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by

Sunil Bharitkar and Jerry M. Mendel

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Signal and Image Processing Institute
UNIVERSITY OF SOUTHERN CALIFORNIA
Department of Electrical Engineering-Systems
3740 McClintock Avenue, Room 404
Los Angeles, CA 90089-2564 U.S.A.

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Sunil Bharitkar and Jerry M. Mendel

Department of Electrical Engineering-Systems. EEB 400

University of Southern California

3740 McClintock Avenue, Los Angeles, CA 90089-2564

E-mail: mendel@sipi.usc.edu

Tel: 213-740-4456. Fax: 213-740-4651

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Abstract: **Several neuron activation functions have been proposed (e.g., linear, binary, sigmoid) for recurrent and multilayer Artificial Neural Networks. In this report we present a hysteretic neuron activation function for optimization and learning. We include this neuron within the framework of the Hopfield network to form the Hysteretic Hopfield Neural Network (HHNN). We then propose a dynamical equation, and an energy equation for this model using the well known Cohen-Grossberg theorem. Finally, these equations are used to prove Lyapunov stability of the HHNN.**

1 Introduction

In this report, we propose a new neuron model that is based on a phenomenon found widely in nature, namely *hysteresis*. Recall that hysteresis [1] is defined as a lagging effect due to a change of force acting on a body. Hysteresis manifests itself in the structures of many co-operative dynamical systems (formed of local interactions). In a study of prey selections by frogs (snapping of the frog to sensory stimulus, e.g., flies) [2], a phenomenon related to hysteresis was observed. Hysteresis was also discovered in the stretch receptor organs (SRO) of the crayfish [3]. Many engineering systems display hysteresis. Among these are mechanical structures stressed beyond the elastic range, and aerostructures (aircraft, missiles) subjected to acoustic or aerodynamic loads. Hysteretic and eddy current losses also appear in the core of a 3-phase transformer [4]. Marshall, et. al. [5] give a detailed explanation of hysteresis as it occurs in ferromagnetic materials (relating the magnetic field intensity and flux density as the current is varied). The identification of hysteretic type nonlinearities is important in earthquake resistant designs of buildings (e.g., [6], [7]).

Taga [8] developed a network containing 6 coupled oscillators (that functioned as neurons) to control bipedal locomotion. Each oscillator received feedback about the state of the biped's limbs; thus, biped velocity could be controlled, and, by varying the amplitude of the network's activation function, abrupt transitions between walking and running could be obtained. These transitions exhibited hysteresis (i.e., the walk-to-run transition occurred at a faster progression speed than the reverse transition). Similar hysteretic behavior can also be observed in humans [9]. Hoffman and Benson [10] demonstrated that a single cell-level neuron model, based on an analogy between the immune system and the central nervous system, exhibits hysteresis.

Models of hysteresis appear, for example, in [11],[12],[13], and [14]. Hysteretic neuron models using *signum* functions have been proposed by Yanai & Sawada [15], and Keeler et al. [16], for associative memory. They demonstrated that their hysteretic models performed better than non-hysteretic neuron models, in terms of capacity, signal-to-noise ratio, recall ability, etc. Takefuji & Lee [17] proposed a two state (binary) hysteretic neuron model such that: (1) if the input to a neuron exceeds a threshold (upper trip point) the neuron fires (i.e., the output of the neuron is unity); (2) if the input is below a certain threshold (lower trip point), the output of the neuron is zero; and, (3) if the input to the neuron is between these trip points, the output equals its previous value.

In this paper, we describe an hysteretic neuron model that differs from those in [15],[16], and [17] in the following ways: it (1) is multivalued, (2) has memory, and (3) is adaptive.

In Section 2, we present the hysteretic neuron and the Hysteretic Hopfield Neural Network (HHNN) along with its circuit dynamical equations. In Section 3, we briefly review the concept of Lyapunov stability for nonlinear dynamical systems. We also introduce Cohen-Grossberg (C-G) theory for analyzing stability of systems. As an interesting example, we show that the energy function for the Hopfield neural network can be obtained from C-G theory. Using similar analysis, we then obtain the energy function for the HHNN. Section 4 concludes this report.

