USC–SIPI REPORT #345

Subspace Selection for Partially Adaptive Sensor Array Processing

by

J. Scott Goldstein and Irving S. Reed

September 1995

Signal and Image Processing Institute
UNIVERSITY OF SOUTHERN CALIFORNIA
Department of Electrical Engineering-Systems
3740 McClintock Avenue, Room 400
Los Angeles, CA 90089-2564 U.S.A.

Subspace Selection for Partially Adaptive Sensor Array Processing

J. SCOTT GOLDSTEIN, Senior Member, IEEE
USAF Rome Laboratory and University of Southern California

IRVING S. REED, Fellow, IEEE
University of Southern California

This paper introduces a cross-spectral metric for subspace selection and rank reduction in partially adaptive minimum variance array processing. The counter-intuitive result that it is suboptimal to perform rank reduction via the selection of the subspace formed by the principal eigenvectors of the array covariance matrix is demonstrated. A cross-spectral metric is shown to be the optimal criterion for reduced-rank Wiener filtering.

Manuscript received September 28, 1995; revised January 15, 1996 and May 31, 1996.

IEEE Log No. T-AES/33/2/1/03159.

This research was sponsored in part by ARPA and the USAF Rome Laboratory under contract F30602-95-1-0001 and the USAF Wright Laboratory under contract F33615-94-C-1536.

Authors' address: Dept. of Electrical Engineering, University of Southern California, Los Angeles, CA 90089-2565.

I. INTRODUCTION

The problem addressed in this paper is minimum variance adaptive sensor array processing subject to limitations on the dimension of the adaptive processor. Advances in technology have made it possible for undersea, space-segment and airborne platforms to support arrays composed of many elements for communications, sonar, and radar systems. However, the computational complexity requirements of such sensor arrays, coupled with the desire or requirement for space-time processing, may prohibit full adaptivity. A new technique for rank reduction based upon a cross-spectral performance index is introduced. It is shown that this method results in a lower minimum mean-square error (MMSE) than the principal components method of rank reduction. An example using the popular minimum variance, distortionless response (MVDR) adaptive array is provided which demonstrates that the cross-spectral metric outperforms the largest eigenvalue criterion and may provide excellent performance even when the rank of the processor is reduced beyond the dimension of the noise subspace eigenstructure.

II. MINIMUM VARIANCE SENSOR ARRAY PROCESSING

The minimum variance adaptive array [1] may be implemented in a partitioned form termed a generalized sidelobe canceller (GSC) [2, 3], as shown in Fig. 1. To simplify the discussion in this work, we assume that the propagating signals are narrowband,

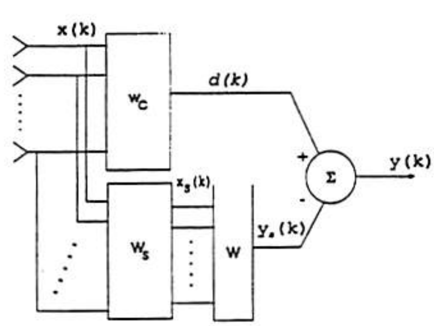

Fig. 1. Full-rank GSC-MVDR processor.

zero-mean wide-sense stationary random processes which impinge on a signal aligned, K-sensor, GSC array. The GSC is implemented by partitioning the received data with filters $\mathbf{w}_c$ and $\mathbf{W}_s$. The conventional beamforming filter $\mathbf{w}_c$ is a vector which enforces the look-direction constraint. For a single MVDR constraint, this $K \times 1$ beamforming filter takes the form $\mathbf{w}_c = (1/K)\mathbf{1}$, where $\mathbf{1}$ represents a vector whose elements are all unity. The desired signal is blocked from the adaptive processor through the signal blocking matrix $\mathbf{W}_s$, which in general is of dimension $N \times K$ where $N < K$. For example, in the single linear constraint case, $N = K - 1$. The full row rank matrix $\mathbf{W}_s$ is composed of rows $\mathbf{a}_i$ such that $\mathbf{a}_i\mathbf{1} = 0$ for $i = 1,2,\ldots K$.

