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Reduced-Rank Adaptive Filtering

by

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Abstract—**A novel rank reduction scheme is introduced for adaptive filtering problems. This rank reduction method uses a cross-spectral metric to select the optimal lower dimensional subspace for reduced-rank adaptive filtering as a function of the basis vectors of the full-rank space.**

I. Introduction

This correspondence is concerned with rank reduction in adaptive signal processing. The goal of reduced-rank adaptive filtering is to find a lower dimensional filter that yields a steady-state performance that is as close as possible to that obtained by the full-rank solution. The motivation for rank reduction can be attributed to many factors. First, it is very common for the problem under consideration to be overmodeled. In this case, the rank may be reduced to the dimension of the signal subspace to suppress the noise. Second, it could be required that the adaptive filter be of a particular order, perhaps lower than the dimension of the signal subspace, due to complexity constraints or other real-time implementation requirements. For this compression problem, it is desired that the steady-state performance of the reduced-rank filter be as close as possible to the full-rank optimal solution for each value of the filter rank. Clearly, the solution to the compression problem satisfies the overmodeling problem when the rank of the filter equals the dimension of the signal subspace. Finally, the popular least squares (LS) class of algorithms converge as a function of the filter order, implying that lower-rank filters converge faster.

Previous work in reduced-rank adaptive filtering has been concerned primarily with the overmodeling problem [1]–[8]. For notational purposes, the full-rank problem is defined to be of dimension N. In addition, let D denote the dimension of the signal subspace. With this notation, the previous work on rank reduction consisted of an estimation of the covariance matrix of the observed data and then a determination of its singular value decomposition (SVD). Those eigenvectors corresponding to the largest D singular values are then retained to form the rank D eigensubspace in which the reduced-rank filter will operate. This method is very effective if the proper dimension D is known exactly. In the event that this dimension is not known, then one must either estimate it or choose a rank large enough to ensure that at least D eigenvectors are retained. If fewer eigenvectors are retained, the performance will suffer greatly.

In this correspondence, a metric is found that relates directly to the data space and provides a measure of the cross-spectral energy projected along each basis vector. Those M bases for which this energy contribution is greatest are retained. It is demonstrated that this cross-spectral metric obtains the best low-rank filter as a function of the basis used. In addition, for the overmodeling problem, this metric provides a more robust criterion than the largest eigenvalue criteria for $M < D$. This counterintuitive result yields a steady-state solution that is the upper bound on the performance of an adaptive filter that operates in the rank M eigensubspace for all $M \leq N$.

II. The Full-Rank LS Problem

Let $\mathbf{X}$ denote an $L \times N$ data observation matrix, and let $\mathbf{d}$ be some desired data vector of dimension L. The goal of the LS problem is to find the best approximation of $\mathbf{d}$ that is solely a weighted linear combination of the N column vectors that compose $\mathbf{X}$. The error ϵ to be minimized is given by

$$\epsilon = \mathbf{d} - \mathbf{X}\mathbf{w} \tag{1}$$

where $\mathbf{w}$ is the N-dimensional weight vector to be determined.

The LS method estimates the $N \times N$ covariance matrix $\mathbf{R}_x = \mathbf{X}^H\mathbf{X}$ and the $N \times 1$ cross-correlation vector between the observed data and the desired signal vector $\mathbf{r}_{xd} = \mathbf{X}^H\mathbf{d}$.The standard LS solution for $\mathbf{w}$ is then provided by $\mathbf{w}_{LS}$, which is computed as

$$\mathbf{w}_{LS} = \mathbf{R}_x^{-1}\mathbf{r}_{xd}. \tag{2}$$

This solution yields $\epsilon_{LS} = \mathbf{d} - \mathbf{X}\mathbf{w}_{LS}$ as the error vector with minimum Euclidean norm. The error ϵ_{LS} is orthogonal to the column space of $\mathbf{X}$.

III. The Reduced-Rank LS Problem

The SVD of the data matrix $\mathbf{X}$ is obtained next as follows:

$$\mathbf{X} = \mathbf{U}\mathbf{S}\mathbf{V}^H \tag{3}$$

where

- $\mathbf{U}$ $L \times N$ orthonormal matrix of left singular vectors
- $\mathbf{S}$ $N \times N$ diagonal matrix of singular values
- $\mathbf{V}$ $N \times N$ orthonormal matrix of right singular vectors of $\mathbf{X}$ (e.g., see [9]).