2 Hysteretic Hopfield Neural Network

2.1 Hysteretic Neuron

Our hysteretic neuron is similar to other neuron models, in that it processes a linear weighted combination of inputs. It differs from other neuron models in that its nonlinear gain (activation) function is the hysteresis function depicted in Fig. 1. Mathematically, our hysteretic neuron gain function is described as:

$$y(x|\dot{x}) = \phi[x - \lambda(\dot{x})] = \tanh[\gamma(\dot{x})(x - \lambda(\dot{x}))] \tag{1}$$

where

$$\gamma(\dot{x}) = \begin{cases} \gamma_\alpha & \dot{x} \geq 0 \\ \gamma_\beta & \dot{x} < 0 \end{cases} \tag{2}$$

$$\lambda(\dot{x}) = \begin{cases} -\alpha & \dot{x} \geq 0 \\ \beta & \dot{x} < 0 \end{cases} \tag{3}$$

and $\beta > -\alpha$, and, $(\gamma_\alpha, \gamma_\beta) > 0$.

The mapping that is effected by this transformation is $y : \Re^2 \to \Re$. Note that, in the special case when $\alpha = \beta$, and $\gamma_\alpha = \gamma_\beta$, the activation function becomes the conventional sigmoid. Observe that this neuron's output not only depends on its input, x, but also on derivative information, namely, $\dot{x}$. It is the latter information that provides the neuron with memory and distinguishes it from other neurons. If, for example, x is positive at one time point and increases in value at the next time point, the activation function remains along segment $C - A$. On the other hand, if x is positive at one time point and decreases at the next time point, then the activation function jumps from hysteretic segment $C - A$ to segment $B - D$. So, this hysteretic neuron has the potential for much more *action* than the usual neuron.

One interpretation of the two segments in the hysteretic neuron is that continued learning occurs along the top segment $C - A$, but forgetting or reduced learning drops us to segment $B - D$.

Note that the hysteretic neuron's activation function has four parameters associated with it, namely, $\alpha, \beta, \gamma_\alpha, \gamma_\beta$. Usually, one does not tune a neuron's activation function because, for the most part, there are no parameters to tune (or there is, at most, one parameter, the slope of the sigmoid). The hysteretic neuron is different in this sense, and we can think about tuning all of its parameters in order to maximize its performance. So, it seems that the hysteretic neuron provides us with much more flexibility than the usual neuron.

2.2 Hysteretic Hopfield Neural Network

A circuit-based noiseless dynamical model of the hysteretic neuron is depicted in Fig. 2a. Important considerations in the design of such an analog circuit are that its individual components have a negligible propagation time, and the differentiator and integrator have a unity RC-time constant. In Fig. 2b, we assume a negligible effect of the capacitance, and a high internal resistance of the operational amplifiers in the differentiator-integrator pair. The net resistance in the two branches consisting of current sources $I_{s1} \triangleq \alpha_j / R_j$ and $I_{s2} \triangleq \beta_j / R_j$ is denoted R_j. Accordingly, we can place a resistance of the same value (i.e., R_j) in the branch which is in parallel with the capacitor C_j, as shown in Fig. 2a. The logic device denoted by an arrow after node J, switches to the upper branch if $\dot{x_j} \geq 0$, and to the lower branch if $\dot{x_j} < 0$. Applying Kirchoff's Current Law (KCL) to node J, we obtain the following circuit equation:

$$C_j \frac{dx_j}{dt} = -\frac{x_j}{R_j} + \frac{\lambda_j(\dot{x_j})}{R_j} + \sum_{i=1}^{N} w_{ji} y_i + I_j \quad j = 1, 2, ..., N \tag{4}$$

where y_i is a shorthand notation for $y_i(x_i|\dot{x_i})$ given by (1), and $\lambda_j(\dot{x_j})$ is as defined in (3).

We refer to the interconnection of the N nonlinear equations in (4) as a *Hysteretic Hopfield Neural Network*. Unlike the usual Hopfield Neural Network (HNN), the HHNN includes memory.

3 Stability of the HHNN

An important consideration in the theory of nonlinear systems is to define and analyze stability. Aleksander Mikhailovich Lyapunov, a Russian mathematician from the late nineteenth and early twentieth centuries, developed an approach for doing this that is widely used in the control theory literature, and is now known as *the direct method of Lyapunov.* Its key feature is that it leads to conclusions about stability of nonlinear systems without having to explicitly solve the system's nonlinear differential equation. Lyapunov used this approach to investigate the stability of systems that involve the rotation of heavy fluids. It has also become an important tool for establishing the stability of HNN's [18] or more complicated nonlinear feedback neural networks [19]. We use it to study the stability of the HHNN.