The $K \times 1$-dimensional received signal vector present on the antenna elements at time k is denoted by $\mathbf{x}(k)$ and the associated $K \times K$ received input data covariance matrix is denoted by $\mathbf{R}_x = \mathbf{E}[\mathbf{x}(k)\mathbf{x}^H(k)]$. The N-dimensional noise subspace data vector $\mathbf{x}_s(k)$, the scalar beamformed output $d(k)$, and the scalar beamformed noise estimator $y_s(k)$ are given by

$$\begin{aligned} \mathbf{x}_s(k) &= \mathbf{W}_s\mathbf{x}(k) \\ d(k) &= \mathbf{w}_c^H\mathbf{x}(k) \\ y_s(k) &= \mathbf{w}^H\mathbf{x}_s(k) \end{aligned} \tag{1}$$

where $\mathbf{w}$ is the N-dimensional weight vector. Finally, the array output is

$$y(k) = (\mathbf{w}_c^H - \mathbf{w}^H\mathbf{W}_s)\mathbf{x}(k). \tag{2}$$

Evidently, the mean-square error of the processor is given by the mean-square value of $y(k)$.

The GSC array in Fig. 1 converges to the discrete-time Wiener–Hopf solution given by

$$\mathbf{w} = \mathbf{R}_{x_s}^{-1}\mathbf{r}_{x_s d} \tag{3}$$

where the observation data covariance matrix is expressed as

$$\mathbf{R}_{x_s} = \mathbf{E}[\mathbf{x}_s(k)\mathbf{x}_s^H(k)] = \mathbf{W}_s\mathbf{R}_x\mathbf{W}_s^H \tag{4}$$

and the cross-correlation vector between the noise subspace data vector and the beamformer output is given by

$$\mathbf{r}_{x_s d} = \mathbf{E}[\mathbf{x}_s(k)d^*(k)] = \mathbf{W}_s\mathbf{R}_x\mathbf{w}_c. \tag{5}$$

The MMSE, denoted by P, is found by substituting (3) into (2) and evaluating the mean-square value of $y(k)$:

$$P = \mathbf{E}[|y(k)|^2] = \sigma_d^2 - \mathbf{r}_{x_s d}^H\mathbf{R}_{x_s}^{-1}\mathbf{r}_{x_s d} \tag{6}$$

where $\sigma_d^2 = \mathbf{w}_c^H\mathbf{R}_{xx}\mathbf{w}_c$ is the variance of the conventional beamformer output.

The observation data covariance matrix $\mathbf{R}_{x_s}$ is expressed next in terms of its eigenvectors and eigenvalues by

$$\mathbf{R}_{x_s} = \mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^H \tag{7}$$

where $\mathbf{U}$ is a unitary $N \times N$ matrix composed of the eigenvectors $\{\nu_i\}_{i=1}^N$ and $\boldsymbol{\Lambda}$ is the diagonal matrix of associated eigenvalues $\{\lambda_i\}_{i=1}^N$. In terms of the principal coordinates of the problem, we define the process $\mathbf{p}(k) = \mathbf{U}^H\mathbf{x}_s(k)$. A normal component covariance matrix $\mathbf{R}_p$, cross-correlation vector $\mathbf{r}_{pd}$, and Wiener filter $\mathbf{w}_N$ are defined now as follows:

$$\begin{aligned} \mathbf{R}_p &= \mathbf{E}[\mathbf{p}(k)\mathbf{p}^H(k)] = \mathbf{U}^H\mathbf{R}_{x_s}\mathbf{U} = \boldsymbol{\Lambda} \\ \mathbf{r}_{pd} &= \mathbf{E}[\mathbf{p}(k)d^*(k)] = \mathbf{U}^H\mathbf{r}_{x_s d} \\ \mathbf{w}_N &= \mathbf{R}_p^{-1}\mathbf{r}_{pd} = \mathbf{U}^H\mathbf{w}. \end{aligned} \tag{8}$$

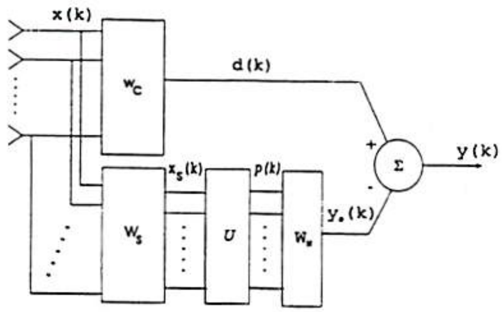

Fig. 2. Full-rank GSC-MVDR processor in principal coordinates.

