulations of 100 realizations. In each realization, the memory size of both α- and γ-estimators is 100, i.e. $M = 100$ samples in one block, total of 200 blocks (20000 samples) are collected. Compared with the true values of $\alpha = 1.5$ and $\gamma_y = 3.4215$, our simulation results are: $\hat{\alpha} = 1.5024$ and $\hat{\gamma_y} = 3.4317$, with standard deviations $\sigma_\alpha = 0.0289$ and $\sigma_\gamma = 0.1316$, respectively. Fig. 2 shows the performance of this new method.

3 Blind Channel Identification

In this section, we present several approaches to blind channel identification when the input is i.i.d $S\alpha S$ process from output covariation (notice covariation is only defined for stable processes with $\alpha > 1$). By introducing a new spectrum for impulsive environments: the α-Spectrum, we prove blind identifiability. Without loss of generality, we assume the channel input is a standard white $S\alpha S$ process.

3.1 Time Domain Covariation Approach

The output covariation is related to the channel impulse response coefficients by [Nikias and Shao, 1995]:

$$\begin{aligned} c_{-j} = [Y_n, Y_{n-j}]_\alpha &= \sum_{i=0}^{q-j} \frac{h_{j+i}}{h_i} |h_i|^\alpha, \text{ for } j = 1, \cdots, q; \\ c_j = [Y_n, Y_{n+j}]_\alpha &= \sum_{i=0}^{q-j} \frac{h_i}{h_{j+i}} |h_{j+i}|^\alpha, \text{ for } j = 1, \cdots, q. \end{aligned} \tag{44}$$

The dispersion of the output is determined by:

$$\gamma_y = [Y_n, Y_n]_\alpha = \sum_{i=0}^{q} |h_i|^\alpha. \tag{45}$$

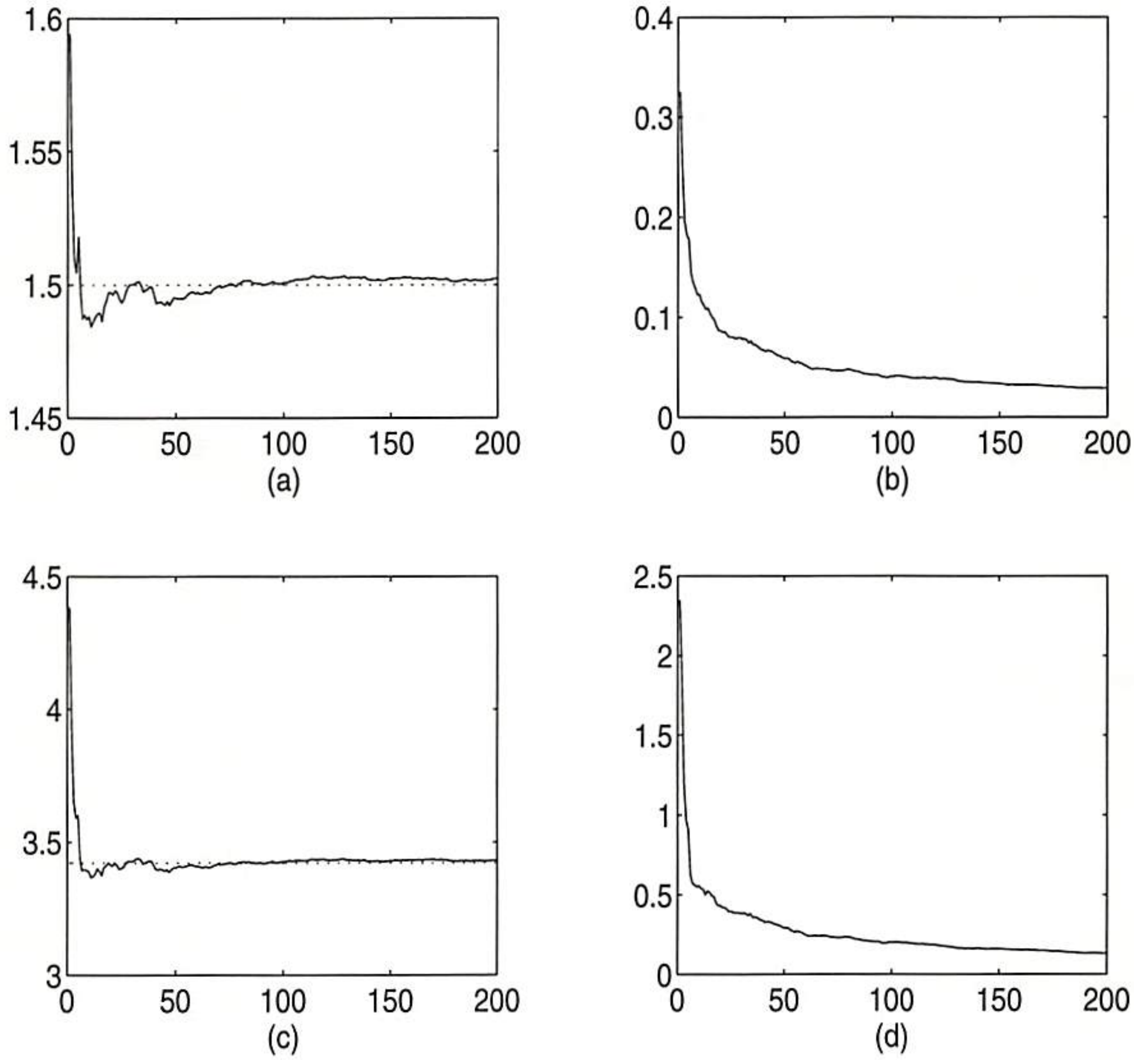

Figure 2: Performance of the iterative estimator of α, γ_y, total number of sample blocks is 200 with 100 samples in each block. (a). Average of $\hat{\alpha}$ over 100 realizations, compared with true value: $\alpha = 1.5$ (dotted). (b). Standard deviation of $\hat{\alpha}$ over 100 realizations. (c). Average of $\hat{\gamma_y}$ over 100 realizations, compared with true value: $\gamma_y = 3.4215$ (dotted). (d). Standard deviation of $\hat{\gamma_y}$ over 100 realizations.