To begin, we recall the definition of a Lyapunov function and stability in the sense of Lyapunov (e.g., [20]). For a function $E(y_1, y_2, ..., y_N)$ to be a Lyapunov function, it must satisfy the following three properties: let $\mathbf{y}^*$=col$(y_1^*, y_2^*, ..., y_N^*)$ be an equilibrium point for a dynamical system; then, (1) $E(y_1^*, y_2^*, ..., y_N^*) = 0$; (2) $E(y_1, y_2, ..., y_N) > 0$, $\mathbf{y} \neq \mathbf{y}^*$; and, (3) $E(y_1, y_2, ..., y_N)$ should have partial derivatives with respect to all y_j. Given that $E(y_1, y_2, ..., y_N)$ is a Lyapunov function, and if

$$\frac{dE(y_1, y_2, ..., y_N)}{dt} \leq 0 \tag{5}$$

then $\mathbf{y}^*$ is stable in the sense of Lyapunov.

The stability of the HHNN can be demonstrated by either of two approaches: (1) use the Cohen-Grossberg theory [19] that is already based on Lyapunov stability theory, or (2) a direct approach in which we must establish an energy function, show that it is a Lyapunov function, and then demonstrate the truth of (5) for it. Here we take the former approach.

3.1 Cohen-Grossberg Theory and Its Relation to Hopfield Neural Network Theory

We now review the Hopfield neural network, and the Cohen-Grossberg (C-G) theory pertaining to the stability of a general class of nonlinear dynamical systems. We also describe the existence of a well known relationship between the C-G theory and Hopfield neural network theory.

The dynamics for the Hopfield neural network can be stated, as

$$C_j \frac{dx_j(t)}{dt} = -\frac{x_j(t)}{R_j} + \sum_{i=1}^{N} w_{ji} y_i(t) + I_j \quad j = 1, 2, ..., N, \tag{6}$$

where $w_{j1}, w_{j2}, ..., w_{jN}$ are the weights representing *conductances* between the N neurons (including a self-feedback conductance) and neuron j. The input to a neuron is denoted by $\{x_j, j = 1, 2, ..., N\}$, and the output from a neuron is denoted by $\{y_j, j = 1, 2, ..., N\}$, and both have their units represented in Volts. The output y_i and input x_i satisfy the following nonlinear relation,

$$y_i = \phi_i(x_i) = \tanh(\gamma_i x_i) \tag{7}$$

The energy function associated with (6) can be expressed as [21, 22]

$$E(y_1, y_2, ..., y_N) = -\frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} w_{ji} y_i y_j + \sum_{j=1}^{N} \frac{1}{R_j} \int_0^{y_j} \phi_j^{-1}(\nu)\, d\nu - \sum_{j=1}^{N} I_j y_j \tag{8}$$

Hopfield and Tank have shown, that: 1) $E(y_1, y_2, ..., y_N)$ is a Lyapunov function, and 2) The model in (6) and (7) is stable according to (5). We shall now state the C-G theory, and then demonstrate that this theory encompasses the Hopfield neural network theory [23].

In 1983, Cohen and Grossberg [19] presented a general principle for assessing the stability of a certain class of neural networks that are described by the following system of coupled nonlinear differential equations,

$$\frac{du_j}{dt} = a_j(u_j)[b_j(u_j) - \sum_{i=1}^{N} c_{ji}\psi_i(u_i)], \quad j = 1, 2, ..., N \tag{9}$$

According to Cohen and Grossberg, this class of neural networks admits a Lyapunov function, defined as

$$E(u_1, u_2, ..., u_N) = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} c_{ji}\psi_i(u_i)\psi_j(u_j) - \sum_{j=1}^{N}\int_0^{u_j} b_j(\xi_j)\psi_j'(\xi_j)\, d\xi_j \tag{10}$$

where,

$$\psi_j'(\xi_j) = \frac{d[\psi_j(\xi_j)]}{d\xi_j} \tag{11}$$

For $E(u_1, u_2, ..., u_N)$ to be a valid Lyapunov function, the following *conditions* must hold:

1. The synaptic weights of the network should be *symmetric*, i.e.,

$$c_{ij} = c_{ji}; \tag{12}$$

2. The function $a_j(u_j)$ must satisfy the *nonnegativity* condition

$$a_j(u_j) \geq 0; \tag{13}$$

and,

3. The nonlinear input-output function (activation function), $\psi_j(u_j)$, must be monotonic [a function $f : X \rightarrow Y$ is *monotone* if $x \leq y \Rightarrow f(x) \leq f(y)$].