The GSC in these normal coordinates, depicted in Fig. 2, is equivalent to the GSC in Fig. 1 in terms of its steady-state characteristics. The array output of the GSC in normal coordinates is given by

$$y = (\mathbf{w}_c^H - \mathbf{w}_N^H\mathbf{U}^H\mathbf{W}_s)\mathbf{x}(k) = (\mathbf{w}_c^H - \mathbf{w}^H\mathbf{W}_s)\mathbf{x}(k). \tag{9}$$

Note that the MMSE,

$$P = \sigma_d^2 - \mathbf{r}_{pd}^H\mathbf{R}_p^{-1}\mathbf{r}_{pd} = \sigma_d^2 - \mathbf{r}_{x_s d}^H\mathbf{R}_{x_s}^{-1}\mathbf{r}_{x_s d} \tag{10}$$

is conserved by any unitary transformation, including that realized by the operator $\mathbf{U}$.

III. PARTIALLY ADAPTIVE PROCESSING

The problem of reducing the degrees of freedom for an array processor involves selecting a subset or some combination of the elements to be adaptively weighted. For notational purposes, we let the space spanned by the columns of the fully adaptive array covariance matrix be denoted by $\mathcal{C}^N$, implying that the observation covariance matrices $\mathbf{R}_{x_s}$ and $\mathbf{R}_p$ are of dimension $N \times N$ and the vectors $\mathbf{r}_{x_s d}$, $\mathbf{r}_{pd}$, $\mathbf{w}$, and $\mathbf{w}_N$ are $N \times 1$-dimensional vectors. The partially adaptive GSC shown in Fig. 3 utilizes an $N \times M$ transformation operator $\mathcal{U}$, in place of $\mathbf{U}$ in Fig. 2, to form the M-dimensional reduced-rank observation data vector

$$\mathbf{z}(k) = \mathcal{U}^H\mathbf{x}_s(k) \tag{11}$$

where $M < N$. The associated $M \times M$ reduced-rank covariance matrix is given by

$$\mathbf{R}_z = \mathcal{U}^H\mathbf{R}_{x_s}\mathcal{U}. \tag{12}$$

The data vector $\mathbf{z}(k)$ is then processed by the reduced-rank weight vector $\mathbf{w}_M$, which is of dimension $M \times 1$. It is the selection of the rank reducing operator $\mathcal{U}$ which serves as the present topic of interest.

The most popular technique for subspace selection is based on the principal components method [4–6]. This method determines the singular value decomposition of the $N \times N$-dimensional covariance matrix $\mathbf{R}_{x_s}$ and selects the M largest eigenvectors (those corresponding to the largest eigenvalues) to form the M-dimensional eigen-subspace $\Psi \subset \mathcal{C}^N$ in

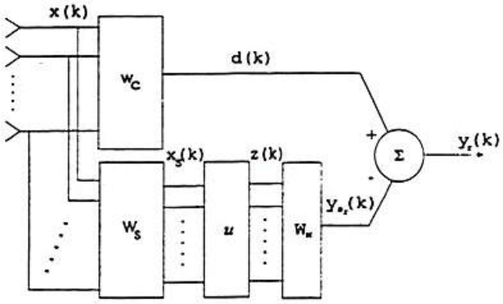

Fig. 3. Reduced-rank GSC-MVDR processor.

which the adaptive processor operates. However, this technique does not directly consider the MMSE performance measure, which is a function of not only the space spanned by the noise covariance matrix but also of the cross-correlation between the desired signal and the noise process. It is noted that Byerly [7] discovered that the eigenvectors corresponding to the largest eigenvalues were not necessarily the best selection, but there was no derivation provided and the approach obtained herein provides a more general solution.

Following Scharf [8, ch. 8], we examine the problem of reducing the rank of the Wiener filter. In Fig. 3, the rank reducing $N \times M$ transformation matrix $\mathcal{U}$ is composed of some M columns from $\mathbf{U}$ which are to be selected. The operator $\mathcal{U}$ is constrained therefore to be a subset of M of the N possible eigenvectors of $\mathbf{R}_{x_s}$. This particular constraint allows a direct comparison with the principal component technique, which chooses the rank reducing transform to be composed of those M eigenvectors corresponding to the largest M eigenvalues. Thus, the particular problem at hand is to choose the subspace spanned by a set of M eigenvectors out of the N available such that the resulting M-dimensional Wiener filter yields the lowest MMSE out of all $\binom{N}{M}$ possible combinations of eigenvectors.