The above equations show that the output covariations do not depend on n, but only depend on the time lag $j, (j = 0, \pm1, \pm2, \cdots \pm q)$. Since the covariation of two $S\alpha S$ random variables is related to FLOM by [Cambanis and Miller, 1981] :

$$[X, Y]_\alpha = \frac{\mathbf{E}(XY^{<p-1>})}{\mathbf{E}(|Y|^p)} \gamma_y, \tag{46}$$

where we used the notation:

$$Y^{<p-1>} = \begin{cases} |Y|^{p-2}Y^* & Y\text{: complex} \\ |Y|^{p-1}\text{sign}(Y) & Y\text{: real,} \end{cases} \tag{47}$$

One way to obtain the coefficients $h_0, h_1, \cdots h_q$ is to solve the above over-determined nonlinear equation system by the least square method, i.e., to minimize the cost function:

$$J(h_0, h_1, \cdots h_q) = \sum_{j=-q}^{j=q} (c_j - [Y_n, \widehat{Y_{n+j}}]_\alpha)^2. \tag{48}$$

However, the cost function $J(h_0, h_1, \cdots h_q)$ is not unimodal, it may converge to different local minima from different initial guess of $(h_0, h_1, \cdots h_q)$. As an example, consider a FIR(3) channel with i.i.d $S\alpha S$ $(\alpha = 1.5)$ input,

$$Y_n = X_n + 0.3X_{n-1} - 0.4X_{n-2}, \tag{49}$$

even in the deterministic case, i.e., when all the parameters (α and output covariations) are exactly known, the least square method still leads to non-unique solutions. Table (7) shows the simulation results. Obviously, least square method does not provide the accurate solution for this problem.

For FIR(2) (two unknown impulse response coefficients) and FIR(3) (three unknown impulse response coefficients) channels, it is possible to obtain closed form solutions of the coefficients in terms of the output covariations. For FIR(2) channels, we have:

$$h_0 = \left([Y_n, Y_{n+1}]_\alpha \cdot [Y_n, Y_{n-1}]_\alpha^{<1-\alpha>}\right)^{\frac{1}{2\alpha-\alpha^2}}, \alpha \neq 2, \tag{50}$$

Initial Guess	h_0	h_1	h_2
$(h_0, h_1, h_2) = (1.5, 1, -2)$	0.987641	0.314898	-0.408127
$(h_0, h_1, h_2) = (-0.5, 1, 2)$	0.443550	-0.687386	-0.673024

Table 7: Least square method yields initial-dependent results even for the deterministic case.

$$h_1 = [Y_n, Y_{n-1}]_\alpha \cdot h_0^{<1-\alpha>}. \tag{51}$$

For FIR(3) channels, we have:

$$h_0 = \left([Y_n, Y_{n+2}]_\alpha \cdot [Y_n, Y_{n-2}]_\alpha^{<1-\alpha>}\right)^{\frac{1}{2\alpha-\alpha^2}}, \alpha \neq 2, \tag{52}$$

$$h_1 = \frac{\frac{[Y_n,Y_{n-1}]_\alpha}{h_2} - \frac{[Y_n,Y_{n+1}]_\alpha}{h_0}}{\frac{[Y_n,Y_{n-2}]_\alpha}{h_2^2} - \frac{[Y_n,Y_{n+2}]_\alpha}{h_0^2}}, \tag{53}$$

$$h_2 = [Y_n, Y_{n-2}]_\alpha \cdot h_0^{<1-\alpha>}, \tag{54}$$

For FIR(4) channels, the closed form solution also exists. However, the error propagation from the output covariation estimation is so severe that the closed form approach is virtually impractical. For longer FIR channels (channels with more than three impulse response coefficients), the closed form solution is unknown. Nevertheless, the closed form expression approach shows that Gaussian ($\alpha = 2$) driven FIR channels are not blindly identifiable, which is a well known fact, but we show it from a different perspective. On the other hand, for non-Gaussian $S\alpha S$ input, the impulse response coefficients are uniquely determined by output covariation (the lower-order statistics) and characteristic exponent α.

3.2 Frequency Domain Approach with α-Spectrum

Consider the generalized form of the output covariation: $[Y_n, W_n]_\alpha$, where

$$W_n = \sum_{i=-q}^{q} a_i Y_{n-i}, \tag{55}$$

and $\{a_i, i = -q, \cdots, -1, 0, 1, \cdots, q\}$ are arbitrary constants taking on real or complex values. Then, Eq.(44) is just a set of special cases of this generalized form.

Then, the covariation between Y_n and W_n is:

$$\begin{aligned}
[Y_n, W_n]_\alpha &= \left[Y_n, \sum_{i=-q}^{q} a_i Y_{n-i} \right]_\alpha \\
&= \left[\sum_{k=0}^{q} h_k X_{n-k}, \sum_{i=-q}^{q} a_i (\sum_{l=0}^{q} h_l X_{n-i-l}) \right]_\alpha \\
&= \sum_{k=0}^{q} h_k \left[X_{n-k}, \sum_{i=-q}^{q} \sum_{l=0}^{q} a_i h_l X_{n-i-l} \right]_\alpha \\
&= \gamma_x \sum_{k=0}^{q} h_k \left(\sum_{l=0}^{q} a_{k-l} h_l \right)^{<\alpha-1>}, \qquad (56)
\end{aligned}$$

where we used the pseudo-linearity of the covariation operation:

$$[aX_m, bX_n]_\alpha = ab^{<\alpha-1>} \delta(m-n) \gamma_x, \qquad (57)$$

where X_m, X_n are i.i.d. *real or isotropic complex* $S\alpha S$ random variables with dispersion γ_x and $\delta(\cdot)$ is the Kronecker delta function. a, b are arbitrary *real or complex* constants.

Comments on Eq.(57): When X_m, X_n are isotropic complex $S\alpha S$ random variables, a, b are real or complex constants, or X_m, X_n are real $S\alpha S$ random variable, a, b are also real constants, Eq.(57) is one of the properties of covariation operation [Cambanis and Miller, 1981; Cambanis, 1983]. When X_m, X_n are real $S\alpha S$ random variables, a, b are complex constants, Eq.(57) is based on the following theorem [Cambanis, 1983]:

Theorem 2 *If $X = (X_1, \cdots, X_n)$ is real $S\alpha S$ with spectral measure Γ_x, and a_k, b_k are complex numbers, then*

$$\left[\sum_{k=1}^{n} a_k X_k, \sum_{j=1}^{n} b_j X_j \right]_\alpha = \int_{S_n} \left(\sum_{k=1}^{n} a_k X_k \right) \left(\sum_{j=1}^{n} b_j X_j \right)^{<\alpha-1>} d\Gamma_x(x). \qquad (58)$$

∎

Since the choice of $a_{-q}, \cdots, a_{-1}, a_0, a_1, \cdots, a_q$ is arbitrary, let

$$a_n = z^n, \ \forall z \in \mathbf{C}, z \neq 0, n = -q, \cdots, -1, 0, 1, \cdots, q, \tag{59}$$

and note that:

$$(z^k)^{<\alpha-1>} = (z^{<\alpha-1>})^k = \left((\frac{1}{z})^{<\alpha-1>} \right)^{-k}, \tag{60}$$

then Eq.(56) becomes:

$$S_\alpha(z) = [Y_n, W_n(z)]_\alpha = \gamma_x H\left((\frac{1}{z})^{<\alpha-1>} \right) (H(z))^{<\alpha-1>}, \tag{61}$$

where

$$W_n(z) = \sum_{i=-q}^{i=q} Y_{n-i} z^i, \tag{62}$$

which is the window z-transform of the channel output Y_n, and

$$H(z) = \sum_{n=0}^{q} h_n z^{-n}, \ \forall z \in \mathbf{C}, z \neq 0. \tag{63}$$

which is simply the z-transform of the filter.