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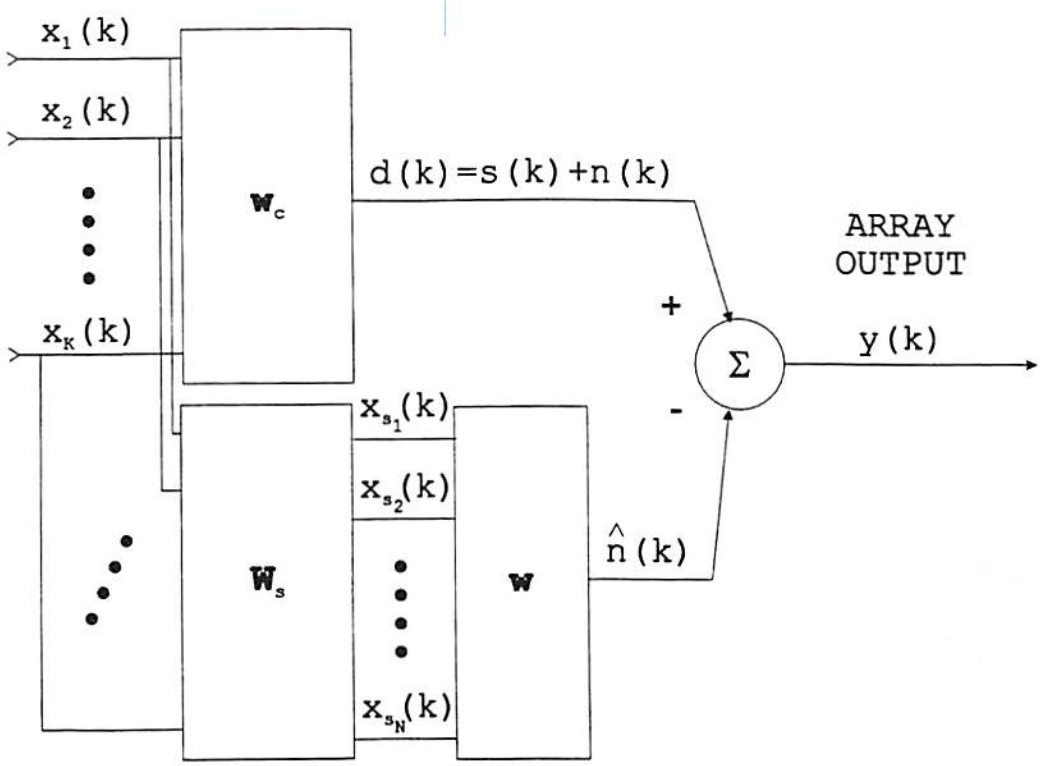

Fig. 1. GSC form sensor array processor

The matrix $\mathbf{V}$ contains the eigenvectors of $\mathbf{R}_x = \mathbf{V}\Lambda\mathbf{V}^H = \mathbf{V}\mathbf{S}^2\mathbf{V}^H$, where $\Lambda = \mathbf{S}^2$ is the corresponding matrix of eigenvalues whose diagonal elements are denoted $\{\lambda_k, 1 \le k \le N\}$. Note that the column vectors $\{\mathbf{u}_k, 1 \le k \le N\}$ of $\mathbf{U}$ form the orthonormal singular basis of $\mathbf{R}_x$. In the reduced-rank problem, we wish to select a number $M < N$ of these column vectors so that the mean-square error, which is given by the Euclidean norm of ϵ, is minimized. This minimization is equivalent to maximizing the norm of the approximation $\mathbf{y} = \mathbf{X}\mathbf{w}_{LS}$. Denote the Euclidean norm of $\mathbf{y}$ by $E_y = \mathbf{y}^H\mathbf{y}$. Then, it is easily verified that the contributions to E_y in the N dimensions of the data space can be quantified as follows:

$$E_y = \mathbf{w}_{LS}^H \mathbf{R}_x \mathbf{w}_{LS} = \mathbf{r}_{xd}^H \mathbf{R}_x^{-1} \mathbf{r}_{xd} = \mathbf{r}_{xd}^H \mathbf{V}\Lambda^{-1}\mathbf{V}^H \mathbf{r}_{xd}. \quad (4)$$

Next, introduce the transformed cross-correlation vector

$$\rho = \mathbf{V}^H \mathbf{r}_{xd} \quad (5)$$

with elements $\{\rho_k, 1 \le k \le N\}$ so that one obtains

$$E_y = \sum_{k=1}^{N} \frac{\rho_k^2}{\lambda_k} = \sum_{k=1}^{N} \theta_k. \quad (6)$$