Under these conditions, the Cohen-Grossberg theorem states that: *For the system of nonlinear differential equations (9) that satisfy the conditions of symmetry, non-negativity, and monotonicity, the Lyapunov function E, defined by (10), satisfies the condition*

$$\frac{dE}{dt} \leq 0 \tag{14}$$

To demonstrate that the C-G model includes the Hopfield-Tank model as a special case, we form a correspondence table (Table 1) between the quantities in (6) and (9). So that we may demonstrate that the Hopfield-Tank energy function (8) can be obtained from the C-G Lyapunov function (10), the following mathematical facts will be required:

1.

$$y_j = \phi_j(x_j) \Rightarrow x_j = \phi_j^{-1}(y_j) \tag{15}$$

2.

$$\int_0^{x_j} \phi_j'(x_j)\,dx_j = \int_0^{y_j} dy_j = y_j \tag{16}$$

3.

$$\int_0^{x_j} x_j\phi_j'(x_j)\,dx_j = \int_0^{y_j} \phi_j^{-1}(y_j)\,dy_j \tag{17}$$

The proofs of (16) and (17) are given in Appendix A.

Substituting the quantities from Table 1 into (10), we derive the Hopfield-Tank energy function as follows:

$$E(x_1, x_2, ..., x_N) = -\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} w_{ji}y_iy_j - \sum_{j=1}^{N}\int_0^{x_j}(-\frac{x_j}{R_j} + I_j)\phi_j'(x_j)\,dx_j \tag{18}$$

$$E(x_1, x_2, ..., x_N) = -\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} w_{ji}y_iy_j + \sum_{j=1}^{N}\frac{1}{R_j}\int_0^{x_j} x_j\phi_j'(x_j)\,dx_j - \sum_{j=1}^{N}\int_0^{x_j} I_j\phi_j'(x_j)\,dx_j \tag{19}$$

$$E(y_1, y_2, ..., y_N) = -\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} w_{ji}y_iy_j + \sum_{j=1}^{N}\frac{1}{R_j}\int_0^{y_j} \phi_j^{-1}(y_j)\,dy_j - \sum_{j=1}^{N} I_jy_j \tag{20}$$

where we used (16) and (17) (involving transformation of variables from x_j to y_j) to obtain (20) from (19). Equation(20) can be rewritten, as

$$E(y_1, y_2, ..., y_N) = -\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} w_{ji}y_iy_j + \sum_{j=1}^{N}\frac{1}{R_j}\int_0^{y_j}\phi_j^{-1}(\nu)\,d\nu - \sum_{j=1}^{N} I_jy_j \tag{21}$$

Comparing (8) and (21), we see that the Hopfield-Tank energy function has indeed been obtained from C-G theory.

3.2 Lyapunov Stability of the HHNN Model

We now demonstrate Lyapunov stability of the HHNN by creating the correspondence table between the C-G model (9) and the HHNN model (4), forming the energy function for the HHNN using the correspondence table and (10), and, invoking the C-G Theorem (14).

Comparing the C-G dynamical model (9), and the HHNN model (4), we form Table 2.

In the same manner as we derived (15), (16), and (17), we can obtain the following:

1.

$$y_j = y_j(x_j|\dot{x_j}) = \phi_j[x_j - \lambda_j(\dot{x_j})] \Rightarrow [x_j - \lambda_j(\dot{x_j})] = \phi_j^{-1}(y_j) \tag{22}$$

2.

$$\int_0^{x_j}\phi_j'[x_j - \lambda_j(\dot{x_j})]\,dx_j = \int_0^{y_j} dy_j = y_j \tag{23}$$

3.

$$\int_0^{x_j} x_j\phi_j'[x_j - \lambda_j(\dot{x_j})]\,dx_j = \int_0^{y_j}\phi_j^{-1}(y_j)\,dy_j + \lambda_j(\dot{x_j})y_j \tag{24}$$

The proof of (24) is given in Appendix A.