Eq.(61) is of fundamental importance in this paper. We coin the term *α-Spectrum* for $S_\alpha(z)$. Given the measurement of the α-Spectrum: $S_\alpha(z)$, we will show how to identify the magnitude response as well as the phase response of the channel. Without loss of generality, we assume the input is of unit dispersion, i.e., $\gamma_x = 1$.

3.3 Channel Magnitude Response Estimation

When $|z| = 1$, i.e., the α-Spectrum $S_\alpha(z)$ is evaluated on the unit circle, then

$$S_\alpha(e^{j\omega}) = H(e^{j\omega}) \left(H(e^{j\omega}) \right)^{<\alpha-1>} = \left| H(e^{j\omega}) \right|^\alpha, \tag{64}$$

which provides an easy way to estimate the channel magnitude response from the output data.

3.4 Channel Phase Response Estimation

To obtain channel phase response from the output α-Spectrum S_α, we should evaluate Eq.(61) on the z-plane other than the unit circle. By taking logarithm of both sides of Eq.(61), we have:

$$\begin{aligned}\log S_\alpha(z) &= \log|S_\alpha(z)| + j\Psi(z) \\ &= \log|H(r^{1-\alpha}e^{j\omega})| + (\alpha-1)\log|H(re^{j\omega})| + j\{\Phi(r^{1-\alpha}e^{j\omega}) - \Phi(re^{j\omega})\},\end{aligned} \tag{65}$$

where $|H(re^{j\omega})|$ and $\Phi(re^{j\omega})$ are the channel magnitude and phase responses evaluated on a circle with radius $r(r \neq 1)$, respectively.

The z-transform of any FIR channel can be written as:

$$H(z) = A_0 z^{-d} \prod_{i=1}^{L_1}(1 - a_i z^{-1}) \prod_{i=1}^{L_2}(1 - b_i z), \tag{66}$$

where A_0 is the constant gain, d is a constant integer (time delay), $\{a_i, |a_i| < 1\}$ are the zeros inside the unit circle and $\{1/b_i, |b_i| < 1\}$ are zeros outside the unit circle. $z = re^{j\omega}$. Because the α-Spectrum S_α in Eq.(61) will suppress the constant delay d and the sign of the constant gain A_0. Therefore, without loss of generality, we assume $d = 0$ and $A_0 > 0$, or equivalently, we have *a priori* knowledge of d and the sign of A_0. It is well known [Oppenheim and Schafer, 1989]:

$$\log H(re^{j\omega}) = \log|H(re^{j\omega})| + j\Phi(re^{j\omega}) \tag{67}$$

where

$$\log|H(re^{j\omega})| = \log(A_0) - \sum_{m=1}^{\infty} \frac{A^{(m)}r^{-m} + B^{(m)}r^m}{m}\cos(m\omega), \tag{68}$$

$$\Phi(re^{j\omega}) = \sum_{m=1}^{\infty} \frac{A^{(m)}r^{-m} - B^{(m)}r^m}{m}\sin(m\omega), \tag{69}$$

where

$$A^{(m)} = \sum_{i=1}^{L_1} a_i^m, \tag{70}$$

$$B^{(m)} = \sum_{i=1}^{L_2} b_i^m, \tag{71}$$

and the region of convergence (ROC) is:

$$\max_i\{|a_i|\} < |z| < \min_i\{1/|b_i|\}. \tag{72}$$

Especially, if $r = 1$, then

$$\log|H(e^{j\omega})| = \log(A_0) - \sum_{m=1}^{\infty} \frac{A^{(m)} + B^{(m)}}{m} \cos(m\omega). \tag{73}$$

$$\Phi(e^{j\omega}) = \sum_{m=1}^{\infty} \frac{A^{(m)} - B^{(m)}}{m} \sin(m\omega). \tag{74}$$

Clearly, $\frac{A^{(m)}+B^{(m)}}{m}$ determines to the magnitude response of the channel, and $\frac{A^{(m)}-B^{(m)}}{m}$ determines to the phase response.

Substituting Eqs. (68) and (69) to Eq.(65), we have:

$$\log|S_\alpha(re^{j\omega})| = \alpha\log(A_0) - \sum_{m=1}^{\infty} \frac{A^{(m)}\mu_m(r) + B^{(m)}\mu_m(\frac{1}{r})}{m} \cos(m\omega). \tag{75}$$

$$\Psi(re^{j\omega}) = \sum_{m=1}^{\infty} \frac{A^{(m)}\nu_m(r) - B^{(m)}\nu_m(\frac{1}{r})}{m} \sin(m\omega), \tag{76}$$

where

$$\begin{aligned} \mu_m(r) &= r^{m(\alpha-1)} + (\alpha-1)r^{-m}, \\ \nu_m(r) &= r^{m(\alpha-1)} - r^{-m}, \end{aligned} \tag{77}$$

with ROC:

$$\max_i\{|a_i|, |b_i|^{1/(\alpha-1)}\} < r < \min_i\{1/|b_i|, (1/|a_i|)^{1/(\alpha-1)}\}. \tag{78}$$

As shown in Eq.(74), the channel phase is determined by $\frac{A^{(m)}-B^{(m)}}{m}$. In the following, we shall show that $\frac{A^{(m)}-B^{(m)}}{m}$ can be obtained from the *magnitude* of the α-Spectrum: $|S_\alpha(re^{j\omega})|$ (Eq.(75)) or from the *phase* of the α-Spectrum: $\Psi(re^{j\omega})$ (Eq.(76)).