Now denote the reduced-rank processor output by $y_r(k)$ and the reduced-rank noise estimator by $y_{s_r}(k)$. This is illustrated in Fig. 3. A study of Figs. 2 and 3 suggest that the reduced-rank processor output can be expressed as

$$y_r(k) = [1 \quad \mathbf{w}_N^H\mathbf{U}^H - \mathbf{w}_M^H\mathcal{U}^H]\begin{bmatrix} y(k) \\ \mathbf{x}_s(k)\end{bmatrix}. \tag{13}$$

Denote the weight error vector between the full-rank weight vector and its reduced-rank version by

$$\mathbf{e} = \mathbf{U}\mathbf{w}_N - \mathcal{U}\mathbf{w}_M. \tag{14}$$

The mean-square value of the reduced-rank processor output $y_r(k)$ in this notation is computed to be

$$\mathbf{E}[|y_r(k)|^2] = P + \mathbf{e}^H\mathbf{R}_{x_s}\mathbf{e} \tag{15}$$

where P is the full-rank MMSE and $P + \mathbf{e}^H\mathbf{R}_{x_s}\mathbf{e}$ is the reduced-rank MMSE. It is now desired to choose the rank reducing operator $\mathcal{U}$ in such a manner that $\mathbf{w}_M$ minimizes the additional mean-square error incurred by rank reduction, namely the scalar term $\mathbf{e}^H\mathbf{R}_{x_s}\mathbf{e}$ in (15). Define the $N \times M$ index matrix $\mathbf{J}$ such that its N rows are composed of M orthonormal unit vectors and N-M null vectors in the order corresponding to the selection of the M columns of $\mathbf{U}$ retained to form the rank reducing operator $\mathcal{U}$. Then the additional mean-square error incurred by rank reduction is minimized as follows:

$$\min[\mathbf{e}^H\mathbf{R}_{x_s}\mathbf{e}] = \min[(\mathbf{w}_N^H\mathbf{R}_p^{1/2} - \mathbf{w}_M^H\mathbf{R}_z^{1/2}\mathbf{J}^H) \times (\mathbf{R}_p^{1/2}\mathbf{w}_N - \mathbf{J}\mathbf{R}_z^{1/2}\mathbf{w}_M)]. \tag{16}$$

Then evidently the best solution for (16) is to choose $\mathcal{U}$ such that $\mathbf{J}\mathbf{R}_z^{1/2}\mathbf{w}_M$ is the best low rank approximation to the vector $\mathbf{R}_p^{1/2}\mathbf{w}_N$.

The Wiener–Hopf relationship for the full-rank case,

$$\mathbf{R}_p\mathbf{w}_N = \mathbf{r}_{pd} \tag{17}$$

implies that

$$\mathbf{R}_p^{1/2}\mathbf{w}_N = \mathbf{R}_p^{-1/2}\mathbf{r}_{pd} = \Lambda^{-1/2}\mathbf{U}^H\mathbf{r}_{x_sd} = \begin{bmatrix} \dfrac{\nu_1^H\mathbf{r}_{x_sd}}{\sqrt{\lambda_1}} \\ \dfrac{\nu_2^H\mathbf{r}_{x_sd}}{\sqrt{\lambda_2}} \\ \vdots \\ \dfrac{\nu_N^H\mathbf{r}_{x_sd}}{\sqrt{\lambda_N}} \end{bmatrix}. \tag{18}$$

Thus, in order to make the vector $\mathbf{J}\mathbf{R}_z^{1/2}\mathbf{w}_M$ be the best low rank approximation to the vector $\mathbf{R}_p^{1/2}\mathbf{w}_N$, it is necessary to rank order the terms in (18) by their magnitude. With this ranking of the eigenvectors of $\mathbf{U}$, the index matrix $\mathbf{J}$ takes the form

$$\mathbf{J} = \begin{bmatrix}\mathbf{I} \\ \mathbf{0}\end{bmatrix}$$

where $\mathbf{I}$ is the $M \times M$ identity matrix and $\mathbf{0}$ is the $(N-M) \times M$ null matrix. Then the rank reducing operator $\mathcal{U}$ is selected by choosing those M eigenvectors which correspond with the largest M values of the sequence of nonnegative terms

$$\left|\frac{\nu_i^H\mathbf{r}_{x_sd}}{\sqrt{\lambda_i}}\right|^2 \tag{19}$$

for $i = 1,2,\ldots N$. With this selection, the columns of the reduced-rank covariance matrix $\mathbf{R}_z$ span the M-dimensional cross-spectral subspace $\Omega \subset C^N$ to provide the lowest MMSE of any M-dimensional subspace which is spanned by M of the N columns of $\mathbf{U}$. It is noted that this solution is similar to the singular value decomposition (SVD) technique described in [8, ch. 8.4]. Also it is of interest physically that the term in (19) measures the cross-spectral energy projected along the ith eigenvector.