On using the energy function (10) from the C-G theory and the results in Table 2, we find

$$E(x_1, x_2, ..., x_N) = -\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} w_{ji}y_iy_j - \sum_{j=1}^{N}\int_0^{x_j}(-\frac{x_j}{R_j} + I_j + \frac{\lambda_j(\dot{x_j})}{R_j})\phi_j'[x_j - \lambda_j(\dot{x_j})]\,dx_j \tag{25}$$

$$E(x_1, x_2, ..., x_N) = -\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} w_{ji}y_i y_j + \sum_{j=1}^{N}\int_0^{x_j} \frac{x_j}{R_j}\phi_j'[x_j - \lambda_j(\dot{x_j})]\,dx_j$$

$$-\sum_{j=1}^{N}\int_0^{x_j} I_j\phi_j'[x_j - \lambda_j(\dot{x_j})]\,dx_j - \sum_{j=1}^{N}\int_0^{x_j} \frac{\lambda_j(\dot{x_j})}{R_j}\phi_j'[x_j - \lambda_j(\dot{x_j})]\,dx_j \qquad (26)$$

Substituting (23) and (24) (involving transformation of variables) into (26), and making use of (22), we find

$$E(y_1, y_2, ..., y_N) = -\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} w_{ji}y_i y_j + \sum_{j=1}^{N}\frac{1}{R_j}\int_0^{y_j} \phi_j^{-1}[y_j(x_j|\dot{x_j})]\,dy_j - \sum_{j=1}^{N} I_j y_j \qquad (27)$$

Observe that (8) and (27) are structurally the same, the difference being in the complicated nature of the hysteretic activation function $\phi_j^{-1}[y_j(x_j|\dot{x_j})]$ in (27). Also, the difference in the representation of the Hopfield neuron model in (6) and the HHNN model (4) is the presence of a "hysteresis generating" term $\lambda_j(\dot{x})/R_j$ and a switching mechanism. Consequently the HHNN is stable in the sense of Lyapunov via the C-G theory.

4 Conclusions

In this report we have addressed theoretical issues pertaining to the HHNN (Hysteretic Hopfield Neural Network). In essence, we have demonstrated Lyapunov stability for the HHNN using the established C-G theory. To achieve this, we have compared the general dynamical equation (9) from C-G theory, to the HHNN equation given by (4), so as to establish a link between the individual parameters in the two equations. We then substituted these quantities in the formula for the Lyapunov function (10), to obtain the Lyapunov function for the HHNN. Finally, we invoke the C-G theorem (14) to prove stability of the HHNN.

References

[1] Merriam-Webster. *Webster's Ninth New Collegiate Dictionary.* Merriam-Webster Inc., Springfield, Massachusetts, 1990.

[2] R. L. Didday. "A model of visuomotor mechanisms in the frog optic tectum". *Mathematical Bioscience*, 30:169–180, 1976.

[3] J. P. Segundo and O. D. Martinez. "Dynamic and static hysteresis in crayfish receptors". *Biological Cybernetics*, 52:291–296, 1985.

[4] E. Napieralska, J. R. Grzybowski, and J. F. Brudny. "Modelling of losses due to eddy currents and histeresis in converter transformer cores during failure". *IEEE Transactions on Magnetics*, 31:1718–1721, 1995.

[5] S. V. Marshall, R. E. DuBroff, and G. G. Skitek. *Electromagnetic Concepts and Applications.* Prentice Hall, Upper Saddle River. NJ., 1996.

[6] A. M. Andronikou, G. A. Bekey, and F. Y. Hadaegh. "Identifiability of nonlinear systems with hysteretic elements". *Journal of Dynamic Systems, Measurement, and Control*, 105:209–214, 1983.

[7] T. Balendra. *Vibration of Buildings to Wind and Earthquake Loads.* Springer-Verlag, New York, 1993.

[8] G. Taga, Y. Yamaguchi, and H. Shimizu. "Self-organized control of bipedal locomotion by neural oscillators in unpredictable environment". *Biological Cybernetics*, 65:147–159, 1991.