Multiplying both sides of Eq.(75) by $\cos(n\omega)$, $n = 1, 2, 3, \cdots$, and integrating with respect to ω from $-\pi$ to π, we have:

$$-\frac{2}{\pi}\int_0^{\pi} \log|S_\alpha(re^{j\omega})|\cos(n\omega)d\omega = \frac{A^{(n)}\mu_n(r) + B^{(n)}\mu_n(\frac{1}{r})}{n}, \tag{79}$$

where we used the following orthogonality:

$$\int_0^{\pi} \cos(m\omega)\cos(n\omega)d\omega = \frac{\pi}{2}\delta(m-n). \tag{80}$$

Replacing r by $1/r$ in Eq.(79), we have:

$$-\frac{2}{\pi}\int_0^{\pi} \log|S_\alpha(\frac{1}{r}e^{j\omega})|\cos(n\omega)d\omega = \frac{A^{(n)}\mu_n(\frac{1}{r}) + B^{(n)}\mu_n(r)}{n}. \tag{81}$$

Subtracting Eq.(81) from Eq.(79), we have:

$$\frac{A^{(n)} - B^{(n)}}{n} = \frac{\frac{2}{\pi}\int_0^{\pi} \log\frac{|S_\alpha(\frac{1}{r}e^{j\omega})|}{|S_\alpha(re^{j\omega})|}\cos(n\omega)d\omega}{\mu_n(r) - \mu_n(\frac{1}{r})}. \tag{82}$$

Similarly, the term $\frac{A^{(n)}-B^{(n)}}{n}$ can also be solved through the *phase* of the α-Spectrum: $\Psi(re^{j\omega})$ in Eq.(76),

$$\frac{A^{(n)} - B^{(n)}}{n} = \frac{\frac{2}{\pi}\int_0^{\pi}\left(\Psi(re^{j\omega}) + \Psi(\frac{1}{r}e^{j\omega})\right)\sin(n\omega)d\omega}{\nu_n(r) + \nu_n(\frac{1}{r})}. \tag{83}$$

Since there is always a phase wrapping ambiguity associated with Eq.(83), i.e., we can only obtain the principle value of $\Psi(re^{j\omega})$ from the estimation. Before applying Eq.(83), we need to unwrap $\Psi(re^{j\omega})$ into a continuous function. This is difficult to achieve in practice. Therefore, Eq.(82) should be used instead to avoid the phase unwrapping problem. Once $\frac{A^{(n)}-B^{(n)}}{n}$ has been estimated, we can use Eq.(74) to estimate the channel phase response.

$$\Phi(e^{j\omega}) = \sum_{n=1}^{\infty}\left(\frac{\frac{2}{\pi}\int_0^{\pi} \log\frac{|S_\alpha(\frac{1}{r}e^{j\Omega})|}{|S_\alpha(re^{j\Omega})|}\cos(n\Omega)d\Omega}{\mu_n(r) - \mu_n(\frac{1}{r})}\right)\sin(n\omega),\ r \neq 1;\ \alpha \neq 2. \tag{84}$$

Eq.(84) implies that we can extract the *channel phase* information from the *magnitude of α-Spectrum* evaluated on circles other than the unit circle. This does not come as a surprise. As a matter of fact, if we apply the above method to Eq.(68), we have:

$$\frac{A^{(n)} - B^{(n)}}{n} = \frac{\frac{2}{\pi}\int_0^\pi \log\frac{|H(re^{j\omega})|}{|H(1/re^{j\omega})|}\cos(n\omega)d\omega}{r^n - r^{-n}}, \ r \neq 1. \tag{85}$$

Consequently, the channel phase response is (from Eq.(69)):

$$\Phi(e^{j\omega}) = \sum_{n=1}^{\infty}\left(\frac{\frac{2}{\pi}\int_0^\pi \log\frac{|H(re^{j\Omega})|}{|H(1/re^{j\Omega})|}\cos(n\Omega)d\Omega}{r^n - r^{-n}}\right)\sin(n\omega), \ r \neq 1, \tag{86}$$

which means we can extract the channel phase response from the *magnitude response* of the channel evaluated on two circles with radii reciprocal to each other.

The channel magnitude response is related to $\frac{A^{(m)}+B^{(m)}}{m}$, which can be obtained by adding Eq.(79) and Eq.(81), we have:

$$\begin{aligned}\frac{A^{(n)} + B^{(n)}}{n} &= \frac{-\frac{2}{\pi}\int_0^\pi \left(\log|S_\alpha(\frac{1}{r}e^{j\omega})| + \log|S_\alpha(re^{j\omega})|\right)\cos(n\omega)d\omega}{\mu_n(r) + \mu_n(\frac{1}{r})} \\ &\overset{r=1}{\equiv} -\frac{2}{\pi\alpha}\int_0^\pi \log|S_\alpha(e^{j\omega})|\cos(n\omega)d\omega.\end{aligned} \tag{87}$$

Although we do not need Eq.(87) to estimate the channel magnitude response, which can be easily estimated directly by Eq.(64), nevertheless, Eq.(87) can reduce the computation intensity. From the equality shown in Eq.(87), we have:

$$\begin{aligned}&\int_0^\pi \left(\log|S_\alpha(\frac{1}{r}e^{j\omega})| + \log|S_\alpha(re^{j\omega})|\right)\cos(n\omega)d\omega \\ &= \frac{\mu_n(r) + \mu_n(\frac{1}{r})}{\alpha}\int_0^\pi \log|S_\alpha(e^{j\omega})|\cos(n\omega)d\omega, \ \forall r.\end{aligned} \tag{88}$$

Therefore,

$$\int_0^\pi \log\frac{|S_\alpha(\frac{1}{r}e^{j\omega})|}{|S_\alpha(re^{j\omega})|}\cos(n\omega)d\omega = -2\int_0^\pi \log|S_\alpha(re^{j\omega})|\cos(n\omega)d\omega$$

$$+\frac{\mu_n(r)+\mu_n(\frac{1}{r})}{\alpha}\int_0^{\pi}\log|S_\alpha(e^{j\omega})|\cos(n\omega)d\omega, \tag{89}$$

so, from Eq.(82), we have:

$$\frac{A^{(n)}-B^{(n)}}{n}=\frac{2}{\pi\alpha}\int_0^{\pi}\Big(\zeta(n,r)\log|S_\alpha(e^{j\omega})|-\eta(n,r)\log|S_\alpha(re^{j\omega})|\Big)\cos(n\omega)d\omega, \tag{90}$$

where

$$\begin{aligned}\zeta(n,r) &= \frac{\cosh(n(\alpha-1)\log r)+(\alpha-1)\cosh(n\log r)}{\sinh(n(\alpha-1)\log r)-(\alpha-1)\sinh(n\log r)},\\ \eta(n,r) &= \frac{\alpha}{\sinh(n(\alpha-1)\log r)-(\alpha-1)\sinh(n\log r)},\quad \forall r\neq 1,\forall\alpha\neq 2.\end{aligned} \tag{91}$$

Similarly, the equivalent expressions for Eq. (85) is:

$$\frac{A^{(n)}-B^{(n)}}{n}=\frac{2}{\pi}\int_0^{\pi}\Big(\mathrm{csch}(n\log r)\log|H(re^{j\omega})|-\coth(n\log r)\log|H(e^{j\omega})|\Big)\cos(n\omega)d\omega, \forall r\neq 1. \tag{92}$$

In general, we can recover any channel entirely (magnitude response and phase response) with the *magnitude* of its z-transform evaluated on the unit circle and another arbitrary circle within the region of convergence of $\log H(z)$.