Hence the reduced-rank Wiener filter in the subspace Ω is given by

$$\mathbf{w}_M = (\mathcal{U}^H \mathbf{R}_{x_s} \mathcal{U})^{-1} \mathcal{U}^H \mathbf{r}_{x_s d} = \Lambda_M^{-1} \mathcal{U}^H \mathbf{r}_{x_s d} \qquad (20)$$

where Λ_M is the diagonal matrix composed of the M eigenvalues corresponding to the eigenvectors which form $\mathcal{U}$. Clearly, the subspace Ω spanned by the eigenvectors corresponding with the M largest values of the cross-spectral metric is not the same as the subspace Ψ which is spanned by the eigenvectors corresponding with the M largest eigenvalues. This means that the Wiener filter in the cross-spectral subspace Ω yields a lower MMSE than the Wiener filter in the subspace Ψ.

To demonstrate that the cross-spectral metric is optimal for each rank $M \leq N$, consider the decomposition of the MMSE performed by the full-rank matrix of eigenvectors $\mathbf{U}$. The full-rank MMSE is given by

$$P = \sigma_d^2 - \mathbf{r}_{pd}^H \mathbf{R}_p^{-1} \mathbf{r}_{pd} = \sigma_d^2 - \mathbf{r}_{x_s d}^H \mathbf{U} \Lambda^{-1} \mathbf{U}^H \mathbf{r}_{x_s d}$$

$$= \sigma_d^2 - \sum_{i=1}^{N} \frac{|\nu_i^H \mathbf{r}_{x_s d}|^2}{\lambda_i}. \qquad (21)$$

Finally, a comparison of (19) and (21) demonstrate that the selection of the subspace which provides the largest cross-spectral contribution also results in the lowest MMSE as a function of the rank of the Wiener filter.

IV. EXAMPLE

We now examine the performance of the fully adaptive GSC, the partially adaptive eigen-subspace GSC, and the partially adaptive cross-spectral subspace GSC processors. For the purpose of this analysis, five narrowband interference signals and one narrowband desired signal impinge a linear array consisting of 16 elements with half-wavelength spacing. The linear constraint imposed is a 0 dB gain at broadside. The dimension of the fully adaptive processor is $N = 15$. To evaluate the performance of the subspace selection techniques, the dimension of the adaptive processor is reduced to $M = 2$. The dimension of the noise subspace eigenstructure is 5, which is supposedly the lower bound for rank reduction [4, 5].

The signal environment consists of a desired signal (D) which arrives broadside (0°) with an input signal-to-noise ratio (SNR_{in}) of 0 dB and five jammers. Jammer 1 ($J1$) has a direction of arrival (DOA) of $-61°$ and $\text{SNR}_{\text{in}} = 40$ dB, the second jammer ($J2$) has a DOA of $-30°$ and $\text{SNR}_{\text{in}} =$ 43.5 dB, the third ($J3$) has a DOA of $-10°$ with $\text{SNR}_{\text{in}} = 34$ dB, the fourth ($J4$) has a DOA of 10° with $\text{SNR}_{\text{in}} = 38$ dB, and the fifth jammer ($J5$) has a DOA of 22° and $\text{SNR}_{\text{in}} = 40$ dB.

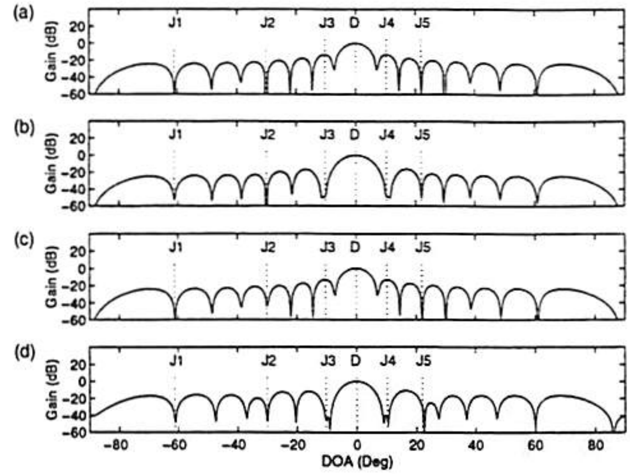

Fig. 4. Array power gain patterns as function of direction of arrival are shown for quiescent response and three 16-sensor GSC realizations using the optimal Wiener–Hopf weight vector for each. (a) Conventional beamformer pattern. (b) Fully adaptive 15-weight GSC. (c) Partially adaptive 2-weight eigen-subspace GSC. (d) Partially adaptive 2-weight cross-spectral subspace CSC.