[9] R. M. Alexander. "Optimization and gaits in the locomotion of vertebrates". *Phys. Rev.*, 69:1199–1227, 1989.

[10] G. W. Hoffman and M. W. Benson. "Neurons with hysteresis form a network that can learn without any changes in synaptic strengths". In *Proceedings of the American Institute of Physics Conference on Neural Networks and Computing (Ed. J. S. Denker)*, 1986.

[11] L. O. Chua and S. C. Bass. "A Generalized Hysteresis Model". *IEEE Transactions on Circuit Theory*, CT-19(1):36–48, 1972.

[12] V. Volterra. *Theory of Functionals and of Integral and Integro-Differential Equations.* Dover, New York, 1959.

[13] J. R. Whiteman. "A mathematical model depicting the stress-strain diagram and the hysteresis loop". *Transactions of the ASME*, 26, 1959.

[14] G. Biorci and A. Ferro. "Hysteresis losses along open transformations". *J. Phys. Radium (Paris)*, 20:237–240, 1959.

[15] H. Yanai and Y. Sawada. "Associative memory network composed of neurons with hysteretic property". *Neural Networks*, 3:223–228, 1990.

[16] J. D. Keeler, E. E. Pichler, and J. Ross. "Noise in neural networks: thresholds, hysteresis and neuromodulation of signal-noise". *Proceedings of the National Academy of Sciences (U.S.A)*, 86:1712–1716, 1989.

[17] Y. Takefuji and K. C. Lee. "An hysteresis binary neuron: a model suppressing the oscillatory behaviour of neural dynamics". *Biological Cybernetics*, 64:353–356, 1991.

[18] J. J. Hopfield. "Neurons, dynamics, and computations". *Physics Today*, 47(2):40–46, Feb. 1994.

[19] M. A. Cohen and S. Grossberg. "Absolute Stability of Global Pattern Formation and Parallel Memory Storage by Competitive Neural Networks". *IEEE Transactions on Systems, Man, and Cybernetics*, SMC–13(5):815–825, 1983.

[20] W. L. Brogan. *Modern Control Theory.* Quantum Publishers Inc., New York, 1974.

[21] J. J. Hopfield. "Neural Networks and Physical Systems with emergent collective computational abilities". *Proceedings of the National Academy of Sciences (U.S.A)*, 79:2554–2558, 1982.

[22] J. J. Hopfield and D. Tank. "Computing with neural circuits: A model". *Science*, 233:625–633, 1986.

[23] S. Grossberg G. Carpenter, M. A. Cohen. "Technical Comments: Computing with Neural Circuits". *Science*, 235:1226–1227, March 6, 1987.

Appendix A: Proofs

A.1: Equation (16)

$$\int_0^{x_j} \phi_j'(x_j)\,dx_j = \int_0^{y_j} dy_j = y_j$$

Proof:

$$\frac{dy_j}{dx_j} = \frac{d\phi_j(x_j)}{dx_j} = \phi_j'(x_j)$$
$$dy_j = \phi_j'(x_j)dx_j$$
$$\int_0^{x_j} \phi_j'(x_j)\,dx_j = \int_0^{y_j} dy_j = y_j$$

A.2: Equation (17)

$$\int_0^{x_j} x_j\phi_j'(x_j)\,dx_j = \int_0^{y_j} \phi_j^{-1}(y_j)\,dy_j$$

Proof:

$$x_j\frac{dy_j}{dx_j} = x_j\phi_j'(x_j)$$
$$x_j dy_j = x_j\phi_j'(x_j)dx_j$$
$$\int_0^{x_j} x_j\phi_j'(x_j)\,dx_j = \int_0^{y_j} x_j\,dy_j = \int_0^{y_j} \phi_j^{-1}(y_j)\,dy_j$$

where we have used (15) to obtain the last equality.