3.5 Simulations for Deterministic Signals

As we have shown in Eq.(64), the α-Spectrum $S_\alpha(re^{j\omega})$ evaluated on the unit circle gives us the channel magnitude response,

$$|H(e^{j\omega})|=\Big(S_\alpha(e^{j\omega})\Big)^{\frac{1}{\alpha}} \tag{93}$$

together with $S_\alpha(re^{j\omega})$ evaluated on another circle in the Region of Convergence (ROC), we can extract the channel phase response:

$$\Psi(e^{j\omega})=\sum_{n=1}^{\infty}\left(\frac{2}{\pi\alpha}\int_0^{\pi}\Big(\zeta(n,r)\log|S_\alpha(e^{j\Omega})|-\eta(n,r)\log|S_\alpha(re^{j\Omega})|\Big)\cos(n\Omega)d\Omega\right)\sin(n\omega), \tag{94}$$

with $\zeta(n,r)$ and $\eta(n,r)$ given in Eq.(91).

The integration in Eq.(82) can be approximated to arbitrary accuracy by *Filon's cosine formula* [Hildebrand, 1974] , which states that:

$$\int_0^\pi f(\Omega)\cos(k\Omega)d\Omega \approx h[\mu(\theta)C_e + \nu(\theta)C_o], \tag{95}$$

where

$$\begin{aligned}
h &= \frac{\pi}{N},\ N:\text{number of equally spaced points in}[0,\pi],\\
\theta &= \frac{k\pi}{N},\\
\mu(\theta) &= \frac{\theta(3+\cos 2\theta)-2\sin 2\theta}{\theta^3},\\
\nu(\theta) &= \frac{4(\sin\theta-\theta\cos\theta)}{\theta^3},\\
C_e &= \frac{1}{2}(f(0)+(-1)^k f(\pi)) + \sum_{i(\text{even})=2}^{N-2} f(ih)\cos(kih),\ (\text{sum of even terms})\\
C_o &= \sum_{i(\text{odd})=1}^{N-1} f(ih)\cos(kih),\ (\text{sum of odd terms})
\end{aligned} \tag{96}$$

Let us consider the same nonminimum phase FIR channel that was used earlier:

$$Y_n = X_n + 0.5X_{n-1} - 1.3X_{n-2} + 0.7X_{n-3}, \tag{97}$$

where X is a white $S\alpha S$ process with $\alpha = 1.5$ and channel impulse response coefficients are: $h_0 = 1.0$, $h_1 = 0.5$, $h_2 = -1.3$, $h_3 = 0.7$, with z-transform:

$$H(z) = 1.5924z^{-1}(1-(0.5462+0.3758j)z^{-1})(1-(0.5462-0.3758j)z^{-1})(1+0.628z) \tag{98}$$

Fig.(3) shows the magnitude estimation result when the α-Spectrum is known exactly. The phase response estimation result is shown in Fig.(4), where $N = 50$ was used in Filon's cosine formula to approximate the integration in Eq.(82).

Comments:

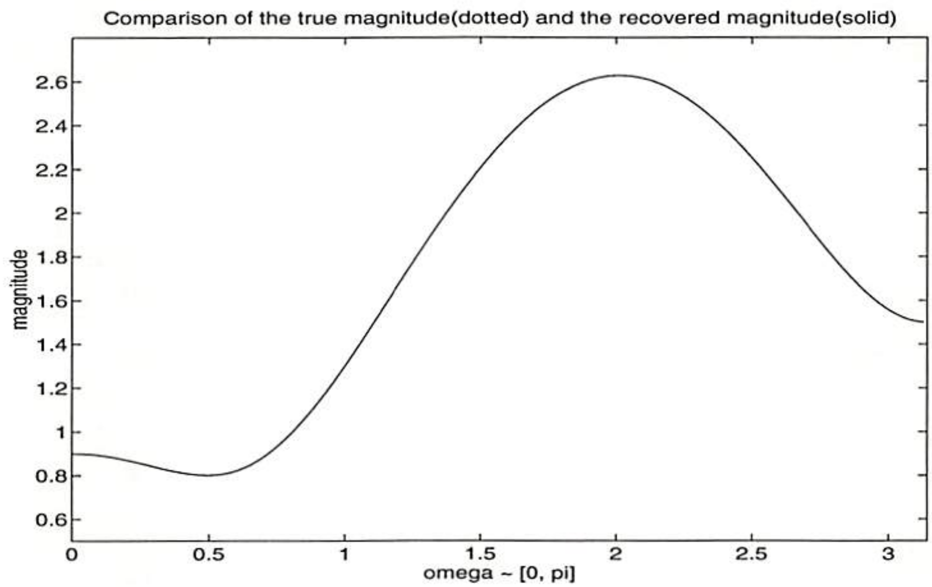

Figure 3: Estimation of the channel magnitude response from *exactly known* output α-Spectrum.

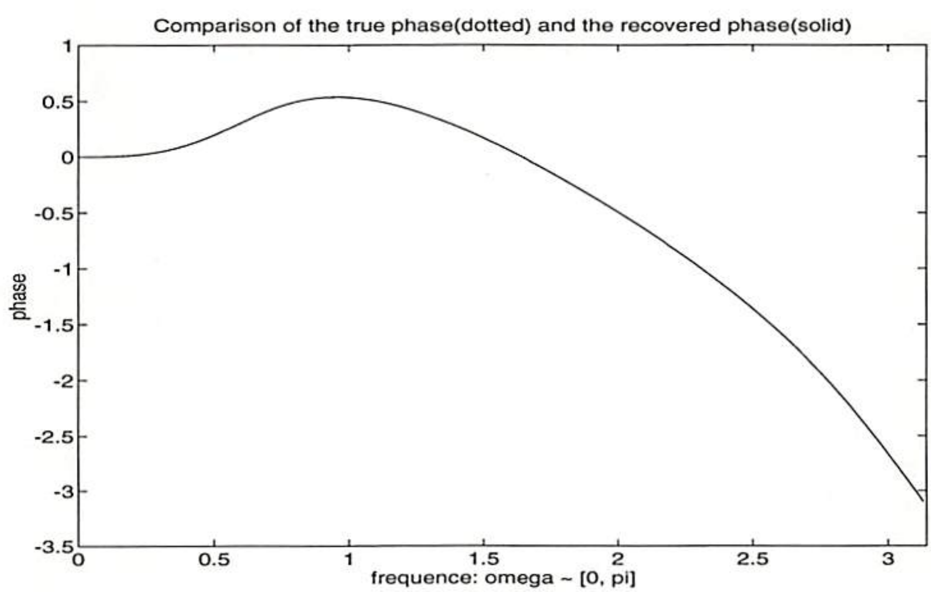

Figure 4: Estimation of the channel phase response from *exactly known* output α-Spectrum.

1. Blind identification of channel driven by Gaussian ($\alpha = 2$) process is impossible. In this case, we can only identify the channel magnitude response, but not the phase response. This is shown clearly in Eq.(91).