Clearly, an optimal low-rank basis should consist of those eigenvectors that correspond to large θ_k. If the norm in (4) is considered to be the total energy of the approximation $\mathbf{y}$, then θ_k in (6) is a measure of the energy projected along the kth basis vector of the space spanned by the columns of $\mathbf{R}_x$. Thus, it is natural to call θ_k the cross-spectral energy or metric. The rank-M subspace selected by choosing the set of M basis vectors that correspond to the M largest values of θ_k is termed the cross-spectral subspace. To verify that a minimization of the error term in (1) is equivalent to a maximization of the Euclidean norm E_y in (4), note that

$$\epsilon_{LS}^H \epsilon_{LS} = \sigma_d^2 - E_y = \sigma_d^2 - \sum_{k=1}^{N} \theta_k \quad (7)$$

where $\sigma_d^2 = \mathbf{d}^H\mathbf{d}$. Thus, the choice of those eigenvectors that correspond to the maximum values of θ_k minimize the mean-squared error as a function of rank. It is noted that the values θ_k correspond to the signal subspace eigenvalues for $1 \le k \le D$ so that the weight solution is unique and the cross-spectral subspace is stable [1].

TABLE I
SIGNAL GEOMETRY

SIGNAL	DOA	SNR_{in}
desired	0°	0 dB
jammer 1	−61°	40 dB
jammer 2	−30°	44 dB
jammer 3	−10°	34 dB
jammer 4	10°	38 dB
jammer 5	22°	40 dB

IV. GEOMETRICAL INTERPRETATION

The Euclidean norm E_y in (4) may be alternatively expressed in terms of the SVD in (3), i.e.,

$$E_y = \mathbf{d}^H \mathbf{U}\mathbf{U}^H \mathbf{d}. \quad (8)$$

Now, introduce the normalized data vector $\overline{\mathbf{d}} = (\mathbf{d}^H\mathbf{d})^{-1/2}\mathbf{d}$, and rewrite (8) as follows:

$$E_y = \sigma_d^2 \overline{\mathbf{d}}^H \mathbf{U}\mathbf{U}^H \overline{\mathbf{d}}. \quad (9)$$