A.3: Equation (24)

$$\int_0^{x_j} x_j\phi_j'[x_j - \lambda_j(\dot{x}_j)]\,dx_j = \int_0^{y_j} \phi_j^{-1}(y_j)\,dy_j + \lambda_j(\dot{x}_j)y_j$$

Proof:

$$x_j\frac{dy_j}{dx_j} = x_j\frac{d\phi_j[x_j - \lambda_j(\dot{x}_j)]}{dx_j} = x_j\phi_j'[x_j - \lambda_j(\dot{x}_j)]$$

$$x_j dy_j = x_j \phi_j'[x_j - \lambda_j(\dot{x}_j)]dx_j$$

$$\int_0^{x_j} x_j \phi_j'[x_j - \lambda(\dot{x}_j)]dx_j = \int_0^{y_j} x_j dy_j = \int_0^{y_j} [\phi_j^{-1}(y_j) + \lambda_j(\dot{x}_j)]dy_j = \int_0^{y_j} \phi_j^{-1}(y_j)dy_j + \lambda_j(\dot{x}_j)y_j$$

The last relation was obtained using (22).

Table 1: Correspondences between C-G and Hopfield-Tank model

Cohen-Grossberg model	Hopfield-Tank model
u_j	x_j
$a_j(u_j)$	C_j^{-1}
$b_j(u_j)$	$-x_j/R_j + I_j$
c_{ji}	$-w_{ji}$
$\psi_j(u_j)$	$y_j = \phi_j(x_j)$

Table 2: Correspondences between C-G and HHNN models

Cohen-Grossberg model	HHNN model
u_j	x_j
$a_j(u_j)$	C_j^{-1}
$b_j(u_j)$	$-x_j/R_j + I_j + \lambda_j(\dot{x_j})/R_j$
c_{ji}	$-w_{ji}$
$\psi_j(u_j)$	$y_j = y_j(x_j\|\dot{x_j}) = \phi_j[x_j - \lambda_j(\dot{x_j})]$