2. The above approach does not require the exact knowledge of the order q of the FIR channel. This is because the channel is estimated in the frequency domain (magnitude response and phase response), not the time domain. When the order q is unknown, the α-Spectrum can be written as:

$$\tilde{S}_\alpha(z) = [Y_n, \tilde{W}_n(z)]_\alpha, \tag{99}$$

where

$$\tilde{W} = \sum_{i=-\tilde{q}}^{i=\tilde{q}} Y_{n-i} z^i, \tag{100}$$

if we take $\tilde{q}$ large enough such that $\tilde{q} \geq q$, from the fact the Y_n is an $MA(q)$ process, Y_n and Y_m are independent when $|n-m| > q$, and by the property of the covariation: $[Y_n, Y_m] = 0$, therefore,

$$\tilde{S}_\alpha(z) = S_\alpha(z) = \gamma_x H\left((\frac{1}{z})^{<\alpha-1>} \right) (H(z))^{<\alpha-1>} . \tag{101}$$

3.6 Simulations for Stochastic Signals and α-Spectrum Estimator

The remaining task is to find an appropriate yet practical estimator for the α-Spectrum estimator. Notice that the FLOM estimator for covariation in Eq.(46) is applicable *if and only if* X and Y are real or *isotropic* complex stable random variables, and γ_y is the dispersion of Y:

$$\gamma_y = \begin{cases} (C_1(p,\alpha)\mathbf{E}(|Y|^p))^{\alpha/p} & \text{if } Y \text{ is real} \\ (C_2(p,\alpha)\mathbf{E}(|Y|^p))^{\alpha/p} & \text{if } Y \text{ is } \textit{isotropic} \text{ complex} \end{cases} \tag{102}$$

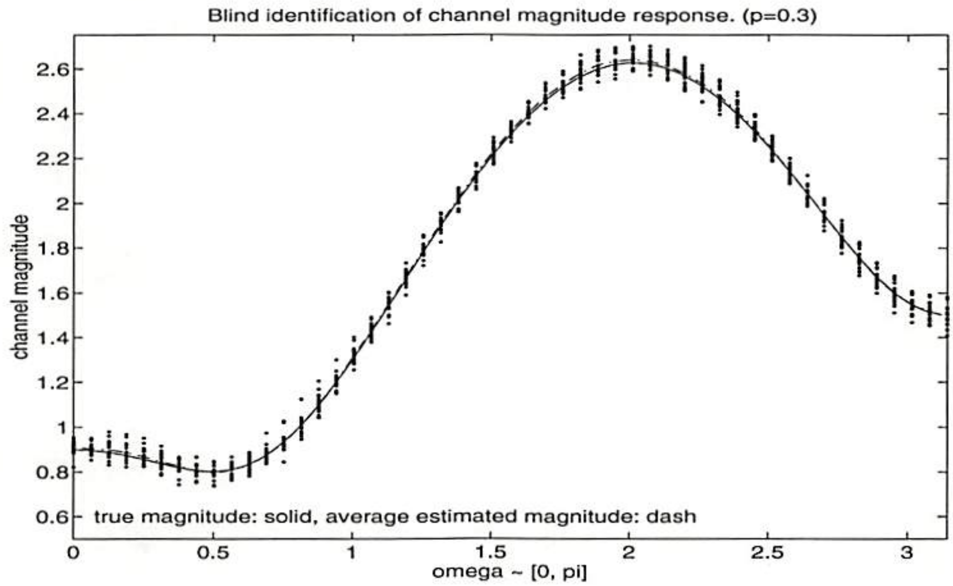

Figure 5: Magnitude response estimation of mixed-phase FIR channel driven by white *isotropic complex* $S\alpha S$ process. (20000 samples)

When the input X_n is i.i.d isotropic complex $S\alpha S$ random variable (see Appendix for isotropic $S\alpha S$ random variable generator), then any finite linear combination with real or complex coefficients of X_n is also isotropic complex $S\alpha S$ random variable, and the FLOM estimator applies. Figures 5 and 6 show simulation results of blind identification of the channel (Eq.(98)) magnitude and phase responses, respectively. 20000 data samples were collected in each of the 20 independent realizations and $p = 0.3$ in the FLOM estimator. [1] The performance is significantly increased with increased sample size. Fig.s (7) and (8) show the simulation results with 200000 samples.

In the above simulations, we kept 15 terms of $\frac{A^{(n)}-B^{(n)}}{n}$ because the zeros of Eq.(98) are not very far from the unit circle. If the zeros of the channel are far from the unit

[1]Most papers claim that p in FLOM estimator should be $1 \leq p < \alpha$. However, we have found covariation estimator with $p < 1$ often has smaller variance. Detailed analysis will be announced later.

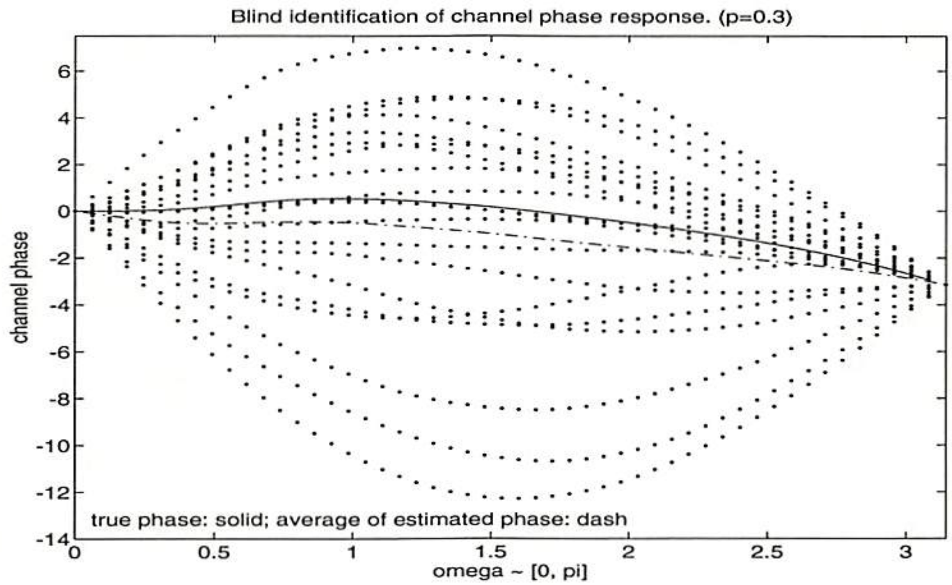

Figure 6: Phase response estimation of mixed-phase FIR channel driven by white *isotropic complex* $S\alpha S$ process. (20000 samples)