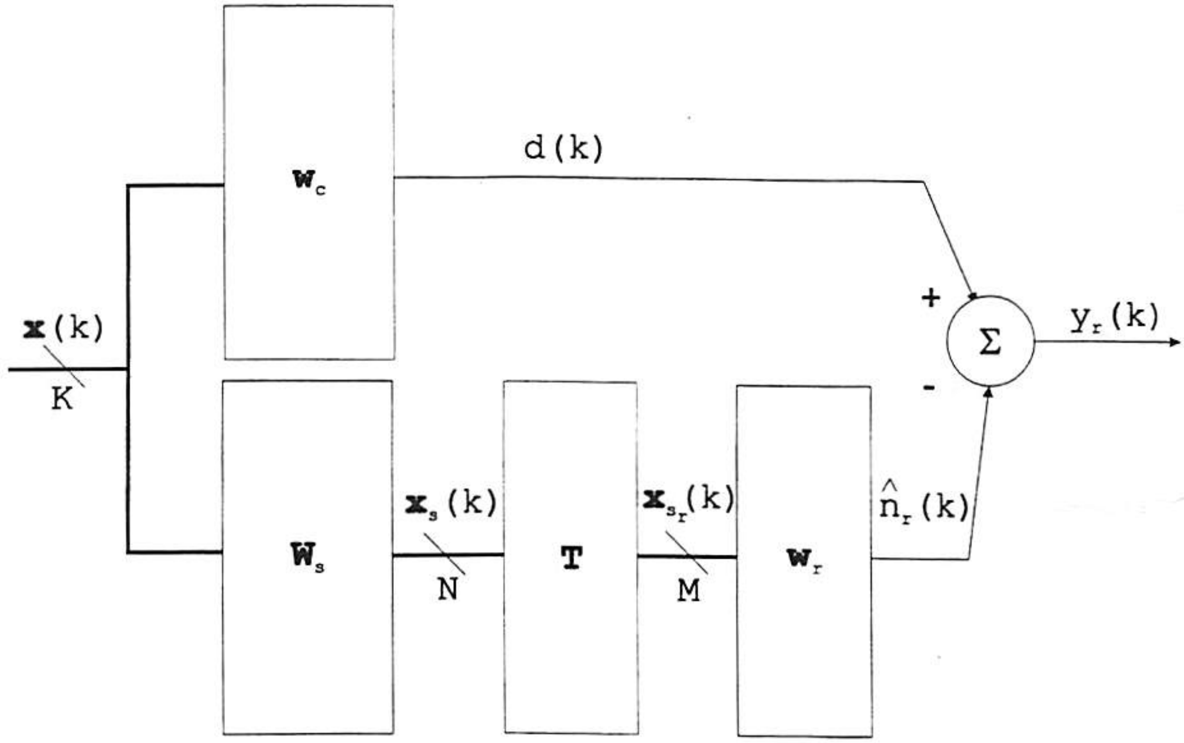

Fig. 2. Reduced-rank GSC form sensor array processor.

Note that $\cos \vartheta = \overline{\mathbf{d}}^H \mathbf{u}_k$ is simply the cosine of the angle between the data vector $\mathbf{d}$ and the basis $\mathbf{u}_k$. Thus, the norm of $\mathbf{y}$ is given by

$$E_y = \sigma_d^2 \sum_{k=1}^{N} (\cos \vartheta_k)^2 \tag{10}$$

which expresses the fact that one wants to select only those basis vectors for the low-rank approximation that are closest to the data in terms of the angles between the basis vectors and the data vector.

V. Example

An example is considered now of both overmodeling and compression using the application of minimum variance distortionless response (MVDR) beamforming in a narrowband signal environment. Extensions to the wideband case can be accomplished by an application of the results presented herein as well. In this example, it is assumed that there is only one target signal present and that the array is signal-aligned in the desired-signal look direction.

Such an MVDR sensor array may be realized in a partitioned form, which is termed a generalized sidelobe canceler (GSC) [10], [11], as depicted in Fig. 1 for an array composed of K sensor elements. The conventional beam forming matrix $\mathbf{W}_c = (1/K)\mathbf{1}$ enforces the look-direction constraint. The look-direction signal is blocked from the adaptive processor by means of a signal blocking matrix $\mathbf{W}_s$, which, in general, is of dimension $N \times K$, where $N < K$. For the single linear constraint case of interest, one takes $N = K - 1$. Thus, the full row-rank matrix $\mathbf{W}_s$ is composed of N rows $\mathbf{a}_i$ such that $\mathbf{a}_i\mathbf{1} = 0$ for $i = 1, 2, \cdots, N$.

Next, the K-dimensional signal vector received by the antenna elements at time k is denoted by $\mathbf{x}(k)$, and the conventional beam former signal is $d(k) = \mathbf{W}_c^H \mathbf{x}(k)$. The noise subspace data vector of dimension N is $\mathbf{x}_s(k) = \mathbf{W}_s \mathbf{x}(k)$. The full-rank adaptive weight vector $\mathbf{w}$ is also of dimension N, and the array output is given by $y(k) = (\mathbf{W}_c^H - \mathbf{w}^H \mathbf{W}_s)\mathbf{x}(k)$, which serves as the error signal for the GSC processor.

For the sensor array processing application, the cross-spectral metric takes a very intuitive form. Any conventional beamformer attenuates the signals coming from other than the look direction. This tapering of the quiescent response has an impact on the performance of the GSC. The upper branch of the GSC results in an output $d(k)$, which is composed of the beamformed signal plus interference. The goal of the lower branch is to estimate the interference present in $d(k)$ at the output of the adaptive weighting network. The Wiener–Hopf solution is a vector $\mathbf{w}$ that designates to the location of minimum mean-square error (MMSE) in the noise subspace. A strong interference source that is attenuated by the quiescent tapering clearly has a reduced effect in the determination of the subspace that contains the location of the MMSE. However, this interference also passes through the signal blocking matrix and results in a large eigenvector in the noise subspace. If this interference signal is completely attenuated by $\mathbf{w}_c$, then this large eigenvector is orthogonal to the cross-correlation vector. A weaker interference source may be less attenuated by the conventional beam former tapering, and as a consequence, the eigensubspace technique disregards the corresponding eigenvector in the noise subspace. The exclusion of the basis vector corresponding to the weaker interference source will yield a greater loss in the MMSE than the gain that is realized through the inclusion of the larger eigenvector. That is, the subspace selected by the inclusion of the strongest eigenvector is further away from the location that provides the optimal MMSE unless all noise subspace eigenvectors are retained.