It is important to note that the jammers $J1$, $J2$, and $J5$ all fall within nulls of the conventional beamformer pattern, as shown in Fig. 4(a). These constitute the most powerful sources of interference impinging the array. It is expected therefore that the eigen-subspace processor will assign its two degrees of freedom to cancel jammers which would normally be attenuated anyway. It is seen soon that this is the case, and that the cross-spectral metric corrects this deficiency. It is emphasized that in this example, there are only two eigenbeams required to cancel the interference. The only difference between the eigen-subspace and the cross-spectral subspace is the choice of the two eigenvectors out of the set of 15 available.

The fully adaptive GSC with fifteen weights provides an MMSE of -11.1 dB. The corresponding array pattern with the optimal 15-dimensional weight vector is provided in Fig. 4(b). It can be seen that all jammers are nulled and that the desired signal receives the expected 0 dB gain.

The eigen-subspace GSC provides an MMSE of -2.8 dB, representing a loss of 8.3 dB in performance due to a reduction in the degrees of freedom from fifteen to two adaptive weights. The corresponding array pattern with the optimal 2-dimensional weight vector in the eigen-subspace is depicted in Fig. 4(c). It can be seen that the weaker jammers, $J3$ and $J4$, are not attenuated. The selection of the eigen-subspace, whose basis vectors are composed of the largest principal axes of $\mathcal{C}^N$, yields a subspace in which the weight vector cannot affect the two jammers which most greatly determine the array performance.

The poor performance of the Wiener filter in the subspace based on the largest eigenvectors is now contrasted to the performance obtained in the cross-spectral subspace. The cross-spectral subspace GSC provides an MMSE of -10.75 dB, indicating that the performance loss incurred in reducing the degrees of freedom from fifteen weights to two is only 0.35 dB. The array pattern constructed with the 2-dimensional cross-spectral subspace optimal weight vector is shown in Fig. 4(d). There is no loss in sidelobe performance, and all jammers are attenuated.

V. CONCLUSIONS

A cross-spectral metric for subspace selection in partially adaptive array processing is derived and a proof is provided to show that this metric is the optimal performance measure to use in deciding which eigenvectors to keep for rank reduction. Also it is demonstrated that an MVDR array, operating in the subspace selected by the cross-spectral subspace estimator, exceeds the performance realized by operation in the subspace which is based upon the principal components method. The cross-spectral subspace provides better performance than the eigen-subspace when the conventional beamformer pattern provides any attenuation of the interference. A narrowband example and analysis are provided on the assumption of exact signal knowledge, although estimated statistics also may be utilized in conjunction with an adaptive algorithm.

REFERENCES

[1] Frost, O. L. (1972)
An algorithm for linearly constrained adaptive array processing.
Proceedings of the IEEE, **60**, 8 (Aug. 1972), 926–935.

[2] Griffiths, L. J., and Jim, C. W. (1982)
An alternative approach to linearly constrained adaptive beamforming.
IEEE Transactions on Antennas and Propagation, **AP-30**, 1 (Jan. 1982), 27–34.

[3] Applebaum, S. P., and Chapman, D. J. (1976)
Adaptive arrays with main beam constraints.
IEEE Transactions on Antennas and Propagation, **AP-24**, 5 (Sept. 1976), 650–662.

[4] Gabriel, W. F. (1986)
Using spectral estimation techniques in adaptive processing antenna systems.
IEEE Transactions on Antennas and Propagation, **AP-34**, 3 (Mar. 1986), 291–300.

[5] Van Veen, B. D. (1988)
Eigenstructure based partially adaptive array design.
IEEE Transactions on Antennas and Propagation, **36**, 3 (Mar. 1988), 357–362.

[6] Carhoun, D. O., Games, R. A., and Williams, R. T. (1990)
A principal components sidelobe cancellation algorithm.
Technical report M90-82, MITRE, Dept. D082, Nov. 1990.

[7] Byerly, K. A., and Roberts, R. A. (1989)
Output power based partially adaptive array design.
In *Proceedings of 23rd Asilomar Conference on Signals, Systems and Computers*, Pacific Grove, CA, Nov. 1989, 576–580.

[8] Scharf, L. L. (1991)
Statistical Signal Processing.
Reading, MA: Addison-Wesley, 1991.