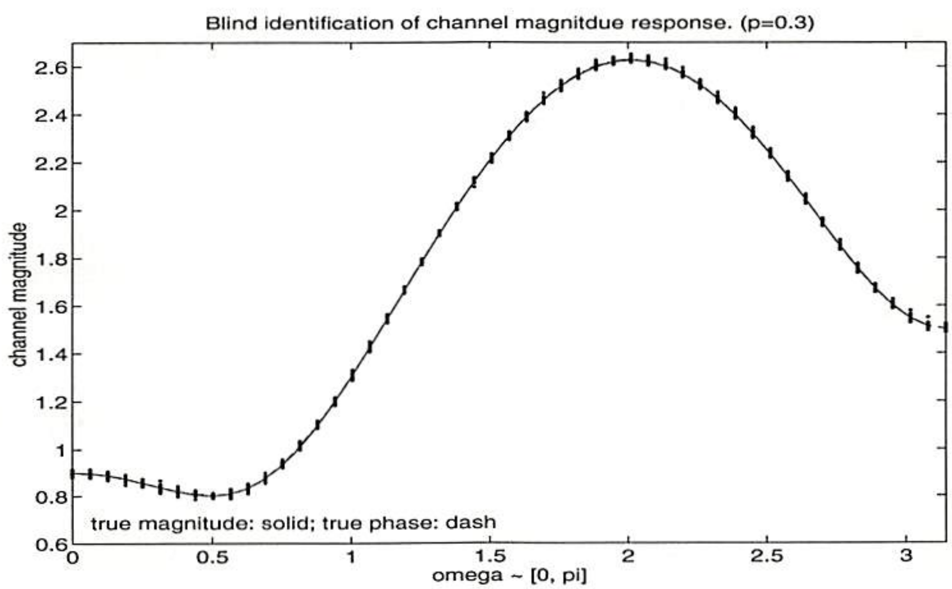

Figure 7: Magnitude response estimation of mixed-phase FIR channel driven by white *isotropic complex* $S\alpha S$ process. (200000 samples)

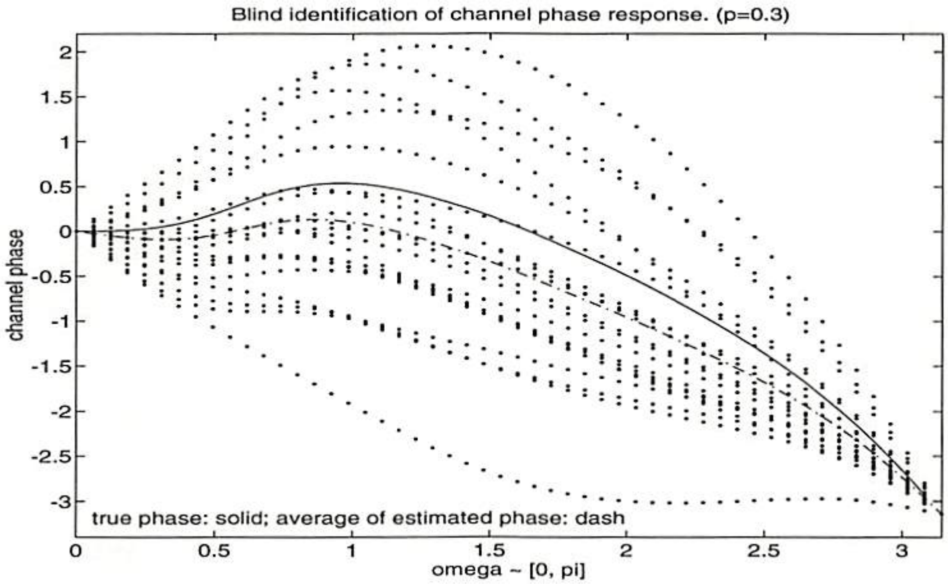

Figure 8: Phase response estimation of mixed-phase FIR channel driven by white *isotropic complex* $S\alpha S$ process. (200000 samples)

circle, $\frac{A^{(n)}-B^{(n)}}{n}$ will decay much faster. Consider another example:

$$Y_n = X_n - 4.4X_{n-1} + 1.68X_{n-2} - 0.32X_{n-3}, \tag{103}$$

where X is a white $S\alpha S$ process with $\alpha = 1.5$ and channel impulse response coefficients are: $h_0 = 1.0$, $h_1 = -4.4$, $h_2 = 1.68$, $h_3 = -0.32$, with z-transform:

$$H(z) = -4z^{-1}(1-(0.2-0.2j)z^{-1})(1-(0.2+0.2j)z^{-1})1-0.25z). \tag{104}$$

In this example, we only need to keep 5 terms of $\frac{A^{(n)}-B^{(n)}}{n}$. Fig.s (9) and (6) show simulation results of blind identification of the channel (Eq.(104)) magnitude and phase responses, respectively. 500000 data samples were collected in each of the 20 independent realizations and $p = 0.3$ in the FLOM estimator.

When the input is real $S\alpha S$ random variable, an appropriate estimator for the α-Spectrum $S_\alpha(z) = [Y_n, W_n(z)]_\alpha$ (the covariation of a real $S\alpha S$ random variable with a

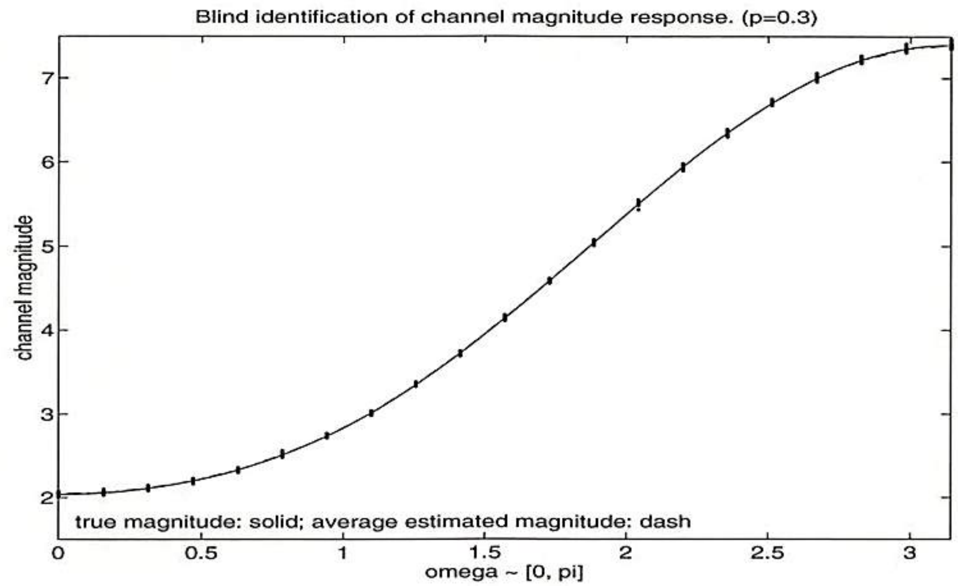

Figure 9: Magnitude response estimation of mixed-phase FIR channel driven by white *isotropic complex* $S\alpha S$ process. (500000 samples)