Consider a linear array with 16 sensors spaced at a half wavelength. The signal environment is composed of one desired signal and the five jammers described in Table I. The full-rank Wiener filter is of dimension 15×1, and we reduce the rank to 2×1 in order to compare the performance of the eigensubspace and cross-spectral

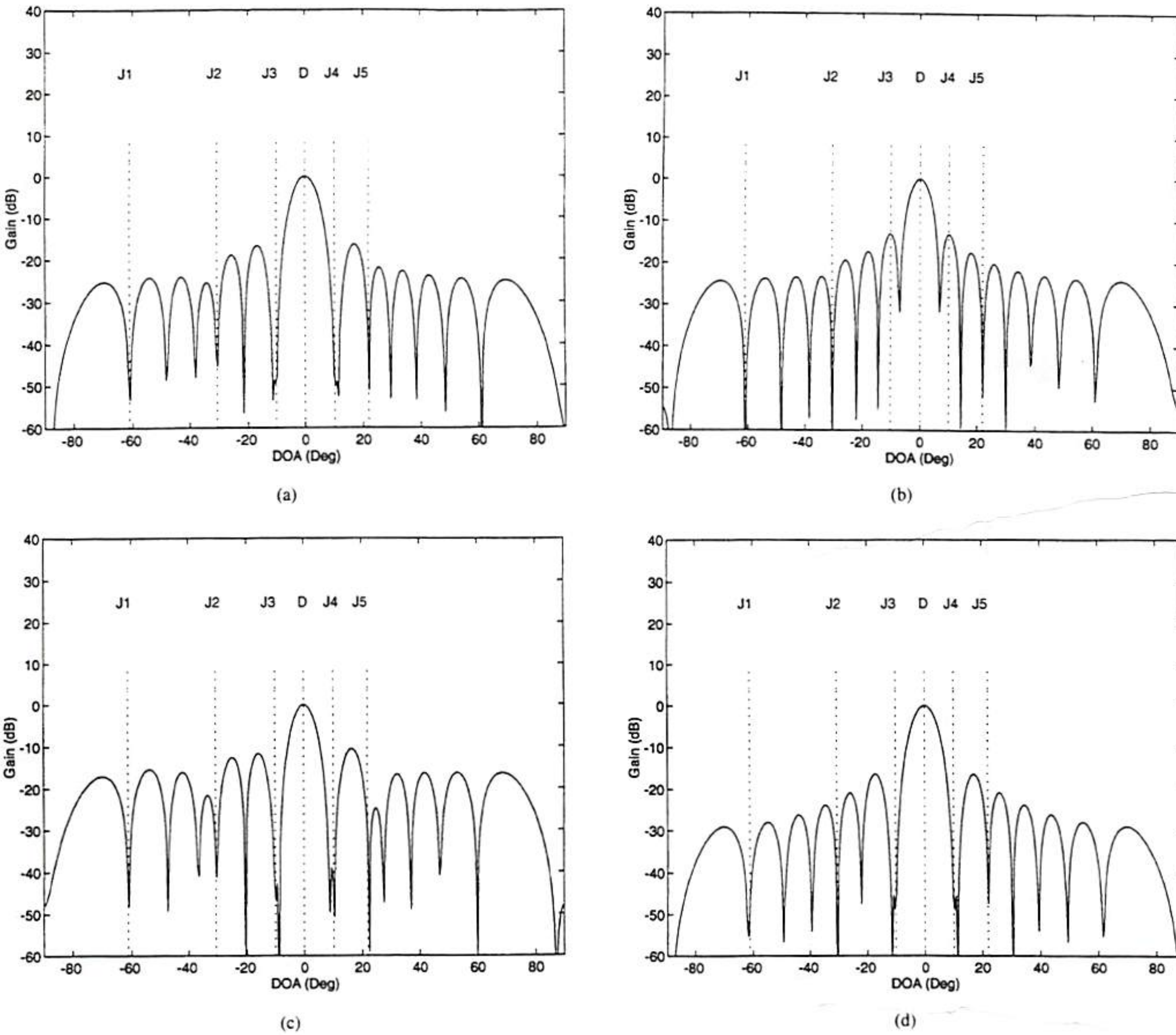

Fig. 3. Array power gain as a function of direction of arrival are shown for four 16–sensor narrowband GSC realizations using the optimal Wiener-Hopf weight vector for each. (a) Fully adaptive 15–weight GSC. (b) Partially adaptive 2-weight eigenspace GSC. (c) Partially adaptive 2-weight cross-spectral subspace GSC using the eigenvector basis. (d) Partially adaptive 2-weight cross-spectral subspace GSC using the DCT basis.

subspace processors using both the eigenvectors and the columns of the DCT matrix as basis vectors for the noise subspace. In the case of the cross-spectral metric approximation using the DCT basis, the ith eigenvalue is estimated by the power in the ith DCT frequency bin. The reduced-rank GSC is depicted in Fig. 2, where the matrix $\mathbf{T}$ is composed of two 15×1 basis vectors to be selected.