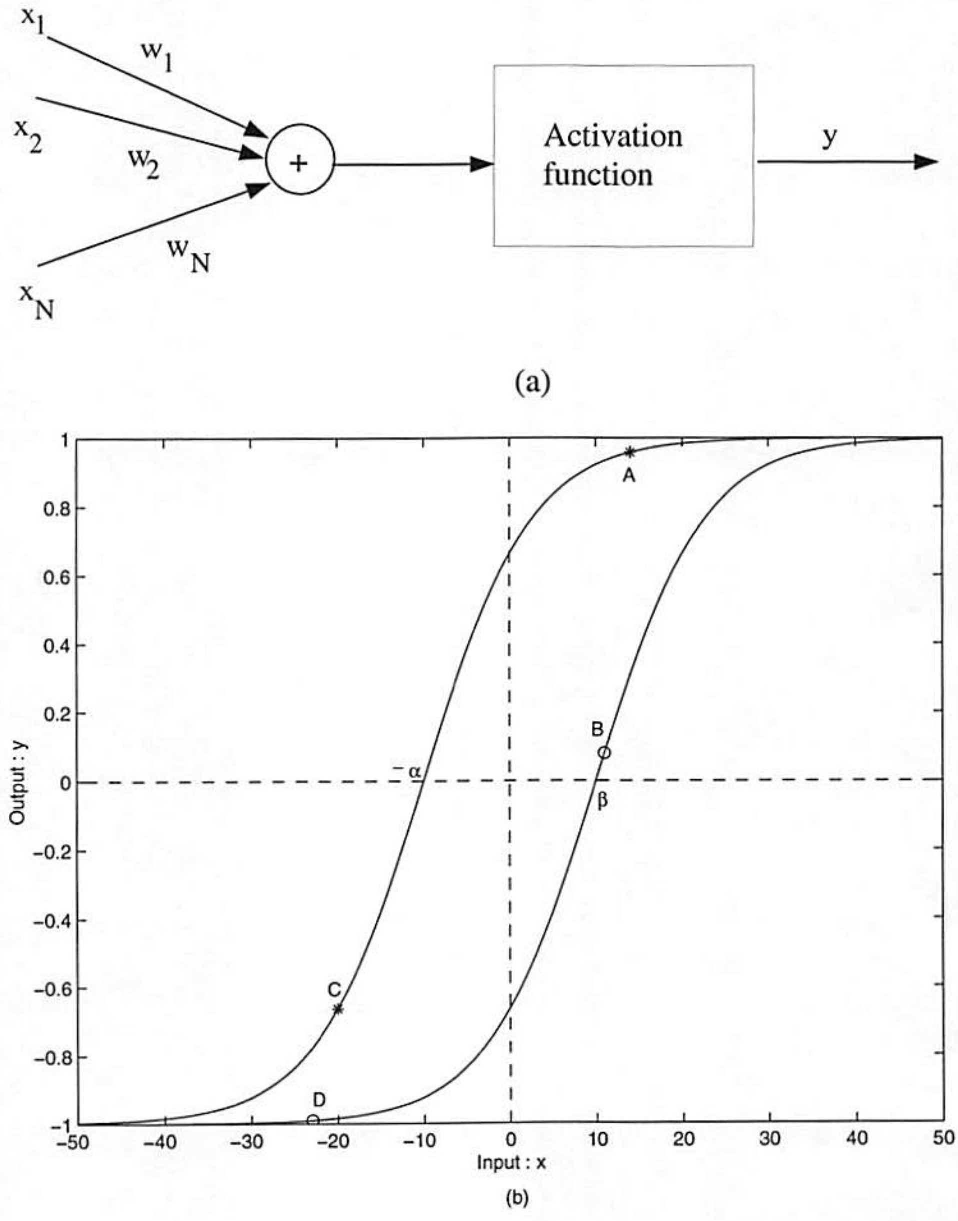

Figure 1: (a) A neuron model. (b) Hysteresis neuron activation function.

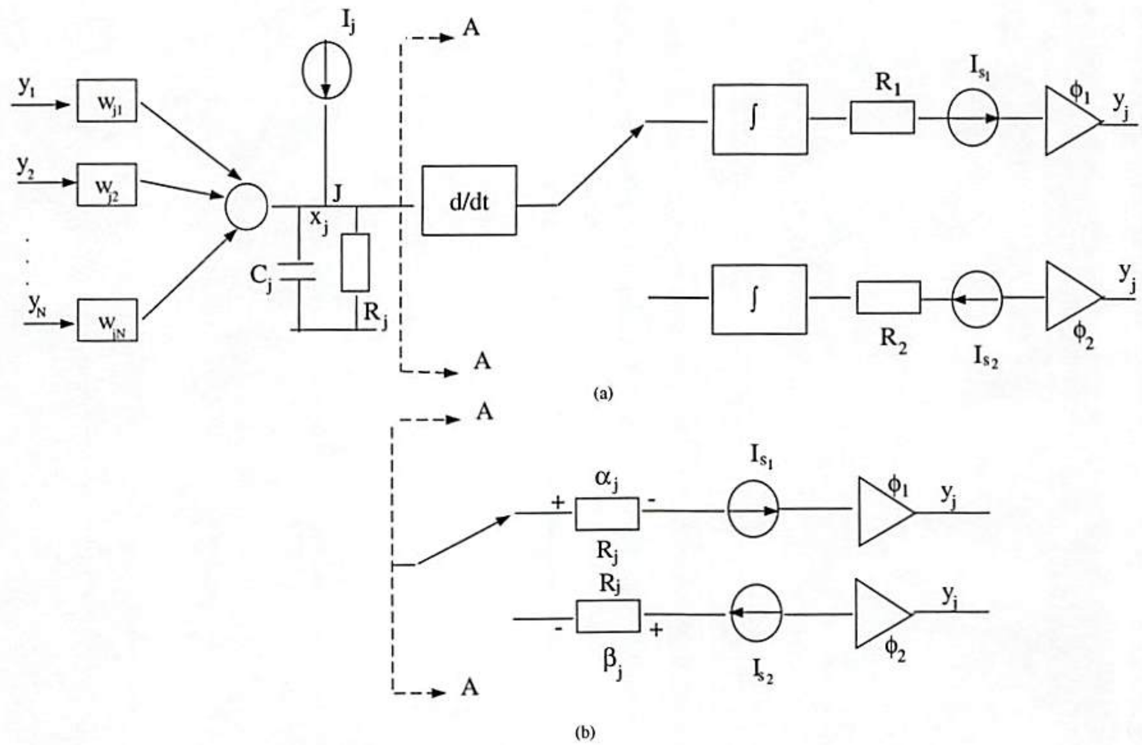

Figure 2: (a) Hysteresis neuron circuit; (b) Differentiator-Integrator pair in series, and modeled as net resistance.