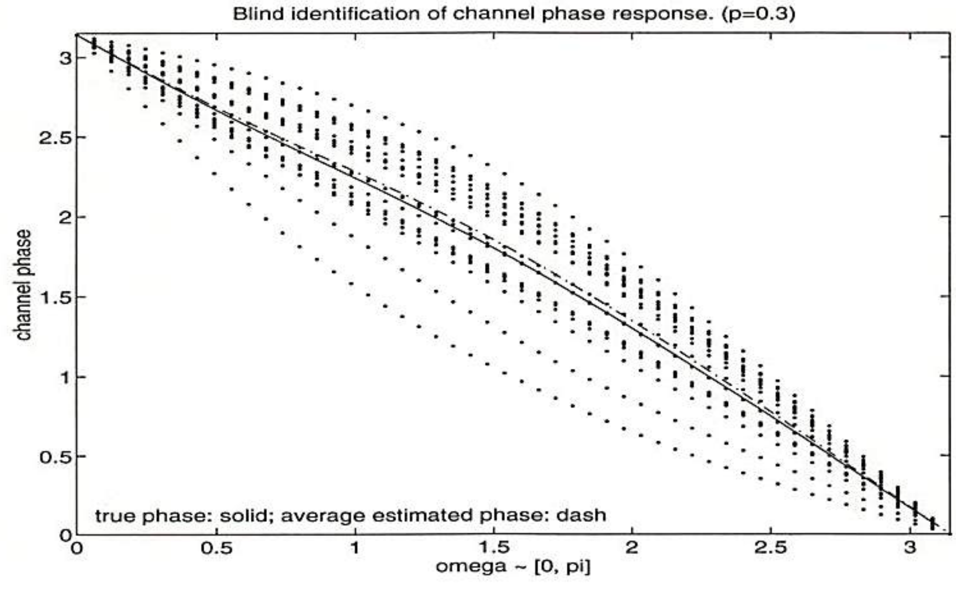

Figure 10: Phase response estimation of mixed-phase FIR channel driven by white *isotropic complex* $S\alpha S$ process. (500000 samples)

complex $S\alpha S$ one) is yet to be found. Also the efficiency of the α-Spectrum needs to improved.

4 α-Spectrum for Blind Equalization

Since blind identification and blind equalization are closely related, the above methods also apply to blind equalization in impulsive noise environments, where the model is:

$$Y_n = h_n * X_n + N_n, \tag{105}$$

where h_n is the FIR channel, X_n is the transmitted signal, and N_n is the additive white $S\alpha S$ noise (AWαN for abbreviation). Assuming both X_n and N_n are white random processes and they are independent from each other. The α-Spectrum approach yields:

$$S_\alpha(z) = \gamma_x H\left((\frac{1}{z})^{<\alpha-1>}\right)(H(z))^{<\alpha-1>} + \gamma_N, \tag{106}$$

where γ_x is the dispersion of the transmitted signal and γ_N is the dispersion of the additive white $S\alpha S$ noise, both of whom are generally assumed to be known. Therefore, given the estimate of the α-Spectrum: $S_\alpha(z)$, we should be able to identify the channel H and construct the inverse filter H^{-1}. Note that the transmitted signal X is not a $S\alpha S$ process, however, we can define its dispersion γ_x according to Eq.(102. The received signal Y_n is no longer a $S\alpha S$ process, nevertheless, we can still estimate its α-Spectrum according to Eq.(46).

5 Conclusion

We introduced the fractional lower-order moments with both positive and negative orders and their application for parameter estimation. Further, we developed an iterative method for estimating the characteristic exponent α and the dispersion γ from an FIR channel output by introducing the $\log|S\alpha S|$ process. Then we discussed several approaches for blind identification of an FIR channel impulse response coefficients driven by a non-Gaussian $S\alpha S$ process. We formulated the α-Spectrum, a new spectral representation based on the output covariation, with which, we proved the blind identifiability of any FIR channels driven by white $S\alpha S$ processes. Simulation results verified our theory.

Acknowledgment

We are thankful to Dr. Min Shao for many helpful discussions.

Appendix: Isotropic $S\alpha S$ Random Number Generator

An isotropic $S\alpha S$ random variable $X = X_r + jX_i$ has a characteristic function:

$$\phi(\omega_1, \omega_2) = \mathbf{E}\{e^{j(\omega_1 X_r + \omega_2 X_i)}\} = e^{-\gamma(\omega_1^2+\omega_2^2)^{\frac{\alpha}{2}}}, \tag{107}$$

where α is the characteristic exponent and γ is the dispersion.

Theorem 3 *A standard ($\gamma = 1$) isotropic complex $S\alpha S$ random variable can be represented as:*

$$X = A^{1/2}(G_1 + jG_2), \tag{108}$$

where A is a positive ($\tilde{\alpha} = \frac{\alpha}{2}, \beta = 1$)-stable random variable and can be generated by:

$$A = \left(\frac{a(\Theta)}{W}\right)^{\frac{1-\tilde{\alpha}}{\tilde{\alpha}}}, \tag{109}$$

where

$$a(\Theta) = \frac{\sin((1-\tilde{\alpha})\Theta)(\sin\tilde{\alpha}\Theta)^{\frac{\tilde{\alpha}}{1-\tilde{\alpha}}}}{(\sin\Theta)^{\frac{1}{1-\tilde{\alpha}}}}, \tag{110}$$

Θ is uniform on $(0, \pi)$ and W is standard exponential random variable. Θ and W are independent. G_1, G_2 are standard $S2S$ (Gaussian) random variables. A, G_1, G_2 are independent from each other. ∎

Proof: Kanter [1975] showed that A as defined above is a positive stable random variable, with the characteristic exponent $\tilde{\alpha} < 1$ and the skewness $\beta = 1$. The Laplace transform of A is:

$$\mathbf{E}(e^{-sA}) = e^{-s^{\tilde{\alpha}}}, s > 0. \tag{111}$$

Therefore, the characteristic function of X is:

$$\begin{aligned}\phi(\omega_1, \omega_2) &= \mathbf{E}\{e^{j(\omega_1 A^{1/2} G_1 + \omega_2 A^{1/2} G_2)}\} = \mathbf{E}\{\mathbf{E}\{e^{j(\omega_1 A^{1/2} G_1 + \omega_2 A^{1/2} G_2)} | A\}\} \\ &= \mathbf{E}\{e^{-(\omega_1^2 + \omega_2^2)A}\} = e^{-(\omega_1^2+\omega_2^2)^{\frac{\alpha}{2}}}.\end{aligned} \tag{112}$$

This shows that X generated by the above method is indeed an isotropic complex $S\alpha S$ random variable. ∎

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