Fig. 3(a) presents the full-rank Wiener solution for this signal environment, where the rank of the Wiener filter is $N = 15$. The full-rank MMSE is -11.1 dB, and all jammers are attenuated.

We now reduce the rank of the Wiener filter to $M = 2$, which is less than the dimension of the noise subspace eigenstructure. Fig. 3(b) demonstrates that the rank 2 Wiener filter in the subspace formed by the eigenvectors corresponding to the largest two eigenvalues is not capable of attenuating two of the intereference signals J_3 and J_4. The MMSE in the eigensubspace is -2.8 dB, reflecting a loss of 8.3 dB. The rank 2 Wiener filter in the cross-spectral subspace with an eigenvector basis, which corresponds with the pattern provided in Fig. 3(c), successfully attenuates all jammers and obtains an MMSE of -10.6 dB. Thus, by simply choosing two different eigenvectors—those selected by the cross-spectral metric—the loss of performance in reducing the rank of the Wiener filter from 15 to 2 is only 0.5 dB. Fig. 3(d) presents the pattern corresponding with the rank 2 Wiener filter in the subspace selected by the cross-spectral metric relative to the DCT basis vectors. The jammers are all attenuated, and the resulting MMSE is -11.05 dB. Thus, for this example, the rank 2 DCT cross-spectral Wiener filter obtains an MMSE within 0.05 dB of the optimal full-rank Wiener filter and does not require an eigendecomposition.

To demonstrate that the performance of the Wiener solution in the cross-spectral subspace is an upper bound for that of the Wiener solution in the eigensubspace, Fig. 4 presents the MMSE as a function of the Wiener filter rank. From Fig. 4, it is clear that the GSC Wiener filter operating in the eigensubspace requires five coefficients to obtain performance close to that realized by the full-rank solution. This is expected since five is the number of narrowband interferers. Conversely, the Wiener filter in the cross-spectral subspace with an eigenvector basis roughly obtains the full-rank solution with about three to four coefficients. This same level of performance is reached

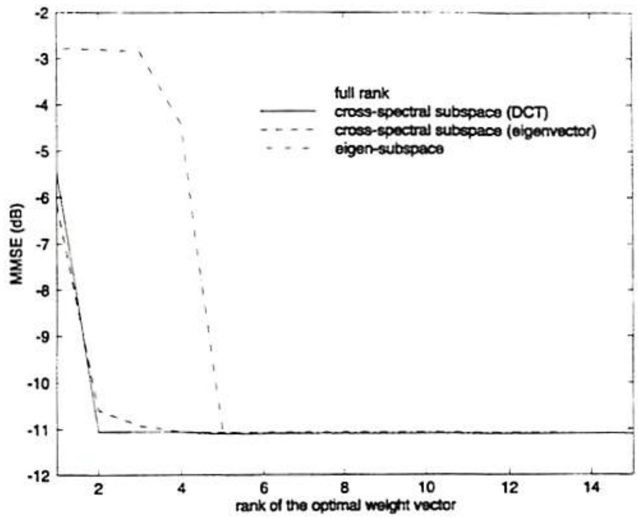

Fig. 4. MMSE as a function of rank for the GSC form sensor array processor.

by a two-coefficient filter in the cross-spectral subspace with a DCT basis. The MMSE performance for the cross-spectral GSC is acceptable at all lower dimensions, obtaining an MMSE loss greater than 0.51 dB only when the weight vector dimension is reduced to unity. The performance of the eigensubspace GSC degrades rapidly for all weight vectors of dimension less than 5.

VI. Conclusions

A cross-spectral metric is derived directly in the data space for reduced-rank adaptive filtering. This metric chooses an optimal lower dimensional subspace as a function of the basis representation for the full-rank space. It is also demonstrated that the steady-state solution in the subspace chosen by the cross-spectral metric provides an upper bound for that solution found in the space spanned by the eigenvectors that correspond with the largest eigenvalues of the full-rank correlation matrix.

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